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# Algebraic Partial equations

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We generalized the concept of hyperbolic and trigonometric functions to the third and the fourth case which gives rise to the parametrization of the orbifold defined by  $x^3 + y^3 + z^3 - 3xyz = 1$  in the third case.

## Introduction

The trigonometric functions (cosinus and sinus) were historically discovered as real and imaginary parts of the complex exponential function which solves the first order partial equations. In addition, the hyperbolic functions are the results of the trigonometric function with imaginary arguments. Even if the trigonometric functions were find before the hyperbolic ones, try to imagine that this is the trigonometric which comes from the trigonometric ones.

Because the exponential function is defined as

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad (0.1)$$

It seems to be natural to generalized the exponential function as

$$e_p(z) = \sum_{k=0}^{\infty} \frac{z^{pk}}{(pk)!} \quad (0.2)$$

that we have call the p-exponential function in [1]. Here we recognize the hyperbolic cosinus if we take  $p = 2$ . In comparasion with the hyperbolic functions, we remarks that the p-th exponential (0.2) can be written as

$$e_p(z) = \frac{1}{p} \left( \sum_{k=0}^{p-1} e^{\omega_p^k z} \right) \quad (0.3)$$

where  $\omega_p$  is the p-th root of unity.

Hence the hyperbolic and the trigonometric functions each satisfie a partial equation

$$y^2 - \left( \frac{\partial y}{\partial z} \right)^2 = 1 \quad (0.4)$$

for the hyperbolic functions and the sames with a minus for the trigonometric ones.

By the way those functions are viewed as the unique solutions of two manifolds whom defining equations are

$$x^2 - y^2 = 1 \quad (0.5)$$

which defines a hyperbol for the hyperbolic functions and the sames with a minus which defines a circle for the trigonometric ones.

As a consequence, we searched for the partial equation solved by the third exponential.

To this end, we showed the equivalent addition relation for the case  $p = 3$ . In the same way we find (0.4) when we put  $b = -a$  in the trigonometric addition formulae ( $\cosh(a + b)$ ), we have put  $b = ja, c = j^2a$  in this equivalent trigonometric addition formulae ( $\epsilon_3(a + b + c)$ ) and we have find the partial equation

$$\left(y(z)\right)^3 + \left(\frac{\partial y}{\partial z}\right)^3 + \left(\frac{\partial^2 y}{\partial z^2}\right)^3 - 3\left(y(z)\right)\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial^2 y}{\partial z^2}\right) = 1 \quad (0.6)$$

Thereby, the third exponential is the unique parametric solution of the orbifold defined by the implicit equation

$$x^3 + y^3 + z^3 - 3xyz = 1 \quad (0.7)$$

The generalization of (0.6) leads to the equation

$$\prod_{q=0}^{p-1} \left( \sum_{k=0}^{p-1} \omega_p^{-kq} y^{(k)}(z) \right) = 1 \quad (0.8)$$

for each  $p \in \mathbb{N}$  with  $y(z) = e_p(z)$ .

In a first time, we recall definitions of the third exponential and we give some expressions of this one. In a second time we give generalized hyperbolic parity and addition formulae of this third exponential. In a third time, we arbitrarily put 0 in the argument of the generalized hyperbolic addition relations to obtain the defining equation of the emerging orbifold and the partial equation solved by the third exponential. In a fourth time, we give a defintion of the hypercomplex number which are a generalization of the complex ones. Finally, we completely solve the case of the fourth exponential and give the orbifold and the partial equation solved in the general case.

# 1 The p-exponential

We recall the definition of the p-exponential given in [1] by :

$$e_p(z) = \sum_{k=0}^{\infty} \frac{z^{pk}}{(pk)!} = \frac{1}{p} \left( \sum_{k=0}^{p-1} e^{\omega_p^k z} \right) \quad (1.1)$$

where  $\omega_p$  is the p-th root of unity. We have shown in [1] that the p-exponential is solution of the partial equation

$$\frac{\partial^p e_p}{\partial z^p} = e_p(z) \quad (1.2)$$

and

$$\sum_{k=0}^{p-1} \frac{\partial^k e_p}{\partial z^k} = e^z \quad (1.3)$$

Example : For p=3, the set of 3-exponential are

$$e_3(z) = \sum_{k=0}^{\infty} \frac{z^{3k}}{(3k)!} \quad (1.4)$$

$$e'_3(z) = \sum_{k=1}^{\infty} \frac{z^{3k-1}}{(3k-1)!} \quad (1.5)$$

$$e''_3(z) = \sum_{k=1}^{\infty} \frac{z^{3k-2}}{(3k-2)!} \quad (1.6)$$

which can also be written

$$e_3(z) = \frac{e^z + e^{jz} + e^{j^2z}}{3} \quad (1.7)$$

$$e'_3(z) = \frac{e^z + je^{jz} + j^2e^{j^2z}}{3} \quad (1.8)$$

$$e''_3(z) = \frac{e^z + j^2e^{jz} + je^{j^2z}}{3} \quad (1.9)$$

or

since  $j = e^{\frac{2i\pi}{3}} = \frac{-1+i\sqrt{3}}{2}$  is the 3th root of unity

# 2 Trigonometric relations for the third exponential

For p=3, we generalized the parity relations for the third exponential

**Proposition 1.** *The parity relations are given by*

$$e_3(jz) = e_3(z) \quad e_3(j^2z) = e_3(z) \quad (2.10)$$

$$e'_3(jz) = j^2e'_3(z) \quad e'_3(j^2z) = je'_3(z) \quad (2.11)$$

$$e''_3(jz) = je''_3(z) \quad e''_3(j^2z) = j^2e''_3(z) \quad (2.12)$$

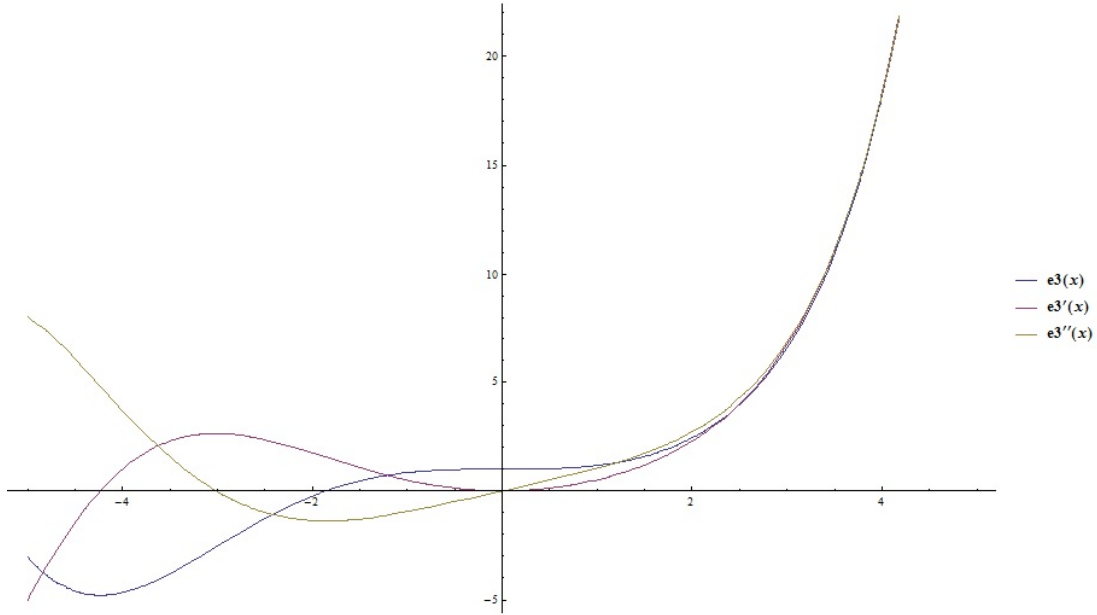


FIGURE 1 – The 3-exponential and its two derivatives

Proof :

From de definition (5.54), we obtain these relations.

For example, to show  $e_3''(jz) = je_3''(z)$

$$\begin{aligned}
 3e_3''(jz) &= e^{jz} + j^2 e^{j(jz)} + j e^{j^2(jz)} \\
 &= e^{jz} + j^2 e^{j^2 z} + j e^z \\
 &= j(j^2 e^{jz} + j e^{j^2 z} + e^z) \\
 3e_3''(jz) &= 3j e_3''(z)
 \end{aligned} \tag{2.13}$$

From (2.13), we obtain the other relation of (2.12) by successive derivations or by doing the same reasonment.

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Now, we generalized the addtion relations for the third exponential

**Proposition 2.** *The addition relations are given by*

$$e_3(a + b) = e_3(a)e_3(b) + e_3'(a)e_3''(b) + e_3''(a)e_3'(b) \tag{2.14}$$

$$e_3'(a + b) = e_3''(a)e_3''(b) + e_3'(a)e_3(b) + e_3(a)e_3'(b) \tag{2.15}$$

$$e_3''(a + b) = e_3'(a)e_3'(b) + e_3(a)e_3''(b) + e_3''(a)e_3(b) \tag{2.16}$$

Proof :

Developping the expression of  $e_3, e'_3, e''_3$  from their definition (5.54), we have the relations

$$\begin{aligned}
9e_3(a)e_3(b) &= e^{a+b} + e^{a+jb} + e^{a+j^2b} + e^{ja+b} + e^{ja+jb} + e^{ja+j^2b} + e^{j^2a+b} + e^{j^2a+jb} + e^{j^2a+j^2b} \\
9e'_3(a)e'_3(b) &= e^{a+b} + je^{a+jb} + j^2e^{a+j^2b} + je^{ja+b} + j^2e^{ja+jb} + e^{ja+j^2b} + j^2e^{j^2a+b} + e^{j^2a+jb} + je^{j^2a+j^2b} \\
9e''_3(a)e''_3(b) &= e^{a+b} + j^2e^{a+jb} + je^{a+j^2b} + j^2e^{ja+b} + je^{ja+jb} + e^{ja+j^2b} + je^{j^2a+b} + e^{j^2a+jb} + j^2e^{j^2a+j^2b}
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
9e_3(a)e'_3(b) &= e^{a+b} + je^{a+jb} + j^2e^{a+j^2b} + e^{ja+b} + je^{ja+jb} + j^2e^{ja+j^2b} + e^{j^2a+b} + je^{j^2a+jb} + j^2e^{j^2a+j^2b} \\
9e_3(a)e''_3(b) &= e^{a+b} + j^2e^{a+jb} + je^{a+j^2b} + e^{ja+b} + j^2e^{ja+jb} + je^{ja+j^2b} + e^{j^2a+b} + j^2e^{j^2a+jb} + je^{j^2a+j^2b} \\
9e'_3(a)e''_3(b) &= e^{a+b} + j^2e^{a+jb} + je^{a+j^2b} + je^{ja+b} + e^{ja+jb} + j^2e^{ja+j^2b} + j^2e^{j^2a+b} + je^{j^2a+jb} + e^{j^2a+j^2b}
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
9e'_3(a)e_3(b) &= e^{a+b} + e^{a+jb} + je^{a+j^2b} + je^{ja+b} + je^{ja+jb} + je^{ja+j^2b} + j^2e^{j^2a+b} + j^2e^{j^2a+jb} + j^2e^{j^2a+j^2b} \\
9e''_3(a)e_3(b) &= e^{a+b} + e^{a+jb} + e^{a+j^2b} + j^2e^{ja+b} + j^2e^{ja+jb} + j^2e^{ja+j^2b} + je^{j^2a+b} + je^{j^2a+jb} + je^{j^2a+j^2b} \\
9e'_3(a)e'_3(b) &= e^{a+b} + je^{a+jb} + j^2e^{a+j^2b} + j^2e^{ja+b} + e^{ja+jb} + je^{ja+j^2b} + je^{j^2a+b} + j^2e^{j^2a+jb} + e^{j^2a+j^2b}
\end{aligned} \tag{2.19}$$

Using (2.18) and (2.19), we have

$$\begin{aligned}
9(e'_3(a)e''_3(b) + e''_3(a)e'_3(b)) &= 2e^{a+b} - e^{a+jb} - e^{a+j^2b} - e^{ja+b} \\
&\quad + 2e^{ja+jb} - e^{ja+j^2b} - e^{j^2a+b} - e^{j^2a+jb} + 2e^{j^2a+j^2b} \\
&= 3(e^{a+b} + e^{ja+jb} + e^{j^2a+j^2b}) - (e^{a+b} + e^{a+jb} + e^{a+j^2b} \\
&\quad + e^{ja+b} + e^{ja+jb} + e^{ja+j^2b} + e^{j^2a+b} + e^{j^2a+jb} + e^{j^2a+j^2b}) \\
9(e'_3(a)e''_3(b) + e''_3(a)e'_3(b)) &= 9e_3(a+b) - 9e_3(a)e_3(b)
\end{aligned}$$

which gives

$$e_3(a+b) = e_3(a)e_3(b) + e'_3(a)e''_3(b) + e''_3(a)e'_3(b) \tag{2.20}$$

which the first relation of (2.16).

In the same way, using (2.18) and (2.19), we have

$$\begin{aligned}
9(e_3(a)e''_3(b) + e''_3(a)e_3(b)) &= 2e^{a+b} - j^2e^{a+jb} - je^{a+j^2b} - j^2e^{ja+b} \\
&\quad + 2je^{ja+jb} - e^{ja+j^2b} - je^{j^2a+b} - e^{j^2a+jb} + 2j^2e^{j^2a+j^2b} \\
&= 3(e^{a+b} + je^{ja+jb} + j^2e^{j^2a+j^2b}) - (e^{a+b} + j^2e^{a+jb} + \\
&\quad je^{a+j^2b} + j^2e^{ja+b} + je^{ja+jb} + e^{ja+j^2b} + je^{j^2a+b} + e^{j^2a+jb} + j^2e^{j^2a+j^2b}) \\
9(e_3(a)e'_3(b) + e'_3(a)e_3(b)) &= 9e'_3(a+b) - 9e''_3(a)e''_3(b)
\end{aligned}$$

which gives

$$e'_3(a+b) = e''_3(a)e''_3(b) + e'_3(a)e_3(b) + e_3(a)e'_3(b) \tag{2.21}$$

which the second relation of (2.16).

In the same way, using (2.18) and (2.19), we have

$$\begin{aligned}
9(e_3(a)e_3''(b) + e_3''(a)e_3(b)) &= 2e^{a+b} - je^{a+jb} - j^2e^{a+j^2b} - je^{ja+b} \\
&\quad + 2j^2e^{ja+jb} - e^{ja+j^2b} - j^2e^{j^2a+b} - e^{j^2a+jb} + 2je^{j^2a+j^2b} \\
&= 3(e^{a+b} + j^2e^{ja+jb} + je^{j^2a+j^2b}) - (e^{a+b} + je^{a+jb} + j^2e^{a+j^2b} + je^{ja+b} \\
&\quad + j^2e^{ja+jb} + e^{ja+j^2b} + j^2e^{j^2a+b} + e^{j^2a+jb} + je^{j^2a+j^2b}) \\
9(e_3(a)e_3''(b) + e_3''(a)e_3(b)) &= 9e_3''(a+b) - 9e_3'(a)e_3'(b)
\end{aligned}$$

which gives

$$e_3''(a+b) = e_3'(a)e_3'(b) + e_3(a)e_3''(b) + e_3''(a)e_3(b) \quad (2.22)$$

which the third relation of (2.16).

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### 3 Orbifold emerging of the trigonometric relations

Example :

For  $p=3$ , we have

**Proposition 3.** *The vector  $(x, y, z) = (e_3, e_3', e_3'')$  is the solution of the equation such that*

$$x^3 + y^3 + z^3 - 3xyz = 1 \quad (3.23)$$

*The equation (3.23) defines an orbifold.*

Proof :

From the addition relation, we have

$$\begin{aligned}
e_3(a+b+c) &= e_3(a)e_3(b+c) + e_3'(a)e_3''(b+c) + e_3''(a)e_3'(b+c) \\
&= e_3(a) \left( e_3(b)e_3(c) + e_3'(b)e_3''(c) + e_3''(b)e_3'(c) \right) \\
&\quad + e_3'(a) \left( e_3'(b)e_3'(c) + e_3(b)e_3''(c) + e_3''(b)e_3(c) \right) \\
&\quad + e_3''(a) \left( e_3''(b)e_3''(c) + e_3'(b)e_3(c) + e_3(b)e_3'(c) \right)
\end{aligned} \quad (3.24)$$



If we impose  $b = ja$  and  $c = j^2a$  and using (2.12), we find

$$\begin{aligned}
 e_3(a + ja + j^2a) &= e_3(a) \left( e_3(a)e_3(a) + je'_3(a)e''_3(a) + j^2e''_3(a)e'_3(a) \right) \\
 &\quad + e'_3(a) \left( e'_3(a)e'_3(a) + j^2e_3(a)e''_3(a) + je''_3(b)e_3(c) \right) \\
 &\quad + e''_3(a) \left( e''_3(a)e''_3(a) + j^2e'_3(a)e_3(a) + je_3(b)e'_3(c) \right) \\
 1 &= \left( e_3(a) \right)^3 + \left( e'_3(a) \right)^3 + \left( e''_3(a) \right)^3 + (j + j^2 + j + j^2 + j + j^2)e_3(a)e'_3(a)e'_3(a)
 \end{aligned}$$

So we find

$$\left( e_3(a) \right)^3 + \left( e'_3(a) \right)^3 + \left( e''_3(a) \right)^3 - 3e_3(a)e'_3(a)e'_3(a) = 1 \quad (3.25)$$

So  $y(z) = e_3(z)$  is the solution of the partial equation

$$\left( y(z) \right)^3 + \left( \frac{\partial y}{\partial z} \right)^3 + \left( \frac{\partial^2 y}{\partial z^2} \right)^3 - 3 \left( y(z) \right) \left( \frac{\partial y}{\partial z} \right) \left( \frac{\partial^2 y}{\partial z^2} \right) = 1 \quad (3.26)$$

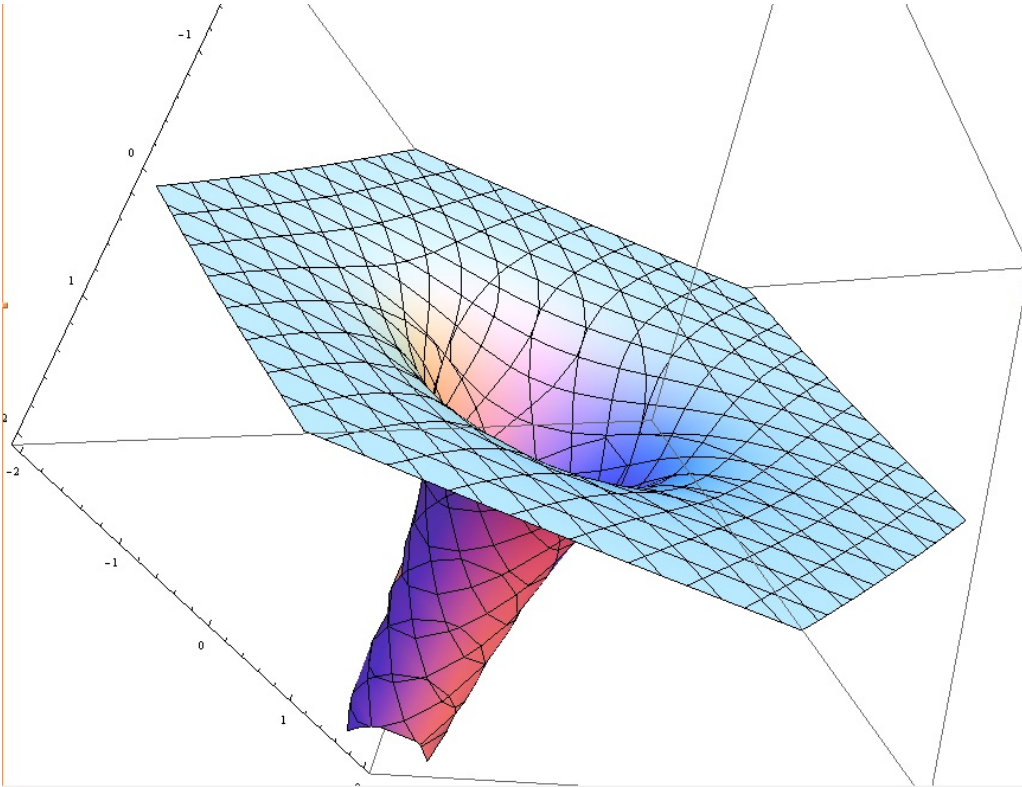


FIGURE 2 – Implicit plot of  $x^3 + y^3 + z^3 - 3xyz = 1$

## 4 An interesting factorization

From the definition of the orbifold emerging of the third exponential

$$x^3 + y^3 + z^3 - 3xyz = 1 \quad (4.27)$$

and because of the identity

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2} (x + y + z) \left[ (x - y)^2 + (y - z)^2 + (z - x)^2 \right] \quad (4.28)$$

we obtain

$$\frac{1}{2} (x + y + z) \left[ (x - y)^2 + (y - z)^2 + (z - x)^2 \right] = 1 \quad (4.29)$$

Now we expand the second factor which gives

$$\begin{aligned} (x + y + z) \left[ x^2 + y^2 + z^2 - xy - yz - xz \right] &= 1 \\ (x + y + z) \left[ x^2 + y^2 + z^2 + (j + j^2)xy + (j + j^2)yz + (j + j^2)xz \right] &= 1 \\ (x + y + z) \left[ x^2 + j^2 xy + j xz + y^2 + j yx + j^2 yz + z^2 + j^2 zx + j zy \right] &= 1 \\ (x + y + z) \left[ (x + j y + j^2 z)(x + j^2 y + j z) \right] &= 1 \end{aligned} \quad (4.30)$$

So if we parametrize

$$\begin{cases} x &= e_3(\theta) \\ y &= e'_3(\theta) \\ z &= e''_3(\theta) \end{cases} \quad (4.31)$$

We can write (4.30) as

$$e^\theta e^{j\theta} e^{j^2\theta} = 1 \quad (4.32)$$

since

$$e^\theta = e_3(\theta) + e'_3(\theta) + e''_3(\theta) \quad (4.33)$$

$$e^{j\theta} = e_3(\theta) + j^2 e'_3(\theta) + j e''_3(\theta) \quad (4.34)$$

$$e^{j^2\theta} = e_3(\theta) + j e'_3(\theta) + j^2 e''_3(\theta) \quad (4.35)$$

$$(4.36)$$

## 5 Generalization

**Theorem 1.** *The p-exponential  $y(z) = e_p(z)$  is the solution the  $p - 1$  order partial equation*

$$\prod_{q=0}^{p-1} \left( \sum_{k=0}^{p-1} \omega_p^{-kq} y^{(k)}(z) \right) = 1 \quad (5.37)$$

*This relation defines an orbifold  $\mathcal{O}$  :*

$$\prod_{q=0}^{p-1} \left( \sum_{k=0}^{p-1} \omega_p^{-kq} x_k \right) = 1 \quad (5.38)$$

where  $(x_0, \dots, x_{n-1}) = \left( e_p(z), e'_p(z), \dots, e_p^{(n)}(z) \right)$ .

*This orbifold is  $n$  minus the number of prime factor of  $n$ .*

Proof : From the definition of the p-exponential, we have

$$\begin{aligned} e_p(z) &= \frac{1}{p} \sum_{k=0}^{p-1} e^{\omega_p^k z} \\ e'_p(z) &= \frac{1}{p} \sum_{k=0}^{p-1} \omega_p^k e^{\omega_p^k z} \\ &\vdots \\ e_p^{(q)}(z) &= \frac{1}{p} \sum_{k=0}^{p-1} \omega_p^{qk} e^{\omega_p^k z} \\ &\vdots \\ e_p^{(n)}(z) &= \frac{1}{p} \sum_{k=0}^{p-1} \omega_p^{nk} e^{\omega_p^k z} \end{aligned} \quad (5.39)$$

with  $\omega_p$  the p-th root of unity. Now we deduce from (5.39) the generalized parity relation

$$e_p^{(q)}(\omega_p z) = \omega_p^{q(p-1)} e_p^{(q)}(z) \quad (5.40)$$

or

$$e_p^{(q)}(\omega_p z) = \omega_p^{-q} e_p^{(q)}(z) \quad (5.41)$$

From (5.41), we obtain

$$\begin{aligned}
e^z &= \sum_{k=0}^{p-1} e_p^{(k)}(z) \\
e^{\omega_p z} &= \sum_{k=0}^{p-1} \omega_p^{-k} e_p^{(k)}(z) \\
&\vdots \\
e^{\omega_p^q z} &= \sum_{k=0}^{p-1} \omega_p^{-qk} e_p^{(k)}(z) \\
&\vdots \\
e^{\omega_p^n z} &= \sum_{k=0}^{p-1} \omega_p^{-nk} e_p^{(k)}(z)
\end{aligned} \tag{5.42}$$

Using (5.59) we obtain

$$\begin{aligned}
e^{\left(\frac{1-\omega_p^p}{1-\omega_p}\right)z} &= 1 \\
e^{\sum_{q=0}^{p-1} \omega_p^q z} &= 1 \\
\prod_{q=0}^{p-1} e^{\omega_p^q z} &= 1 \\
\prod_{q=0}^{p-1} \left( \sum_{k=0}^{p-1} \omega_p^{-qk} e_p^{(k)}(z) \right) &= 1
\end{aligned} \tag{5.43}$$

which is the relation (5.37).

The orbifold (5.43) is  $n - 1$  dimensional when  $n$  is prime. When  $n$  is not prime we have the two equivalent decomposition if  $n = qq'$  has two prime factors :

$$\frac{1 - \omega_p^{qq'}}{1 - \omega_p} = 1 + \omega_p^q + (\omega_p^q)^2 + \dots + (\omega_p^q)^{q'-1} \tag{5.44}$$

which has  $q'$  term or

$$\frac{1 - \omega_p^{qq'}}{1 - \omega_p} = 1 + \omega_p^{q'} + (\omega_p^{q'})^2 + \dots + (\omega_p^{q'})^{q-1} \tag{5.45}$$

which has  $q$  term.

In doing this by recurrence, we see that the dimension of the orbifold (5.43) is  $n$  minus the number of prime factor of  $n$  because only the number of term matters.

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**Remark 1.** If we define the Armian of a vector  $\mathbf{e} = (e_1, \dots, e_n) \in \mathbb{R}_n$  :

$$\text{Arm}(\mathbf{e}) = \prod_{q=1}^n e_q \quad (5.46)$$

we can write (5.37) as

$$\text{Arm}(\mathbf{W} \mathbf{e}_p) = 1 \quad (5.47)$$

with

$$\mathbf{e}_p = \begin{pmatrix} e_p(z) \\ e_p'(z) \\ \vdots \\ e_p^{(q)}(z) \\ \vdots \\ e_p^{(n)}(z) \end{pmatrix} \quad (5.48)$$

and  $W = (\omega_p^{-(k-1)(q-1)})_{1 \leq k, q \leq p}$  or

$$W = \begin{pmatrix} 1 & \dots & 1 & \dots & 1 \\ 1 & \dots & \omega_p^{-(k-1)} & \dots & \omega_p^{-(p-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \omega_p^{-(k-1)(q-1)} & \dots & \omega_p^{-(p-1)(q-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \omega_p^{-(k-1)(p-1)} & \dots & \omega_p^{-(p-1)(p-1)} \end{pmatrix} \quad (5.49)$$

Using the Vondermonde determinant formula, we can check

$$\det(W) = \prod_{1 \leq i < j \leq p} (\omega_p^{-(j-1)} - \omega_p^{-(i-1)}) \quad (5.50)$$

Example :

For  $p=4$ , the set of 4-exponential are

$$e_4(\theta) = \frac{e^\theta + e^{i\theta} + e^{-\theta} + e^{-i\theta}}{4} \quad (5.51)$$

$$e_4'(\theta) = \frac{e^\theta + ie^{i\theta} - e^{-\theta} - ie^{-i\theta}}{4} \quad (5.52)$$

$$e_4''(\theta) = \frac{e^\theta - e^{i\theta} + e^{-\theta} - e^{-i\theta}}{4} \quad (5.53)$$

$$e_4'''(\theta) = \frac{e^\theta - ie^{i\theta} - e^{-\theta} + ie^{-i\theta}}{4} \quad (5.54)$$

or

$$e_4(\theta) = \frac{1}{2} (\cos(\theta) + \cosh(\theta)) \quad (5.55)$$

$$e'_4(\theta) = \frac{1}{2} (-\sin(\theta) + \sinh(\theta)) \quad (5.56)$$

$$e''_4(\theta) = \frac{1}{2} (-\cos(\theta) + \cosh(\theta)) \quad (5.57)$$

$$e'''_4(\theta) = \frac{1}{2} (\sin(\theta) + \sinh(\theta)) \quad (5.58)$$

We have

$$\begin{aligned} e^\theta &= e_4(\theta) + e'_4(\theta) + e''_4(\theta) + e'''_4(\theta) \\ e^{i\theta} &= e_4(\theta) - ie'_4(\theta) - e''_4(\theta) + ie'''_4(\theta) \\ e^{-\theta} &= e_4(\theta) - e'_4(\theta) + e''_4(\theta) - e'''_4(\theta) \\ e^{-i\theta} &= e_4(\theta) + ie'_4(\theta) - e''_4(\theta) - ie'''_4(\theta) \end{aligned} \quad (5.59)$$

Here we can see the W matrix

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \quad (5.60)$$

Using

$$\begin{cases} x &= e_4(\theta) \\ y &= e'_4(\theta) \\ z &= e''_4(\theta) \\ t &= e'''_4(\theta) \end{cases} \quad (5.61)$$

we can write

$$\begin{aligned} e^\theta e^{-\theta} &= 1 \\ (x + y + z + t)(x - y + z - t) &= 1 \\ (x + z)^2 - (y + t)^2 &= 1 \end{aligned} \quad (5.62)$$

and

$$\begin{aligned} e^{i\theta} e^{-i\theta} &= 1 \\ (x - iy + z + it)(x + iy + z - it) &= 1 \\ (x - z)^2 + (y - t)^2 &= 1 \end{aligned} \quad (5.63)$$

Combining (5.62) and (5.64), we obtain

$$\begin{aligned} e^\theta e^{-\theta} e^{i\theta} e^{-i\theta} &= 1 \\ [(x - z)^2 + (y - t)^2] [(x + z)^2 - (y + t)^2] &= 1 \end{aligned}$$

the defining equation of the fourth exponential

$$(x^2 - z^2)^2 + (y^2 - t^2)^2 + 4(xy - zt)(zy - xt) = 1 \quad (5.64)$$

## 6 Arm Algebras

Let  $a$  be a generator of the  $n$  order cyclic group  $C_n \simeq \mathbb{Z}/n\mathbb{Z}$  which is in the symmetric group  $S_n$ .

**Proposition 4.** *We define an  $n$ -order Arm algebra as the commutative algebra constituted with all power of a generator of the  $n$  order cyclic group.*

All the following theorems work with each generator of the cyclic group, but we decide to consider the easiest generator of the cyclic group :  $(123\dots n)$  which matrix representation is

$$a = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \end{pmatrix} \quad (6.65)$$

**Proposition 5.** *We have that the exponential of the generator  $a$*

$$\exp(ax) = \sum_{k=0}^n e_n^{(k)}(x) a^k \quad (6.66)$$

for  $\forall x \in \mathbb{R}$  defines a group.

**Theorem 2.** *The relation (5.37) is equivalent to*

$$\det(\exp(ax)) = 1 \quad (6.67)$$

and a general solution is given by

$$\det \left( \exp \left( \sum_{k=1}^m a^k x_k \right) \right) = 1 \quad (6.68)$$

where  $m$  is  $n$  minus the number of prime factors of  $n$ .

Example :

For  $p = 2$ , the realization of the 2-order cyclic group is the (12) permutation matrix :

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (6.69)$$

The Arm algebra is the algebra  $\{a\}$  and the corresponding Arm group is :

$$\exp(ax) = e_2(x) \text{Id}_2 + e_2'(x) a \quad (6.70)$$

with  $w \in \mathbb{R}$ . The relation (6.70) can be expressed as

$$\exp(ax) = \begin{pmatrix} e_2(x) & e_2'(x) \\ e_2'(x) & e_2(x) \end{pmatrix} \quad (6.71)$$

or

$$\exp(ax) = \begin{pmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{pmatrix} \quad (6.72)$$

Here (6.81), we recognize that the 2-Arm Group is homeomorph to the group  $O(1,1)$ .

For  $p = 3$ , the realization of the 3-order cyclic group in the symmetric group are the matrices of the permutations (123) and (132) :

$$a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad a^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad (6.73)$$

So, in the case  $p = 3$ , an element of the 3-Arm Group is

$$\exp(ax) = e_3(x)\text{Id}_3 + e'_3(x)a + e''_3(x)a^2 \quad (6.74)$$

or

$$\exp(ax) = \begin{pmatrix} e_3(x) & e''_3(x) & e'_3(x) \\ e'_3(x) & e_3(x) & e''_3(x) \\ e''_3(x) & e'_3(x) & e_3(x) \end{pmatrix} \quad (6.75)$$

Here we can note that  $\det(\exp(ax)) = 1$  is equivalent as

$$(e_3(x))^3 + (e'_3(x))^3 + (e''_3(x))^3 - 3e_3(x)e'_3(x)e''_3(x) = 1 \quad (6.76)$$

which is the relation (3.25).

Then we can see that the Arm algebra is  $\{a, a^2\}$  gives the general solution of an element of the 3-Arm Group :

$$\exp(ax_1 + a^2x_2) = \begin{pmatrix} e_3(x_1) & e''_3(x_1) & e'_3(x_1) \\ e'_3(x_1) & e_3(x_1) & e''_3(x_1) \\ e''_3(x_1) & e'_3(x_1) & e_3(x_1) \end{pmatrix} \begin{pmatrix} e_3(x_2) & e'_3(x_2) & e''_3(x_2) \\ e''_3(x_2) & e_3(x_2) & e'_3(x_2) \\ e'_3(x_2) & e''_3(x_2) & e_3(x_2) \end{pmatrix} \quad (6.77)$$

for  $\forall x_1, x_2 \in \mathbb{R}$ . Calculating (6.77), we obtain the general parametrization of the relation (3.23)

$$y_1^3 + y_2^3 + y_3^3 - 3y_1y_2y_3 = 1 \quad (6.78)$$

which is 2 dimensional

$$\begin{cases} y_1 = e^{\frac{x_1+x_2}{3}} + \frac{2}{3}e^{-\frac{x_1+x_2}{2}} \cos\left(\frac{\sqrt{3}}{2}(x_1 - y)\right) \\ y_2 = e^{\frac{x_1+x_2}{3}} - \frac{e^{-\frac{x_1+x_2}{2}}}{3} \left( \cos\left(\frac{\sqrt{3}}{2}(x_1 - x_2)\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}(x_1 - x_2)\right) \right) \\ y_3 = e^{\frac{x_1+x_2}{3}} - \frac{e^{-\frac{x_1+x_2}{2}}}{3} \left( \cos\left(\frac{\sqrt{3}}{2}(x_1 - x_2)\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}(x - x_2)\right) \right) \end{cases} \quad (6.79)$$

for  $\forall x_1, x_2 \in \mathbb{R}$ .

For  $p = 4$ , the 4-order cyclic group in the symmetric group  $S_4$  are ((1234), (1243), (1324), (1342), (1423), (1432)) where we arbitrarily choose  $a$  as the realization of the easiest permutation(1234) :

$$a = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad a^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad a^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (6.80)$$



So, in the case  $p = 4$ , an element of the 4-Arm Group is

$$\exp(ax) = e_4(x)\text{Id}_4 + e'_4(x)a + e''_4(x)a^2 + e'''_4(x)a^3 \quad (6.81)$$

or

$$\exp(ax) = \begin{pmatrix} e_4(x) & e'''_4(x) & e''_4(x) & e'_4(x) \\ e'_4(x) & e_4(x) & e'''_4(x) & e''_4(x) \\ e''_4(x) & e'_4(x) & e_4(x) & e'''_4(x) \\ e'''_4(x) & e''_4(x) & e'_4(x) & e_4(x) \end{pmatrix} \quad (6.82)$$

Here we can note that  $\det(\exp(ax)) = 1$  is equivalent as

$$(y_1^2 - y_3^2)^2 + (y_2^2 - y_4^2)^2 + 4(y_1y_2 - y_3y_4)(y_3y_2 - y_1y_4) = 1 \quad (6.83)$$

with  $y_1 = e_4(x), y_2 = e'_4(x), y_3 = e''_4(x), y_4 = e'''_4(x)$  which is the relation (5.64).

## Conclusion

I choose to give the generalized theorem for each  $p \in \mathbb{N}$  to say that it is true for each  $p$  but I only formally prove it only for the third and the fourth exponential.

By the way, if somebody want to study the topology or the metric of the orbifold  $x^3+y^3+z^3-3xyz = 1$ , please contact me, I would be happy to work on it.

## Références

- [1] Arm B. N., The p-Arm Theory