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# Takayasu cofibrations revisited

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## 1 Introduction

Given a natural number  $n$ , let  $\tilde{\rho}_n$  be the reduced real regular representation of the elementary abelian 2-group  $V_n := (\mathbb{Z}/2)^n$ . Let  $BV_n^{k\tilde{\rho}_n}$ ,  $k \in \mathbb{N}$ , denote the Thom space over the classifying space  $BV_n$  associated to the direct sum of  $k$  copies of the representation  $\tilde{\rho}_n$ . Following S. Takayasu [14], let  $M(n)_k$  denote the stable summand of  $BV_n^{k\tilde{\rho}_n}$  which corresponds to the Steinberg module of the general linear group  $GL_n(\mathbb{F}_2)$  [12].

Takayasu constructed in [14] a cofibration of the following form:

$$\Sigma^k M(n-1)_{2k+1} \rightarrow M(n)_k \rightarrow M(n)_{k+1}.$$

This generalised the splitting of Mitchell and Priddy  $M(n) \simeq L(n) \vee L(n-1)$ , where  $M(n) = M(n)_0$  and  $L(n) = M(n)_1$  [12]. Takayasu also considered the spectra  $M(n)_k$  associated to the virtual representations  $k\tilde{\rho}_n$ ,  $k < 0$ , and proved that the above cofibrations are still valid for these spectra. Here and below, all spectra are implicitly completed at the prime two.

Note that the spectra  $M(n)_k$ ,  $k \geq 0$ , are used in the description of layers of the Goodwillie tower of the identity functor evaluated at spheres [2, 1], and the above cofibrations can also be deduced by combining Goodwillie calculus with the James fibration, as described by M. Behrens in [3, Chapter 2].

The purpose of this note is to give another proof for the existence of the above cofibrations for the cases  $k \in \mathbb{N}$ . This will be carried out by employing techniques in the category of unstable modules over the Steenrod algebra [13]. Especially, the action of Lannes' T-functor on the Steinberg unstable modules (see §4), will play a crucial role in studying the vanishing of some extension groups of modules over the Steenrod algebra.

## 2 Algebraic short exact sequences

In this section, we recall the linear structure of the mod 2 cohomology of  $M(n, k)$  and the short exact sequences relating these  $\mathcal{A}$ -modules. Recall that the general linear group  $GL_n := GL_n(\mathbb{F}_2)$  acts on  $H^*V_n \cong \mathbb{F}_2[x_1, \dots, x_n]$  by the rule:

$$(gF)(x_1, \dots, x_n) := F\left(\sum_{i=1}^n g_{i,1}x_i, \dots, \sum_{i=1}^n g_{i,n}x_i\right),$$

where  $g = (g_{i,j}) \in GL_n$  and  $F(x_1, \dots, x_n) \in \mathbb{F}_2[x_1, \dots, x_n]$ . This action commutes with the action of the Steenrod action on  $\mathbb{F}_2[x_1, \dots, x_n]$ .

By definition, the Thom class of the vector bundle associated to the reduced regular representation  $\tilde{\rho}_n$  is given by the top Dickson invariant:

$$\omega_n = \omega_n(x_1, \dots, x_n) := \prod_{0 \neq x \in \mathbb{F}_2 \langle x_1, \dots, x_n \rangle} x.$$

Recall also that the Steinberg idempotent  $e_n$  of  $\mathbb{F}_2[GL_n]$  is given by

$$e_n := \sum_{b \in B, \sigma \in \Sigma_n} b\sigma,$$

where  $B_n$  is the subgroup of upper triangular matrices in  $GL_n$  and  $\Sigma_n$  the subgroup of permutation matrices.

Let  $M_{n,k}$  denote the mod 2 cohomology of the spectrum  $M(n, k)$ . By Thom isomorphism, we have an isomorphism of  $\mathcal{A}$ -modules:

$$M_{n,k} \cong \text{Im}[\omega_n^k H^* BV_n \xrightarrow{e_n} \omega_n^k H^* BV_n].$$

We note that  $M_{n,k}$  is invariant under the action of the group  $B_n$ .

**Proposition 2.1** ([5]). *A basis for the graded vector space  $M_{n,k}$  is given by*

$$\{e_n(\omega_1^{i_1-2i_2} \dots \omega_{n-1}^{i_{n-1}-2i_n} \omega_n^{i_n}) \mid i_j > 2i_{j+1} \text{ for } 1 \leq j \leq n-1 \text{ and } i_n \geq k\}.$$

**Theorem 2.2** (cf. [14]). *Let  $\alpha : M_{n,k+1} \rightarrow M_{n,k}$  be the natural inclusion and let  $\beta : M_{n,k} \rightarrow \Sigma^k M_{n-1,2k+1}$  be the map given by*

$$\beta(\omega_{i_1, \dots, i_n}) = \begin{cases} 0, & i_n > k, \\ \Sigma^k \omega_{i_1, \dots, i_{n-1}}, & i_n = k. \end{cases}$$

Then

$$0 \rightarrow M_{n,k+1} \xrightarrow{\alpha} M_{n,k} \xrightarrow{\beta} \Sigma^k M_{n-1,2k+1} \rightarrow 0$$

is a short exact sequence of  $\mathcal{A}$ -modules:

The exactness of the sequence can be proved by using the following:

**Lemma 2.3** ([6, Proposition 1.2]). *We have*

$$\omega_{i_1, \dots, i_n} = \omega_{i_1, \dots, i_{n-1}} x_n^{i_n} + \text{terms } \omega_{j_1, \dots, j_{n-1}} x_n^j \text{ with } j > j_n.$$

Note also that a minimal generating set for the  $\mathcal{A}$ -module  $M_{n,k}$  was constructed in [5], generating the work of Inoue [8].

### 3 Existence of the cofibrations

A spectrum  $X$  is said to be of finite type if its mod 2 cohomology,  $H^*X$ , is finite-dimensional in each degree. Recall that given a sequence  $X \rightarrow Y \rightarrow Z$  of spectra of finite type, if the composite  $X \rightarrow Z$  is homotopically trivial and the induced sequence  $0 \rightarrow H^*Z \rightarrow H^*Y \rightarrow H^*X \rightarrow 0$  is a short exact sequence of  $\mathcal{A}$ -modules, then  $X \rightarrow Y \rightarrow Z$  is a cofibration.

We wish to realise the algebraic short sequence

$$0 \rightarrow M_{n,k+1} \xrightarrow{\alpha} M_{n,k} \xrightarrow{\beta} \Sigma^k M_{n-1,2k+1} \rightarrow 0.$$

by a cofibration of spectra

$$\Sigma^k M(n-1)_{2k+1} \rightarrow M(n)_k \rightarrow M(n)_{k+1}.$$

The inclusion of  $k\tilde{\rho}_n$  into  $(k+1)\tilde{\rho}_n$  induces a natural map of spectra

$$i : M(n)_k \rightarrow M(n)_{k+1}.$$

It is clear that this map realises the inclusion of  $\mathcal{A}$ -modules  $\alpha : M_{n,k+1} \rightarrow M_{n,k}$ . We wish now to realise the  $\mathcal{A}$ -linear map  $\beta : M_{n,k} \rightarrow \Sigma^k M_{n-1,2k+1}$  by a map of spectra

$$j : \Sigma^k M(n-1)_{2k+1} \rightarrow M(n)_k$$

such that the composite  $i \circ j$  is homotopically trivial. The existence of such a map is an immediate consequence of the following result.

**Theorem 3.1.** *For all  $k \geq 0$ , we have*

1. *The natural map  $[\Sigma^k M(n-1)_{2k+1}, M(n)_k] \rightarrow \text{Hom}_{\mathcal{A}}(M_{n,k}, \Sigma^k M_{n-1,2k+1})$  is onto.*
2. *The group  $[\Sigma^k M(n-1)_{2k+1}, M(n)_{k+1}]$  is trivial.*

The theorem is proved by using the Adams spectral sequence

$$\text{Ext}_{\mathcal{A}}^s(H^*Y, \Sigma^t H^*X) \implies [\Sigma^{t-s} X, Y].$$

For the first part, it suffices to prove that

$$\text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{k+t} M_{n-1,2k+1}) = 0 \quad \text{for } s \geq 0 \text{ and } t - s < 0, \quad (1)$$

so that the non-trivial elements in  $\text{Hom}_{\mathcal{A}}(M_{n,k}, \Sigma^k M_{n-1,2k+1})$  are permanent cycles. For the second part, it suffices to prove that

$$\text{Ext}_{\mathcal{A}}^s(M_{n,k+1}, \Sigma^{k+s} M_{n-1,2k+1}) = 0, \quad \text{for } s \geq 0. \quad (2)$$

Here and below,  $\mathcal{A}$ -linear maps are of degree zero, and so  $\text{Ext}_{\mathcal{A}}^s(M, \Sigma^t N)$  is the same as the group denoted by  $\text{Ext}_{\mathcal{A}}^{s,t}(M, N)$  in the traditional notation.

The vanishing of the above extension groups will be proved in the next section.

## 4 On the vanishing of $\text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{i+s}M_{m,j})$

In this section, we establish a sufficient condition for the vanishing of the extension groups  $\text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{i+s}M_{m,j})$ . Note that we always consider the modules  $M_{n,k}$  with  $k \geq 0$ .

Below we consider separately two cases for the vanishing of the groups  $\text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{i+s}M_{m,j})$ : Proposition 4.1 gives a condition for the case  $j = 0$  and Proposition 4.2 gives a condition for the case  $j > 0$ .

**Proposition 4.1.** *Suppose  $n > m \geq 0$  and  $-\infty < i < |M_{n-m,k}|$ . Then*

$$\text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{i+s}M_m) = 0, \quad s \geq 0.$$

Here  $|M|$  denotes the connectivity of  $M$ , i.e. the minimal degree in which  $M$  is non-trivial.

To consider the case  $j > 0$ , put  $\varphi(j) = 2j - 1$  and

$$F(i, j, q) = i + j + \varphi(j) + \varphi^2(j) + \cdots + \varphi^{q-1}(j),$$

where  $\varphi^t$  is the  $t$ -fold composition of  $\varphi$ . Explicitly,

$$F(i, j, q) = i + (j - 1)(2^q - 1) + q.$$

Note that  $F(i + j, 2j - 1, q) = F(i, j, q + 1)$  and  $F(i, j', q) \leq F(i, j, q)$  if  $j' \leq j$ .

**Proposition 4.2.** *Suppose  $n > m \geq 0$ ,  $j > 0$  and  $F(i, j, q) < |M_{n-m+q,k}|$  for  $0 \leq q \leq m$ . Then*

$$\text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{i+s}M_{m,j}) = 0, \quad s \geq 0.$$

Recall that Lannes' T-functor is left adjoint to the tensoring with  $H := H^*B\mathbb{Z}/2$  in the category  $\mathcal{U}$  of unstable modules over the Steenrod algebra [9]. We need the following result, observed by Harris and Shank [7], to prove Proposition 4.1.

**Proposition 4.3** (Carlisle-Kuhn [4, 6.1] combined with Harris-Shank [7, 4.19]). *There is an isomorphism of unstable modules*

$$\mathbb{T}(L_n) \cong L_n \oplus (H \otimes L_{n-1}).$$

Here  $L_n = M_{n,1}$ .

**Corollary 4.4.** *For  $n \geq m$ , we have  $|\mathbb{T}^m(M_{n,k})| = |M_{n-m,k}|$ .*

*Proof.* By iterating the action of  $\mathbb{T}$  on  $L_n$ , we see that there is an isomorphism of unstable modules

$$\mathbb{T}^m(L_n) \cong \bigoplus_{i=0}^m [H^{\otimes i} \otimes L_{n-i}]^{\oplus a_i},$$

where  $a_i$  are certain positive integers depending only on  $m$ . By using the exactitude of  $\mathbb{T}^m$  and the short exact sequences

$$0 \rightarrow M_{n,k+1} \xrightarrow{\alpha} M_{n,k} \xrightarrow{\beta} \Sigma^k M_{n-1,2k+1} \rightarrow 0,$$

it is easy to prove by induction that there is an isomorphism of graded vector spaces

$$\mathbb{T}^m(M_{n,k}) \cong \bigoplus_{i=0}^m [H^{\otimes i} \otimes M_{n-i,k}]^{\oplus a_i}.$$

The corollary follows.  $\square$

*Proof of Proposition 4.1.* Fix  $i, s$  and take a positive integer  $q$  big enough such that  $i + s + q$  is positive. We have

$$\text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{i+s} M_m) = \text{Ext}_{\mathcal{A}}^s(\Sigma^q M_{n,k}, \Sigma^{i+s+q} M_m).$$

Using the Grothendieck spectral sequence, we need to prove that

$$\text{Ext}_{\mathcal{U}}^{s-j}(\mathbb{D}_j \Sigma^q M_{n,k}, \Sigma^{i+s+q} M_m) = 0, \quad 0 \leq j \leq s.$$

Here  $\mathbb{D}_j$  is the  $j$ th-derived functor of the destabilisation functor

$$\mathbb{D} : \mathcal{A}\text{-mod} \rightarrow \mathcal{U}$$

from the category of  $\mathcal{A}$ -modules to the category of unstable  $\mathcal{A}$ -modules [11].

As  $M_n$  is  $\mathcal{U}$ -injective, it is easily seen that  $\Sigma^\ell M_m$  has a  $\mathcal{U}$ -injective resolution  $I^\bullet$  where  $I^t$  is a direct sum of  $M_m \otimes J(a)$  with  $a \leq \ell - t$ , where  $J(a)$  is the Brown-Gitler module [10]. So we need to prove that, for  $a \leq (i + s + q) - (s - j) = i + j + q$ , we have

$$\text{Hom}_{\mathcal{U}}(\mathbb{D}_j \Sigma^q M_{n,k}, M_m \otimes J(a)) = 0.$$

By Lannes-Zarati [11], we have

$$\mathbb{D}_j \Sigma^q M_{n,k} = \Sigma R_j \Sigma^{j-1+q} M_{n,k} \subset \Sigma^{j+q} H^{\otimes j} \otimes M_{n,k},$$

where  $R_j$  is the Singer functor. It follows that  $\text{Hom}_{\mathcal{U}}(\mathbb{D}_j \Sigma^q M_{n,k}, M_m \otimes J(a))$  is a quotient of

$$\text{Hom}_{\mathcal{U}}(\Sigma^{j+q} H^{\otimes j} \otimes M_{n,k}, M_m \otimes J(a))$$

which is in turn a subgroup of

$$\text{Hom}_{\mathcal{U}}(\Sigma^{j+q} H^{\otimes j} \otimes M_{n,k}, H^{\otimes m} \otimes J(a)) = ((\mathbb{T}^m(\Sigma^{j+q} H^{\otimes j} \otimes M_{n,k}))^a)^*.$$

This group is trivial because, by Corollary 4.4, we have

$$|\mathbb{T}^m(\Sigma^{j+q} H_i \otimes M_{n,k})| = |\Sigma^{j+q} M_{n-m,k}| = |M_{n-m,k}| + j + q > i + j + q \geq a.$$

The proposition follows.  $\square$

*Proof of Proposition 4.2.* We prove the proposition by induction on  $m \geq 0$ . By noting that  $M_{0,j} = \mathbb{Z}/2$ , the case  $m = 0$  is a special case of Proposition 4.1.

Suppose  $m > 0$ . For simplicity, put  $E^s(\Sigma^i M_{m,j}) = \text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{i+s} M_{m,j})$ . The short exact sequence of  $\mathcal{A}$ -modules  $M_{m,j} \hookrightarrow M_{m,j-1} \twoheadrightarrow \Sigma^{j-1} M_{m-1,2j-1}$  induces a long exact sequence in cohomology

$$\cdots \rightarrow E^{s-1}(\Sigma^{i+j} M_{m-1,2j-1}) \rightarrow E^s(\Sigma^i M_{m,j}) \rightarrow E^s(\Sigma^i M_{m,j-1}) \rightarrow \cdots$$

So from the cofiltration of  $M_{m,j}$

$$\begin{array}{ccccccc} \Sigma^i M_{m,j} & \hookrightarrow & \Sigma^i M_{m,j-1} & \hookrightarrow & \cdots & \hookrightarrow & \Sigma^i M_{m,1} & \hookrightarrow & \Sigma^i M_m \\ & & \downarrow & & & & \downarrow & & \downarrow \\ & & \Sigma^{i+j-1} M_{m-1,2j-1} & & & & \Sigma^{i+1} M_{m-1,3} & & \Sigma^i M_{m-1,1} \end{array}$$

we see that, in order to prove  $E^s(\Sigma^i M_{m,j}) = 0$ , it suffices to prove that the groups  $E^{s-1}(\Sigma^{i+j'} M_{m-1,2j'-1})$ ,  $1 \leq j' \leq j$ , and  $E^s(\Sigma^i M_m)$ , are trivial.

By Proposition 4.1,  $E^s(\Sigma^i M_m)$  is trivial since  $i = F(i, j, 0) < |M_{n,k}|$ . For  $1 \leq j' \leq j$  and  $0 \leq q \leq m-1$ , we have

$$F(i + j', 2j' - 1, q) = F(i, j', q + 1) \leq F(i, j, q + 1) < |M_{n-m+1+q,k}|.$$

By inductive hypothesis for  $m-1$ , we have  $E^{s-1}(\Sigma^{i+j'} M_{m-1,2j'-1}) = 0$ . The proposition is proved.  $\square$

We are now ready to prove Theorem 3.1. Recall that the connectivity of  $M_{n,k}$  is given by

$$|M_{n,k}| = 1 + 3 + \cdots + (2^{n-1} - 1) + (2^n - 1)k.$$

*Proof of Theorem 3.1 (1).* Using the Adams spectral sequence, it suffices to prove that

$$\text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{k+t} M_{n-1,2k+1}) = 0 \quad \text{for } s \geq 0 \text{ and } t - s < 0.$$

For  $q \geq 0$ , we have

$$F(k + t - s, 2k + 1, q) = k + t - s + 2k(2^q - 1) + q < (2^{q+1} - 1)k + q \leq |M_{q+1,k}|.$$

The vanishing of the extension groups follows from Proposition 4.2.  $\square$

*Proof of Theorem 3.1 (2).* Using the Adams spectral sequence, it suffices to prove that

$$\text{Ext}_{\mathcal{A}}^s(M_{n,k+1}, \Sigma^{k+s} M_{n-1,2k+1}) = 0, \quad \text{for } s \geq 0.$$

For  $q \geq 0$ , we have

$$F(k, 2k + 1, q) = k + 2k(2^q - 1) + t = (2^{q+1} - 1)k + q < |M_{q+1,k+1}|.$$

The vanishing of the extension groups follows from Proposition 4.2.  $\square$

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