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Dang Ho Hai Ndh Nguyen, Lionel Schwartz

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Takayasu cofibrations revisited

Nguyen D.H. Hai and Lionel Schwartz

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1 Introduction

Given a natural number n , let $\tilde{\rho}_n$ be the reduced real regular representation of the elementary abelian 2-group $V_n := (\mathbb{Z}/2)^n$. Let $BV_n^{k\tilde{\rho}_n}$, $k \in \mathbb{N}$, denote the Thom space over the classifying space BV_n associated to the direct sum of k copies of the representation $\tilde{\rho}_n$. Following S. Takayasu [14], let $M(n)_k$ denote the stable summand of $BV_n^{k\tilde{\rho}_n}$ which corresponds to the Steinberg module of the general linear group $GL_n(\mathbb{F}_2)$ [12].

Takayasu constructed in [14] a cofibration of the following form:

$$\Sigma^k M(n-1)_{2k+1} \rightarrow M(n)_k \rightarrow M(n)_{k+1}.$$

This generalised the splitting of Mitchell and Priddy $M(n) \simeq L(n) \vee L(n-1)$, where $M(n) = M(n)_0$ and $L(n) = M(n)_1$ [12]. Takayasu also considered the spectra $M(n)_k$ associated to the virtual representations $k\tilde{\rho}_n$, $k < 0$, and proved that the above cofibrations are still valid for these spectra. Here and below, all spectra are implicitly completed at the prime two.

Note that the spectra $M(n)_k$, $k \geq 0$, are used in the description of layers of the Goodwillie tower of the identity functor evaluated at spheres [2, 1], and the above cofibrations can also be deduced by combining Goodwillie calculus with the James fibration, as described by M. Behrens in [3, Chapter 2].

The purpose of this note is to give another proof for the existence of the above cofibrations for the cases $k \in \mathbb{N}$. This will be carried out by employing techniques in the category of unstable modules over the Steenrod algebra [13]. Especially, the action of Lannes' T-functor on the Steinberg unstable modules (see §4), will play a crucial role in studying the vanishing of some extension groups of modules over the Steenrod algebra.

2 Algebraic short exact sequences

In this section, we recall the linear structure of the mod 2 cohomology of $M(n, k)$ and the short exact sequences relating these \mathcal{A} -modules. Recall that the general linear group $GL_n := GL_n(\mathbb{F}_2)$ acts on $H^*V_n \cong \mathbb{F}_2[x_1, \dots, x_n]$ by the rule:

$$(gF)(x_1, \dots, x_n) := F\left(\sum_{i=1}^n g_{i,1}x_i, \dots, \sum_{i=1}^n g_{i,n}x_i\right),$$

where $g = (g_{i,j}) \in GL_n$ and $F(x_1, \dots, x_n) \in \mathbb{F}_2[x_1, \dots, x_n]$. This action commutes with the action of the Steenrod action on $\mathbb{F}_2[x_1, \dots, x_n]$.

By definition, the Thom class of the vector bundle associated to the reduced regular representation $\tilde{\rho}_n$ is given by the top Dickson invariant:

$$\omega_n = \omega_n(x_1, \dots, x_n) := \prod_{0 \neq x \in \mathbb{F}_2 \langle x_1, \dots, x_n \rangle} x.$$

Recall also that the Steinberg idempotent e_n of $\mathbb{F}_2[GL_n]$ is given by

$$e_n := \sum_{b \in B, \sigma \in \Sigma_n} b\sigma,$$

where B_n is the subgroup of upper triangular matrices in GL_n and Σ_n the subgroup of permutation matrices.

Let $M_{n,k}$ denote the mod 2 cohomology of the spectrum $M(n, k)$. By Thom isomorphism, we have an isomorphism of \mathcal{A} -modules:

$$M_{n,k} \cong \text{Im}[\omega_n^k H^* BV_n \xrightarrow{e_n} \omega_n^k H^* BV_n].$$

We note that $M_{n,k}$ is invariant under the action of the group B_n .

Proposition 2.1 ([5]). *A basis for the graded vector space $M_{n,k}$ is given by*

$$\{e_n(\omega_1^{i_1-2i_2} \dots \omega_{n-1}^{i_{n-1}-2i_n} \omega_n^{i_n}) \mid i_j > 2i_{j+1} \text{ for } 1 \leq j \leq n-1 \text{ and } i_n \geq k\}.$$

Theorem 2.2 (cf. [14]). *Let $\alpha : M_{n,k+1} \rightarrow M_{n,k}$ be the natural inclusion and let $\beta : M_{n,k} \rightarrow \Sigma^k M_{n-1,2k+1}$ be the map given by*

$$\beta(\omega_{i_1, \dots, i_n}) = \begin{cases} 0, & i_n > k, \\ \Sigma^k \omega_{i_1, \dots, i_{n-1}}, & i_n = k. \end{cases}$$

Then

$$0 \rightarrow M_{n,k+1} \xrightarrow{\alpha} M_{n,k} \xrightarrow{\beta} \Sigma^k M_{n-1,2k+1} \rightarrow 0$$

is a short exact sequence of \mathcal{A} -modules:

The exactness of the sequence can be proved by using the following:

Lemma 2.3 ([6, Proposition 1.2]). *We have*

$$\omega_{i_1, \dots, i_n} = \omega_{i_1, \dots, i_{n-1}} x_n^{i_n} + \text{terms } \omega_{j_1, \dots, j_{n-1}} x_n^j \text{ with } j > j_n.$$

Note also that a minimal generating set for the \mathcal{A} -module $M_{n,k}$ was constructed in [5], generating the work of Inoue [8].

3 Existence of the cofibrations

A spectrum X is said to be of finite type if its mod 2 cohomology, H^*X , is finite-dimensional in each degree. Recall that given a sequence $X \rightarrow Y \rightarrow Z$ of spectra of finite type, if the composite $X \rightarrow Z$ is homotopically trivial and the induced sequence $0 \rightarrow H^*Z \rightarrow H^*Y \rightarrow H^*X \rightarrow 0$ is a short exact sequence of \mathcal{A} -modules, then $X \rightarrow Y \rightarrow Z$ is a cofibration.

We wish to realise the algebraic short sequence

$$0 \rightarrow M_{n,k+1} \xrightarrow{\alpha} M_{n,k} \xrightarrow{\beta} \Sigma^k M_{n-1,2k+1} \rightarrow 0.$$

by a cofibration of spectra

$$\Sigma^k M(n-1)_{2k+1} \rightarrow M(n)_k \rightarrow M(n)_{k+1}.$$

The inclusion of $k\tilde{\rho}_n$ into $(k+1)\tilde{\rho}_n$ induces a natural map of spectra

$$i : M(n)_k \rightarrow M(n)_{k+1}.$$

It is clear that this map realises the inclusion of \mathcal{A} -modules $\alpha : M_{n,k+1} \rightarrow M_{n,k}$. We wish now to realise the \mathcal{A} -linear map $\beta : M_{n,k} \rightarrow \Sigma^k M_{n-1,2k+1}$ by a map of spectra

$$j : \Sigma^k M(n-1)_{2k+1} \rightarrow M(n)_k$$

such that the composite $i \circ j$ is homotopically trivial. The existence of such a map is an immediate consequence of the following result.

Theorem 3.1. *For all $k \geq 0$, we have*

1. *The natural map $[\Sigma^k M(n-1)_{2k+1}, M(n)_k] \rightarrow \text{Hom}_{\mathcal{A}}(M_{n,k}, \Sigma^k M_{n-1,2k+1})$ is onto.*
2. *The group $[\Sigma^k M(n-1)_{2k+1}, M(n)_{k+1}]$ is trivial.*

The theorem is proved by using the Adams spectral sequence

$$\text{Ext}_{\mathcal{A}}^s(H^*Y, \Sigma^t H^*X) \implies [\Sigma^{t-s} X, Y].$$

For the first part, it suffices to prove that

$$\text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{k+t} M_{n-1,2k+1}) = 0 \quad \text{for } s \geq 0 \text{ and } t - s < 0, \quad (1)$$

so that the non-trivial elements in $\text{Hom}_{\mathcal{A}}(M_{n,k}, \Sigma^k M_{n-1,2k+1})$ are permanent cycles. For the second part, it suffices to prove that

$$\text{Ext}_{\mathcal{A}}^s(M_{n,k+1}, \Sigma^{k+s} M_{n-1,2k+1}) = 0, \quad \text{for } s \geq 0. \quad (2)$$

Here and below, \mathcal{A} -linear maps are of degree zero, and so $\text{Ext}_{\mathcal{A}}^s(M, \Sigma^t N)$ is the same as the group denoted by $\text{Ext}_{\mathcal{A}}^{s,t}(M, N)$ in the traditional notation.

The vanishing of the above extension groups will be proved in the next section.

4 On the vanishing of $\text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{i+s}M_{m,j})$

In this section, we establish a sufficient condition for the vanishing of the extension groups $\text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{i+s}M_{m,j})$. Note that we always consider the modules $M_{n,k}$ with $k \geq 0$.

Below we consider separately two cases for the vanishing of the groups $\text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{i+s}M_{m,j})$: Proposition 4.1 gives a condition for the case $j = 0$ and Proposition 4.2 gives a condition for the case $j > 0$.

Proposition 4.1. *Suppose $n > m \geq 0$ and $-\infty < i < |M_{n-m,k}|$. Then*

$$\text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{i+s}M_m) = 0, \quad s \geq 0.$$

Here $|M|$ denotes the connectivity of M , i.e. the minimal degree in which M is non-trivial.

To consider the case $j > 0$, put $\varphi(j) = 2j - 1$ and

$$F(i, j, q) = i + j + \varphi(j) + \varphi^2(j) + \cdots + \varphi^{q-1}(j),$$

where φ^t is the t -fold composition of φ . Explicitly,

$$F(i, j, q) = i + (j - 1)(2^q - 1) + q.$$

Note that $F(i + j, 2j - 1, q) = F(i, j, q + 1)$ and $F(i, j', q) \leq F(i, j, q)$ if $j' \leq j$.

Proposition 4.2. *Suppose $n > m \geq 0$, $j > 0$ and $F(i, j, q) < |M_{n-m+q,k}|$ for $0 \leq q \leq m$. Then*

$$\text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{i+s}M_{m,j}) = 0, \quad s \geq 0.$$

Recall that Lannes' T-functor is left adjoint to the tensoring with $H := H^*B\mathbb{Z}/2$ in the category \mathcal{U} of unstable modules over the Steenrod algebra [9]. We need the following result, observed by Harris and Shank [7], to prove Proposition 4.1.

Proposition 4.3 (Carlisle-Kuhn [4, 6.1] combined with Harris-Shank [7, 4.19]). *There is an isomorphism of unstable modules*

$$\mathbb{T}(L_n) \cong L_n \oplus (H \otimes L_{n-1}).$$

Here $L_n = M_{n,1}$.

Corollary 4.4. *For $n \geq m$, we have $|\mathbb{T}^m(M_{n,k})| = |M_{n-m,k}|$.*

Proof. By iterating the action of \mathbb{T} on L_n , we see that there is an isomorphism of unstable modules

$$\mathbb{T}^m(L_n) \cong \bigoplus_{i=0}^m [H^{\otimes i} \otimes L_{n-i}]^{\oplus a_i},$$

where a_i are certain positive integers depending only on m . By using the exactitude of \mathbb{T}^m and the short exact sequences

$$0 \rightarrow M_{n,k+1} \xrightarrow{\alpha} M_{n,k} \xrightarrow{\beta} \Sigma^k M_{n-1,2k+1} \rightarrow 0,$$

it is easy to prove by induction that there is an isomorphism of graded vector spaces

$$\mathbb{T}^m(M_{n,k}) \cong \bigoplus_{i=0}^m [H^{\otimes i} \otimes M_{n-i,k}]^{\oplus a_i}.$$

The corollary follows. \square

Proof of Proposition 4.1. Fix i, s and take a positive integer q big enough such that $i + s + q$ is positive. We have

$$\mathrm{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{i+s} M_m) = \mathrm{Ext}_{\mathcal{A}}^s(\Sigma^q M_{n,k}, \Sigma^{i+s+q} M_m).$$

Using the Grothendieck spectral sequence, we need to prove that

$$\mathrm{Ext}_{\mathcal{U}}^{s-j}(\mathbb{D}_j \Sigma^q M_{n,k}, \Sigma^{i+s+q} M_m) = 0, \quad 0 \leq j \leq s.$$

Here \mathbb{D}_j is the j th-derived functor of the destabilisation functor

$$\mathbb{D} : \mathcal{A}\text{-mod} \rightarrow \mathcal{U}$$

from the category of \mathcal{A} -modules to the category of unstable \mathcal{A} -modules [11].

As M_n is \mathcal{U} -injective, it is easily seen that $\Sigma^\ell M_m$ has a \mathcal{U} -injective resolution I^\bullet where I^t is a direct sum of $M_m \otimes J(a)$ with $a \leq \ell - t$, where $J(a)$ is the Brown-Gitler module [10]. So we need to prove that, for $a \leq (i + s + q) - (s - j) = i + j + q$, we have

$$\mathrm{Hom}_{\mathcal{U}}(\mathbb{D}_j \Sigma^q M_{n,k}, M_m \otimes J(a)) = 0.$$

By Lannes-Zarati [11], we have

$$\mathbb{D}_j \Sigma^q M_{n,k} = \Sigma R_j \Sigma^{j-1+q} M_{n,k} \subset \Sigma^{j+q} H^{\otimes j} \otimes M_{n,k},$$

where R_j is the Singer functor. It follows that $\mathrm{Hom}_{\mathcal{U}}(\mathbb{D}_j \Sigma^q M_{n,k}, M_m \otimes J(a))$ is a quotient of

$$\mathrm{Hom}_{\mathcal{U}}(\Sigma^{j+q} H^{\otimes j} \otimes M_{n,k}, M_m \otimes J(a))$$

which is in turn a subgroup of

$$\mathrm{Hom}_{\mathcal{U}}(\Sigma^{j+q} H^{\otimes j} \otimes M_{n,k}, H^{\otimes m} \otimes J(a)) = ((\mathbb{T}^m(\Sigma^{j+q} H^{\otimes j} \otimes M_{n,k}))^a)^*.$$

This group is trivial because, by Corollary 4.4, we have

$$|\mathbb{T}^m(\Sigma^{j+q} H_i \otimes M_{n,k})| = |\Sigma^{j+q} M_{n-m,k}| = |M_{n-m,k}| + j + q > i + j + q \geq a.$$

The proposition follows. \square

Proof of Proposition 4.2. We prove the proposition by induction on $m \geq 0$. By noting that $M_{0,j} = \mathbb{Z}/2$, the case $m = 0$ is a special case of Proposition 4.1.

Suppose $m > 0$. For simplicity, put $E^s(\Sigma^i M_{m,j}) = \text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{i+s} M_{m,j})$. The short exact sequence of \mathcal{A} -modules $M_{m,j} \hookrightarrow M_{m,j-1} \twoheadrightarrow \Sigma^{j-1} M_{m-1,2j-1}$ induces a long exact sequence in cohomology

$$\cdots \rightarrow E^{s-1}(\Sigma^{i+j} M_{m-1,2j-1}) \rightarrow E^s(\Sigma^i M_{m,j}) \rightarrow E^s(\Sigma^i M_{m,j-1}) \rightarrow \cdots$$

So from the cofiltration of $M_{m,j}$

$$\begin{array}{ccccccc} \Sigma^i M_{m,j} & \hookrightarrow & \Sigma^i M_{m,j-1} & \hookrightarrow & \cdots & \hookrightarrow & \Sigma^i M_{m,1} & \hookrightarrow & \Sigma^i M_m \\ & & \downarrow & & & & \downarrow & & \downarrow \\ & & \Sigma^{i+j-1} M_{m-1,2j-1} & & & & \Sigma^{i+1} M_{m-1,3} & & \Sigma^i M_{m-1,1} \end{array}$$

we see that, in order to prove $E^s(\Sigma^i M_{m,j}) = 0$, it suffices to prove that the groups $E^{s-1}(\Sigma^{i+j'} M_{m-1,2j'-1})$, $1 \leq j' \leq j$, and $E^s(\Sigma^i M_m)$, are trivial.

By Proposition 4.1, $E^s(\Sigma^i M_m)$ is trivial since $i = F(i, j, 0) < |M_{n,k}|$. For $1 \leq j' \leq j$ and $0 \leq q \leq m-1$, we have

$$F(i + j', 2j' - 1, q) = F(i, j', q + 1) \leq F(i, j, q + 1) < |M_{n-m+1+q,k}|.$$

By inductive hypothesis for $m-1$, we have $E^{s-1}(\Sigma^{i+j'} M_{m-1,2j'-1}) = 0$. The proposition is proved. \square

We are now ready to prove Theorem 3.1. Recall that the connectivity of $M_{n,k}$ is given by

$$|M_{n,k}| = 1 + 3 + \cdots + (2^{n-1} - 1) + (2^n - 1)k.$$

Proof of Theorem 3.1 (1). Using the Adams spectral sequence, it suffices to prove that

$$\text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{k+t} M_{n-1,2k+1}) = 0 \quad \text{for } s \geq 0 \text{ and } t - s < 0.$$

For $q \geq 0$, we have

$$F(k + t - s, 2k + 1, q) = k + t - s + 2k(2^q - 1) + q < (2^{q+1} - 1)k + q \leq |M_{q+1,k}|.$$

The vanishing of the extension groups follows from Proposition 4.2. \square

Proof of Theorem 3.1 (2). Using the Adams spectral sequence, it suffices to prove that

$$\text{Ext}_{\mathcal{A}}^s(M_{n,k+1}, \Sigma^{k+s} M_{n-1,2k+1}) = 0, \quad \text{for } s \geq 0.$$

For $q \geq 0$, we have

$$F(k, 2k + 1, q) = k + 2k(2^q - 1) + t = (2^{q+1} - 1)k + q < |M_{q+1,k+1}|.$$

The vanishing of the extension groups follows from Proposition 4.2. \square

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