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# Some examples due to H. Hironaka.

Daniel Barlet.\*

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## Introduction

The aim of this paper is to give some comments on the construction by H. Hironaka [H.61] of a holomorphic (in fact "algebraic") family of compact complex manifolds parametrized by  $\mathbb{C}$  such for all  $u \in \mathbb{C} \setminus \{0\}$  the fiber is projective, but such that the fiber at the origin is non kählerian. We also explain why it is not possible to make in the same way such a family with fiber at 0 a simpler example of non kählerian Moishezon manifold which is also due to H. Hironaka (see section 4).

This paper does not give a complete proof of Hironaka's construction. It only tries to give some help for the reader of this famous article and tries to explain some points which are not "explicit" although they are well known to specialists.

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# 1 Some ideas about the construction.

The initial point of H. Hironaka's construction is to consider in the 3-dimensional complex projective space  $\mathbb{P}_3$  the three curves given in the homogeneous coordinates  $Y_0, Y_1, Y_2, Y_3$  by the following equations (see picture 4)

$$\begin{aligned} F_1 &:= \{Y_0 = Y_1 = 0\} \quad (= C_1 \text{ on pictures}) \\ F_2 &:= \{Y_0.Y_1 + Y_1.Y_2 + Y_2.Y_0 = Y_3 = 0\} \quad (= C_2) \\ F_3(u) &:= \{(Y_1 + Y_3).(Y_0 + u.Y_1) + Y_1.Y_3 = Y_2 = 0\} \quad (F_3(0) = C_3) \end{aligned}$$

where  $u \in \mathbb{C}$  is a parameter.

Note that for  $u = 0$  these three smooth curves (a line and two conics) has two common points  $P := (1, 0, 0, 0)$  and  $Q := (0, 1, 0, 0)$ . For  $u \neq 0$  the conic  $F_3(u)$  still contains  $P$  but not  $Q$ . In a first step we shall consider only the case  $u = 0$ .

The first part of the proof is to blow-up in a convenient way the affine open sets  $U_0 := \{Y_0 \neq 0\}$  and  $U_1 := \{Y_1 \neq 0\}$  and to patch the resulting (smooth) manifolds in order to produce a compact complex manifold  $V_0$  which is not kählerian. So it is not projective but still a modification of  $\mathbb{P}_3$ . In a second step, the aim is to do such a construction with the parameter  $u \in \mathbb{C}$  in order to produce an holomorphic family of compact complex manifolds over  $\mathbb{C}$  such that the fibers over  $u \neq 0$  are projective but the fiber over 0 (which is the previously constructed  $V_0$ ) is not kähler.

The proof will go as follow :

First construct the fiber at  $u = 0$  of this family. It is smooth and obtained by blowing up in  $U_0$  the ideal of the lemma 2.0.3, by blowing up in  $U_1$  the product of the reduced ideals of  $F_1, F_2$  and  $F_3(0)$ . And then showing that these smooth complex manifolds patch over the intersection  $U_0 \cap U_1$  in a smooth connected complex compact manifold  $V_0$  which is not kählerian. This last fact is proved by showing that there exists a curve in  $V_0$  which is homologous to zero (see the end of section 3).

The second step is performed with the same construction adding the parameter  $u \in \mathbb{C}$ . This second step is rather analogous to the first one : because on  $\mathbb{C} \times U_0$  we simply replace the inhomogeneous coordinates  $x_1, x_2, x_3$  on  $U_0$  by

$$\tilde{x}_1 := (1 + u.x_1)(x_1 + x_2 + x_3 + x_1.x_2) + x_1.x_3, \quad \tilde{x}_2 := x_2, \quad \tilde{x}_3 := x_3$$

and follows then the same line than in the absolute case. The projectivity of the fibers over  $u \neq 0$  is deduced from the fact that the corresponding manifold built after patching can be obtained directly by blowing up a coherent ideal in  $\{u\} \times \mathbb{P}_3$ . A delicate point is to verify that the fiber at  $u = 0$  of the family constructed is precisely the manifold  $V_0$  constructed in the absolute case. We explain in section 4 that this point is crucial and not obvious. For that purpose we give an example where the construction "in family" do not give the expected fiber at  $u = 0$ .

## 2 Some properties of blowing up coherent ideals.

In the present text, a complex space is always assumed to be reduced and a modification of a complex space is always assumed to be proper.

When  $f : Y \rightarrow X$  is a morphism of complex spaces and  $\mathcal{I}$  a sheaf of ideals in  $\mathcal{O}_X$ , the pull-back by  $f$  of the ideal sheaf  $\mathcal{I}$ , denoted by  $f_{st}^*(\mathcal{I})$ , is the image in  $\mathcal{O}_Y$  of the “usual” pull-back sheaf  $f^*(\mathcal{I}) := f^{-1}(\mathcal{I}) \otimes_{f^{-1}(\mathcal{O}_X)} \mathcal{O}_Y$ . If  $\mathcal{I}$  is locally generated on an open set  $U$  by holomorphic functions  $g_1, \dots, g_k$  then the ideal  $f_{st}^*(\mathcal{I})$  is generated on  $f^{-1}(U)$  by the holomorphic functions  $g_1 \circ f, \dots, g_k \circ f$ .

To begin, recall that if  $X$  is a complex space and  $\mathcal{I}$  a coherent ideal sheaf in  $\mathcal{O}_X$  the blow-up of  $\mathcal{I}$  is, by definition, the (proper) modification  $\tau : \tilde{X} \rightarrow X$  such that the ideal  $\tau_{st}^*(\mathcal{I})$  is locally principal and which satisfies the following universal property:

- For any (proper) modification  $\theta : Y \rightarrow X$  such that the ideal  $\theta_{st}^*(\mathcal{I})$  is a locally principal ideal sheaf, there exists a unique holomorphic map  $\sigma : Y \rightarrow \tilde{X}$  such that  $\tau \circ \sigma = \theta$ .

For any coherent ideal  $\mathcal{I}$  in  $\mathcal{O}_X$  the blow-up exists and is unique. It can be constructed as follows :

Assume that  $\mathcal{I}$  is locally generated on an open set  $U \subset X$  by holomorphic functions  $g_1, \dots, g_k$  ; then, if  $Z$  is the analytic subset of  $U$  defined by  $\{g_1 = \dots = g_k = 0\}$ , we have a holomorphic map  $G : U \setminus Z \rightarrow \mathbb{P}_{k-1}$  defined in homogeneous coordinates by  $x \mapsto (g_1(x), \dots, g_k(x))$ . Then the closure of the graph  $\Gamma_0$  of  $G$  in  $U \times \mathbb{P}_{k-1}$  is an analytic subset and its projection on  $U$  is a proper modification of  $U$ . The easiest way to prove these assertions is, assuming  $U$  irreducible, to consider the irreducible component  $\Gamma$  of the closed analytic subset

$$Y := \{(x, z) \in U \times \mathbb{P}_{k-1} / rk(x, z) = 1\}$$

which contains the graph of  $G$ . Then it is easy to see that it coincides with the closure of  $\Gamma_0$ . It is clearly proper over  $U$  and the pull-back of  $\mathcal{I} := (g_1, \dots, g_k)$  on  $\Gamma$  is locally principal : if  $z_1, \dots, z_k$  are the homogeneous coordinates in  $\mathbb{P}_{k-1}$ , let  $\Omega_i := \{z_i \neq 0\} \subset \mathbb{P}_{k-1}$ . Then on  $(U \times \Omega_i) \cap \Gamma$  we have  $g_j = \frac{z_j}{z_i} \cdot g_i$  for each  $j \in [1, k]$ . So  $\mathcal{I}$  is generated by  $g_i$  on  $(U \times \Omega_i) \cap \Gamma$ .

REMARKS.

1. For  $\mathcal{J}$  a coherent ideal sheaf consider  $\mathcal{I} = \varphi \cdot \mathcal{J}$  where  $\varphi$  is a holomorphic function on  $X$ ; then there is a canonical isomorphism of modifications of  $X$  between the blow-up of  $\mathcal{I}$  and the blow-up of  $\mathcal{J}$ .
2. If  $X$  is a complex manifold and  $\mathcal{I}$  is the (reduced) ideal sheaf of a closed complex submanifold  $V$  in  $X$ , then the blow-up  $\tilde{X}$  of  $\mathcal{I}$  is a complex manifold

and the map  $\tau : \tau^{-1}(V) \rightarrow V$  is the projection of the projectivized normal bundle of  $V$  in  $X$ .

3. The modification  $\tau : \tilde{X} \rightarrow X$  associated to the blow-up of a coherent ideal sheaf is always a projective morphism. Then if  $X$  is projective (resp. kählerian) so is  $\tilde{X}$ .

**Proposition 2.0.1** *Let  $X$  be a complex manifold and let  $\mathcal{I}$  and  $\mathcal{J}$  be two coherent ideals in  $\mathcal{O}_X$ . Then, assuming that the blow-up of the ideal  $\mathcal{I}\mathcal{J}$  is a complex manifold, it is isomorphic to the fiber product of the blow-up of  $\mathcal{I}$  and  $\mathcal{J}$ .*

PROOF. Denote respectively by  $\sigma_1 : \tilde{X}_1 \rightarrow X$  the blow-up of  $\mathcal{I}$  and by  $\sigma_2 : \tilde{X}_2 \rightarrow X$  the blow-up of  $\mathcal{J}$ . Denote by  $\theta_i : Y \rightarrow \tilde{X}_i, i = 1, 2$  the projection of the fiber product of  $\sigma_1$  and  $\sigma_2$ . As both sheaves  $\pi_{st}^*(\mathcal{I})$  and  $\pi_{st}^*(\mathcal{J})$  are locally principal on  $Y$ , where  $\pi : Y \rightarrow X$  is given by  $\pi := \sigma_1 \circ \theta_1 = \sigma_2 \circ \theta_2$ , the pull-back ideal  $\pi_{st}^*(\mathcal{I}\mathcal{J})$  is locally principal also. Then the universal property of the blow-up  $\eta : T \rightarrow X$  of the sheaf  $\mathcal{I}\mathcal{J}$  gives a holomorphic map  $f : Y \rightarrow T$  such that  $\pi = f \circ \eta$ .

Now on  $T$  the ideal  $\eta_{st}^*(\mathcal{I}\mathcal{J}) = \eta_{st}^*(\mathcal{I})\eta_{st}^*(\mathcal{J})$  is locally principal. The lemma 2.0.2 below implies that  $\eta_{st}^*(\mathcal{I})$  and  $\eta_{st}^*(\mathcal{J})$  are locally principal on  $T$ , so we obtain two holomorphic maps  $i : T \rightarrow \tilde{X}_1$  and  $j : T \rightarrow \tilde{X}_2$  such that  $\eta = \theta_1 \circ i = \theta_2 \circ j$ . The universal property of the fiber product shows that the map  $(i, j) : T \rightarrow \tilde{X}_1 \times \tilde{X}_2$  factors through  $Y$  giving the holomorphic inverse to  $f$ . ■

As a consequence, under the hypothesis of the proposition, to blow-up the product  $\mathcal{I}\mathcal{J}$  of two ideals in a complex manifold  $X$  is equivalent to successively blow-up  $\mathcal{I}$  and then to blow-up  $(\tau_{\mathcal{I}})^*(\mathcal{J})$  in  $X_{\mathcal{I}}$  where  $\tau_{\mathcal{I}} : X_{\mathcal{I}} \rightarrow X$  is the blow-up of  $\mathcal{I}$ .

**Lemma 2.0.2** *Let  $X$  be a complex manifold and let  $\mathcal{I}$  and  $\mathcal{J}$  be two coherent ideals in  $\mathcal{O}_X$ . Assume that the ideal  $\mathcal{I}\mathcal{J}$  is locally principal on  $X$ . Then  $\mathcal{I}$  and  $\mathcal{J}$  are locally principal on  $X$ .*

PROOF. The problem is local, so we may assume that  $X$  is an open neighbourhood of 0 in  $\mathbb{C}^n$  and  $\mathcal{I}\mathcal{J} = (\varphi)$ . We shall prove that the germs  $\mathcal{I}_0$  and  $\mathcal{J}_0$  are principal ideals in  $\mathcal{O}_{X,0}$  by induction on the number  $q$  of irreducible factors of the germ  $\varphi_0$ . As the case  $q = 0$  is trivial, assume  $q \geq 1$  and the assertion proved for integers less or equal to  $q - 1$ . Let  $\pi$  be an irreducible factor of  $\varphi_0$ . Assume that there exists  $s \in \mathcal{I}_0$  with  $s \notin (\pi)$ . Then any  $t \in \mathcal{J}_0$  have to be in  $(\pi)$  so we conclude that either  $\mathcal{I}_0$  or  $\mathcal{J}_0$  is contained in  $(\pi)$ . Assume for instance that  $\mathcal{I} = \pi\mathcal{I}'$  near 0. Put  $\varphi = \pi\psi$  near 0. Then we have  $\mathcal{I}'\mathcal{J} = (\psi)$  and by the induction hypothesis we conclude that  $\mathcal{I}'$  and  $\mathcal{J}$  are locally principal near 0. So is  $\mathcal{I}$ . ■

Of course we have the same result than in proposition 2.0.1 assuming that the blow-up of the product ideals is smooth, for a finite product of coherent ideals.

REMARK. The hypothesis of smoothness for the blow-up of the product of ideals may be weakened to the factoriality of the local rings of the corresponding complex space.

I dont know if some weaker hypothesis can also work but note that it is not true in general that on a reduced complex space  $X$  if the product of two coherent ideals is locally principal, then both ideals are locally principal as one can see on the following example :

EXAMPLE. Consider in  $\mathbb{C}^4$  with coordinates  $(x, y, u, v)$  the ideal generated by

$$x.u - y.v, \quad x.v - y.u, \quad y.u - y.v.$$

It define a surface  $S$  which is the union of three 2–planes :

$$P_1 := \{u = v = 0\}, \quad P_2 := \{x = y = 0\}, \quad P_3 := \{x = y \text{ and } u = v\}.$$

The intersection  $P_1 \cap P_2$  is reduced to  $\{0\}$  but  $P_3$  meets  $P_1$  and  $P_2$  respectively in the lines  $\{u = v = x - y = 0\}$  and  $\{u - v = x = y = 0\}$ . Moreover, it is easy to verify that the given ideal is reduced, so  $S$  is a reduced complex surface (but not irreducible). It is immediate to see that the ideals  $\mathcal{I} := (x, y)$  and  $\mathcal{J} := (u, v)$  are not principal in the ring  $\mathcal{O}_{S,0}$  but that their product is principal and generated by the element  $\varphi := x.u = x.v = y.u = y.v$  in the ring  $\mathcal{O}_{S,0}$ .

Now we come back to the example of [H.61]. An important point is to blow-up ideals which are product of (simple) ideals in order to use the proposition 2.0.1. The ideal which appears in the next lemma is precisely the ideal which is used in the first chart for the construction of the manifold  $V_0$ .

**Lemma 2.0.3** [see [H.61]] *Let  $(x_1, x_2, x_3)$  the coordinates in  $\mathbb{C}^3$ . then we have the equality of ideals in  $\mathcal{O}_{\mathbb{C}^3}$  :*

$$(x_2, x_3)^5 \cap (x_1, x_3)^4 \cap (x_1, x_2)^4 \cap (x_1, x_2, x_3)^7 = \\ (x_1.x_2, x_2.x_3, x_1.x_3).(x_1, x_2.x_3).(x_2, x_1.x_3)^2.(x_3, x_1.x_2)^2.$$

PROOF. It is enough to compare monomials in these ideals. A necessary and sufficient condition for  $x_1^a.x_2^b.x_3^c$  to be in the left-handside is given by the following inequalities :

$$a + b + c \geq 7, \quad a + b \geq 4, \quad a + c \geq 4, \quad \text{and} \quad b + c \geq 5. \quad (@)$$

So listing the cases for  $a = 0$  to  $a = 4$  gives the following generators for the left-handside :

$$m_0 := x_2^4.x_3^4, \quad m_1 := x_1.x_2^3.x_3^3, \quad m_2 := x_1^2.x_2^3.x_3^2, \quad m'_2 := x_1^2.x_2^2.x_3^3, \\ m_3 = x_1^3.x_2^4.x_3, \quad m'_3 := x_1^3.x_2.x_3^4, \quad m_4 := x_1^4.x_2^5, \quad m'_4 := x_1^4.x_3^5.$$

Now it is rather easy to verify (and this will be indicate in the verification of the opposite inclusion) that each of these monomials is in the right-handside:

$$\begin{aligned}
m_0 &= x_2^4.x_3^4 = (x_2.x_3).(x_2.x_3).x_2^2.x_3^2 \\
m_1 &= x_1.x_2^3.x_3^3 = (x_2.x_3).x_1.x_2^2.x_3^2 \\
m_2 &= x_1^2.x_2^3.x_3^2 = (x_1.x_2).x_1.x_2^2.x_3^2 \\
m'_2 &= x_1^2.x_2^2.x_3^3 = (x_1.x_3).x_1.x_2^2.x_3^2 \\
m_3 &= x_1^3.x_2^4.x_3 = (x_1.x_2).x_1.x_2^2.(x_1.x_2.x_3) \\
m'_3 &= x_1^3.x_2.x_3^4 = (x_1.x_3).x_1.(x_1.x_2.x_3).x_3^2 \\
m_4 &= x_1^4.x_2^5 = (x_1.x_2).x_1.x_2^2.x_3^2 \\
m'_4 &= x_1^4.x_3^5 = (x_1.x_3).x_1.(x_1.x_3)^2.x_3^2
\end{aligned}$$

For  $a \geq 5$ , as we have  $b + c \geq 5$ , it is easy to check that we obtain multiples of the eight monomials above : if  $b$  or  $c$  is 0, a multiple of  $m_4$  or  $m'_4$  respectively, if  $b$  or  $c$  is equal to 1 a multiple of  $m_3$  or  $m'_3$  respectively and if  $b$  or  $c$  is at least 2 a multiple of  $m_2$  or  $m'_2$ .

It is a little more painful to verify the opposite inclusion because there are *a priori* 54 monomials in the generator of the right-handside.

We shall use the symbol (@) to indicate when we find one of the eight monomials above in the list of these 54 monomials.

The maximal degree in  $x_1$  for such a monomial is 6. In degree 6 there are only two:

$$(x_1.x_2).x_1.(x_1.x_3)^2.(x_1.x_3)^2 = x_1^6.x_2^3.x_3^2$$

and the one obtain by exchanging  $x_2$  and  $x_3$  (we note this by (ex-2-3)). They are multiples of  $m_2$  and  $m'_2$  respectively.

In degree 5 in  $x_1$  we have

$$\begin{aligned}
(x_2.x_3).x_1.(x_1.x_3)^2.(x_1.x_2)^2 &= x_1^5.x_2^3.x_3^3 \in (m_1) \\
(x_1.x_2).(x_2.x_3).(x_1.x_3)^2.(x_1.x_2)^2 &= x_1^5.x_2^4.x_3^3 \text{ (ex - 2 - 3)} \in (m_1) \\
(x_1.x_2).x_1.(x_1.x_2.x_3).(x_1.x_2)^2 &= x_1^5.x_2^4.x_3 \text{ (ex - 2 - 3)} \in (m_3) \text{ and } (m'_3) \\
(x_1.x_2).x_1.(x_1.x_3)^2.(x_1.x_2.x_3) &= x_1^5.x_2^2.x_3^3 \text{ (ex - 2 - 3)} \in (m'_2) \text{ and } (m_2)
\end{aligned}$$

Now consider the monomial of degree 4 in  $x_1$ . First consider these where we have two  $x_1^2$  coming from one of the four terms.

$$(x_2.x_3).(x_2.x_3).(x_1.x_3)^2.(x_1.x_2)^2 = x_1^4.x_2^4.x_3^4 \in (m_1)$$

Now if we have only one  $x_1^2$  coming for one four terms

$$\begin{aligned}
(x_2.x_3).x_1.(x_1.x_2.x_3).(x_1.x_2)^2 &= x_1^4.x_2^4.x_3^2 \in (m_2) \\
(x_1.x_2).(x_2.x_3).(x_1.x_2.x_3).(x_1.x_2)^2 &= x_1^4.x_2^5.x_3^2 \quad (\text{ex} - 2 - 3) \in (m_2) \quad \text{and} \quad (m'_2) \\
(x_1.x_2).x_1.x_2^2.(x_1.x_2)^2 &= x_1^4.x_2^5 \quad (\text{ex} - 2 - 3) \in (m_4) \quad \text{and} \quad (m'_4) \quad (\textcircled{a}) + (\textcircled{a}) \\
(x_1.x_2).x_1.(x_1.x_3)^2.x_3^2 &= x_1^4.x_2.x_3^4 \quad (\text{ex} - 2 - 3) \in (m'_3) \quad \text{and} \quad (m_3) \\
(x_2.x_3).x_1.(x_1.x_3)^2.(x_1.x_2.x_3) &= x_1^4.x_2^2.x_3^4 \in (m'_2) \\
(x_1.x_2).(x_2.x_3).(x_1.x_3)^2.(x_1.x_2.x_3) &= x_1^4.x_2^3.x_3^4 \quad (\text{ex} - 2 - 3) \in (m_1) \quad \text{and} \quad (m_1)
\end{aligned}$$

If we choose one  $x_1$  in each term we find only

$$(x_1.x_2).x_1.(x_1.x_2.x_3).(x_1.x_2.x_3) = x_1^4.x_2^3.x_3^2 \quad (\text{ex} - 2 - 3) \in (m_2) \quad \text{and} \quad (m'_2)$$

In degree 3 in  $x_1$  if we take  $x_1^2$  in one term we will get

$$\begin{aligned}
(x_1.x_2).(x_2.x_3).x_2^2.(x_1.x_2)^2 &= x_1^3.x_2^6.x_3 \quad (\text{ex} - 2 - 3) \in (m_3) \quad \text{and} \quad (m'_3) \\
(x_2.x_3).x_1.x_2^2.(x_1.x_2)^2 &= x_1^3.x_2^5.x_3 \in (m_3) \\
(x_2.x_3).(x_2.x_3).(x_1.x_2.x_3).(x_1.x_2)^2 &= x_1^3.x_2^5.x_3^3 \in (m_1) \\
(x_1.x_2).(x_2.x_3).(x_1.x_3)^2.x_3^2 &= x_1^3.x_2^2.x_3^5 \quad (\text{ex} - 2 - 3) \in (m'_2) \quad \text{and} \quad (m_2) \\
(x_2.x_3).x_1.(x_1.x_3)^2.x_3^2 &= x_1^3.x_2.x_3^5 \in (m'_3) \\
(x_2.x_3).(x_2.x_3).(x_1.x_3)^2.(x_1.x_2.x_3) &= x_1^3.x_2^3.x_3^5 \in (m_1)
\end{aligned}$$

For the degree 3 in  $x_1$ , if we take at most one  $x_1$  in each term, we have four places where to avoid  $x_1$  :

$$\begin{aligned}
(x_2.x_3).x_1.(x_1.x_2.x_3).(x_1.x_2.x_3) &= x_1^3.x_2^3.x_3^3 \in (m_1) \\
(x_1.x_2).(x_2.x_3).(x_1.x_2.x_3).(x_1.x_2.x_3) &= x_1^3.x_2^4.x_3^3 \quad (\text{ex} - 2 - 3) \in (m_1) \\
(x_1.x_2).x_1.x_2^2.(x_1.x_2.x_3) &= x_1^3.x_2^4.x_3 \quad (\text{ex} - 2 - 3) \in (m_3) \quad \text{and} \quad (m'_3) \quad (\textcircled{a}) + (\textcircled{a}) \\
(x_1.x_2).x_1.(x_1.x_2.x_3).x_3^2 &= x_1^3.x_2^2.x_3^3 \quad (\text{ex} - 2 - 3) \in (m'_2) \quad \text{and} \quad (m_2)
\end{aligned}$$

In degree 2 there are 8 choices, modulo exchanging  $x_2$  and  $x_3$ :

$$\begin{aligned}
(x_1.x_2).x_1.x_2^2.x_3^2 &= x_1^2.x_2^3.x_3^2 \quad (\text{ex} - 2 - 3) \in (m_2) \quad \text{and} \quad (m'_2) \quad (\textcircled{a}) + (\textcircled{a}) \\
(x_1.x_2).(x_2.x_3).(x_1.x_2.x_3).x_3^2 &= x_1^2.x_2^3.x_3^4 \quad (\text{ex} - 2 - 3) \in (m_1) \\
(x_1.x_2).(x_2.x_3).x_2^2.(x_1.x_2.x_3) &= x_1^2.x_2^5.x_3^2 \quad (\text{ex} - 2 - 3) \in (m_2) \quad \text{and} \quad (m'_2) \\
(x_2.x_3).x_1.(x_1.x_2.x_3).x_3^2 &= x_1^2.x_2^2.x_3^4 \in (m'_2) \\
(x_2.x_3).x_1.x_2^2.(x_1.x_2.x_3) &= x_1^2.x_2^4.x_3^2 \in (m_2) \\
(x_2.x_3).(x_2.x_3).(x_1.x_2.x_3).(x_1.x_2.x_3) &= x_1^2.x_2^4.x_3^4 \in (m_1) \\
(x_2.x_3).(x_2.x_3).(x_1.x_3)^2.x_3^2 &= x_1^2.x_2^2.x_3^6 \in (m'_2) \\
(x_2.x_3).(x_2.x_3).x_2^2.(x_1.x_2)^2 &= x_1^2.x_2^6.x_3^2 \in (m_2)
\end{aligned}$$



In degree 1 there are 4 choices, modulo exchanging  $x_2$  and  $x_3$ :

$$\begin{aligned} (x_1.x_2).(x_2.x_3).x_2^2.x_3^2 &= x_1.x_2^4.x_3^3 \quad (\text{ex} - 2 - 3) \in (m_1) \\ (x_2.x_3).x_1.x_2^2.x_3^2 &= x_1.x_2^3.x_3^3 \in (m_1) \quad (@) \\ (x_2.x_3).(x_2.x_3).(x_1.x_2.x_3).x_3^2 &= x_1.x_2^3.x_3^5 \in (m_1) \\ (x_2.x_3).(x_2.x_3).x_2^2.(x_1.x_2.x_3) &= x_1.x_2^5.x_3^3 \in (m_1) \end{aligned}$$

For the degree 0 there is only one possibility :  $x_2^4.x_3^4$  which is  $m_0$  (@). So the verification for the 54 monomial is over.  $\blacksquare$

### 3 Smoothness.

As explained in the previous section, it is important to know that the blow-up of the product of two ideals is smooth in order to compare it with the successive blow-up of these two ideals.

We shall consider the blow-up  $X$  of the ideal  $(x_1.x_2, x_2.x_3, x_3.x_1)$  in  $\mathbb{C}^3$  and then the blow up  $Y$  of the ideal  $(x_3, x_1.x_2).(x_1, x_2.x_3).(x_2, x_3.x_1)$  in  $X$ . This corresponds to the blow-up of the ideal described in the lemma 2.0.3 which gives the first piece in the construction of the manifold  $V_0$ .

In  $\mathbb{C}^3 \times \mathbb{P}_2$  we look for the irreducible component of the analytic subset

$$Z := \{((x_1, x_2, x_3), (a, b, c)) / x_1.x_2.b = x_2.x_3.a \quad x_1.x_2.c = x_3.x_1.a \quad x_2.x_3.c = x_3.x_1.b\}$$

which dominates the graph of the corresponding map. By circular permutation on  $x_1, x_2, x_3$  we can restrict our study to the chart  $a \neq 0$  in  $\mathbb{P}_2$ . Then let  $u := b/a$  and  $v := c/a$  be the corresponding coordinates. As  $x_1, x_2, x_3$  are not zero at the generic point of the graph, the equations simplify in

$$x_3 = u.x_1 \quad x_3 = v.x_2 \quad x_2.v = x_1.u$$

and this show that this Zariski open set  $X'$  in  $X$  is isomorphic to the hypersurface  $H := \{x_2.v = x_1.u\} \subset \mathbb{C}^4$  where the coordinates in  $\mathbb{C}^4$  are  $(x_1, x_2, u, v)$ . So we have an unique singular (Morse) point : the origin.

Note that the pre-image of 0 in  $X'$  is the plane defined by  $x_1 = x_2 = 0$ .

In this chart, isomorphic to the hypersurface  $H$  note

$$P_1 := \{x_2 = v = 0\}, \quad P_2 := \{x_1 = v = 0\}, \quad P_3 := \{x_1 = x_2 = 0\}.$$

We have  $P_1 \cap P_2 = \{0\}$ ,  $P_1 \cap P_3 = \{x_1 = x_2 = u = 0\}$ ,  $P_2 \cap P_3 = \{x_1 = x_2 = v = 0\}$ . The pull back of the curves  $C_1 = \{x_1 = x_2 = 0\}$  is  $P_1 \cup P_3$ , the pull-back of the curve

$C_2 = \{x_1 = x_3 = 0\}$  is  $P_2 \cup P_3$  and the pull-back of the curve  $C_3 = \{x_1 = x_2 = 0\}$  is  $P_3$ . See the picture 0.

Remark now that the ideals  $(x_1, x_2.x_3)$  and  $(x_2, x_1.x_3)$  are principal on  $X'$  as we have  $x_3 = u.x_1 = v.x_2$ . So the only blow-up to perform in  $X'$  is now the blow up of the ideal  $(x_3, x_1.x_2)$ . Using  $x_3 = u.x_1$  this is the same as the blow-up of  $(u, x_2)$ . Then it is given in  $X' \times \mathbb{P}_1$  by the equation  $u.\beta = x_2.\alpha$ . Let  $z := \beta/\alpha$  for the first chart in  $\mathbb{P}_1$ . Then we have in this chart  $X''_1$  of this blow-up

$$x_2 = u.z, \quad x_2.v = x_1.u \quad x_3 = v.x_2 = u.x_1$$

which give, as  $x_2$  is generically non zero,  $x_1 = v.z$ ,  $x_2 = u.z$ ,  $x_3 = u.v.z$  and then we have a copy of  $\mathbb{C}^3$  with coordinates  $(u, v, z)$ . The computations on the other charts are analogous as  $x_1$  and  $x_2$  have symmetric roles in  $X'$  and in the ideal blow up.

So we conclude that the blow-up of  $\mathbb{C}^3$  by the ideal

$$(x_1.x_2, x_2.x_3, x_3.x_1).(x_1, x_2.x_3).(x_2, x_3.x_1)^2.(x_3, x_1.x_2)^2$$

is a smooth quasi-projective manifold.

Remark that the pull-back of the maximal ideal of  $\mathbb{C}^3$  on  $X'$  is the ideal  $(x_1, x_2)$  in  $X'$  which is a copy of  $\mathbb{C}^2$  with coordinates  $(u, v)$ . Now, it becomes the ideal  $z.(u, v)$  in  $X''$  which is the union of a  $\mathbb{C}^2 := \{z = 0\}$  and a transversal line which has only its intersection with this plane over the origin in  $\mathbb{C}^3$ .

So our exceptionnal divisor is just this plane  $\mathbb{C}^2$ . Now, as  $u = b/a$  and  $v := c/a$  it is easy to see that the global exceptionnal divisor for the final modification of  $\mathbb{C}^3$  is a copy of  $\mathbb{P}_2$ . This divisor is called  $E_0$  in the pictures 1.

The intersection, in this chart, of this  $\mathbb{P}_2$  which is view as  $\mathbb{C}^2 = \{z = 0\}$ , with the strict transform of the curve  $\{x_1 = x_3 = 0\}$  is  $\{z = v = 0\}$ , and with the strict transform the curve  $\{x_2 = x_3 = 0\}$  is  $\{z = u = 0\}$ .

Using the six charts for the blow-up, we find that each strict transform of the curve  $\{x_3 = x_1 = 0\}$  cuts (generically transversally) the exceptionnal divisor in 3 lines in general position. This corresponds to the picture 1. The visible part in the chart  $X''$  is given by the picture 2.

The pictures 3 explains the situation near the point  $Q$  and the picture 4 the global situation in  $\mathbb{P}_3$  before the blowing-up.

WHY  $V_0$  IS NOT KÄHLERIAN. We can read on pictures 1 and 3 the following algebraic equivalence of curves in  $V_0$  :

$$\begin{aligned} L_1 &\sim L'_1 + L_2 + L_3 && \text{on picture 3} \\ L_1 &\sim L_{1,2} + L_{0,1} + L_{1,3} && \text{on picture 1} \\ L_2 &\sim L_{2,3} + L_{1,2} + L_{0,2} && \text{on picture 1} \\ L_3 &\sim L_{3,1} + L_{2,3} + L_{0,3} && \text{on picture 1} \end{aligned}$$

and this implies,

$$L_{0,1} \sim L'_1 + 2L_{2,3} + L_{0,3}.$$

Now the curve  $L_{0,1}, L_{0,2}, L_{0,3}$  are lines in  $E_2 \simeq \mathbb{P}_2$  so they are algebraically equivalent. This implies that  $L'_1 + 2L_{2,3} + L_{0,2} \sim 0$  in  $V_0$  and prove our claim.

Remark that for  $u \neq 0$  the curve  $F_3(u)$  no longer meets  $F_1$  at  $Q$  so the first relation above becomes  $L_1 \sim L'_1 + L_2$  and now the computation analog to the computation above only gives  $L'_1 + L_{2,3} \sim L_{1,3}$  which does not contradict the projectivity of the fiber at  $u$  of the family.

## 4 Simpler examples.

Let  $\mathcal{B}$  a ball in  $\mathbb{C}^3$  with center 0 and let  $C_1$  and  $C_2$  be smooth connected curves defined in  $\mathcal{B}$  and meeting transversally at 0 and nowhere else. Let  $X_1$  the blow-up of  $\mathcal{B}$  along the reduced ideal of  $C_1$  and denote by  $C'_2$  the strict transform of  $C_2$  in  $X_1$  and  $E_1$  the exceptional divisor in  $X_1$ . If  $\tau_1 : X_1 \rightarrow \mathcal{B}$  is the blow-up map, we have  $E_1 = \tau_1^{-1}(C_1)$  and it is a reduced smooth divisor isomorphic to the projectivized normal bundle to  $C_1$  in  $\mathcal{B}$ . The curve  $C'_2$  meets  $E_1$  transversally in a (unique) point  $\tilde{P}$  which belongs to the curve  $A := \tau_1^{-1}(0)$  which is a smooth rational curve (the projectif space of the fiber at 0 of the normal bundle of  $C_2$  at the origin).

Now we blow-up  $C'_2$  in  $X_1$  to obtain a modification  $\sigma_{1,2} : X_{1,2} \rightarrow X_1$ . Then the strict transform of  $C'_2$  is a smooth divisor  $E_2$  in  $X_{1,2}$  and the strict transform  $E'_1$  of  $E_1$  is, via the map  $\sigma_{1,2}$ , isomorphic to the blow-up of  $E_1$  at the point  $\tilde{P}$ . Let  $B$  be the smooth rational curve  $\sigma_{1,2}^{-1}(\tilde{P})$ . Denote by  $Z_1$  the generic fiber on  $E_1$  (or of  $E'_1$ ) over  $C_1$ . Then in  $X_{1,2}$  we have  $Z_1 \sim A' + B$ , where  $A'$  is the strict transform of  $A$  in  $E'_1$  by the blow-up of  $\tilde{P}$  in  $E_1$ . So  $A' \cup B$  is the pull back of  $A$  in  $X_{1,2}$ . Remark also that  $B$  is homologous in  $X_{1,2}$  to the generic fiber  $Z_2$  of  $E_2$  over  $C'_2$  (or  $C_2$ ).

Let  $C$  be a connected curve in  $\mathbb{P}_3$  with an ordinary double point  $P$  and which is smooth outside  $P$ . Consider now  $\mathcal{B}$  as an open neighbourhood of  $P$  in  $\mathbb{P}_3$  and assume that  $C_1$  and  $C_2$  are the two branches of the curve  $C$  near  $P$ . In the complex manifold  $V := \mathbb{P}_3 \setminus \overline{\mathcal{B}'}$ , where  $\mathcal{B}' \subset \subset \mathcal{B}$  is an open ball with a smaller radius, consider the blow-up  $\sigma : Y_\infty \rightarrow V$  of the smooth connected curve  $C \cap V$ . Then on the open set  $\mathcal{B} \setminus \overline{\mathcal{B}'}$  we have a natural identification between  $\sigma$  and  $\tau_{1,2}$ , because, outside the origin, it is the same thing to blow-up successively  $C_1$  and  $C_2$  or to blow-up  $C$ .

So we can glue these two maps to obtain a complex manifold  $Y$  and a modification  $\tau : Y \rightarrow \mathbb{P}_3$ . Now in  $Y$  we have  $Z_1 \sim A' + B$  and also  $Z_2 \sim B$ . But the generic fiber of the gluing of  $E'_1 \cup E_2$  with the blow-up of  $C$  in  $V$  implies that  $Z_1 = Z_2$ . Then the curve  $A'$  is homologous to 0 in  $Y$ . See picture 5.

If we consider a holomorphic family  $(C_s)_{s \in D}$  of curves parametrized by the unit disc in  $\mathbb{C}$  such that  $C_0 = C$  and such that for  $s \neq 0$  the curve  $C_s$  has no singular point, it seems interesting to blow-up (in a suitable way) the graph  $\Gamma$  of this family of curve

in  $D \times \mathbb{P}_3$  in order to obtain an holomorphic family of complex manifolds  $(Y_s)_{s \in D}$  such that for  $s \neq 0$  the manifold  $Y_s$  is projective, and such that  $Y_0$  is not Kähler. Of course this would give a much more simple example of such a family than the previous example of H. Hironaka from his paper [H.61] "explained" above.

Let me explain why this does not work.

The graph  $\Gamma \subset D \times \mathbb{P}_3$  is a complex submanifold excepted at the point  $\{0\} \times \{P\}$  where we have a normal crossing point for a surface in  $\mathbb{C}^4$ . In order to perform a blowing-up "in family" for the curve  $C_s$ , we want to separate the two branches of this surface near  $\{0\} \times \{P\}$ . Of course, this is possible set theoretically, but we want that the fiber at  $s = 0$  of this "family blow-up" will give us the previous construction at  $t = 0$ . This essentially means that we want that the projection  $\Gamma \rightarrow D$ , which is flat when we consider the reduced structure for  $\Gamma$ , has a **reduced** fiber at  $s = 0$ . But we shall show now that **this is not the case** in our situation. So it is not possible to separate the two irreducible branches of the fiber at 0 near 0 keeping the non reduced structure of this fiber !

**Lemma 4.0.4** *Let  $X \subset D \times M$  be an analytic subset where  $D \subset \mathbb{C}$  is the unit disc and  $M$  a complex manifold. Assume that  $X$  is reduced and flat on  $D$ . Let  $X_0$  be the fiber at 0 of the projection  $\pi : X \rightarrow D$  with its "fiber structure" meaning that  $\mathcal{I}_{X_0} := \mathcal{I}_X / s \cdot \mathcal{I}_X \subset \mathcal{O}_M$ . Assume that this fiber is contained in a smooth hypersurface  $H_0$  in  $M$  near a point  $p \in X_0$ <sup>1</sup>. Then there exists near  $(0, p)$  in  $D \times M$  a smooth hypersurface  $H$  such that for each  $s$  near enough 0 the fiber  $X_s$  is also contained in  $H$  in a fixed open set around  $(0, p)$ .*

PROOF. Let  $f_0$  be a holomorphic function in an open neighbourhood  $U$  of  $p$  such that  $(df_0)_p \neq 0$  and  $f_0$  is a section of  $\mathcal{I}_{X_0}$  on  $U$  and define  $H_0 := \{x \in U / f_0(x) = 0\}$ . Then for  $0 \in D' \subset D$  small enough there exist  $f \in \Gamma(D' \times U, \mathcal{I}_X)$  inducing  $f_0$  on  $U$ . Then we have  $df_{0,p} \neq 0$  and  $H := \{(s, x) \in (D \times M) \cap (D' \times U)\}$  is a smooth hypersurface near  $(0, p)$  which contains  $X$  so  $X_s \cap (D' \times U)$  for  $s \in D'$ . ■

APPLICATION. Assume that in the situation of the previous lemma we know that  $red(X_0)$ , the reduced fiber of  $\pi$  at 0, is contained near  $p$  in a smooth hypersurface in  $M^2$ , and that  $X$  is not contained in a smooth hypersurface in  $D \times M$  near  $(0, p)$ . Then the fiber  $X_0$  cannot be reduced.

EXAMPLE. Let  $M := \mathbb{C}^3$  and define  $X := \{x = y = 0\} \cup \{x - s = z = 0\}$  with its reduced structure in  $D \times M$ . Then  $X$  is not contained in a smooth hypersurface near  $(0, (0, 0, 0))$  because if  $\varphi$  is an holomorphic function near  $(0, (0, 0, 0))$  vanishing on  $X$  we have  $\varphi = a \cdot x + b \cdot y = c \cdot (x - s) + d \cdot z$  with holomorphic functions  $a, b, c, d$  and as  $x, y, z, s$  is a regular sequence, it implies that  $\varphi$  is in the square of the maximal ideal at the origin. This implies that the Zariski tangent space to  $X$  at the origin

<sup>1</sup>This precisely mean that there exists a holomorphic function  $f_0$  in an open neighbourhood  $U$  of  $p$  such that  $(df_0)_p \neq 0$  and  $f_0$  is a section of  $\mathcal{I}_{X_0}$  on  $U$ . Then  $H_0 := \{x \in U / f_0(x) = 0\}$

<sup>2</sup>this is equivalent to the fact that its Zariski tangent space at  $p$  has codimension 1 in  $T_{M,p}$ .

is  $\mathbb{C}^4$ . But remark that  $\text{red}(X_0)$  is contained in  $\{x = 0\}$ . So we conclude that the fiber  $X_0$  is not reduced. A direct (and easy) calculation gives that  $x$  is a non trivial nilpotent element in sheaf  $\mathcal{O}_{X_0} \simeq \mathcal{O}_{\mathbb{C}^3}/(x^2, x.y, x.z, y.z)$ .

REMARK. For any pair of smooth families of smooth curves in  $\mathbb{C}^3$  parametrized by  $D$  such that the graph  $\Gamma$  and  $\Gamma'$  are smooth surfaces in  $D \times \mathbb{C}^3$  which meets transversally in a point  $(0, p)$  the same phenomenon occurs because in  $\mathbb{C}^4$  the union of two transversal 2–planes at 0 (with its reduced structure) is not locally contained in a smooth hypersurface near  $0^3$  and the Zariski tangent space of two transversal curves in  $\mathbb{C}^3$  has always codimension 1 in  $\mathbb{C}^3$ . So the union of two transversal curves at 0 in  $\mathbb{C}^3$  is always locally contained in a smooth (hyper-)surface.

THREE CURVES EXAMPLE. Consider the analytic set

$$Y := \{x = y = 0\} \cup \{y = z = 0\} \cup \{x - s = z = 0\}$$

in  $\mathbb{C}^4$  with its reduced structure. We have  $\mathcal{I}_Y = (y.(x - s), x.z, y.z)$  and the ideal  $\mathcal{I}_{Y_0} = (x.y, x.z, y.z)$  is clearly the reduced ideal of the fiber  $Y_0$  for the projection on  $D$  corresponding to the coordinate  $s$ .

We want to prove that the map  $\pi : Y \rightarrow D$  given by the  $s$ –projection is flat and has a **reduced** fiber at  $s = 0$ . So let us prove that  $\mathcal{O}_Y$  has no  $s$ –torsion. Consider a holomorphic function  $\varphi$  near 0 in  $\mathbb{C}^4$  such that  $s.\varphi$  is in  $\mathcal{I}_Y$  near 0. Then modulo  $(z)$  we have  $s.[\varphi] \in [y.(x - s)]$ , and, as  $s, y(x - s)$  is a regular sequence in  $\mathbb{C}^3$  we obtain that  $\varphi = a.y.(x - s) + z.b$  where  $a, b$  are holomorphic. Now we have  $s.z.b \in \mathcal{I}_Y$  and this gives modulo  $(y)$  that  $s.[z.b] = c.[x.z]$ . But  $z$  does not divide 0 modulo  $(y)$  so  $s.[b] = c.[x]$  and then, as  $s, x, y$  is a regular sequence  $b = u.x + v.y$  and  $\varphi = a.y.(x - s) + u.z.x + v.z.y \in \mathcal{I}_Y$ . So  $Y$  is flat on  $D$  and its fiber at  $s = 0$  is reduced because its ideal is generated by  $x.y, x.z, y.z$ .

This enlight why H. Hironaka considers 3 transversal curves at the point  $Q$  and let one of the 3 curves moving outside this point; this allows to preserve in such a family the construction given at  $s = 0$ . Of course, this does not replace the proof of the section 4 in [H.61] but explains that with three curves it is possible, although it does not work with two.

## 5 A last example.

Choose now in  $\mathbb{P}_3$  two smooth connected curves  $C_1$  and  $C_2$  meeting transversally at two points  $P$  and  $Q$  and perform for each point the previous construction consisting in the blowing-up of one branch after the other in a small ball around each point,

---

<sup>3</sup>Two surfaces in a smooth 3–fold which meet, meet in codimension at most equal to 2.

But remark that in the case we are interested with, they are locally contained in a (singular) hypersurface like  $x.u = y.v$  where  $(x, y, u, v)$  are local coordinates near the origin.

but reversing the order : at  $P$  we blow-up first  $C_1$  and after the strict transform of  $C_2$  but near  $Q$  we blow-up first  $C_2$  and then the strict transform of  $C_1$ . Of course, outside the two balls we just blow-up  $C_1$  and  $C_2$  which are disjoint. So the all thing patch in a complex manifold  $Y$  and in a modification  $\pi : Y \rightarrow \mathbb{P}_3$ . We shall denote by  $L_1$  the generic fiber of  $E_1 := \pi^{-1}(C_1)$  on  $C_1$  and  $L_2$  the generic fiber of  $E_2 := \pi^{-1}(C_2)$  on  $C_2$ .

Then we find over  $P$  two smooth rational curves  $A$  and  $B$  and we have  $L_1 \sim A + B$  and  $L_2 \sim B$ . In a similar way, we find over  $Q$  two smooth rational curves  $C$  and  $D$  which satisfy  $L_2 \sim C + D$  and  $L_1 \sim D$ . This gives  $A + C \sim 0$  in  $Y$ . So we have again a non kähler smooth Moishezon manifold. See picture 6.

The main interest in this example is that we have two disjoint curves, such that the blow-up of one of the two gives back a projective manifold.

**Lemma 5.0.5** *If we blow-up  $A$  or  $C$  in the manifold  $Y$  defined above we obtain a projective manifold.*

PROOF. In fact, as  $A$  is the pull-back of the point  $P$  in the first blow-up, to blow-up  $Y$  along  $A$  will give the same result than to begin by the blow-up of  $P$  and then to blow-up the strict transform of  $C_1$  and then the strict transform of  $C_2$ . But after blowing-up  $P$  the strict transforms of  $C_1$  and  $C_2$  are disjoint in the pull-back of a small ball around  $P$  and the order does not matter then. So after the blow-up of  $P$  we can simply blow-up in a global manner the strict transform of  $C_2$  and then the strict transform of  $C_1$  respecting the order choosen in the ball around  $Q$ . So we have performed globally 3 blow-up of smooth projective subvarieties and the result is projective. The situation is analog for  $C$ . ■

This example shows that, in some sense, the non kählerianity of this manifold is “concentrated around  $A$ ” or “around  $C$ ” but  $A$  and  $C$  are disjoint ...

#### BIBLIOGRAPHY.

1. [H.61] Hironaka, H. *An example of a non-kählerian complex-analytic deformation of kählerian complex structures* Annals of Math. 75, 1, January, (1962) p.190-208.

## 6 Pictures

Remember that these pictures try to represent subsets in a 3–dimensional complex space. So they are necessarily “false” and can only help to understand using imagination.

Après le premier étirement carte  $\{a \neq 0\}$   
 $u = b/a \quad v = c/a$   
 $H := \{x_2 v = x_2 u\} \subset \mathbb{C}^4$

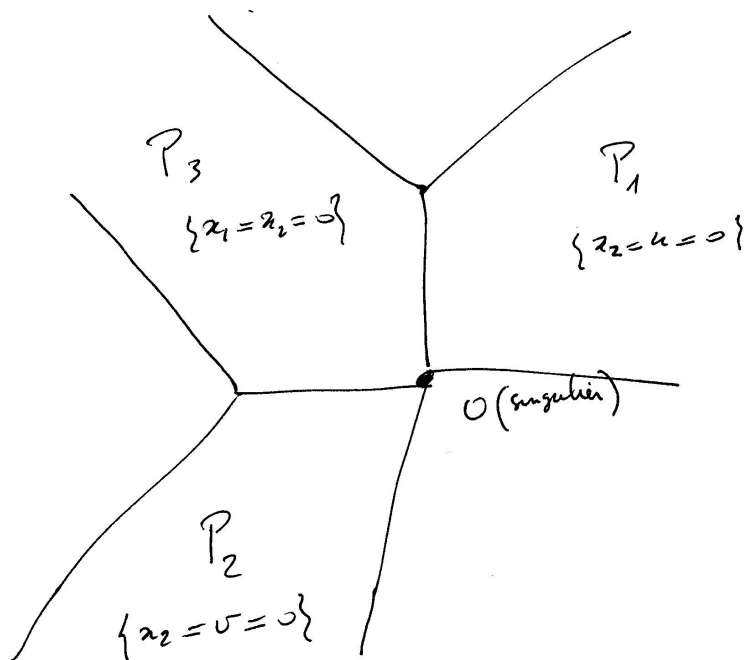


Figure 0

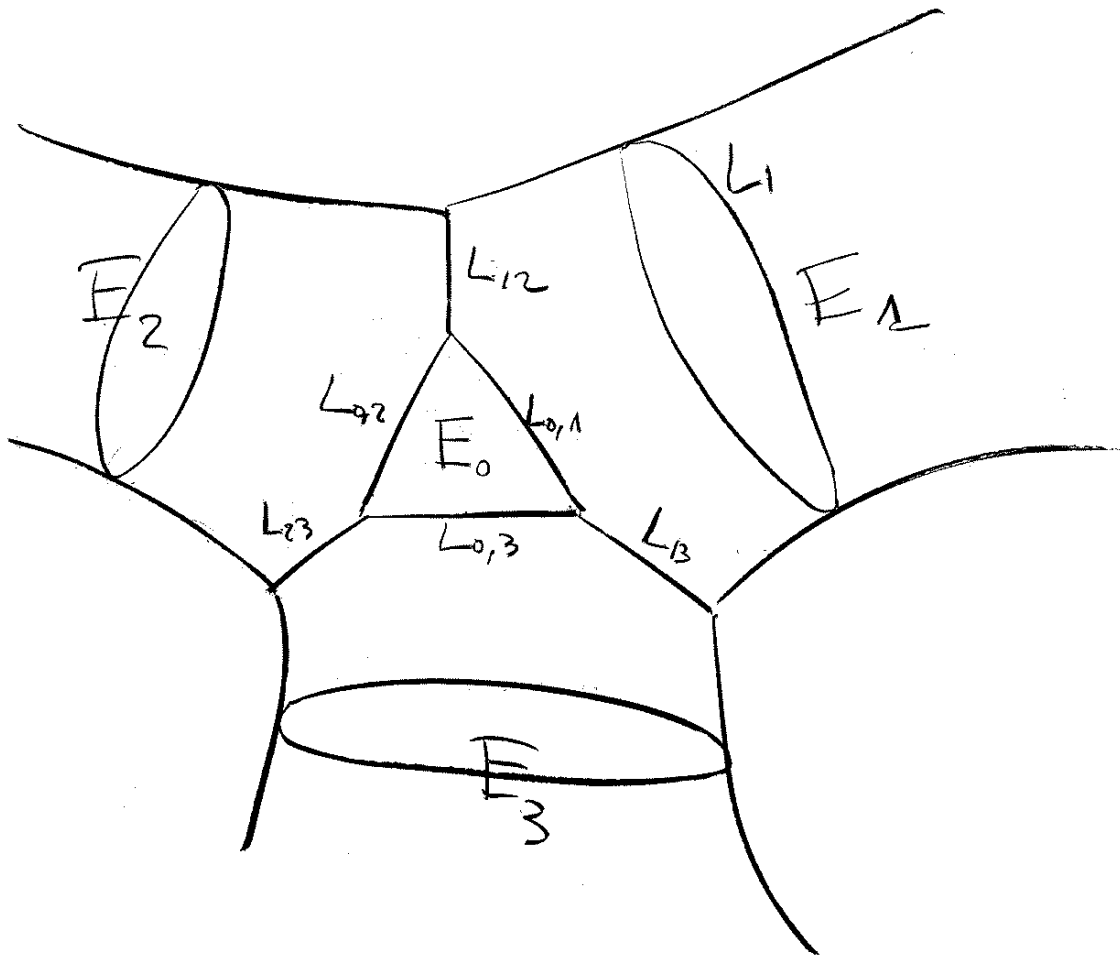


Figure 1.

$$L_1 \sim L_{1,2} + L_{0,1} + L_{1,3}$$

$$L_2 \sim L_{2,3} + L_{1,2} + L_{0,2}$$

$$L_3 \sim L_{3,1} + L_{2,3} + L_{0,3}$$





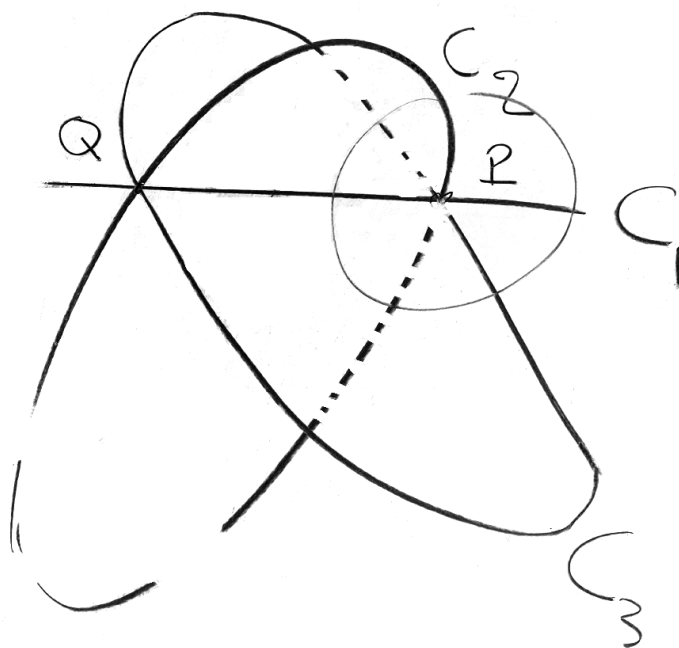


Figure 4.

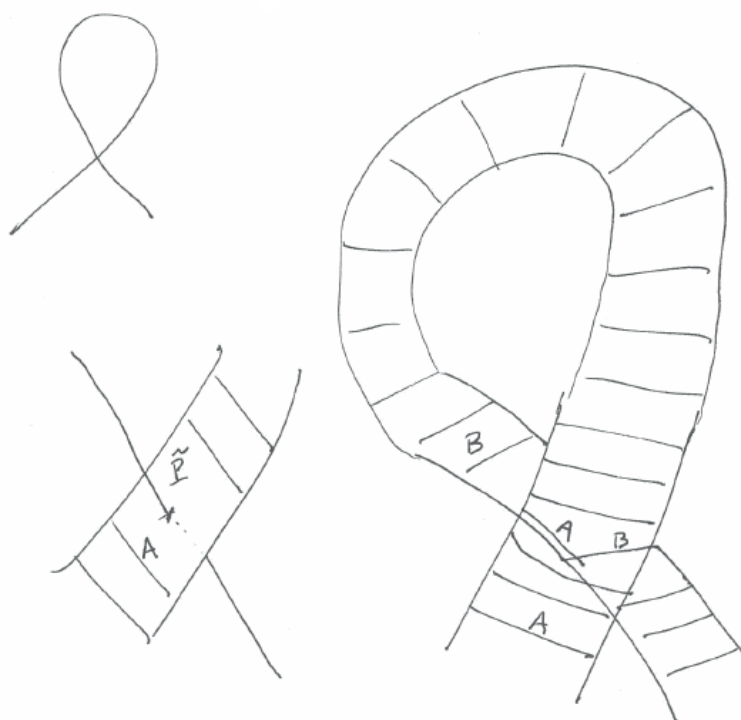


Figure 5.

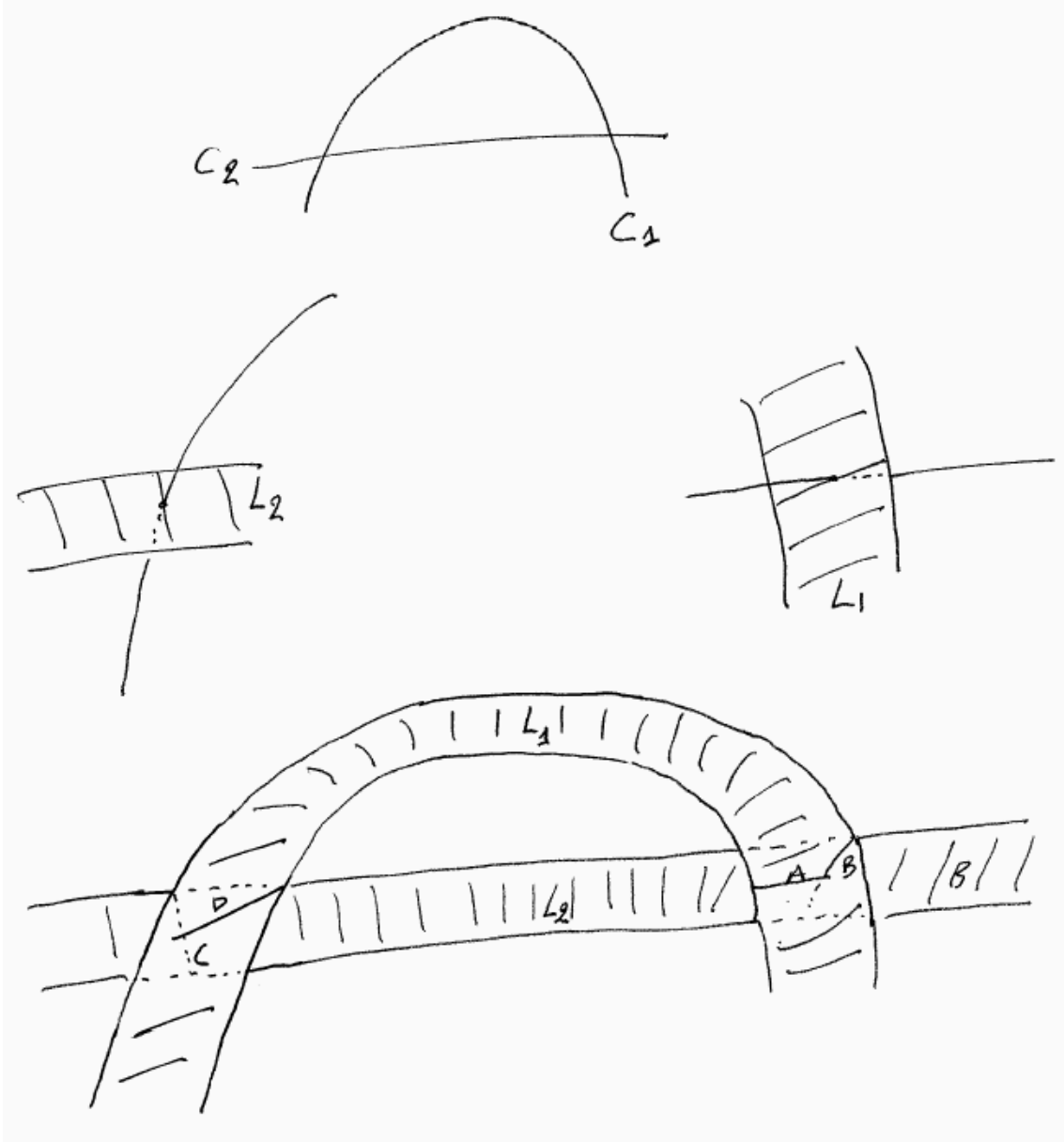


Figure 6.