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LND-FILTRATIONS AND SEMI-RIGID DOMAINS

BACHAR ALHAJJAR

Abstract. We investigate the filtration corresponding to the degree function induced by a non-zero locally nilpotent derivation and its associated graded algebra. We show that this kind of filtration, referred to as the LND-*filtration*, is the ideal candidate to study the structure of *semi-rigid* k-domains, that is, k-domains for which every non-zero locally nilpotent derivation gives rise to the same filtration. Indeed, the LND-filtration gives a very precise understanding of these structure, it is impeccable for the computation of the Makar-Limanov invariant, and it is an efficient tool to determine their isomorphism types and automorphism groups. Then, we construct a new interesting class of semi-rigid k-domains in which we elaborate the fundamental requirement of LND-filtrations. The importance of these new examples is due to the fact that they possess a relatively big set of invariant sub-algebras, which can not be recoverd by known invariants such as the Makar-Limanov and the Derksen invariants. Also, we define a new family of invariant sub-algebras as a generalization of the Derksen invariant. Finally, we introduce an algorithm to establish explicit isomorphisms between cylinders over non-isomorphic members of the new class, providing by that new counter-examples to the cancellation problem.

Introduction

Let k be a field of characteristic zero and let B be a commutative k-domain. A k-derivation $\partial \in \mathrm{Der}_k(B)$ is said to be *locally nilpotent* if for every $a \in B$, there is an integer $n \geq 0$ such that $\partial^n(a) = 0$.

An important invariant of k-domains B admitting non-trivial locally nilpotent derivations is the so called *Makar-Limanov invariant* $\mathrm{ML}(B)$ which was defined by Makar-Limanov as the intersection of the kernels of all locally nilpotent derivations on B ([ML3]). This invariant was initially introduced as a tool to distinguish certain k-domains from polynomial rings but it has many other applications for the study of k-algebras and their automorphism groups ([ML2]). One of the main difficulties in applications is to compute this invariant without a prior knowledge of all locally nilpotent derivations of a given k-domain.

In [K-ML1], S. Kaliman and L. Makar-Limanov developed general techniques to determine the ML-invariant for a class of finitely generated k-domains $B = k[X_1, \ldots, X_n]/I$. The idea, referred to as "homogenization of derivations", is to reduce the problem to the study of homogeneous locally nilpotent derivations on graded algebras $\mathrm{Gr}(B)$ associated to B. For this, one considers appropriate filtrations $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{R}}$ on B generated by real-valued weight degree functions $\omega \in \mathbb{R}^n$, in such a way that every non-zero locally nilpotent derivation on B induces a non-zero homogeneous locally nilpotent derivation on the associated graded algebra $\mathrm{Gr}_{\mathcal{F}}(B)$. Unfortunately, these techniques only work if the associated graded algebra $\mathrm{Gr}(B)$ is in fact a k-domain itself, which will only occur if the ideal $\hat{I}$, generated by top homogenous components relative to ω of all elements in I, is prime.

Finding a new way to tackle similar complications became an inevitable necessity. Therefore, we start inspecting real-valued weight degree functions on $k^{[N]}$ with the following new perspective. The positive integer N is chosen to be bigger than n, the dimension of the ambient space. The considered ring $B = k^{[n]}/I$ is identified in a specific "twisting" way to $k^{[N]}/J \simeq B$. This different point of view allows us to avoid these kind of difficulties. Furthermore, it simplifies the study of homogenous locally nilpotent derivation even in these cases where classical techniques work.

We present a new class of examples for which the "homogenization" method can be effectively applied with the alternative perspective, while all other approaches fail. The new class comes to be a very interesting object due to the fact that it possesses a huge set of invariant sub-algebras, which can not be recoverd by any known invariant such as the Makar-Limanov and the Derksen invariants.

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As a modest outcome, the alternative approach delivers a full description of the filtration induced by any locally nilpotent derivation, with a finitely generated kernel, and its associated graded algebra. In particular, a non-zero locally nilpotent derivation ∂ gives rise to a proper $\mathbb{N}$-filtration $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{N}}$ of B by the sub-spaces $\mathcal{F}_i = \ker\partial^{i+1}$, $i \in \mathbb{N}$. We call it the ∂-*filtration*, which corresponds to the $\mathbb{N}$-degree function $\deg_\partial$ induced by ∂. It turns out that $\deg_\partial$ is nothing but the degree function induced by an $\mathbb{N}$-weight degree function $\omega \in \mathbb{N}^{[N]}$ defined on $k^{[N]}$ for suitable choices of ω and N.

In turn, the ∂-filtration comes out to be the ideal candidate to study the structure of *semi-rigid* k-domains, that is, k-domains for which every non-zero locally nilpotent derivation gives rise to the same filtration that we call the *unique* LND-*filtration.* This unique LND-filtration gives a very precise understanding of the structure of semi-rigid k-domains, it is impeccable for the computation of the ML-invariant, and it is an efficient tool to determine isomorphism types and automorphism groups. Nevertheless, the computation of the ML-invariant, isomorphism types, and automorphism groups of similar classically known structures can be simplified and reduced considering this new point of view.

Another important tool for the study of non-rigid k-domains is the *Derksen invariant* $\mathcal{D}(B)$ which is defined to be the sub-algebra of B generated by $\ker\partial$ for all non-zero locally nilpotent derivations. We generalize this invariant to obtain a new family $\{\mathrm{AL}_i(B)\}_{i\in\mathbb{N}}$ of invariant sub-algebras of B, where for each $i \in \mathbb{N}$ we define $\mathrm{AL}_i(B)$ to be the algebra generated by $\ker\partial^{i+1}$ for all non-zero locally nilpotent derivation of B. We are interested in one particular member of this family of invariants that corresponds to $i = 1$, which we call the ring of all local slices of B and which we denote AL-*invariant.* We show that the new class of k-domains can be realized as an affine modification of the AL-invariant with center (f, I) for certain ideal I in AL and some $f \in I$.

Finally, we propose an algorithm to construct explicit isomorphisms between cylinders over non-isomorphic members of the new class, providing by that new counter-examples to the cancellation problem.

1. **Preliminaries**

In this section we briefly recall basic facts on filtered algebra and their relation with derivation in a form appropriate to our needs.

In the sequel, unless otherwise specified B will denote a commutative domain over a field k of characteristic zero. The set $\mathbb{Z}_{\geqslant 0}$ of non-negative integers will be denoted by $\mathbb{N}$.

1.1. **Filtrations and associated graded algebras.**

Definition 1.1. An $\mathbb{N}$-*filtration* of B is a collection $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ of k-sub-vector-spaces of B with the following properties:

1- $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ for all $i \in \mathbb{N}$.

2- $B = \cup_{i\in\mathbb{N}}\mathcal{F}_i$.

3- $\mathcal{F}_i.\mathcal{F}_j \subset \mathcal{F}_{i+j}$ for all $i, j \in \mathbb{N}$.

The filtration is called *proper* if the following additional property holds:

4- If $a \in \mathcal{F}_i \setminus \mathcal{F}_{i-1}$ and $b \in \mathcal{F}_j \setminus \mathcal{F}_{j-1}$, then $ab \in \mathcal{F}_{i+j} \setminus \mathcal{F}_{i+j-1}$.

There is a one-to-one correspondence between proper $\mathbb{N}$-filtrations and so called $\mathbb{N}$-degree functions:

Definition 1.2. An $\mathbb{N}$-*degree function* on B is a map $\deg : B \longrightarrow \mathbb{N}\cup\{-\infty\}$ such that, for all $a, b \in B$, the following conditions are satisfied:

(1) $\deg(a) = -\infty \Leftrightarrow a = 0$.

(2) $\deg(ab) = \deg(a) + \deg(b)$.

(3) $\deg(a + b) \leq \max\{\deg(a), \deg(b)\}$.

If the equality in (2) replaced by the inequality $\deg(ab) \leq \deg(a)+\deg(b)$, we say that deg is an $\mathbb{N}$-*semi-degree function.*

Indeed, for an $\mathbb{N}$-degree function on B, the sub-sets $\mathcal{F}_i = \{b \in B \mid \deg(b) \leq i\}$ are k-subvector spaces of B that give rise to a proper $\mathbb{N}$-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$. Conversely, every proper $\mathbb{N}$-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$, yields an $\mathbb{N}$-degree function $\omega : B \longrightarrow \mathbb{N}\cup\{-\infty\}$ defined by $\omega(0) = -\infty$ and $\omega(b) = i$ if $b \in \mathcal{F}_i \setminus \mathcal{F}_{i-1}$.

Definition 1.3. Given a k-domain $B = \cup_{i\in\mathbb{N}}\mathcal{F}_i$ equipped with a proper $\mathbb{N}$-filtration, the associated graded algebra $\mathrm{Gr}(B)$ is the k-vector space

$$\mathrm{Gr}(B) = \oplus_{i\in\mathbb{N}}\mathcal{F}_i/\mathcal{F}_{i-1}$$

equipped with the unique multiplicative structure for which the product of the elements $a+\mathcal{F}_{i-1} \in \mathcal{F}_i/\mathcal{F}_{i-1}$ and $b+\mathcal{F}_{j-1} \in \mathcal{F}_j/\mathcal{F}_{j-1}$, where $a \in \mathcal{F}_i$ and $b \in \mathcal{F}_j$, is the element

$$(a+\mathcal{F}_{i-1})(b+\mathcal{F}_{j-1}) := ab+\mathcal{F}_{i+j-1} \in \mathcal{F}_{i+j}/\mathcal{F}_{i+j-1}.$$

Property 4 for a proper filtration in Definition 1.1 ensures that $\mathrm{Gr}(B)$ is a commutative k-domain when B is an integral domain. Since for each $a \in B \setminus \{0\}$ the set $\{n \in \mathbb{N} \mid a \in \mathcal{F}_n\}$ has a minimum, there exists i such that $a \in \mathcal{F}_i$ and $a \notin \mathcal{F}_{i-1}$. So we can define a k-linear map $\mathrm{gr} : B \longrightarrow \mathrm{Gr}(B)$ by sending a to its class in $\mathcal{F}_i/\mathcal{F}_{i-1}$, i.e $a \mapsto a+\mathcal{F}_{i-1}$, and $\mathrm{gr}(0)=0$. We will frequently denote $\mathrm{gr}(a)$ simply by $\overline{a}$. Observe that $\mathrm{gr}(a)=0$ if and only if $a=0$.

Denote by deg the $\mathbb{N}$-degree function $\deg : B \longrightarrow \mathbb{N} \cup \{-\infty\}$ corresponding to the proper $\mathbb{N}$-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$. We have the following properties.

Lemma 1.4. *Given $a, b \in B$ the following holds:*
P1) $\overline{a\,b} = \overline{a}\,\overline{b}$, *that is* gr *is a multiplicative map.*
P2) *If* $\deg(a) > \deg(b)$, *then* $\overline{a+b} = \overline{a}$.
P3) *If* $\deg(a) = \deg(b) = \deg(a+b)$, *then* $\overline{a+b} = \overline{a} + \overline{b}$.
P4) *If* $\deg(a) = \deg(b) > \deg(a+b)$, *then* $\overline{a} + \overline{b} = 0$. *In particular,* gr *is not an additive map in general.*

Proof. Let us assume that $\deg(a)=i$ and $\deg(b)=j$. By definition, $\deg(ab)=i+j$ means that $ab \in \mathcal{F}_{i+j}$ and $ab \notin \mathcal{F}_{i+j-1}$, so $\overline{ab} = ab+\mathcal{F}_{i+j-1} := (a+\mathcal{F}_{i-1})(b+\mathcal{F}_{j-1}) = \overline{a}\,\overline{b}$. Which gives P1. For P2 we observe that since $\deg(a+b)=\deg(a)$, we have $\overline{a+b} = (a+b)+\mathcal{F}_{i-1} = (a+\mathcal{F}_{i-1})+(b+\mathcal{F}_{i-1})$, and since $\mathcal{F}_{j-1} \subset \mathcal{F}_j \subseteq \mathcal{F}_{i-1}$ as $i>j$, we get $b+\mathcal{F}_{i-1}=0$. P3) is immediate, by definition. Finally, assume by contradiction that $\overline{a}+\overline{b} \neq 0$, then $\overline{a}+\overline{b} = (a+\mathcal{F}_{i-1})+(b+\mathcal{F}_{i-1}) = ((a+b)+\mathcal{F}_{i-1}) \neq 0$, which means that $a+b \notin \mathcal{F}_{i-1}$ and $\deg(a+b)=i$, which is absurd. So P4 follows. □

1.2. **Derivations.**
By a *k-derivation* of B, we mean a k-linear map $D : B \longrightarrow B$ which satisfies the Leibniz rule: For all $a, b \in B$; $D(ab) = aD(b)+bD(a)$. The set of all k-derivations of B is denoted by $\mathrm{Der}_k(B)$.
The *kernel* of a derivation D is the subalgebra $\ker D = \{b \in B; D(b)=0\}$ of B.
The *plinth ideal* of D is the ideal $\mathrm{pl}(D) = \ker D \cap D(B)$ of $\ker D$, where $D(B)$ denotes the image of B.
An element $s \in B$ such that $D(s) \in \ker(D) \setminus \{0\}$ is called a *local slice* for D.

Definition 1.5. Given a k-algebra $B = \cup_{i\in\mathbb{N}}\mathcal{F}_i$ equipped with a proper $\mathbb{N}$-filtration, a k-derivation D of B is said to *respect* the filtration if there exists an integer d such that $D(\mathcal{F}_i) \subset \mathcal{F}_{i+d}$ for all $i \in \mathbb{N}$. The smallest integer d, such that $D(\mathcal{F}_i) \subset \mathcal{F}_{i+d}$ for all $i \in \mathbb{N}$, is called the *degree* of D with respect to $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{N}}$ and denoted by $\deg_{\mathcal{F}} D$.
Note that if D respects the filtration $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{N}}$ then $\deg_{\mathcal{F}} D$ is well-defined. Indeed, denote by deg the $\mathbb{N}$-degree function corresponding to $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{N}}$ and let U be the non-empty subset of $\mathbb{Z} \cup \{-\infty\}$ defined by $U := \{\deg(D(b)) - \deg(b)\ ;\ b \in B \setminus \{0\}\}$. Since D respects the filtration $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{N}}$, the set U is bounded above by d. Thus it has a greatest element d_0 which is exactly $\deg_{\mathcal{F}} D$ by definition.
Suppose that D respects the filtration $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{N}}$ and let $d = \deg_{\mathcal{F}} D$, we define a k-linear map $\overline{D} : \mathrm{Gr}(B) \longrightarrow \mathrm{Gr}(B)$ as follows: If $D=0$, then $\overline{D}=0$ the zero map. Otherwise, if $D \neq 0$ then we define

$$\overline{D} : \mathcal{F}_i/\mathcal{F}_{i-1} \longrightarrow \mathcal{F}_{i+d}/\mathcal{F}_{i+d-1}$$

by the rule $\overline{D}(a+\mathcal{F}_{i-1}) = D(a)+\mathcal{F}_{i+d-1}$. Now extend $\overline{D}$ to all of $\mathrm{Gr}(B)$ by linearity. One checks that $\overline{D}$ satisfies the Leibniz rule, therefore it is a homogeneous k-derivation of $\mathrm{Gr}(B)$ of degree d, that is, $\overline{D}$ sends homogeneous elements of degree i to zero or to homogeneous elements of degree $i+d$.

Observe that $\overline{D}=0$ if and only if $D=0$. In addition, $\mathrm{gr}(\ker D) \subset \ker \overline{D}$.

2. LND-Filtrations and New Invariant sub-algebras

In this section we introduce the ∂-filtration associated with a locally nilpotent derivation ∂. We explain how to compute this filtration and its associated graded algebra in certain situations. Also we present new invariants that generalize the Derksen invariant.

Definition 2.1. A k-derivation $\partial \in \mathrm{Der}_k(B)$ is said to be *locally nilpotent* if for every $a \in B$, there exists $n \in \mathbb{N}$ (depending of a) such that $\partial^n(a) = 0$. The set of all locally nilpotent derivations of B is denoted by $\mathrm{LND}(B)$.
In particular, every locally nilpotent derivation ∂ of B gives rise to a proper $\mathbb{N}$-filtration of B by the subspaces $\mathcal{F}_i = \ker\partial^{i+1}$, $i \in \mathbb{N}$, that we call the *∂-filtration.* It is straightforward to check (see [F, Prop. 1.9]) that the ∂-filtration corresponds to the $\mathbb{N}$-degree function $\deg_\partial : B \longrightarrow \mathbb{N} \cup \{-\infty\}$ defined by

$$\deg_\partial(a) := \min\{i \in \mathbb{N} \mid \partial^{i+1}(a) = 0\}, \text{ and } \deg_\partial(0) := -\infty.$$

Note that by definition $\mathcal{F}_0 = \ker\partial$ and that $\mathcal{F}_1 \setminus \mathcal{F}_0$ consists of all local slices for ∂.

Let $\mathrm{Gr}_\partial(B) = \oplus_{i\in\mathbb{N}}\mathcal{F}_i/\mathcal{F}_{i-1}$ denote the associated graded algebra relative to the ∂-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$. Let $\mathrm{gr}_\partial : B \longrightarrow \mathrm{Gr}_\partial(B)$; $a \overset{\mathrm{gr}_\partial}{\longmapsto} \overline{a}$ be the natural map between B and $\mathrm{Gr}_\partial(B)$ defined in 1.3, where $\overline{a}$ denote $\mathrm{gr}_\partial(a)$.

The next Proposition, which is due to Daigle ([F, Theorem 2.11], see also [D3, Theorem 1.7 and Corollary 4.12]), implies in particular that if B is of finite transcendence degree over k, then every non-zero $D \in \mathrm{LND}(B)$ respects the ∂-filtration and therefore induces a non-zero homogeneous locally nilpotent derivation $\overline{D}$ of $\mathrm{Gr}_\partial(B)$.

Proposition 2.2. (**Daigle**) *Suppose that B is a commutative domain, of finite transcendence degree over k. Then for every pair $D \in \mathrm{Der}_k(B)$ and $\partial \in \mathrm{LND}(B)$, D respects the ∂-filtration. Consequently, $\overline{D}$ is a well defined homogeneous derivation of the integral domain $\mathrm{Gr}_\partial(B)$ relative to this filtration, and it is locally nilpotent if D is locally nilpotent.*

2.1. **New Invariants** (AL_i**-invariants**).

Definition 2.3. Let $\partial \in \mathrm{LND}(B)$ be non-zero and let $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ be the ∂-filtration. We denote by L_∂ the sub-algebra of B generated by $\mathcal{F}_1 = \ker\partial^2$ and we call it the *ring of local slices* for ∂. The sub-algebra of B generated by L_∂ for all non-zero locally nilpotent derivation of B will be called the ring of *all local slices* of B. It will be denoted by $\mathrm{AL}(B)$ and referred to as the AL-*invariant*, which is manifested by the fact that $\mathrm{AL}(B)$ is invariant by all algebraic k-automorphisms of B.

In a sense, the AL-invariant is a generalization of the *Derksen invariant* $\mathcal{D}(B)$ which is defined to be the sub-algebra of B generated by $\ker\partial$ for all non-zero $\mathrm{LND}(B)$.

In a more general way we define the following invariants for non-rigid k-domains. Let $\mathrm{AL}_i(B)$ denotes the sub-algebra of B generated by $\ker\partial^{i+1}$ for all non-zero locally nilpotent derivation of B, then it is clear that $\mathrm{AL}_i(B)$ is invariant by all algebraic k-automorphisms of B. Indeed, let $\partial \in \mathrm{LND}(B) \setminus \{0\}$ and $\alpha \in \mathrm{Aut}_k(B)$, then $(\alpha^{-1}\partial\alpha)^n = \alpha^{-1}\partial^n\alpha$ for every $n \in \mathbb{N}$, which implies that $\partial_\alpha := \alpha^{-1}\partial\alpha \in \mathrm{LND}(B)$. Therefore, $\alpha.\partial_\alpha^n = \partial^n.\alpha$ so we get $\deg_{\partial_\alpha}(b) = \deg_\partial(\alpha(b))$ for any $b \in B$. In other words, α respects $\deg_\partial$ and $\deg_{\partial_\alpha}$, that is, α sends an element of degree i relative to $\deg_{\partial_\alpha}$, to an element of the same degree i relative to $\deg_\partial$.
Note that $\mathrm{AL}_0(B) = \mathcal{D}(B)$, $\mathrm{AL}_1(B) = \mathrm{AL}(B)$, and $\mathrm{AL}_i(B) \subseteq \mathrm{AL}_{i+1}(B)$ for all i.

In the case where $B = k[X_1, \ldots, X_n]/I = k[x_1, \ldots, x_n]$ is a finitely generated k-domain that admits a non-zero $\partial \in \mathrm{LND}(B)$, the chain of inclusions $\mathrm{AL}_0(B) \hookrightarrow \mathrm{AL}_1(B) \hookrightarrow \mathrm{AL}_2(B) \hookrightarrow \cdots \hookrightarrow \mathrm{AL}_i(B) \hookrightarrow \cdots$ will eventually stabilize, that is, there exists $d \in \mathbb{N}$ such that $\mathrm{AL}_d(B) = B$. Indeed, by definition of LND, there exist $d_1, \ldots, d_n \in \mathbb{N}$ such that $\partial^{d_i+1}(x_i) = 0$. Denote by $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ the ∂-filtration and let $d = \max_{i\in\{1,\ldots,n\}}\{d_i\}$, then $x_i \in \mathcal{F}_d$ for all i. Since $\mathcal{F}_d \subset \mathrm{AL}_d(B)$ by definition, we see that $B \subseteq \mathrm{AL}_d(B)$ and we are done.

Recall that the *Makar-Limanov invariant* $\mathrm{ML}(B)$ is defined to be the intersection of the kernels of all locally nilpotent derivations on B. One might think that $\mathrm{AL}_{i\in\mathbb{N}}$-invariants, in addition to the ML-invariant, cover all invariant sub-algebras of B. This, however, is not correct, see §4.1.3 below.

2.2. **Computing the ∂-filtration and its associated graded algebra.**

Here, given a finitely generated k-domain B, we describe a general method which enables the computation of the ∂-filtration for a locally nilpotent derivation with finitely generated kernel. First we consider a more general situation where the plinth ideal $\mathrm{pl}(\partial)$ is finitely generated as an ideal in $\ker\partial$ then we deal with the case where $\ker\partial$ is itself finitely generated as a k-algebra.

2.2.1. Let $B = k[X_1, \ldots, X_n]/I = k[x_1, \ldots, x_n]$ be a finitely generated k-domain, and let $\partial \in \mathrm{LND}(B)$ be such that $\mathrm{pl}(\partial)$ is generated by precisely m elements say $f_1, \ldots, f_m$ as an ideal in $\ker \partial$. Denote by $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{N}}$ the ∂-filtration, then: By definition $\mathcal{F}_0 = \ker \partial$. Furthermore, given elements $s_i \in \mathcal{F}_1$ such that $\partial(s_i) = f_i$ for every $i \in \{1, \ldots, m\}$, it is straightforward to check that

$$\mathcal{F}_1 \quad = \quad \mathcal{F}_0 s_1 + \ldots + \mathcal{F}_0 s_m + \mathcal{F}_0.$$

Letting $\deg_\partial(x_i) = d_i$, we denote by H_j the $\mathcal{F}_0$-sub-module in B generated by elements of degree j relative to $\deg_\partial$ of the form $s_1^{u_1} \ldots s_m^{u_m} x_1^{v_1} \ldots x_n^{v_n}$, that is,

$$H_j := \sum_{\sum_l u_l + \sum_i d_i . v_i = j} \mathcal{F}_0 \left(s_1^{u_1} \ldots s_m^{u_m} x_1^{v_1} \ldots x_n^{v_n}\right)$$

where $u_l, v_i \in \mathbb{N}$ for all i and l. The integer $\sum_l u_l + \sum_i d_i v_i$ is nothing but $\deg_\partial(s_1^{u_1} s_2^{u_2} \ldots . s_m^{u_m} x_1^{v_1} x_2^{v_2} \ldots . x_n^{v_n})$. Then we define a new $\mathbb{N}$-filration $\mathcal{G} = \{\mathcal{G}_i\}_{i\in\mathbb{N}}$ of B by setting

$$\mathcal{G}_i = \sum_{j \leq i} H_j.$$

By construction $\mathcal{G}_i \subseteq \mathcal{F}_i$ for all $i \in \mathbb{N}$, with equality for $i = 0$ and $i = 1$. The following result provides a characterization of when these two filtrations coincide:

Lemma 2.4. *The filtrations $\mathcal{F}$ and $\mathcal{G}$ are equal if and only if $\mathcal{G}$ is proper.*

Proof. One direction is clear since $\mathcal{F}$ is proper. Conversely, suppose that $\mathcal{G}$ is proper with the corresponding $\mathbb{N}$-degree function ω on B (see §1.1). Given $f \in \mathcal{F}_i \setminus \mathcal{F}_{i-1}$, $i > 1$, for every local slice $s \in \mathcal{F}_1 \setminus \mathcal{F}_0$, there exist $f_0 \neq 0, a_i \neq 0, a_{i-1}, \ldots, a_0 \in \mathcal{F}_0$ such that $f_0 f = a_i s^i + a_{i-1} s^{i-1} + \cdots + a_0$ (see [ML1, Proof of Lemma 4]). Since $\omega(g) = 0$ (resp. $\omega(g) = 1$) for every $g \in \mathcal{F}_0$ (resp. $g \in \mathcal{F}_1 \setminus \mathcal{F}_0$), we obtain

$$\omega(f) = \omega(f_0 f) = \omega(a_i s^i + a_{i-1} s^{i-1} + \cdots + a_0) = \max\{\omega(a_i s^i), \ldots, \omega(a_0)\} = i,$$

and so $f \in \mathcal{G}_i$. □

2.2.2. The twisting technique.
Next, we determine the ∂-filtration, for a locally nilpotent derivation ∂ with finitely generated kernel, by giving an effective criterion to decide when the $\mathbb{N}$-filtration $\mathcal{G}$ defined above is proper.

Hereafter, we assume that $0 \in \mathrm{Spec}(B)$ and that $\ker(\partial)$ is generated by elements $z_j \in B$ such that $z_j(0, \ldots, 0) = 0$, $j \in \{1, \ldots, r\}$. Since $\ker(\partial)$ is finitely generated k-algebra, the plinth ideal $\mathrm{pl}(\partial)$ is finitely generated. So there exist $s_1, \ldots, s_m \in \mathcal{F}_1$ such that $\mathcal{F}_1 = \mathcal{F}_0 s_1 + \ldots + \mathcal{F}_0 s_m + \mathcal{F}_0$. We can also assume that s_i is irreducible and $s_i(0, \ldots, 0) = 0$ for all i.

Letting $J \subset k^{[r+n+m]} = k[Z_1, \ldots, Z_r][X_1, \ldots, X_n][S_1, \ldots, S_m]$ be the ideal generated by I and the elements $Z_j = z_j(X_1, \ldots, X_n)$, $j \in \{1, \ldots, r\}$, $S_i = s_i(X_1, \ldots, X_n)$, $i \in \{1, \ldots, m\}$, then we have

$$B = k[Z_1, \ldots, Z_r][X_1, \ldots, X_n][S_1, \ldots, S_m]/J.$$

Note that by construction $0 \in \mathrm{Spec}(k^{[r+n+m]}/J)$.

We define an $\mathbb{N}$-weight degree function ω on $k^{[r+n+m]}$ by declaring that $\omega(Z_i) = 0 = \deg_\partial(z_i)$ for all $i \in \{1, \ldots, r\}$, $\omega(S_i) = 1 = \deg_\partial(s_i)$ for all $i \in \{1, \ldots, m\}$, and $\omega(X_i) = \deg_\partial(x_i) = d_i$ for all $i \in \{1, \ldots, n\}$. The corresponding proper $\mathbb{N}$-filtration $\mathcal{Q}_i := \{P \in k^{[n]} \mid \omega(P) \leq i\}$, $i \in \mathbb{N}$, on $k^{[r+n+m]}$ has the form $\mathcal{Q}_i = \oplus_{j \leq i} \mathcal{H}_j$ where

$$\mathcal{H}_j := \oplus_{\sum_j u_j + \sum_i d_i v_i = j} k[Z_1, \ldots, Z_r] S_1^{u_1} \ldots S_m^{u_m} X_1^{v_1} \ldots X_n^{v_n}.$$

By construction $\pi(\mathcal{Q}_i) = \mathcal{G}_i$ where $\pi : k^{[r+n+m]} \longrightarrow B$ denotes the natural projection. Indeed, since

$$\pi(\mathcal{Q}_i) = \sum_{j \leq i} \pi(\mathcal{H}_j)$$

and $\pi(\mathcal{H}_j) = \sum_{\sum_j u_j + \sum_i d_i v_i = j} k[z_1, \ldots, z_r] s_1^{u_1} \ldots . s_m^{u_m} x_1^{v_1} \ldots x_n^{v_n}$, we get

$$\pi(\mathcal{H}_j) = \sum_{\sum_j u_j + \sum_i d_i v_i = j} (\ker \partial)\, s_1^{u_1} \ldots . s_m^{u_m} x_1^{v_1} \ldots . x_n^{v_n}$$

which means precisely that $\pi(\mathcal{Q}_i) = \mathcal{G}_i$.

Let $\hat{J} \subset k^{[r+n+m]}$ be the homogeneous ideal generated by the highest homogeneous components relative to ω of all elements in J. Then we have the following result, which is inspired by the technique developed by S. Kaliman and L. Makar-Limanov:

Proposition 2.5. *The $\mathbb{N}$-filtration $\mathcal{G}$ is proper if and only if $\hat{J}$ is prime.*

Proof. It is enough to show that $\mathcal{G} = \{\pi(\mathcal{Q}_i)\}_{i\in\mathbb{N}}$ coincides with the filtration corresponding to the $\mathbb{N}$-semi-degree function ω_B on B defined by

$$\omega_B(p) := \min_{P\in\pi^{-1}(p)} \{\omega(P)\}.$$

Indeed, if so, the result will follow from [K-ML1, Lemma 3.2] which asserts in particular that ω_B is an $\mathbb{N}$-degree function on B if and only if $\hat{J}$ is prime. Let $\{\mathcal{G}_i^{'}\}_{i\in\mathbb{N}}$ be the filtration corresponding to ω_B. Given $f \in \mathcal{G}_i^{'}$ there exists $F \in \mathcal{Q}_i$ such that $\pi(F) = f$, which means that $\mathcal{G}_i^{'} \subset \pi(\mathcal{Q}_i)$. Conversely, it is clear that $\omega_B(z_i) = \omega(Z_i) = 0$ for all $i \in \{1, \ldots, r\}$. Furthermore $\omega_B(s_i) = \omega(S_i) = 1$ for all $i \in \{1, \ldots, m\}$, for otherwise $s_i \in \ker\partial$ which is absurd. Finally, if $d_i \neq 0$ and $\omega_B(x_i) < \omega(X_i) = d_i \neq 0$, then $x_i \in \pi(\mathcal{Q}_{d_i-1}) \subset \ker\partial^{d_i-1}$ which implies that $\deg_\partial(x_i) < d_i$, a contradiction. So $\omega_B(x_i) = d_i$. Thus $\omega_B(f) \leq i$ for every $f \in \pi(\mathcal{Q}_i)$ which means that $\pi(\mathcal{Q}_i) \subset \mathcal{G}_i^{'}$. □

The next Proposition, which is a reinterpretation of [K-ML1, Prop. 4.1], describes the associated graded algebra $\mathrm{Gr}_\partial(B)$ of the filtered algebra $(B, \mathcal{F})$ in the case where the $\mathbb{N}$-filtration $\mathcal{G}$ is proper:

Proposition 2.6. *If the $\mathbb{N}$-filtration $\mathcal{G}$ is proper then* $\mathrm{Gr}_\partial(B) \simeq k^{[r+n+m]}/\hat{J}$.

Proof. By virtue of ([K-ML1, Prop. 4.1]) the graded algebra associated to the filtered algebra $(B, \mathcal{G})$ is isomorphic to $k^{[r+n+m]}/\hat{J}$. So the assertion follows from Lemma 2.4. □

3. Semi-Rigid k-Domains

3.1. Definitions and basic properties.

In [F-M], D. Finston and S. Maubach considered rings B whose sets of locally nilpotent derivations are "one-dimensional" in the sense that $\mathrm{LND}(B) = \ker(\partial).\partial$ for some non-zero $\partial \in \mathrm{LND}(B)$. They called them *almost-rigid* rings. Hereafter, we consider the following definition which seems more natural in our context (see Prop. 3.3 below for a comparison between the two notions).

Definition 3.1. A commutative domain B over a field k of characteristic zero is called *semi-rigid* if all non-zero locally nilpotent derivations of B induce the same proper $\mathbb{N}$-filtration (equivalently, the same $\mathbb{N}$-degree function).
The unique proper $\mathbb{N}$-filtration of a semi-rigid k-domain B, that corresponds to any non-zero $\partial \in \mathrm{LND}(B)$, will be referred to and called the *unique* LND-*filtration* of B.

Semi-rigid k-domains B can be equivalently characterized in terms of their *Makar-Limanov invariant* $\mathrm{ML}(B) := \cap_{D\in\mathrm{LND}(B)} \ker(D)$ as follows:

Proposition 3.2. *A k-domain B is semi-rigid if and only if* $\mathrm{ML}(B) = \ker(\partial)$ *for any non-zero* $\partial \in \mathrm{LND}(B)$.

Proof. Given $D, E \in \mathrm{LND}(B) \setminus \{0\}$ such that $A := \ker(D) = \ker(E)$, there exist non-zero elements $a, b \in A$ such that $aD = bE$ ([F, Principle 12]) which implies that the D-filtration is equal to the E-filtration. So if $\mathrm{ML}(B) = \ker(\partial)$ for any non-zero $\partial \in \mathrm{LND}(B)$ then B is semi-rigid. The other implication is clear by definition. □

Recall that $D \in \mathrm{Der}_k(B)$ is *irreducible* if and only if $D(B)$ is contained in no proper principal ideal of B, and that B is said to satisfy the ascending chain condition (ACC) on principal ideals if and only if every infinite chain $(b_1) \subset (b_2) \subset (b_3) \subset \cdots$ of principal ideals of B eventually stabilizes. B is said to be a *highest common factor ring*, or HCF-ring, if and only if the intersection of any two principal ideals of B is again principal.

Proposition 3.3. *Let B be a semi-rigid k-domain satisfying the ACC on principal ideals. If* $\mathrm{ML}(B)$ *is an HCF-ring, then there exists a unique irreducible* $\partial \in \mathrm{LND}(B)$ *up to multiplication by unit. Consequently, every* $D \in \mathrm{LND}(B)$ *has the form* $D = f\partial$ *for some* $f \in \ker(\partial)$*, and so B is almost rigid.*

Proof. Existence follows from the fact that since B satisfies the ACC on principal ideals, then for every non-zero $T \in \mathrm{LND}(B)$, there exists an irreducible $T_0 \in \mathrm{LND}(B)$ and $c \in \ker(T)$ such that $T = cT_0$. ([F, Prop. 2.2 and Principle 7]). The argument for uniqueness is similar to that in [F, Prop. 2.2.b], but with a little difference, that is, in [F] it is assumed that B itself is an HCF-ring while here we only require that $\mathrm{ML}(B)$ is an HCF-ring. Namely, let $D, E \in \mathrm{LND}(B)$ be irreducible derivations, and let $A = \mathrm{ML}(B)$. By hypothesis $\ker(D) = \ker(E) = A$, so there exist non-zero $a, b \in A$ such that $aD = bE$ ([F, Principle 12]). Here we can assume that a, b are not units otherwise we are done. Set $T = aD = bE$. Since A is an HCF-ring, there exists $c \in A$ such that $aA \cap bA = cA$. Therefore, $T(B) \subset cB$, and there exists $T_0 \in \mathrm{LND}(B)$ such that $T = cT_0$. Write $c = as = bt$ for $s, t \in B$. Then $cT_0 = asT_0 = aD$ implies $D = sT_0$, and likewise $E = tT_0$. By irreducibility, s and t are units of B, and we are done. □

3.2. **Elementary examples of semi-rigid k-domains.**

3.2.1. **Polynomial rings in one variable over rigid k-domains**. Recall that a k-domain A is called *rigid* if the zero derivation is the only locally nilpotent k-derivation of A. Equivalently, A is rigid if and only if $\mathrm{ML}(A) = A$. The next Proposition, which is due to Makar-Limanov ([ML1, Lemma 21], also [C-ML, Theorem 3.1]), presents the simplest examples of semi-rigid k-domains.

Proposition 3.4. (**Makar-Limanov**) *Let A be a rigid domain of finite transcendence degree over a field k of characteristic zero. Then the polynomial ring $A[x]$ is semi-rigid.*

Proof. For the convenience of the reader, we provide an argument formulated in the LND-filtration language. Let ∂ be the locally nilpotent derivation of $A[x]$ defined by $\partial(a) = 0$ for every $a \in A$ and $\partial(x) = 1$. Then the ∂-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ is given by $\mathcal{F}_i = Ax^i \oplus \mathcal{F}_{i-1}$ where $\mathcal{F}_0 = \ker(\partial) = A$, and the associated graded algebra is $\mathrm{Gr}(A[x]) = \oplus_{i\in\mathbb{N}} A\overline{x}^i$, where $\overline{x} := \mathrm{gr}(x)$ and $\overline{A} = A$. Since $A[x]$ is of finite transcendence degree over k, Proposition 2.2 implies that every non-zero $D \in \mathrm{LND}(A[x])$ respects the ∂-filtration and induces a non-zero homogeneous locally nilpotent derivation $\overline{D}$ of $\mathrm{Gr}(A[x])$ of a certain degree $d = \deg_\partial(D) \geq -1$. It is enough to check that in fact $d = -1$. Indeed, if so then $D = a\partial$, for some $a \in A$ which implies the semi-rigidity of $A[x]$. So suppose for contradiction that $d \geq 0$, then $D(x) \in \mathcal{F}_{d+1} = Ax^{d+1} + \mathcal{F}_d$. Therefore, $\overline{D}$ sends $\overline{x}$ to zero or to $\overline{a}\,\overline{x}^{d+1}$. Either way, we have $\overline{x} \in \ker(\overline{D})$, see Corollary 1.20 [F]. Furthermore, $\overline{D}(\overline{a}) = \begin{cases} 0 \\ \overline{a}_0\overline{x}^d \end{cases}$, so $\overline{D} = \overline{x}^d E$ where $E(\overline{a}) = \begin{cases} 0 \\ \overline{a}_0 \end{cases}$ and $E(\overline{x}) = 0$. This asserts that $E \in \mathrm{LND}(A[\overline{x}])$ by virtue of [F, Principle 7]. Clearly, E restricts to $\mathrm{LND}(A)$, so by hypothesis $E = 0$ which yields $\overline{D} = 0$, a contradiction. □

3.2.2. **Danielewski k-domains**. Let

$$B_{n,P} = k[X, S, Y]/\langle X^n Y - P(X, S)\rangle$$

where $n \geq 1$, $d \geq 2$, $P(X, S) = S^d + f_{d-1}(X)S^{d-1} + \cdots + f_0(X)$, and $f_i(X) \in k[X]$. We call $B_{n,P}$ the Danielewski k-domain corresponding to the pair (n, P). Let x, s, y be the class of X, S, and Y in $B_{n,P}$. It is well known (see [ML1, Section 4] for the case $P \in k[S]$; and [P, Section 2.4] for the case, where $P(X, S) \in k[X, S]$) that if $n \geq 2$, then $\mathrm{ML}(B_{n,P}) = k[x]$ and $\mathrm{LND}(B_{n,P}) = k[x].\partial$, for the locally nilpotent derivation ∂ of B defined by

$$\partial = x^n\partial_s + \frac{\partial P}{\partial s}\partial_y,$$

where $\frac{\partial P}{\partial s} = ds^{d-1} + (d-1)f_{d-1}(x)s^{d-2} + \ldots + f_1(x)$. Hence, $B_{n,P}$ is almost rigid.

We easily recover these previous results using the LND-filtration method as follows:

The ∂-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ of $B_{n,P}$ is given by:

$$\mathcal{F}_{di+j} = k[x]s^j y^i + \mathcal{F}_{di+j-1}.$$

where $i \in \mathbb{N}$ and $j \in \{0, \ldots d-1\}$. The associated graded algebra is $\mathrm{Gr}_\partial(B_{n,P}) = k[X, S, Y]/\langle X^n Y - S^d\rangle$, and $B_{[di+j]} := \mathcal{F}_{di+j}/\mathcal{F}_{di+j-1} = k[\overline{x}]\overline{s}^j\overline{y}^i$ where $i \in \mathbb{N}$ and $j \in \{0, \ldots d-1\}$ (see §4.3 for more details). Corollary 4.6 below provides, in particular, an alternative argument formulated in the LND-filtration language proving directly that $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ is indeed the unique LND-filtration of B.

3.3. Algebraic isomorphisms between semi-rigid k-domains.

Let $\Psi : A \longrightarrow B$ be an algebraic isomorphism between two k-domains. Given $\partial \in \mathrm{LND}(B)$, then for any $n \in \mathbb{N}$ we have $(\Psi^{-1}\partial\Psi)^n = \Psi^{-1}\partial^n\Psi$. So we see that $\partial_\Psi := \Psi^{-1}\partial\Psi \in \mathrm{LND}(A)$. An immediate consequence is that $\Psi(\mathrm{ML}(A)) = \mathrm{ML}(B)$. Furthermore, $\Psi\{\ker(\partial_\Psi)\} = \ker(\partial)$, and more generally, Ψ sends elements of degree n, relative to ∂_Ψ, to elements of the same degree n relative to ∂, that is, $\deg_{\partial_\Psi}(a) = \deg_\partial(\Psi(a))$ for all $a \in A$. So in particular, $\Psi(\mathrm{AL}(A)) = \mathrm{AL}(B)$. These properties, combined with Definition 3.1, give the following result.

Proposition 3.5. *Let $\Psi : A \longrightarrow B$ be an isomorphism between two semi-rigid k-domains. Let $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ (resp. $\{\mathcal{G}_i\}_{i\in\mathbb{N}}$) be the unique* LND-*filtration* of A (resp. B). *Then: $\Psi(\mathcal{F}_i) = \mathcal{G}_i$ for every i.*

In the case where $A = B$, we obtain an action of the group $\mathrm{Aut_k}(B)$ of algebraic k-automorphisms of B by conjugation on $\mathrm{LND}(B)$. As a consequence of Proposition 3.5, every k-automorphism of a semi-rigid k-domain B preserves its unique LND-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$. Letting $\mathrm{Aut_k}\,(B, \mathrm{ML}(B))$ be the sub-group of $\mathrm{Aut_k}(B)$ consisting of elements whose induced action on $\mathrm{ML}(B)$ is trivial, we have the following Corollary which describes the structure of $\mathrm{Aut_k}(B)$.

Corollary 3.6. *For every semi-rigid k-domain B, there exists an exact sequence*

$$0 \to \mathrm{Aut_k}\,(B, \mathrm{ML}(B)) \to \mathrm{Aut_k}(B) \to \mathrm{Aut_k}\,(\mathrm{ML}(B))\,.$$

Furthermore, every element of $\mathrm{Aut_k}\,(B, \mathrm{ML}(B))$ induces for every $i \geq 1$ an automorphism of $\mathcal{F}_0$-module of each $\mathcal{F}_i$.

4. A new class of semi-rigid k-domains

In this section, we introduce a new family of domains $R_{n,e,P,Q}$ of the form

$$R_{n,e,P,Q} := k[X,Y,Z]/\left\langle X^nY - P\left(X, Q(X,Y) - X^eZ\right)\right\rangle$$

where $e \geq 0$, $n \geq 1$, $(n,e) \neq (1,0)$, $d, m \geq 2$,

$$P(X,S) = S^d + f_{d-1}(X)S^{d-1} + \cdots + f_1(X)S + f_0(X)\ ,\ \textit{and}$$

$$Q(X,Y) = Y^m + g_{m-1}(X)Y^{m-1} + \cdots + g_1(X)Y + g_0(X).$$

The trivial case ($e = 0$), corresponds to the Danielewski k-domains $B_{n,P} = k[X,S,Y]/\left\langle X^nY - P(X,S)\right\rangle$. Indeed, the ring $R_{n,0,P,Q} = k[X,Y,Z]/\left\langle X^nY - P\left(X, Q(X,Y) - Z\right)\right\rangle$ is isomorphic to $B_{n,P}$ via an isomorphism induced by $\Phi : k[X,S,Y] \longrightarrow k[X,Y,Z]$, where $\Phi(X) = X$, $\Phi(S) = -Z + Q(X,Y)$, and $\Phi(Y) = Y$. It is clear that $\Phi^* = \pi_{X^nY-P(X,Q(X,Y)-Z)} \circ \Phi$ is surjective, where $\pi_{X^nY-P(X,Q(X,Y)-Z)} : k[X,Y,Z] \to R_{n,e,P,Q}$ is the natural projection. Thus $R_{n,0,P,Q} = \mathrm{Im}\Phi^* \simeq k[X,S,Y]/\ker\Phi^*$. This yields, in particular, that the ideal $\ker\Phi^* \subset k^{[3]}$ is principal. But since $\Phi^*(X^nY - P(X,S)) = 0$, $\langle F\rangle \subset \ker\Phi^*$, and $X^nY - P(X,S)$ is irreducible, we conclude that $\langle X^nY - P(X,S)\rangle = \ker\Phi^*$. Therefore, Φ^* induces an isomorphism between the two rings.

Remark 4.1. Computing the ML-invariant for these examples using known techniques up to date is rather a hopeless task. Indeed, for the non-trivial case of $R_{n,e,P,Q}$ where $e \neq 0$, a real-valued weight degree function ω on $k^{[3]}$ has to be of the form $\omega = (\frac{md-1}{nm-md+1}\lambda, \frac{n}{nm-md+1}\lambda, \lambda)$, where $\lambda \in \mathbb{R}$, to induce a degree function ω_0 on $R_{n,e,P,Q}$. Hence, the associated graded algebra, corresponding to ω_0-filtration, takes the form $\mathrm{Gr}_\omega(R_{n,e,P,Q}) = k[X,Y,Z]/\langle X^nY - (Y^m - X^eZ)^d\rangle$. The latter ring is again another member of the new family that corresponds to $R_{n,e,S^d,Y^m} = \mathrm{Gr}_\omega(R_{n,e,P,Q})$. So any hope of simplifying the study of locally nilpotent derivation of $R_{n,e,P,Q}$, by studying the homogenous locally nilpotent derivation on the associated graded algebra R_{n,e,S^d,Y^m}, collapses. On the other hand, the remaining choices of ω in $\mathbb{R}^{[3]}$ induces a semi-degree function on $R_{n,e,P,Q}$ with the associated graded algebra $\mathrm{Gr}_\omega(R_{n,e,P,Q}) = k[X,Y,Z]/\langle(Y^m - X^eZ)^d\rangle$. This is not an integral domain, which complicates the situation even more.

Nevertheless, the LND-filtration method allows us to pass through these complications as will be shown in the rest of this paper. Indeed, consider $\omega \in \mathbb{N}^{[4]}$ the $\mathbb{N}$-weight degree function on $k^{[4]}$ defined by $\omega(X,S,Y,Z) = (0,1,d,md)$, then it induces $\omega_{R_{n,e,P,Q}}$ a degree function on

$$R_{n,e,P,Q} \simeq k[X,S,Y,Z]/\left\langle X^nY - P\left(X,S\right), Q(X,Y) - X^eZ - S\right\rangle.$$

It turns out that the degree function $\omega_{R_{n,e,P,Q}}$ coincides with $\deg_\partial$ for any non-zero $\partial \in \mathrm{LND}(R_{n,e,P,Q})$.

4.1. **Properties of the new class.**
Here, we point out some properties of $R_{n,e,P,Q}$ that we will establish in the rest of this section:

4.1.1. **Algebraic construction:** Consider the following k-domain

$$B_{n,P} = k[X,S,Y]/\langle X^nY - P(X,S)\rangle,$$

which is the Danielewski k-domain corresponding to the pair (n,P). Let us extend this ring by taking the sub-algebra of $k[X^{\pm 1},S]$ generated by $B_{n,P} \subset k[X^{\pm 1},S]$ and $z \in k[X^{\pm 1},S]$, where z is an algebraic element over $k[X,S]$ that has a dependence relation of the form

$$X^{nm+e}Z - [P(X,S)]^m - Xg_{m-1}(X)[P(X,S)]^{m-1} - \cdots - X^{m-1}g_1(X)P(X,S) - X^m g_0(X) + X^{nm}S.$$

By sending S to $Q(X,Y) - X^eZ$ we immediately see that:

$$B_{n,P} \subset R_{n,e,P,Q} \text{ , and } B_{nm+e,F} \subset R_{n,e,P,Q} \text{ ,}$$

where $B_{nm+e,F}$ is the Danielewski k-domain corresponding to the pair $(nm+e,F)$:

$$B_{nm+e,F} = k[X,S,Z]/\langle X^{nm+e}Z - F(X,S)\rangle,$$

and $F(X,S) = [P(X,S)]^m + Xg_{m-1}(X)[P(X,S)]^{m-1} + \cdots + X^{m-1}g_1(X)P(X,S) + X^m g_0(X)$.

Clearly, we have $B_{n,P}.B_{nm+e,F} = R_{n,e,P,Q}$, which simply means that $R_{n,e,P,Q}$ can be realized as the sub-algebra of $k[X^{\pm 1},S]$ generated by both $B_{n,P}$ and $B_{nm+e,F}$.

These new rings $R_{n,e,P,Q}$, for $e \neq 0$, are not isomorphic to any of Danielewski rings, see Proposition 4.13. Nevertheless, they share with them the property to come naturally equipped with an irreducible locally nilpotent derivation. But in contrast with the Danielewski rings, the corresponding derivation on $k[X,Y,Z]$ are no longer triangular, in fact not even triangulable by virtue of the characterization due to Daigle [D1]. For instance: let D be the locally nilpotent (triangular) derivation of $k[X,S,Y,Z]$ defined by:

$$\partial(X) = 0,\ \partial(S) = X^{n+e},\ \partial(Y) = X^e\tfrac{\partial P}{\partial S}, \text{ and } \partial(Z) = \tfrac{\partial Q}{\partial Y}\tfrac{\partial P}{\partial S} - X^n$$

where $\frac{\partial P}{\partial S} = dS^{d-1} + (d-1)f_{d-1}(X)S^{d-2} + \cdots + f_1(X)$, and $\frac{\partial Q}{\partial Y} = mY^{m-1} + (m-1)g_{m-1}(X)Y^{m-2} + \cdots + g_1(X)$. Then ∂ induces a non-zero irreducible locally nilpotent derivation of $R_{n,e,P,Q}$. Let x,s,y,z be the class of X, $S := Q(X,Y) - X^eZ$, Y, and Z in $R_{n,e,P,Q}$, then:

$$\partial = x^e\frac{\partial P}{\partial s}\partial_y + (\frac{\partial Q}{\partial y}\frac{\partial P}{\partial s} - x^n)\partial_z \in \mathrm{LND}(R_{n,e,P,Q})$$

Furthermore, $\mathrm{ML}(R_{n,e,P,Q}) = k[x]$ whenever $(n,e) \neq (1,0)$, see Corollary 4.6. This implies that $R_{n,e,P,Q}$ is semi-rigid, even almost rigid by virtue of Proposition 3.3. Hence $\mathrm{AL}(R_{n,e,P,Q}) = k[X,S]$. In addition, every non-zero locally nilpotent derivation of $R_{n,e,P,Q}$ restricts to a non-zero locally nilpotent derivation on $B_{n,P}$. And most importantly, every k-automorphism of $R_{n,e,P,Q}$ restricts to an automorphism of $B_{n,P}$. Also, it restricts to an k-automorphism of $\mathrm{AL}(R_{n,e,P,Q})$ (resp. $\mathrm{ML}(R_{n,e,P,Q})$). So in particular, every k-automorphism of $B_{n,P}$ ($\simeq R_{n,0,P,Q}$) restricts to a k-automorphism of $\mathrm{AL}(R_{n,e,P})$ (resp. $\mathrm{ML}(R_{n,e,P,Q})$). But of course this is not the full picture, see 4.1.3.

4.1.2. **Affine modification of the AL-invariant:** Here, we present another point of view about the construction of the new class.
The affine modification of the AL-invariant $\mathrm{AL}(R_{n,e,P,Q}) = k[X,S]$ along X^{nm+e} with center

$$I_1 = \langle X^{nm+e}, X^{n(m-1)+e}P(X,S), F(X,S)\rangle,$$

see [K-Z, Definition 1.1], coincides by virtue of [K-Z, Proposition 1.1] with

$$k[X,S]\left[I_1/X^{nm+e}\right] = k[X,S][P(X,S)/X^n, F(X,S)/X^{nm+e}] = k[X,S,y,z] \simeq R_{n,e,P,Q}.$$

Also, the affine modification of the AL-invariant $\mathrm{AL}(R_{n,e,P,Q}) = k[X,S]$ along X^n with center $I_2 = \langle X^n, P(X,S)\rangle$ coincides with

$$k[X,S]\left[I_2/X^n\right] = k[X,S][P(X,S)/X^n] = k[X,S,y] \simeq B_{n,P}.$$

Finally, the affine modification of $B_{n,P}$ along X^e with center $I_3 = \langle X^e, Q(X,Y) - S\rangle$ coincides with

$$B_{n,P}\left[I_3/X^e\right] = B_{n,P}[(Q(X,Y) - S)/X^e]) = B_{n,P}[z] \simeq R_{n,e,P,Q}.$$

We put together previous observations in the following Proposition.

Proposition 4.2. *with the above notation we have:*
(1) $R_{n,e,P,Q}$ *is the affine modification of the* AL*-invariant along* X^{nm+e} *with center* I_1.
(2) $B_{n,P}$ *is the affine modification of the* AL*-invariant along* X^n *with center* I_2.
(3) $R_{n,e,P,Q}$ *is the affine modification of* $B_{n,P}$ *along* X^e *with center* I_3.

4.1.3. **Invariant sub-algebras of $R_{n,e,P,Q}$:** For simplicity let $Q(X,Y) = Y^m$. Denote $R_{n,e,P} := R_{n,e,P,Y^m}$ and $B_{n,P} \simeq R_{n,0,P}$. Consider the two chains of inclusions:

The first chain of inclusions, which is realized by sending S to $Y^m - XZ$ for the first inclusion and by sending Z to XZ for the rest steps

$$B_{n,P} \hookrightarrow R_{n,1,P} \hookrightarrow \cdots \hookrightarrow R_{n,e,P}.$$

The second chain of inclusions, which is realized by sending Y to XY for every step

$$B_{1,P} \hookrightarrow B_{2,P} \hookrightarrow \cdots \hookrightarrow B_{n,P}.$$

Together they produce the following chain of inclusions

$$B_{1,P} \hookrightarrow B_{2,P} \hookrightarrow \cdots \hookrightarrow B_{n,P} \hookrightarrow R_{n,1,P} \hookrightarrow \cdots \hookrightarrow R_{n,e,P}$$

with the following properties.

Theorem 4.3. *With the above notation the following holds:*
(a) *Every non-zero* $\partial \in \mathrm{LND}(R_{n,e,P})$ *restricts to a non-zero* $\mathrm{LND}(R_{n,e_0,P})$ *for any* $e_0 \in \{1,\dots,e\}$. *Also, it restricts to a non-zero* $\mathrm{LND}(B_{n_0,P})$ *for any* $n_0 \in \{0,\dots,n\}$.
(b) *Every algebraic* k*-automorphism of* $R_{n,e,P}$ *restricts to an algebraic* k*-automorphism of* $R_{n,e_0,P}$ *for any* $e_0 \in \{1,\dots,e\}$. *Also, it restricts to a* k*-automorphism of* $B_{n_0,P}$ *for any* $n_0 \in \{1,\dots,n\}$.
(c) $R_{n_1,e_1,P} \simeq R_{n_2,e_2,P}$ *if an only if* $n_1 = n_2$ *and* $e_1 = e_2$. *Hence these* k*-domains are not algebraically isomorphic to each other (pairwise).*
(d) *Every element of the set*

$$\{\mathrm{ML}(R_{n,e,P}) = k[X],\ \mathrm{AL}(R_{n,e,P}) = k[X,S],\ B_{n_0,P}\ ,\ R_{n,e_0,P};\ n_0 \in \{1,\dots,n\}, e_0 \in \{1,\dots,e\}\}$$

represents an invariant sub-algebra of $R_{n,e,P}$.
(e) $\mathrm{AL}_0(R_{n,e,P}) = \mathcal{D}(B) = \mathrm{ML}(R_{n,e,P}) \hookrightarrow \mathrm{AL}(R_{n,e,P}) = \mathrm{AL}_1(R_{n,e,P}) = \cdots = \mathrm{AL}_{d-1}(R_{n,e,P}) \hookrightarrow B_{n,P} = \mathrm{AL}_d(R_{n,e,P}) = \cdots = \mathrm{AL}_{md-1}(R_{n,e,P}) \hookrightarrow \mathrm{AL}_{md}(R_{n,e,P}) = R_{n,e,P}$.

Proof. (a) Is immediate by Corollary 4.6 below. (b) is an immediate consequence of Theorem 4.10 below. (c) a consequence of Proposition 4.12 and Proposition 4.13 below. (a), (b), and Corollary 4.6 implies (d). Finally, (e) is a trivial consequence of Theorem 4.5, and Definition 2.3. □

4.2. **A toy example.**
We will begin with a very elementary example illustrating the steps needed to determine the LND-filtration and its associated graded algebra, and then we proceed to the general case. We let

$$R = k[X,Y,Z]/\langle X^2Y - (Y^2 - XZ)^2\rangle$$

and we let x, y, z be the class of X, Y, and Z in R. A direct computation reveals that the derivation

$$2XS\partial_Y + (4YS - X^2)\partial_Z$$

of $k[X,Y,Z]$ where $S := Y^2 - XZ$ is locally nilpotent and annihilates the polynomial $X^2Y - (Y^2 - XZ)^2$. Therefore, it induces a locally nilpotent derivation ∂ of R for which we have $\partial(x) = 0$, $\partial^3(y) = 0, \partial^5(z) = 0$. Furthermore, the element $s = y^2 - xz$ is a local slice for ∂ with $\partial(s) = x^3$. So we have $\deg_\partial(x) = 0$, $\deg_\partial(y) = 2$, $\deg_\partial(z) = 4$, $\deg_\partial(s) = 1$. The kernel of ∂ is $k[x]$ and the plinth ideal is the principal ideal generated by x^3.

Proposition 4.4. *With the notation above, we have:*
(1) *The* ∂*-filtration* $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ *is given by :*

$$\mathcal{F}_{4i+2j+l} = k[x]s^l y^j z^i + \mathcal{F}_{4i+2j+l-1}$$

where $i \in \mathbb{N}$, $j \in \{0,1\}$, $l \in \{0,1\}$.

(2) *The associated graded algebra* $\mathrm{Gr}_\partial(R) = \oplus_{i\in\mathbb{N}} R_{[i]}$, *where* $R_{[i]} = \mathcal{F}_i/\mathcal{F}_{i-1}$, *is generated by* $\overline{x} = \mathrm{gr}_\partial(x)$, $\overline{y} = \mathrm{gr}_\partial(y)$, $\overline{z} = \mathrm{gr}_\partial(z)$, $\overline{s} = \mathrm{gr}_\partial(s)$ *as an algebra over* k *with relations* $\overline{x}^2\overline{y} = \overline{s}^2$ *and* $\overline{x}\,\overline{z} = \overline{y}^2$, *that is* $\mathrm{Gr}_\partial(R) \simeq k[X,Y,Z,S]/\langle X^2Y - S^2, X\,Z - Y^2\rangle$. *Furthermore:*

$$R_{[4i+2j+l]} = k[\overline{x}]\overline{s}^l\overline{y}^j\overline{z}^i$$

where $i \in \mathbb{N}$, $j \in \{0,1\}$, $l \in \{0,1,2,3\}$.

Proof. 1) First, the ∂-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ is given by $\mathcal{F}_r = \sum_{h\leq r} H_h$ where $H_h := \sum_{u+2v+4w=h} k[x]\,(s^u y^v z^w)$ and $u, v, w, h \in \mathbb{N}$. To show this, let J be the ideal in $k^{[4]} = k[X,Y,Z,S]$ defined by

$$J = \langle X^2Y - S^2, Y^2 - XZ - S\rangle.$$

Define an $\mathbb{N}$-weight degree function ω on $k^{[4]}$ by declaring that $\omega(X) = 0$, $\omega(S) = 1$, $\omega(Y) = 2$, and $\omega(Z) = 4$. By Proposition 2.5, the $\mathbb{N}$-filtration $\{\mathcal{G}_r\}_{i\in\mathbb{N}}$ where $\mathcal{G}_r = \sum_{h\leq r} H_h$ is proper if and only if $\hat{J}$ is prime. Which is the case since $\hat{J} = \left\langle X^2Y - S^2, Y^2 - XZ\right\rangle$ is prime. Thus by Lemma 2.4 we get the desired description.

Second, let $l \in \{0,1\}$ and $j \in \{0,1,2,3\}$ be such that $l := r \bmod 2$, $j := r - l \bmod 4$, and $i := \frac{r-2j-l}{4}$. Then we get the following unique expression $r = 4i + 2j + l$. Since $\mathcal{F}_r = \sum_{u+2v+4w=r} k[x]\,(s^u y^v z^w) + \mathcal{F}_{r-1}$, we conclude in particular that $\mathcal{F}_r \supseteq k[x]s^l y^j z^i + \mathcal{F}_{r-1}$. For the other inclusion, the relation $x^2y = s^2$ allows to write $s^u y^v z^w = x^e s^l y^{v_0} z^w$ and from the relation $y^2 = s + zx$ we get $x^e s^l y^{v_0} z^w = x^e s^l y^j (s+xz)^n z^w$. Since the monomial with the highest degree relative to $\deg_\partial$ in $(s+xz)^n$ is $x^n.z^n$, we deduce that $x^e s^l y^j (s+xz)^n z^w = x^{e+n} s^l y^j z^{w+n} + \sum M_\beta$ where M_β is monomial in x, y, s, z of degree less than r. Since the expression $r = 4i + 2j + l$ is unique, we get $w + n = i$. So $s^u y^v z^w = x^{e+n} s^l y^j z^i + f$ where $f \in \mathcal{F}_{r-1}$. Thus $k[x]\,(s^u y^v z^w) \subseteq k[x]s^l y^j z^i + \mathcal{F}_{r-1}$ and finally $\mathcal{F}_r = k[x]s^l y^j z^i + \mathcal{F}_{r-1}$.

2) By part (1), an element f of degree r can be written as $f = g(x)s^l y^j z^i + f_0$ where $f_0 \in \mathcal{F}_{r-1}$, $l = r \bmod 2$, $j = r - l \bmod 4$, $i = \frac{r-2j-l}{4}$, and $i \in \mathbb{N}$, $j \in \{0,1\}$, $l \in \{0,1\}$. So by Lemma 1.4, P2, P1, and P3, respectively we get

$$\overline{f} = \overline{g(x)s^l y^j z^i + h} = \overline{g(x)s^l y^j z^i} = \overline{g(x)}\overline{s}^l\overline{y}^j\overline{z}^i = g(\overline{x})\overline{s}^l\overline{y}^j\overline{z}^i$$

and therefore $\overline{B}_{[4i+2j+l]} = k[\overline{x}]\overline{s}^l\overline{y}^j\overline{z}^i$.

Finally, by Proposition 2.6, $\mathrm{Gr}_\partial(B) = k[X,Y,Z,S]/\langle X^2Y - S^2, X\,Z - Y^2\rangle$. □

4.3. **The general case.**

We now consider more generally rings $R_{n,e,P,Q}$ of the form

$$k[X,Y,Z]/\left\langle X^nY - P\left(X, Q(X,Y) - X^eZ\right)\right\rangle$$

where $e \geq 0$, $n \geq 1$, $(n,e) \neq (1,0)$, $d, m \geq 2$,

$$P(X,S) = S^d + f_{d-1}(X)S^{d-1} + \cdots + f_1(X)S + f_0(X)\ ,\ \textit{and}$$
$$Q(X,Y) = Y^m + g_{m-1}(X)Y^{m-1} + \cdots + g_1(X)Y + g_0(X)$$

Up to a change of variable of the form $Y \mapsto Y - c$ where $c \in k$, we may assume that $0 \in \mathrm{Spec}(R_{n,e,P,Q})$. Let x, y, z be the class of X, Y, and Z in $R_{n,e,P,Q}$. Define ∂ by: $\partial(x) = 0$, $\partial(s) = x^{n+e}$ where $s := Q(x,y) - x^e z$. Considering the relation $x^n y = P(x, Q(x,y) - x^e z)$, a simple computation leads to $\partial(y) = x^e\frac{\partial P}{\partial s}$,$\partial(z) = \frac{\partial Q}{\partial y}\frac{\partial P}{\partial s} - x^n$, that is

$$\partial := x^e\frac{\partial P}{\partial s}\partial_y + (\frac{\partial Q}{\partial y}\frac{\partial P}{\partial s} - x^n)\partial_z$$

where $\frac{\partial P}{\partial s} = ds^{d-1} + (d-1)f_{d-1}(x)s^{d-2} + \cdots + f_1(x)$, and $\frac{\partial Q}{\partial y} = my^{m-1} + (m-1)g_{m-1}(x)y^{m-2} + \cdots + g_1(x)$. Since $\partial(x^n y - P(x, Q(x,y) - x^e z)) = 0$ and $\partial^{d+1}(y) = 0, \partial^{md+1}(z) = 0$, ∂ is a well-defined locally nilpotent derivation of $R_{n,e,P,Q}$. The element s is a local slice for ∂ by construction, and a direct computation shows that $\deg_\partial(x) = 0$, $\deg_\partial(y) = d$, $\deg_\partial(z) = md$ and $\deg_\partial(s) = 1$. The kernel of ∂ is equal to $k[x]$. One checks further that the plinth ideal is equal to $\mathrm{pl}(\partial) = \langle x^{n+e}\rangle$. Furthermore, the exact same method as in the proof of Proposition 4.4 provides a full description of the ∂-filtration and its associated graded algebra, that is:

Theorem 4.5. *Let* ∂ *be defined as above, then we have:*

(1) *The* ∂*-filtration* $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{N}}$ *is given by :*

$$\mathcal{F}_{mdi+dj+l} = k[x]s^l y^j z^i + \mathcal{F}_{mdi+dj+l-1}$$

where $i \in \mathbb{N}$, $j \in \{0,\ldots,m-1\}$, $l \in \{0,\ldots,d-1\}$.

(2) *The associated graded algebra* $\mathrm{Gr}(R_{n,e,P,Q}) = \oplus_{i\in\mathbb{N}} R_{[i]}$, *where* $R_{[i]} = \mathcal{F}_i/\mathcal{F}_{i-1}$, *is generated by* $\overline{x} = \mathrm{gr}_\partial(x)$, $\overline{y} = \mathrm{gr}_\partial(y)$, $\overline{z} = \mathrm{gr}_\partial(z)$, $\overline{s} = \mathrm{gr}_\partial(s)$ *as an algebra over* k *with relations* $\overline{x}^n\overline{y} = \overline{s}^d$ *and* $\overline{x}^e\,\overline{z} = \overline{y}^m$ *, that is,* $\mathrm{Gr}(R_{n,e,P,Q}) = k[X,Y,Z,S]/\langle X^nY - S^d, X^eZ - Y^m\rangle$. *In addition, we have:*

$$R_{[mdi+dj+l]} = k[\overline{x}]\overline{s}^l\overline{y}^j\overline{z}^i$$

where $i \in \mathbb{N}$, $j \in \{0,\dots,m-1\}$, $l \in \{0,\dots,d-1\}$.

Proof. Consider $\omega \in \mathbb{N}^{[4]}$ the $\mathbb{N}$-weight degree function on $k^{[4]} = k[X,S,Y,Z]$ defined by $\omega(X,S,Y,Z) = (0,1,d,md)$. Let $J = \langle X^nY - P(X,S), Q(X,Y) - X^eZ - S\rangle$, it is clear that we can identify $R_{n,e,P,Q}$ with $R_{n,e,P,Q} \simeq k[X,S,Y,Z]/\langle X^nY - P(X,S), Q(X,Y) - X^eZ - S\rangle$. Since $\hat{J} = \langle X^nY - S^d, Y^m - X^eZ\rangle$ and $\langle X^nY - S^d, Y^m - X^eZ\rangle$ is a prime ideal in $k^{[4]}$, ω induces $\omega_{R_{n,e,P,Q}}$ (defined as in Proposition 2.5) a degree function on $k[X,S,Y,Z]/\langle X^nY - P(X,S), Q(X,Y) - X^eZ - S\rangle$. The latter coincides with $\deg_\partial$ by virtue of Lemma 2.4. Hence by Proposition 2.6, the associated graded algebra is given by $\mathrm{Gr}(R_{n,e,P,Q}) = k[X,Y,Z,S]/\langle X^nY - S^d, X^eZ - Y^m\rangle$. The explicit description of $\mathcal{F}_{mdi+dj+l}$ and $R_{[mdi+dj+l]}$, using the exact same method as in the proof of Proposition 4.4, is left to the reader. □

Corollary 4.6. *With the above notation, the following hold:*

(1) $\mathrm{ML}(R_{n,e,P,Q}) = k[x]$. *Consequently,* $R_{n,e,P,Q}$ *is semi-rigid, and its unique* LND*-filtration is the* ∂*-filtration.*

(2) *Every* $D \in \mathrm{LND}(R_{n,e,P,Q})$ *has the form* $D = f(x)\partial$. *Consequently,* $R_{n,e,P,Q}$ *is almost-rigid.*

Proof. (1) Given a non-zero $D \in \mathrm{LND}(R_{n,e,P,Q})$. By Proposition 2.2, D respects the ∂-filtration and induces a non-zero locally nilpotent derivation $\overline{D}$ of $\mathrm{Gr}(R_{n,e,P,Q})$. Suppose that $f \in \ker(D)\setminus k$, then $\overline{f} \in \ker(\overline{D})\setminus k$ is an homogenous element of $\mathrm{Gr}(R_{n,e,P,Q})$. So there exists $i \in \mathbb{N}$ such that $\overline{f} \in R_{[i]}$.

Assume that $\overline{f} \notin k[\overline{x}] = R_{[0]}$, then one of the elements $\overline{s}$, $\overline{y}$, $\overline{z}$ must divides $\overline{f}$ by Theorem 4.5. This leads to a contradiction as follows:

If $\overline{s}$ divides $\overline{f}$, then $\overline{s} \in \ker(\overline{D})$ as $\ker(\overline{D})$ is factorially closed, and for the same reason $\overline{x}, \overline{y} \in \ker(\overline{D})$ due to the relation $\overline{x}^n\overline{y} = \overline{s}^d$. Then by the relation $\overline{x}^e\,\overline{z} = \overline{y}^m$, we must have $\overline{z} \in \ker(\overline{D})$, which means $\overline{D} = 0$, a contradiction. In the same way, we get a contradiction if $\overline{y}$ divides $\overline{f}$.

Finally, if $\overline{z}$ divides $\overline{f}$, then $\overline{D}(\overline{z}) = 0$. So $\overline{D}$ induces in a natural way a locally nilpotent derivation $\widetilde{D}$ of the ring $\widetilde{R} = k(Z)[X,Y,S]/\langle X^n\,Y - S^d, X^e\,Z - Y^m\rangle$. It follows from the Jacobian criterion that $0 \in \mathrm{Spec}(\widetilde{\mathrm{R}})$ is a singular point, therefore $\widetilde{R}$ is rigid, see [F, Corollary 1.29]. Hence $\widetilde{D} = 0$, which implies $\overline{D} = 0$, a contradiction.

So the only possibility is that $\overline{f} \in k[\overline{x}]$, and this means $\deg_\partial(f) = 0$, thus $f \in k[x]$ and $\ker(D) \subset k[x]$. Finally, $k[x] = \ker(D)$ because $\mathrm{tr.deg_k}(\ker(D)) = 1$ and $k[x]$ is algebraically closed in $R_{n,e,P,Q}$. So $\mathrm{ML}(R_{n,e,P,Q}) = k[x]$.

(2) follows immediately from Proposition 3.3. □

As a direct consequence of Theorem 4.6, we have the following two Corollary.

Corollary 4.7. *The* AL_i*-invariant of* $R_{n,e,P,Q}$ *are given by:*

(1) $\mathrm{AL}_0(R_{n,e,P,Q}) = \mathcal{D}(B) = \mathrm{ML}(R_{n,e,P,Q}) = k[x]$.
(2) $\mathrm{AL}(R_{n,e,P,Q}) = \mathrm{AL}_1(R_{n,e,P,Q}) = \cdots = \mathrm{AL}_{d-1}(R_{n,e,P,Q}) = k[x,s]$.
(3) $B_{n,P} = \mathrm{AL}_d(R_{n,e,P,Q}) = \cdots = \mathrm{AL}_{md-1}(R_{n,e,P,Q}) = k[x,s,y] \simeq B_{n,P}$.
(4) $\mathrm{AL}_{md}(R_{n,e,P,Q}) = R_{n,e,P,Q}$.

Also, we have an interesting fact. Consider the following chain of inclusions, realized by the identity for the first inclusion and by sending S to $Q(X,Y) - X^eZ$ for the second one.

$$\mathrm{AL}(R_{n,e,P,Q}) = k[X,S] \hookrightarrow B_{n,P} = k[X,S,Y]/\langle X^nY - P(X,S)\rangle \hookrightarrow R_{n,e,P,Q}.$$

Then every non-zero locally nilpotent derivation of $R_{n,e,P,Q}$ restricts to a non-zero locally nilpotent derivation of $k[x,s,y] \simeq B_{n,P}$ ($\simeq R_{n,0,P,Q}$). Also, it restricts to a non-zero locally nilpotent derivation of the sub-algebra $k[x,s] \simeq k^{[2]}$.

Corollary 4.8. *Every non-zero* $D \in \mathrm{LND}(R_{n,e,P,Q})$ *restricts to a non-zero locally nilpotent derivation of* $B_{n,P}$ *(resp.* $\mathrm{AL}(R_{n,e,P,Q})$*).*

4.4. $\mathrm{Aut_k}$ **for the new class 4.3.**
For simplicity we only deal with the case where $Q(X,Y)=Y^m$. Up to change of variable of the form S by $S-\frac{f_{d-1}(X)}{d}$ we may assume without loss of generality that $f_{d-1}(X)=0$.

Let $R_{n,e,P}$ denote the ring

$$R_{n,e,P}:=R_{n,e,P,Y^m}=k[X,Y,Z]/\langle X^nY-P(X,Y^m-X^eZ)\rangle=k[x,y,z]$$

where $P(X,S)=S^d+f_{d-2}(X)S^{d-2}+\cdots+f_1(X)S+f_0(X)$, $e\geq 0$, $n\geq 1$, $(n,e)\neq(1,0)$, and $d,m\geq 2$. Let $\mathcal{F}=\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ be its unique LND-filtration.

As an immediate consequence of Corollary 4.7, we have the following Corollary that shows how the computation of the algebraic k-automorphism group of $R_{n,e,P}$ can be simplified by consider the following chain of inclusions, realized by sending S to Y^m-X^eZ for the last one.

$$\mathrm{ML}(R_{n,e,P})=k[x]\hookrightarrow \mathrm{AL}(R_{n,e,P})=k[x,s]\hookrightarrow \mathrm{AL}_d(R_{n,e,P})=B_{n,P}\hookrightarrow R_{n,e,P}.$$

Corollary 4.9. *Every algebraic k-automorphism of $R_{n,e,P}$ restricts to:*
(1) *a k-automorphism of* $\mathrm{AL}_d(R_{n,e,P})=B_{n,P}$ $(\simeq R_{n,0,P})$.
(2) *a k-automorphism of* $\mathrm{AL}(R_{n,e,P})=k[x,s]$.
(3) *a k-automorphism of* $\mathrm{ML}(R_{n,e,P})=k[x]$.

Nevertheless, for those who are not familiar with the algebraic k-automorphism group of $B_{n,P}$, we present a complete proof for the next Theorem 4.10 without implicitly using the previous Corollary 4.9. Let $\lambda,\mu\in k^*$, and $a(x)\in k[x]$. Denote $s=y^m-x^ez$ and $W:=\frac{P(\lambda x,\mu s+x^{n+e}a(x))-\mu^dP(x,s)}{\lambda^nx^n}$.

Theorem 4.10. *Every algebraic k-automorphism α of $R_{n,e,P}$ has the form:*

$$\alpha(x,s,y,z)=(\lambda x,\ \mu s+x^{n+e}a(x),\ \frac{\mu^d}{\lambda^n}y+W,\ \frac{\mu^{dm}}{\lambda^{nm+e}}z+\frac{(\frac{\mu^d}{\lambda^n}y+W)^m-\frac{\mu^{dm}}{\lambda^{nm}}y^m+x^{n+e}a(x)}{\lambda^ex^e})$$

where $\lambda,\mu\in k^*$ verify both: $\frac{\mu^{dm-1}}{\lambda^{nm}}=1$ and $f_{d-i}(\lambda x)\equiv\mu^if_{d-i}(x)\mod x^{n+e}$ for every $i\in\{2,\ldots,d\}$.

Proof. By Proposition 3.5, α preserve the unique LND-filtration of B, described in Theorem 4.5.

Thus we must have $\alpha(x)\in\mathcal{F}_0=k[x]$, $\alpha(s)\in\mathcal{F}_1=k[x]s+k[x]$, $\alpha(y)\in\mathcal{F}_d=k[x]y+\mathcal{F}_{d-1}$ and $\alpha(z)\in\mathcal{F}_{md}=k[x].z+\mathcal{F}_{md-1}$. In addition, α restricts to an automorphism of $\mathcal{F}_0=k[x]$. Therefore, $\alpha(x)=\lambda x+c$ where $\lambda\in k^*$, and $c\in k$.

Since α is invertible we get $\alpha(s)=\mu s+b(x)$, $\alpha(y)=\varepsilon y+h(x,s)$, and $\alpha(z)=\xi z+g(x,s,y)$ for some $\mu,\varepsilon,\xi\in k^*$, $b(x)\in k[x]$, $h(x,s)\in k[x,s]$, and $g(x,s,y)\in k[x,s,y]$.

By Corollary 4.6 (2) every $D\in\mathrm{LND}(R_{n,e,P})$ has the form $D=f(x)\partial$. In particular, $\partial_\alpha:=\alpha^{-1}\partial\alpha=f(x)\partial$ for some $f(x)\in k[x]$. Since $\alpha\partial_\alpha=\partial\alpha$ we have $\partial(\alpha(s))=\alpha(f(x)\partial(s))=f(\alpha(x))\alpha(x^{n+e})$ where $(\partial(s)=x^{n+e})$. So we get $\partial(\mu s+b(x))=f(\alpha(x))\,(\lambda x+c)^{n+e}$. Since $\partial(\mu y+b(x))=\mu x^{n+e}$, x divides $(\lambda x+c)^{n+e}$ in $k[x]$, and this is possible only if $c=0$, so we finally get $\alpha(x)=\lambda x$.

Applying α to the relation $x^ny=P(x,s)$ in $R_{n,e,P}$, we get $\lambda^nx^n\alpha(y)=P(\lambda x,\mu s+b(x))=\mu^dP(x,s)+d\mu^{d-1}s^{d-1}b(x)+H(x,s)$ where $\deg_sH\leq d-2$. Since x^n divides both x^ny and $P(x,s)$, and $\deg_sH\leq m-2$, we conclude that x^n divides $d\mu^{d-1}s^{d-1}b(x)+H(x,s)$ in $k[x,s]$. So x^n divides $b(x)$, that is $\alpha(s)=\mu s+x^na(x)$.

In addition, x^n divides every coefficient of H as a polynomial in s, so x^n divides $-\mu^df_{d-i}(x)+\mu^{d-i}f_{d-i}(\lambda x)$ because coefficients of $H(s)$ are of the form $q(x,s)b(x)-\mu^df_{d-i}(x)+\mu^{d-i}f_{d-i}(\lambda x)$. Since x^n divides $b(x)$, it divides also $-\mu^if_{d-i}(x)+f_{d-i}(\lambda x)$ for every i.

Now $\alpha(x)$ and $\alpha(s)$ fully determine $\alpha(y)$:

$$\alpha(y)=\frac{\mu^d}{\lambda^n}y+\frac{P(\lambda x,\mu s+x^na(x))-\mu^dP(x,s)}{\lambda^nx^n}.$$

Apply α to $x^ez=y^m-s$ to get $\lambda^ex^e\alpha(z)=(\frac{\mu^h}{\lambda^n}y+W)^m-\mu s-x^na(x)$ where $W=\frac{P(\lambda x,\mu s+x^na(x))-\mu^nP(x,s)}{\lambda^nx^n}\in k[x,s]$. So we have $\lambda^ex^e\alpha(z)=[\frac{\mu^{nm}}{\lambda^{nm}}y^m-\frac{\mu^{nm}}{\lambda^{nm}}s]+(\frac{\mu^{nm}}{\lambda^{nm}}-\mu)s+m(\frac{\mu^n}{\lambda^n}y)^{m-1}W+\ldots+W^m-x^na(x)$. Since $\frac{\mu^{nm}}{\lambda^{nm}}y^m-\frac{\mu^{nm}}{\lambda^{nm}}s=\frac{\mu^{nm}}{\lambda^{nm}}x^ez$, we see that x^e divides $G:=(\frac{\mu^{nm}}{\lambda^{nm}}-\mu)s+m(\frac{\mu^n}{\lambda^n}y)^{m-1}W+\ldots+W^m-x^na(x)$ in $k[x,s,y]\subset R_{n,e,P}$ because $\deg_sG\leq\deg_s(z)-1=md-1$.

Note that $W=\frac{d\mu^{d-1}s^{d-1}x^na(x)+H(x,s)}{\lambda^nx^n}$, thus $\deg_sW<d$. So by applying the map $\mathrm{gr}_{\mathcal{F}}$ we get

$$\overline{G}=\overline{m(\frac{\mu^n}{\lambda^n}y)^{m-1}W}=\overline{m(\frac{\mu^n}{\lambda^n}y)^{m-1}}\,\overline{d\frac{\mu^{d-1}}{\lambda^n}s^{d-1}a(x)}.$$

Since x^e divides G, x^e divides $a(x)$. Thus x^e divides every coefficients of G as a polynomial in y. So x^e divides W and $\frac{\mu^{nm}}{\lambda^{nm}} - \mu = 0$. This means that x^{n+e} divides $f_{d-i}(\lambda x) - \mu^i f_{d-i}(x)$ for all $i \in \{2, \dots, d\}$, and $\frac{\mu^{nm}}{\lambda^{nm}} = \mu$. Finally, by the relation $s = y^m - x^e z$, we get $\alpha(z) = \frac{\mu^{nm}}{\lambda^{nm+e}} z + \frac{m(\frac{\mu^n}{\lambda^n} y)^{m-1} W + \ldots + W^m + x^n a(x)}{\lambda^e x^e}$, and we are done. □

The next Corollary describes the algebraic k-automorphism group of $R_{n,e,P}$ in terms of the algebraic k-automorphism group of the AL-invariant. Denote by $\mathcal{A}_1$ the sub-group of $\mathrm{Aut}_k\left(\mathrm{AL}(R_{n,e,P})\right) = \mathrm{Aut}_k\left(k[X,S]\right)$ of automorphisms which preserve the ideals $\langle X \rangle$ and $I_1 = \langle X^{nm+e}, X^{n(m-1)+e} P(X,S), F(X,S) \rangle$. Also, denote by $\mathcal{A}_2$ the sub-group of $\mathrm{Aut}_k\left(\mathrm{AL}(R_{n,e,P})\right)$ of automorphisms which preserve the ideals $\langle X \rangle$ and $I_2 = \langle X^n, P(X,S) \rangle$. Finally, denote by $\mathcal{A}_3$ the sub-group of $\mathrm{Aut}_k\left(B_{n,P}\right)$ of automorphisms which preserve the ideals $\langle X \rangle$ and $I_3 = \langle X^e, Q(X,Y) - S \rangle$. Then,

Corollary 4.11. *In the case $e \neq 0$, $\mathrm{Aut}_k(R_{n,e,P}) \cong \mathcal{A}_1 = \mathcal{A}_3$. In the case $n \neq 1$, $\mathrm{Aut}_k\left(B_{n,P}\right) \cong \mathcal{A}_2$. The isomorphism of $\mathrm{Aut}_k(R_{n,e,P})$ to $\mathcal{A}_1 = \mathcal{A}_3$ is induced by restriction of any automorphism of $R_{n,e,P}$ to the* AL*-invariant.*

Proof. Theorem 4.10 implies that every algebraic k-automorphism of $R_{n,e,P}$ restricts to an algebraic k-automorphism of $\mathrm{AL}(R_{n,e,P}) = k[X,S]$ that preserves the ideals $\langle X \rangle$ and I_1 (resp. every algebraic k-automorphism of $B_{n,P}$ restricts to an algebraic k-automorphism of $\mathrm{AL}(R_{n,e,P}) = k[X,S]$ that preserves the ideals $\langle X \rangle$ and I_2). Finally, since $R_{n,e,P}$ is the affine modification of the AL-invariant along X^{nm+e} with center I_1 (resp. $B_{n,P}$ is the affine modification of the AL-invariant along X^n with center I_2), see Proposition 4.2, every algebraic k-automorphism of $\mathrm{AL}(R_{n,e,P})$ that preserves the ideals $\langle X \rangle$ and I_1 (resp. preserves the ideals $\langle X \rangle$ and I_2) extends in a unique way to an algebraic k-automorphism of the affine modification $R_{n,e,P}$ (resp. $B_{n,P}$), see [K-Z, Corollary 2.2]. □

4.5. **Isomorphism class for the new family .**
Again, we only deal with the case where $Q(X,Y) = Y^m$. First, in Proposition 4.12 we deal with the non-trivial case of $R_{n,e,P}$ where $(e \neq 0)$. We deliberately exclude the trivial case $e \neq 0$, which correspond to Danielewski k-domains of the form $B_{n,P}$ $(\simeq R_{n,0,P,m})$. Then, in Proposition 4.13 we compare the non-trivial case of $R_{n,e,P}$ $(e \neq 0)$ with the trivial case $B_{n,P}$ $(\simeq R_{n,0,P,m})$. The reason for doing that is to elaborate the importance of the non-trivial case $R_{n,e,P}$ where $(e \neq 0)$, that is, they are not isomorphic to any of Danielewski k-domains.

4.5.1. The case $R_{n,e,P}$ where $(e \neq 0)$. Let $R_{n,e,P}$ be the ring defined as

$$R_{n,e,P} := k[X,Y,Z] / \langle X^n Y - P(X, Y^m - X^e Z) \rangle$$

where $n, e \geq 1$, $P(X,S) = S^d + f_{d-2}(X) S^{d-2} + \cdots + f_1(X) S + f_0(X)$, and $d, m > 1$.

We give necessary and sufficient conditions for two rings, of the form $R_{n,e,P}$, to be isomorphic. Let $P_1(X,S) = S^{d_1} + f_{d-2}(X) S^{d_1-2} + \cdots + f_1(X) S + f_0(X)$ and $P_2(X,S) = S^{d_2} + g_{d-2}(X) S^{d_2-2} + \cdots + g_1(X) S + g_0(X)$. Then we have the following

Proposition 4.12. *$R_{n_1,e_1,P_1,m} \simeq R_{n_2,e_2,P_2,m}$ if an only if $n = n_1 = n_2$, $e = e_1 = e_2$, $d = d_1 = d_2$, and there exist $\lambda, \mu \in k^*$ such that $f_{d-i}(\lambda X) \equiv \mu^i g_{d-i}(X) \mod X^{n+e}$, and $\frac{\mu^{dm}}{\lambda^{nm}} = \mu$ for every $i \in \{2, \dots, d\}$.*

Proof. Let x_i, s_i, y_i, z_i be the class of X, $S = Y^m - X^{e_i} Z$, Y, and Z in $R_{n_i,e_i,P_i,m}$ for $i \in \{1,2\}$. Let $\Psi : R_{n_1,e_1,P_1,m} \longrightarrow R_{n_2,e_2,P_2,m}$ be an isomorphism between the two semi-rigid rings. Then it induces ψ an automorphism of $R_{n_2,e_2,P_2,m}$, which restricts by Corollary 4.9 to an automorphism of

$$k[x_2, s_2, y_2] = B_{n_2,P_2} := k[X,S,Y] / \langle X^{n_2} Y - P_2(X,S) \rangle \subset R_{n_2,e_2,P_2,m}.$$

Also, ψ restricts to an automorphism of $k[x_2, s_2]$ (resp. $k[x_2]$).

Proposition 3.5 shows that Ψ respects the semi-rigid structure, that is, $\Psi\left(\mathcal{F}_j\right) = \mathcal{G}_j$ for every j, where $\{\mathcal{F}_j\}_{j \in \mathbb{N}}$ (resp. $\{\mathcal{G}_j\}_{j \in \mathbb{N}}$) is the unique LND-filtration of $R_{n_1,e_1,P_1,m}$ (resp. $R_{n_2,e_2,P_2,m}$). Therefore, Ψ restricts to an isomorphism between $k[x_1, s_1, y_1] = B_{n_1,P_1}$ and $k[x_2, s_2, y_2] = B_{n_2,P_2}$ Also, Ψ restricts to an isomorphism between $k[x_1, s_1]$ and $k[x_2, s_2]$ (resp. $k[x_1]$ and $k[x_2]$). So we conclude that $\Psi(x_1) = \psi(x_2)$, $\Psi(s_1) = \psi(s_2)$, $\Psi(y_1) = \psi(y_2)$, and $\Psi(z_1) = \psi(z_2)$. This directly implies that $n = n_1 = n_2$, and $d = d_1 = d_2$.

In addition, Theorem 4.10 fully describes ψ, so we get the following form of Ψ:

$$\begin{cases} \Psi(x_1) = \psi(x_2) = \lambda x_2 \\ \Psi(s_1) = \psi(s_2) = \mu s_2 + x_2^{n+e_2} a(x_2) \\ \Psi(y_1) = \psi(y_2) = \frac{\mu^d}{\lambda^n} y_2 + W \\ \Psi(z_1) = \psi(z_2) = \frac{\mu^{dm}}{\lambda^{nm+e_2}} z_2 + \frac{(\frac{\mu^d}{\lambda^n} y_2 + W)^m - \frac{\mu^{dm}}{\lambda^{nm}} y_2^m + x_2^{n+e_2} a(x_2)}{\lambda^{e_2} x_2^{e_2}} \end{cases}$$

for certain $\mu, \lambda \in k*$ such that $\frac{\mu^{dm}}{\lambda^{nm}} = \mu$ and $g_{d-i}(\lambda x_2) \equiv \mu^i g_{d-i}(x_2) \mod x_2^{n+e_2}$ for every $i \in \{2, \ldots, d_2\}$, $a(x_2) \in k[x_2]$, and $W := \frac{P_2(\lambda x_2, \mu s_2 + x_2^{n+e_2} a(x_2)) - \mu^d P_2(x_2, s_2)}{\lambda^n x_2^n}$.

Finally, applying Ψ to the relation $x_1^{e_1} z_1 = y_1^m - s_1$ in $R_{n,e_1,P_1,m}$, we get $\lambda^{e_1} x_2^{e_1} \Psi(z_1) = \frac{\mu^{md}}{\lambda^{mn}} [y_2^m - s_2] + b = \frac{\mu^{md}}{\lambda^{mn}} [x_2^{e_2} z_2] + b$ where $b \in \mathcal{G}_{md-1}$.

Comparing top homogeneous components, relative to the filtration $\{\mathcal{G}_j\}_{j \in \mathbb{N}}$, for the last equation, we obtain $e = e_1 = e_2$. And we are done. □

4.5.2. Comparison with Danielewski k-domains.
The next Proposition shows that rings of the new family $R_{n,e,P}$ ($e \neq 0$) are not isomorphic to any of Danielewski k-domains.

Proposition 4.13. *The ring $R_{n_1,e,P,Q} = k[X,Y,Z]/\langle X^{n_1} Y - P(X, Q(X,Y) - X^e Z) \rangle$ is not isomorphic to $B_{n_2,F} = k[X,S,Y]/\langle X^{n_2} Y - F(X,S) \rangle$ for any $n_1, n_2, e > 0$, and $P(X,S), Q(X,S), F(X,S) \in k[X,S]$ such that $\deg_S F, \deg_S Q, \deg_S P \geq 2$.*

Proof. The case where $n_2 = 1$ is obvious since $\mathrm{ML}(B_{1,P}) = k$ which yields by Proposition 3.2 that $B_{1,P}$ is not semi-rigid, while $R_{n_1,F,Q}$ is for any $n_1 \geq 1$, see Corollary 4.6.

Suppose that $n_2 \geq 2$, then both $B_{n_2,P}$ and $R_{n_1,e,F,Q}$ are semi-rigid k-domains. Let x_1, s_1, y_1, z be the class of X, $S = Q(X,Y) - X^e Z$, Y, and Z in $R_{n_1,F,Q}$, and let x_2, s_2, y_2 be the class of X, S, and Y in $B_{n_2,P}$. denote by $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$ (resp. $\{\mathcal{G}_i\}_{i \in \mathbb{N}}$) the unique proper $\mathbb{N}$-filtration of $R_{n_1,e,F,Q}$ (resp. $B_{n_2,P}$).

Let $\Psi : B_{n_2,P} \longrightarrow R_{n_1,e,F,Q}$ be an isomorphism between the two rings, then Ψ must respect the semi-rigid structure of the two rings, that is, $\Psi(\mathcal{G}_i) = \mathcal{F}_i$ for every i, see 4.5. This immediately implies that Ψ restricts to an isomorphism between AL-invariants $k[x_2, s_2] \underset{\Psi}{\simeq} k[x_1, s_1]$.

On the other hand, we have $y_2 \in \mathcal{G}_l$, $y_1 \in \mathcal{F}_d$, $z \in \mathcal{F}_{md}$, where $\deg_S F = l, \deg_S Q = m, \deg_S P = d$. Assume that $d \lneq l$, then there exists an element $b \in k[x_2, s_2]$ such that $\Psi(b) = y_1$, which means that $y_1 \in k[x_1, s_1]$, a contradiction. In the same way we get a contradiction if we assumed that $l \lneq d$. So the only possibility is $d = l$, thus we conclude that $k[x_2, s_2, y_2] \underset{\Psi}{\simeq} k[x_1, s_1, y_1]$. Finally, let $b \in B_{n_2,P}$ such that $\Psi(b) = z$. Since $\Psi(b) \in k[x_1, y_1, s_1]$, we get $z \in k[x_1, y_1, s_1]$, which is a contradiction ($e \geq 1$). And we are done. □

5. Cylinders over the new class

In this section we are interested in finding an algorithm to construct an explicit isomorphism between cylinders over certain member of the new family rather than stating that such cylinder are isomorphic. The latter is known to be true in the abstract due to the classic Danielewski argument.

We will create explicit isomorphisms between cylinders over rings of the form $R_{n,e}$ defined by:

$$R_{n,e} := R_{n,e,S^2+1,Y^2} = k[X,Y,Z]/\langle X^n Y - (Y^2 - X^e Z)^2 - 1 \rangle$$

where $e \geq 0$, $n \geq 1$, and $(n,e) \neq (1,0)$.

5.1. Basic strategy .

Let $\Phi : k^{[N]} \longrightarrow k^{[N]}$ be an endomorphism of $k^{[N]} = k[X_1, \ldots, X_N]$ and let $F \in k^{[N]}$ be an irreducible polynomial. Let G be an irreducible factor of $\Phi(F)$ in $k^{[N]}$, so we have $\Phi(F) \in \langle G \rangle$. Consider the induced homomorphism of algebras $\Phi^* : k^{[N]} \longrightarrow k^{[N]}/\langle G \rangle$ given by $\Phi^* = \pi_G \circ \Phi$ where $\pi_G : k^{[N]} \longrightarrow k^{[N]}/\langle G \rangle$ is the natural projection. Notice that $\mathrm{Im}(\Phi^*) \simeq k^{[N]}/\ker(\Phi^*)$.

Now, suppose that Φ^* is surjective, then $\ker(\Phi^*) = \langle F \rangle$. Indeed, if Φ^* is surjective then $\mathrm{Im}(\Phi^*) \simeq k^{[N]}/\langle G \rangle \simeq k^{[N]}/\ker(\Phi^*)$ which implies in particular that the ideal $\ker(\Phi^*) \subset k^{[N]}$ is principal. But since $\Phi^*(F) = 0$, $\langle F \rangle \subset \ker \Phi^*$, and F is irreducible, we conclude that $\langle F \rangle = \ker \Phi^*$.

The latter shows that an isomorphism between $k^{[N]}/\langle F\rangle$ and $k^{[N]}/\langle G\rangle$ can be obtained if we find an endomorphism of $k^{[N]}$ that verify: first $\Phi(F) \in \langle G\rangle$ (or simply $\Phi(F) = G$), and second $\Phi^* = \pi_G \circ \Phi$ is surjective.

5.2. **The case $e \neq 0$.**
First we will start with a simple case and then we proceed to the general case by induction. We should mention that following results are known abstractly, however here we give an algorithm.

Lemma 5.1. $R_{1,1} \otimes_k k[T] \simeq R_{1,2} \otimes_k k[T]$.

Proof. (Which is also an **algorithm** to construct isomorphisms)
Let $\Phi : k^{[5]} \longrightarrow k^{[5]}$ be the endomorphism of $k^{[5]} = k[X, S, Y, Z, T]$ defined as follows:

$$\begin{cases} \Phi(X) = X \\ \Phi(S) = S + H(X,T) \\ \Phi(Y) = Y + L(X,S,T) \\ \Phi(Z) = XZ + F(X,S,Y,T) \\ \Phi(T) = T(X,S,Y,Z,T) \end{cases}$$

We choose $H = H(X,T)$ such that:
a) $\Phi(XY - S^2 - 1) = XY - S^2 - 1$. This gives the following relation between H and $L = L(X,S,T)$:

$$XL = 2HS + H^2$$

which implies that X divides H. So $H = XH_1$, hence $L = 2H_1S + XH_1^2$.
b) $\Phi(Y^2 - XZ - S) = Y^2 - X^2Z - S$. This gives the following relation between H, and $F = F(X,S,Y,T)$:

$$XF = 2Y(2H_1S + XH_1^2) + (2H_1S + XH_1^2)^2 - XH_1$$

which directly implies that X divides H_1, and we obtain $H = X^2H_2$ for some $H_2 \in k[X,T]$.
Note that if X^3divides H, then we immediately notice that $\Phi(Z)$ is divisible by X which implies that Φ does not induce an isomorphism. Therefore, The only choice for H is such that X^2 divides H but X^3 does not.
Let for instance $H(X,T) = X^2T$ (any other choice such that $H_2 \in k[T]$ will do), then it reminds to determine $\Phi(T)$ to fully describe Φ.
Choose $\Phi(T)$ to be such that the following holds

$$\Phi(YZ - XT) = 4T(-XY) + 4TS(Y^2 - X^2Z)$$

which simply means that

$$\Phi(YZ - XT) \equiv -4T \mod \langle XY - S^2 - 1, Y^2 - X^2Z - S\rangle.$$

Note that such a choice of $\Phi(T)$ is always possible even in a more complicated situation where $P(X,S)$ can be any polynomial in $k[X,S]$.
An elementary computation can determine $\Phi(T)$ to reach to the following form of Φ:
(1) $\Phi(X) = X$
(2) $\Phi(S) = S + X^2T$
(3) $\Phi(Y) = Y + 2XST + X^3T^2$
(4) $\Phi(Z) = XZ + 4SYT - XT + 2X^2YT^2 + 4XS^2T^2 + 4X^3ST^3 + X^5T^4$
(5) $\Phi(T) = ZY + 6XSZT + 3YT + 2XY^2T^2 + 12S^2YT^2 + X^3ZT^2 - X^3T^3 + 12X^2SYT^3 + 8XS^3T^3 + 3X^4YT^4 + 12X^3S^2T^4 + 6X^5ST^5 + X^7T^6$
Now, define $\phi : k^{[4]} \longrightarrow k^{[4]}$ to be the endomorphism of $k^{[4]} = k[X,Y,Z,T]$ given by: $\phi(X) = \Phi(X)$, $\phi(Y) = \Phi(Y)$, $\phi(Z) = \Phi(Z)$, $\phi(T) = \Phi(T)$ where we substitute S by $S = Y^2 - X^2Z$. Then we have $\phi(Y^2 - XZ) = S + X^2T$, $\phi\left(XY - (Y^2 - XZ)^2 - 1\right) = XY - (Y^2 - X^2Z)^2 - 1$, and $\phi(YZ - XT) = -4T$ mod $\langle XY - (Y^2 - X^2Z)^2 - 1\rangle$.
As discussed before §5.1, to prove that ϕ induces an isomorphism between $R_{1,1} \otimes_k k[T]$ and $R_{1,2} \otimes_k k[T]$, it is enough to show that $\phi^* = \pi_{XY-(Y^2-X^2Z)^2-1} \circ \phi$ is surjective. For that, denote by x, s, y, z, t the class of X, S, Y, Z, and T in $R_{1,2} \otimes_k k[T]$, then (1) immediately shows that $x \in \mathrm{Im}\phi^*$. By construction $t \in \mathrm{Im}\phi^*$. Therefore, (2) implies $s \in \mathrm{Im}\phi^*$, and (3) implies that $y \in \mathrm{Im}\phi^*$. So (4) provides $xz \in \mathrm{Im}\phi^*$, and (5) ensures that $yz \in \mathrm{Im}\phi^*$. Since $s = y^2 - x^2z$, we get $sz = y(yz) - (xz)^2$. This means that $sz \in \mathrm{Im}\phi^*$. Finally, since

$z = z(xy - s^2) = x(yz) - s(sz)$ where all terms in the second part of the last equation belong to $\mathrm{Im}\phi^*$, we deduce that $z \in \mathrm{Im}\phi^*$. In conclusion, ϕ^* is surjective. $\square$

The exact same algorithm, as in the proof of Lemma 5.1, can be applied to construct an isomorphism between $R_{1,e} \otimes_k k[T]$ and $R_{1,e+1} \otimes_k k[T]$ for every $e > 0$. The only different step is that $\Phi(T)$ must be chosen to verify:

$$\Phi(YZ - XT) = 4T(-XY) + 4TS(Y^2 - X^{e+1}Z).$$

Also, the same algorithm can be used to establish an isomorphism between $R_{n,1} \otimes_k k[T]$ and $R_{n,2} \otimes_k k[T]$ for every $n > 0$, where $\Phi(T)$ is chosen to hold:

$$\Phi(YZ - XT) = 4T(-X^nY) + 4TS(Y^2 - X^2Z).$$

We put together the previous observation to obtain the following.

Lemma 5.2. *$R_{1,e} \otimes_k k[T] \simeq R_{1,e+1} \otimes_k k[T]$, and $R_{n,1} \otimes_k k[T] \simeq R_{n,2} \otimes_k k[T]$.*

Finally, by induction we get:

Theorem 5.3. *$R_{n,e_1} \otimes_k k[T] \simeq R_{n,e_2} \otimes_k k[T]$ for every $n, e_1, e_2 > 0$.*

In addition, if ϕ_{e_1+i,e_1+i+1} is the endomorphisms, as determined in Lemma 5.1, of $k^{[4]}$ that induces an isomorphism between $R_{n,e_1+i}[T]$ and $R_{n,e_1+i+1}[T]$, then the endomorphisms $\phi_{e_2-1,e_2} \circ \cdots \circ \phi_{e_1,e_1+1}$ of $k^{[4]}$ induces an isomorphism between $R_{n,e_1}[T]$ and $R_{n,e_2}[T]$.

5.2.1. A counter-example of the cancellation problem.
Consider the following chains of inclusions, which is realized by sending Z to XZ in every step.

$$R_{n_0,1} \hookrightarrow R_{n_0,2} \hookrightarrow \cdots \hookrightarrow R_{n_0,e}$$

for every $n_0 \in \{1, \ldots, n\}$.

They are pairwise not isomorphic to each other by Proposition 4.12, whereas, Theorem 5.3 indicates

$$R_{n_0,1} \otimes_k k[T] \simeq R_{n_0,2} \otimes_k k[T] \simeq \cdots \simeq R_{n_0,e} \otimes_k k[T]$$

for every $n_0 \in \{1, \ldots, n\}$.

5.3. **Cylinders over Danielewski k-domains, the case $e = 0$.**
Here, we will show how to create an isomorphism between cylinders over k-domains of the form:

$$B_{n,P} = k[X, S, Y]/\langle X^nY - P(X, S)\rangle$$

for every $n \geq 1$. Where $P(X, S) = S^d + XQ(X, S) + c$, $c \in k - \{0\}$, and $Q(X, S) \in k[X, S]$,

First, we illustrate how the algorithm, presented in the proof of Lemma 5.1, can be modified to establish isomorphisms between the below mentioned rings and then we proceed to the general case.

Lemma 5.4. *$B_{1,P} \otimes_k k[T] \simeq B_{2,P} \otimes_k k[T]$, where $P(X, S) = S^4 + X^2S^2 + 1$.*

Proof. In a similar way as in the proof of Lemma 5.1, we will establish an isomorphism between $B_{1,P}[T]$ and $B_{2,P}[T]$.

Let $\Phi : k^{[4]} \longrightarrow k^{[4]}$ be the endomorphism of $k^{[4]}$ defined as follows:

$$\begin{cases} \Phi(X) = X \\ \Phi(S) = S + H(X, T) \\ \Phi(Y) = XY + L(X, S, T) \\ \Phi(T) = T(X, S, Y, T) \end{cases}$$

We choose $H = H(X, T)$ such that $\Phi(XY - S^4 - X^2S^2 - Xf(X) - 1) = X^2Y - S^4 - X^2S^2 - Xf(X) - 1$. This gives the following relation between H and $L = L(X, S, T)$:

$$XL = H^4 + 4H^3S + 6H^2S^2 + 4HS^3 + H^2X^2 + 2HSX^2$$

which directly implies that H is divisible by X. Note that if X^2divides H, then we immediately obtain $\Phi(Y)$ is divisible by X which implies that Φ will never induces an isomorphism. Therefore, The only choice for H is such that X divides H but X^2 does not. So let $H(X, T) = XT$, then a simple computation leads to

$$L(X, S, T) = X^3T^4 + 4X^2T^3S + 6XT^2S^2 + 4TS^3 + T^2X^3 + 2TSX^2.$$

Choose $\Phi(T)$ such that

$$\Phi(YS - XT) = 4T(-X^2Y + S^4 + X^2S^2 + Xf(X)),$$

which can be done by virtue of the condition (X divides every coefficient of $P(X,S) - S^4 - 1$). Notice that this choice is made to get

$$\Phi(YS - XT) \equiv -4T \mod (X^2Y - S^4 - X^2S^2 - Xf(X) - 1),$$

which simply implies that $T + \langle X^2Y - P\rangle \in \mathrm{Im}\pi_{X^2Y-P} \circ \Phi$. An elementary computation can determine $\Phi(T)$ to reach to the following form of Φ:

(1) $\Phi(X) = X$

(2) $\Phi(S) = S + XT$

(3) $\Phi(Y) = XY + X^3T^4 + 4X^2T^3S + 6XT^2S^2 + 4TS^3 + T^2X^3 + 2TSX^2$

(4) $\Phi(T) = SY + 5XYT - 4f(X)T - 2S^2TX + 3ST^2X^2 + 10S^3T^2 + T^3X^3 + 10S^2T^3X + 5ST^4X^2 + T^5X^3$

Now, as we discussed before §5.1, to prove that Φ induces an isomorphism between $B_{1,P}$ and $B_{2,P}$, it is enough to show that $\Phi^* = \pi_{X^2Y-P} \circ \Phi$ is surjective. For that, denote by x, s, y, t the class of X, S, Y, and T in $R_{2,0,P}$, then immediately we see that $x \in \mathrm{Im}\Phi^*$. By construction $t \in \mathrm{Im}\Phi^*$, therefore (2) implies $s \in \mathrm{Im}\Phi^*$. Thus (3) provides $xy \in \mathrm{Im}\Phi^*$, again using (4) we see that $sy \in \mathrm{Im}\Phi^*$. Finally, since $y = y.1 = y(x^2y - s^4 - s^2x^2 - xf(x)) = (xy)^2 - (ys)s^3 - (ys)sx^2 - (xy)f(x)$ where all terms in the second part of the last equation belong to $\mathrm{Im}\Phi^*$, we deduce that $y \in \mathrm{Im}\Phi^*$. Since x, s, y, t are generators of $B_{2,P}$, we conclude that Φ^* is surjective. □

The exact same method, as in the proof of Lemma 5.4, can be used to create an isomorphism between k-domains presented in the following Proposition. Where we imposed conditions, X divides every coefficient of $P(X,S) - S^d - c$ as a polynomial in S, and $c \neq 0$, which are essential to enable us to proceed.

Lemma 5.5. $B_{n,P} \otimes_k k[T] \simeq B_{n+1,P} \otimes_k k[T]$.

where $n \geq 1$, $d \geq 2$, $c \in k \setminus \{0\}$, $P(X,S) = S^d + XQ(X,S) + c$, and $Q(X,S) \in k[X,S]$ with no restriction on the degree of $Q(0,S)$.

Finally, by induction we have the following.

Theorem 5.6. $B_{n,P} \otimes_k k[T] \simeq B_{m,P} \otimes_k k[T]$.

where $n, m \geq 1$, $d \geq 2$, $c \in k \setminus \{0\}$, $P(X,S) = S^d + XQ(X,S) + c$, and $Q(X,S) \in k[X,S]$.

In addition, if $\Phi_{i,i+1}$ is the endomorphisms, as determined in Lemma 5.4, of $k^{[4]}$ that induces an isomorphism between $B_{i,P}[T]$ and $B_{i+1,P}[T]$, then the endomorphisms $\Phi_{m-1,m} \circ \cdots \circ \Phi_{n,n+1}$ of $k^{[4]}$ induces an isomorphism between $B_{n,P}[T]$ and $B_{m,P}[T]$.

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