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A theory of many-body localization in periodically driven systems

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We present a theory of periodically driven, many-body localized (MBL) systems. We show that many-body localization persists under periodic driving at high enough driving frequency. The Floquet operator (evolution operator over one driving period) can be represented as an exponential of an effective time-independent Hamiltonian, which is a sum of local terms and is itself MBL. The Floquet eigenstates in this case have area-law entanglement entropy, and there exists an extensive set of local integrals of motion. We argue that at sufficiently low frequency, there is always delocalization, owing to a large number of many-body level crossings and non-diabatic Landau-Zener transition between them. We propose a phase diagram of driven MBL systems.

Introduction. Recently, there has been much interest in quantum many-body localized (MBL) systems and their properties [1–17]. MBL phase is characterized by an extensive set of emergent local integrals of motion [12, 13], which lead to quantum ergodicity breaking, and in particular, absence of thermalization. Therefore, MBL systems cannot be described by conventional statistical mechanics. Previous works explored experimental manifestations of MBL systems, and predicted universal dynamical properties following a sudden quantum quench, including logarithmic growth of entanglement entropy [6, 8–10, 12, 13], as well as characteristic decay [18] and revivals [19] of local observables.

In this paper, we study the behaviour of MBL systems under periodic driving. Our motivation is twofold. First, studying response of many-body systems to periodically varying fields is a conventional experimental probe in systems of cold atoms in optical lattices [20, 21], which are promising candidates for realizing MBL phase [22, 23]. Second, theoretically little is known about general properties of quantum many-body systems under time-varying fields (beyond linear-response theory).

We consider a time-dependent periodic Hamiltonian $H(t) = H(t+T)$ and we split it in its mean and oscillating parts:

$$H(t) = H^{(0)} + gV(t), \quad H^{(0)} = \frac{1}{T} \int_0^T dt H(t). \quad (1)$$

We study the case when the time-averaged Hamiltonian $H^{(0)}$ is fully MBL (meaning that all its eigenstates are localized), and present two main results:

(I) We show that at high driving frequency, the system remains many-body localized, failing to absorb energy. To reach our conclusion, we develop a scheme similar to that devised by Imbrie [15] to establish the existence of a localized phase for time-independent Hamiltonians (see also [24]). We apply it to show that the Floquet operator $U(T)$ (evolution operator over one period) can be represented as $U(T) = e^{-iH_*T}$, where H_* is a time-independent quasi-local effective Hamiltonian, which is

MBL. We emphasize that $H^{(0)}$ can be fully MBL even when the instantaneous Hamiltonian $H(t)$ is ergodic for most or even all $t \in [0, T]$.

(II) We argue that at sufficiently low frequency, the Floquet operator strongly mixes states with a very different spatial structure, thus inducing delocalization. To show this, we invoke an analogy with the multi-level Landau-Zener problem. Combining the results (I,II), we propose a phase diagram of driven MBL systems.

Previous works on driven many-body systems focused mostly on the translationally invariant case [25–30]. In particular, D’Alessio and Polkovnikov [27] conjectured that, if the dynamics is generated by switching between an ergodic and an integrable (but translationally-invariant) Hamiltonian, a transition will be observed in function of the driving frequency: At low frequency, the system shows heating to an infinite temperature, while at high frequency the dynamics is described by an effective Hamiltonian written as a sum of local terms, so that there is an integral of motion in the system, which leads to localization in the energy space. For translationally-invariant systems, this conjecture remains unproven (and moreover, it was argued that driven ergodic systems typically delocalize and heat up to an infinite temperature [28, 30, 31]).

Our theory demonstrates that localization in energy space is indeed possible in driven MBL systems in the high-frequency regime, as there exists a local Floquet Hamiltonian. Furthermore, not only there is energy localization, but also there exists a complete set of emergent local integrals of motion (LIOM).

Model. For concreteness, we assume that our system is a one-dimensional spins- $\frac{1}{2}$ chain of size L . We make the following assumptions: (a) The Hamiltonian $H^{(0)}$ is fully MBL, and therefore it has a complete set of LIOMs. Choosing the LIOM’s to be local spins $\sigma_j = \sigma_j^z$ (third Pauli matrix), $H^{(0)}$ takes the form

$$H^{(0)} = \sum_i h_j \sigma_j + \sum_{i<j} J_{i,j} \sigma_i \sigma_j + \dots \quad (2)$$

where $\sum_K |J_{i,K,j}| \sim e^{-(j-i)/\xi}$ with K ranging over sets of sites between i and $j > i$, and with $\xi \lesssim 1$ the localization length of $H^{(0)}$. (b) The Hamiltonian $H^{(0)}$ is disordered: $h_j, J_{i,j}, \dots$ are random. We assume that a local spin flip results in a typical energy change of order W (the strength of the disorder). (c) The driving is of the form

$$V(t) = \sum_i \hat{V}_i(t), \quad (3)$$

with g the coupling strength, $\hat{V}_i(t) = \hat{V}_i(t+T)$ such that $\hat{V}_i(t)$ acts on a few sites around site i (and hence also on a few LIOM's around i) $\|\hat{V}_i(t)\| \leq 1$ for all t and such that $\frac{1}{T} \int_0^T ds V(s) = 0$, e.g. $\hat{V}_i(t) = \cos(\omega t) \hat{V}_i$ with $\omega = 2\pi/T$. The ultimate aim of our analysis is to analyze the one-cycle evolution operator $U(T)$, where the evolution operator $U(\cdot)$ is the solution of the equation

$$i \frac{d}{dt} U(t) = H(t)U(t), \quad U(0) = 1.$$

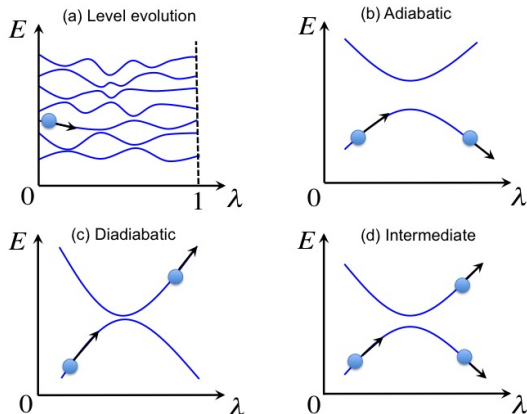


FIG. 1. (a) Spectrum of the many-body localized system as a function of parameter λ . (b-d) Three kinds of level crossings: (b) Adiabatic, when the system follows instantaneous eigenstate, (c) Diabatic, when the system ends up in the original eigenstate, and (d) Intermediate, when the system is in a superposition of two states at long times.

Localization at high frequency. We will start our analysis from the limit of large driving frequency $\nu \equiv 1/T$. We will show that, for large enough ν , the unitary $U(T)$ is still fully MBL, i.e. that the eigenvectors of $U(T)$ are locally close to the eigenvectors of $H^{(0)}$ (that is, they can be obtained from the eigenvectors of $H^{(0)}$ by quasi-local unitary transformation) and that one can construct a full set of local conserved quantities for $U(T)$.

An effective Hamiltonian H_* is defined by $U(T) = e^{-iH_*T}$. We can decompose $U(t) = P(t)e^{-iH_*t}$ where $P(t)$ is a unitary so that $P(t) = P(t+T)$. Since

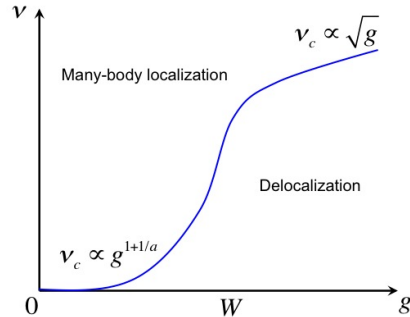


FIG. 2. Phase diagram of the driven MBL system at given W , depending on driving frequency ν and driving strength g . At small g , the critical frequency is given by (19), while at large g it is estimated from the condition (10), which gives $\nu_c \sim \sqrt{gW}$.

$i \frac{d}{dt} U(t) = H(t)U(t)$, it follows that

$$P^*(t) \left(H(t) - i \frac{d}{dt} \right) P(t) = H_*. \quad (4)$$

Obviously, H_* is not given to us, but we can now start from this relation to determine it: we try to find $P(t)$ such that H_* in (4) is time independent. While at $g = 0$, we can immediately solve (4) with $P(t) = 1$ and $H_* = H^{(0)}$, this equation admits in general no explicit solution for $g > 0$. Nevertheless, we show that for g small enough compared to ν and $\sqrt{\nu W}$ (see (10) below), we can solve it approximately and start an iterative procedure to eventually determine H_* .

Let us write $H(t) =: H_1(t)$ ($H(t)$ is the Hamiltonian we start with). We look for a unitary $P_1(t)$ “close to the identity”, meaning that $P_1^*(t)O_iP_1(t)$ stays a quasi-local observable around the site i if O_i was so, and so as to remove the driving in first order in the perturbative parameter ($g/\nu \ll 1$, see below):

$$P_1^*(t) \left(H_1^{(0)} + gV_1(t) - i \frac{d}{dt} \right) P_1(t) = H_1^{(0)} + \frac{g^2}{\nu} \tilde{V}(t) =: H_2(t) \quad (5)$$

We then decompose

$$H_2(t) = H_2^{(0)} + \frac{g^2}{\nu} V_2(t), \quad H_2^{(0)} := \frac{1}{T} \int_0^T ds H_2(s).$$

The main point is that, with $P_1(t)$ close to the identity and assuming $g^2/\nu W \ll 1$, $H_2^{(0)}$ can be seen as a small perturbation of $H_1^{(0)}$, and is therefore still MBL. Decomposing next $P(t) = \tilde{P}(t)P_1(t)$, (4) becomes

$$\tilde{P}^*(t) \left(H_2(t) - i \frac{d}{dt} \right) \tilde{P}(t) = H_*.$$

From this point, we see that the procedure can be iterated. For g small enough (compared to ν and $\sqrt{\nu W}$), we expect the Hamiltonians $H_n^{(0)}$ to converge super-exponentially fast to H_* , each step bringing super-exponentially small corrections. Therefore, the Hamiltonian H_* will only be a small perturbation of $H^{(0)}$ and will stay MBL.

We now carry out in more details a general step of this scheme. Mostly for notational convenience, we consider the first step. Doing so we will determine conditions on g , ν and W so that convergence can indeed be expected (but we do not claim that they generally allow to correctly locate the transition).

For simplicity, let us think of g as the perturbative parameter. We write $P_1(t) = e^{-gA_1(t)}$ where $A_1(t) = A_1(t+T)$ is anti-hermitian. Expanding the l.h.s. of (5) in powers of g , we find in first order

$$P_1^*(t)(H_1^{(0)} + gV_1(t) - i\frac{d}{dt})P_1(t) = H_1^{(0)} + g\left(V_1(t) + [A_1(t), H_1^{(0)}] + i\frac{dA_1}{dt}(t)\right) + \mathcal{O}(g^2/\nu). \quad (6)$$

In this first step, the Hamiltonian $H^{(0)}$ is diagonal in the basis of classical configurations where each spin is up/down, denoted by $|\eta\rangle \in \{\pm 1\}^L$. Expanding $V_1(t)$ and $A_1(t)$ in Fourier series, $V_1(t) = \sum_{k \neq 0} V_1(k)e^{i\nu kt}$ and similarly for $A_1(t)$, the first order in g in the r.h.s. of (6) will vanish if

$$\langle \eta' | A_1(k) | \eta \rangle = \frac{\langle \eta' | V_1(k) | \eta \rangle}{(H_1^{(0)}(\eta') - H_1^{(0)}(\eta)) + \nu k}, \quad k \neq 0 \quad (7)$$

with $H_1^{(0)}(\eta) = \langle \eta | H^{(0)} | \eta \rangle$. Since $V_1(t)$ is a sum of local terms, $A_1(t)$ is also a sum of local terms. The denominator in the r.h.s. of (7) is typically of order ν . Therefore, for the unitary $P_1(t)$ to be close to the identity and for the expansion in g to make sense, we impose

$$g/\nu \ll 1. \quad (8)$$

We observe that it was here crucial that $V_1(t)$ was local in the basis that diagonalizes $H_1^{(0)}$; we do not expect the theory to extend to the case where $H^{(0)}$ is ergodic.

From (6), we can compute $H_2(t)$ in second order in g , taking (7) into account:

$$H_2(t) = H^{(0)} + \frac{g^2}{2\nu}[A_1(t), V_1(t)] + \mathcal{O}(g^3/\nu^2).$$

Since the new perturbation does not in general average to zero over one cycle, we further impose that

$$g^2/\nu W \ll 1 \quad (9)$$

to ensure that $H_2^{(0)} = \frac{1}{T} \int_0^T ds H_2(s)$ is still MBL.

From (8) and (9), we expect the scheme to converge towards H_* given that

$$\max(g^2/(\nu W), g/\nu) \ll 1. \quad (10)$$

Note that (10) allows for $g \gg W$ which means that we can easily find $V(t)$ such that for all times t , the local terms in $V(t)$ are of order 1 and hence the Hamiltonian $H(t)$ is in the ergodic phase, i.e. it does not have MBL itself.

Let us finally remark that, due to fluctuations of the disorder, the perturbative approach will fail in some places. A resonance occurs for example in (7) when an energy difference $H^{(0)}(\eta') - H^{(0)}(\eta)$ becomes approximately equal to νk for some k such that $\hat{V}_1(k) \sim 1$. In general, non-perturbative effects may be responsible for delocalization, as it is probably the case in disorder-free systems [32] and in classical disordered systems [36, 37]. Here however, the location of resonances is determined by the disorder; as in [15], in the spots where $P_1(t)$ cannot be determined via perturbation, a solution to (4) can always be given by $P(t) = U(t)e^{iH_*t}$.

Delocalization at low frequency. Next, we will argue that at sufficiently low driving frequencies, the properties of the Floquet operator $U(T)$ change dramatically compared to the high-frequency limit considered above: $U(T)$ strongly mixes eigenstates of $H^{(0)}$, and the Floquet eigenstates become delocalized.

Our argument relies on the analogy with the multi-level Landau-Zener problem. It is convenient to introduce parameter $\lambda = \nu t$, and to slightly change notation by setting $H(\lambda) := H(t)$,

$$H(\lambda) = H^{(0)} + gV(\lambda), \quad (11)$$

For simplicity, we shall assume that $g \ll W$, and therefore for any $\lambda \in [0, 1]$, the Hamiltonian $H(\lambda)$ is in the MBL phase. For the sake of exposition, let us split $V = V_d + V_{od}$ such that V_d is that part of the perturbation that commutes with $H^{(0)}$. Then we write α and $E_\alpha = E_\alpha(\lambda)$ for the eigenvectors and energies of $H^{(0)} + gV_d(\lambda)$. As λ goes through a cycle, these levels can cross, whereas the levels of $H(\lambda)$ have an avoided crossing (see Fig.1(a)). The character of a pairwise level crossing is determined by (i) the matrix element of the operator $V(\lambda)$ between the energy levels $|\alpha\rangle, |\beta\rangle$ that undergo the crossing, $M_{\alpha\beta} = \langle \beta | V_d(\lambda_c) | \alpha \rangle$, where λ_c is the value of parameter λ at which crossing takes place, and by (ii) the speed at which the crossing is passed: $v_{\alpha\beta} = \frac{d(E_\alpha(\lambda) - E_\beta(\lambda))}{d\lambda} \nu$. In the Landau-Zener problem (crossing of just two levels), the transition amplitude is given by (see, e.g. Ref. [33]:

$$S_{\alpha \rightarrow \alpha} = \exp(-C_{\alpha\beta}), \quad C_{\alpha\beta} \equiv \pi \frac{|M_{\alpha\beta}|^2}{v_{\alpha\beta}}, \quad (12)$$

and therefore one can distinguish three regimes: (1) *adiabatic*, when parameter $C_{\alpha\beta} \gg 1$. In this case, the system ends up in eigenstate β after the crossing is passed, and the probability to stay in the ‘‘excited’’ state α is exponentially small; (2) *diabatic*, when $C_{\alpha\beta} \ll 1$; in this case, the system stays in state α , and (3) *intermediate*, when

$C_{\alpha\beta} \sim 1$; in this case, the system ends up in a superposition of states α and β at long times, with approximately similar weights. The three regimes are illustrated in Fig. 1(b-d).

As we will now argue, in our problem the relevant crossings, which lead to delocalization at low frequency, occur between levels that differ only by a small number of LIOM's. Let us consider two levels α, β , which have different values of local integrals of motion only in a region R of size $x \ll L$. We first show that there is a scale \bar{x} , such that at $x \ll \bar{x}$ the crossings between states which differ in region R are very rare, while at $x \gg \bar{x}$ there are many such crossings. There are $n(x) = 2^x$ different levels which have different value of LIOM's in region R , and are identical outside R . The typical level spacing for this group of levels can be estimated as:

$$\Delta(x) \approx W \frac{\sqrt{x}}{2^x}. \quad (13)$$

On the other hand, the typical change of *energy difference* between two levels α, β when $\delta\lambda \sim 1$ can be estimated as:

$$\delta E_{\alpha\beta}(x) \sim g(\langle \alpha | V | \alpha \rangle - \langle \beta | V | \beta \rangle) \sim g\sqrt{x}, \quad (14)$$

where we used $\|\hat{V}_i\| \sim 1$. If $\delta E_{\alpha\beta}(x)$ is much smaller than the level spacing $\Delta(x)$, the levels in this group typically do not cross. In the opposite limit, $\delta E_{\alpha\beta} \gg \Delta(x)$, there are multiple level crossings of this kind. The scale \bar{x} can therefore be estimated from the condition $\delta E_{\alpha\beta}(\bar{x}) \approx \Delta(\bar{x})$, which gives:

$$\bar{x} = \log_2 \frac{W}{g}. \quad (15)$$

At $x \gtrsim \bar{x}$, each level α therefore crosses multiple other levels which differ from α by changing values of some or all LIOMs in (any) region of size x .

Next, let us understand the character of crossings between levels α, β that have different LIOMs only in a region R or size x (assuming such crossing is encountered as λ is varied). First, we estimate the speed at which crossing is passed: $v_{\alpha\beta}(x) \sim \delta E_{\alpha\beta} \nu \sim g\nu\sqrt{x}$. Second, we note that the typical matrix element of a local operator between two MBL eigenstates which differ in region R , is given by:

$$M_{\alpha\beta}(x) \sim g\langle \alpha | V | \beta \rangle \sim g\sqrt{x}e^{-x/\xi}. \quad (16)$$

The value of the parameter $C_{\alpha\beta}(x)$ characterizing the crossing is then given by:

$$C_{\alpha\beta}(x) \sim \frac{g\sqrt{x}}{\nu} e^{-2x/\xi}. \quad (17)$$

The crossing is in the intermediate regime (the two crossing levels mix strongly at long times) at scale x_* , which can be estimated from the relation $C_{\alpha\beta}(x_*) \sim 1$:

$$x_* \approx \frac{\xi}{2} \log \frac{g}{\nu}. \quad (18)$$

At $x \gg x_*$, (nearly) all crossings are in the diabatic regime, while at $x \ll x_*$ crossings are adiabatic.

The properties of the Floquet operator, most importantly, the way it mixes states with very different spatial structure, depend on the relation between length scales x_*, \bar{x} (15,18). If $x_* \gg \bar{x}$, during one period of driving, each level experiences multiple crossings which are in the intermediate or adiabatic regime. This means that the operator $U(T)$ cannot be considered a small perturbation of $U_{g=0}(T) = \exp(-iT H^{(0)})$, as it changes the values of most LIOM's. Hence, in this case we expect that the eigenstates of $U(T)$ are delocalized.

Using Eqs.(15,18) the condition $x_* \gg \bar{x}$ under which the system delocalizes, can be rewritten in terms of frequency ν :

$$\nu \ll \nu_*, \quad \nu_* = g \left(\frac{g}{W} \right)^{1/a}, \quad a \equiv \frac{\xi \log 2}{2}. \quad (19)$$

Thus, at sufficiently low driving frequency, the system *always delocalizes*.

As we showed above, in the limit of high frequency (Eqs.(8,9)), the system is in the MBL phase. Therefore, we expect a localization-delocalization transition to take place at some critical frequency ν_c . We conjecture that at $g \ll W$ (the assumption under which the analogy with Landau-Zener problem can be invoked), the transition takes place when $x_* \approx \bar{x}$, that is, $\nu_c \approx \nu_*$, although we were not able to prove this fact. Further, in the limit of large driving strength, $g \gg W$, the perturbation theory shows that MBL persists at $\nu \gg \sqrt{gW}$. The perturbation theory breaks down at $\nu \sim \sqrt{gW}$, and we conjecture that in this limit delocalization occurs at $\nu_c \sim \sqrt{gW}$. Combining these two results, we propose a phase diagram in Fig.2.

Discussion. In summary, we have analytically established that many-body localization persists under periodic driving, if the driving frequency is high enough. The MBL phase in driven systems is characterized by (i) the existence of an extensive number of local conservation laws; (ii) area-law for all, but an exponentially small fraction of Floquet eigenstates; (iii) logarithmic spreading of entanglement entropy of initial product states. At sufficiently low driving frequencies, the system undergoes a transition into the delocalized (almost certainly ergodic) phase. In the future, it will be interesting to explore the nature of the MBL-delocalization transition in driven systems, and in particular, to understand whether it belongs to the same universality class as the MBL-delocalization transition for static Hamiltonians.

We also note that the existence of MBL phase at large driving frequency is confirmed by two recent numerical studies [34, 35]. Ponte et al. [34] studied an MBL spin model, periodically "kicked" with a delocalizing perturbation, finding an MBL-delocalization transition as a function of driving strength. Lazarides et al. [35] studied a periodically driven system of disordered bosons, and

presented numerical evidence for the persistence of MBL phase at high driving frequency.

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