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DICHOTOMY THEOREMS FOR FAMILIES OF NON-COFINAL ESSENTIAL COMPLEXITY

JOHN D. CLEMENS, DOMINIQUE LECOMTE, AND BENJAMIN D. MILLER

Abstract. We prove that for every Borel equivalence relation E, either E is Borel reducible to $\mathbb{E}_0$, or the family of Borel equivalence relations incompatible with E has cofinal essential complexity. It follows that if F is a Borel equivalence relation and $\mathscr{F}$ is a family of Borel equivalence relations of non-cofinal essential complexity which together satisfy the dichotomy that for every Borel equivalence relation E, either $E \in \mathscr{F}$ or F is Borel reducible to E, then $\mathscr{F}$ consists solely of smooth equivalence relations, thus the dichotomy is equivalent to a known theorem.

Introduction

A *reduction* of an equivalence relation E on a set X to an equivalence relation F on a set Y is a function $\pi \colon X \to Y$ with the property that $\forall x_1, x_2 \in X \ (x_1 \ E \ x_2 \iff \pi(x_1) \ F \ \pi(x_2))$. A topological space is *Polish* if it is second countable and completely metrizable, a subset of such a space is *Borel* if it is in the σ-algebra generated by the underlying topology, and a function between such spaces is *Borel* if pre-images of open sets are Borel. Over the last few decades, the study of Borel reducibility of Borel equivalence relations on Polish spaces has emerged as a central theme in descriptive set theory.

The early development of this area was dominated by dichotomy theorems. There are several trivial ones, such as the fact that if n is a natural number, then for every Borel equivalence relation E on a Polish space, either E is Borel reducible to equality on n, or equality on $n+1$ is Borel reducible to E. Similarly, either there is a natural number n for which E is Borel reducible to equality on n, or equality on $\mathbb{N}$ is Borel reducible to E.

There are also non-trivial results of this form. By [Sil80], either E is Borel reducible to equality on $\mathbb{N}$, or equality on $2^{\mathbb{N}}$ is Borel reducible to E. And by [HKL90, Theorem 1.1], either E is Borel reducible to

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equality on $2^{\mathbb{N}}$, or $\mathbb{E}_0$ is Borel reducible to E, where $\mathbb{E}_0$ is the relation on $2^{\mathbb{N}}$ given by $x \mathrel{\mathbb{E}_0} y \iff \exists n \in \mathbb{N} \forall m \geq n \; x(m) = y(m)$.

Whereas the results we have mentioned thus far concern the global structure of the Borel reducibility hierarchy, [KL97, Theorem 1] yields a local dichotomy of this form. Namely, that for every Borel equivalence relation E on a Polish space which is Borel reducible to $\mathbb{E}_1$, either E is Borel reducible to $\mathbb{E}_0$, or $\mathbb{E}_1$ is Borel reducible to E, where $\mathbb{E}_1$ is the relation on $(2^{\mathbb{N}})^{\mathbb{N}}$ given by $x \mathrel{\mathbb{E}_1} y \iff \exists n \in \mathbb{N} \forall m \geq n \; x(m) = y(m)$.

At first glance, one might hope the assumption that E is Borel reducible to $\mathbb{E}_1$ could be eliminated, thereby yielding a new global dichotomy theorem. Unfortunately, [KL97, Theorem 2] ensures that if E is not Borel reducible to $\mathbb{E}_0$, then there is a Borel equivalence relation with which it is incomparable under Borel reducibility. It follows that only the pairs (F, F') discussed thus far (up to Borel bi-reducibility) satisfy both (1) there is a Borel reduction of F to F' but not vice versa, and (2) for every Borel equivalence relation E on a Polish space, either E is Borel reducible to F, or F' is Borel reducible to E.

As the latter result rules out further global dichotomies of the sort discussed thus far, it is interesting to note that its proof hinges upon the previously mentioned local dichotomy, as well as Harrington's unpublished theorem that the family of orbit equivalence relations induced by Borel actions of Polish groups on Polish spaces is unbounded in the Borel reducibility hierarchy. Here we utilize strengthenings of these results to provide a substantially stronger anti-dichotomy theorem.

Given a property P of Borel equivalence relations, we say that a Borel equivalence relation is *essentially* P if it is Borel reducible to a Borel equivalence relation on a Polish space with the given property. A *Wadge reduction* of a set $A \subseteq X$ to a set $B \subseteq Y$ is a continuous function $\pi \colon X \to Y$ such that $\forall x \in X \; (x \in A \iff \pi(x) \in B)$. We say that a Borel equivalence relation E has *essential complexity at least the complexity* of a set $B \subseteq 2^{\mathbb{N}}$ if B is Wadge reducible to every Borel equivalence relation to which E is Borel reducible. We say that a family $\mathscr{F}$ of Borel equivalence relations has *cofinal essential complexity* if for every Borel set $B \subseteq 2^{\mathbb{N}}$, there is a Borel equivalence relation $E \in \mathscr{F}$ with essential complexity at least the complexity of B.

Much as in [KL97], we obtain our anti-dichotomy theorem as a consequence of a result yielding the existence of incomparable Borel equivalence relations, albeit one considerably stronger than that given there.

Theorem 1. *Suppose that X is a Polish space and E is a Borel equivalence relation on X. Then exactly one of the following holds:*

(1) *The relation E is Borel reducible to $\mathbb{E}_0$.*

(2) *The family of Borel equivalence relations on Polish spaces which are incomparable with E under Borel reducibility has cofinal essential complexity.*

We say that a family $\mathscr{F}$ of Borel equivalence relations on Polish spaces is *dichotomical* if there is a *minimum* Borel equivalence relation F on a Polish space which is not in $\mathscr{F}$, meaning that whenever E is a Borel equivalence relation on a Polish space, either $E \in \mathscr{F}$ or there is a Borel reduction of F to E. The following consequence of Theorem 1 implies that the only such families are those associated with the dichotomies mentioned earlier.

Theorem 2. *Suppose that $\mathscr{F}$ is a dichotomical class of Borel equivalence relations on Polish spaces of non-cofinal essential complexity. Then every equivalence relation in $\mathscr{F}$ is smooth.*

In §1, we briefly review the preliminaries needed throughout the paper. In §2, we introduce a property of graphs G ensuring that if a Borel equivalence relation E on a Polish space, whose classes are all countable, is Borel reducible to the equivalence relation generated by G, then it is Borel reducible to $\mathbb{E}_0$. In §3, we introduce a family of ideals on $\mathbb{N} \times \mathbb{N}$, and show that they are cofinal under Wadge reducibility. In §4, we introduce a family of trees on $\mathbb{N} \times \mathbb{N}$, and show that the graphs determined by their branches interact nicely with equivalence relations induced by ideals. In §5, we consider a subfamily of these trees satisfying an appropriate density condition, and show that the graphs determined by their branches interact particularly nicely with equivalence relations induced by the ideals introduced earlier. And in §6, we establish our primary results.

1. Preliminaries

Two sets $M, N \subseteq \mathbb{N}$ are *almost disjoint* if $|M \cap N| < \aleph_0$.

Proposition 1.1. *There is a continuous injection $\pi\colon 2^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ into a family of pairwise almost disjoint infinite sets.*

Proof. It is sufficient to observe that the function $\pi\colon 2^{\mathbb{N}} \to \mathcal{P}(2^{<\mathbb{N}})$, given by $\pi(c) = \{c \restriction n \mid n \in \mathbb{N}\}$, is a continuous injection into a family of pairwise almost disjoint infinite sets. ⊠

A set is *comeager* if it contains an intersection of countably many dense open sets.

Proposition 1.2. *Suppose that X and Y are Polish spaces and the map $\pi\colon X \to Y$ is Borel. Then there is a comeager set $C \subseteq X$ on which π is continuous.*

Proof. See, for example, [Kec95, Theorem 8.38]. ⊠

A subset of a Polish space is *analytic* if it is the continuous image of a Borel subset of a Polish space. A subset of a Polish space is *co-analytic* if its complement is analytic.

Theorem 1.3 (Souslin)**.** *Suppose that X is a Polish space and $B \subseteq X$. Then B is Borel if and only if B is both analytic and co-analytic.*

Proof. See, for example, [Kec95, Theorem 14.11]. ⊠

The *projection* from $X \times Y$ to X is given by $\mathrm{proj}_X(x, y) = x$. A *partial uniformization* of a set $R \subseteq X \times Y$ is a function whose graph is contained in R. A *uniformization* of a set $R \subseteq X \times Y$ is a partial uniformization of R whose domain is $\mathrm{proj}_X(R)$.

Theorem 1.4 (Lusin-Novikov)**.** *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is a Borel set whose vertical sections are all countable. Then $\mathrm{proj}_X(R)$ is Borel, and R is a countable union of Borel uniformizations.*

Proof. See, for example, [Kec95, Theorem 18.10]. ⊠

For each $x \in X$, the xth *vertical section* of a set $R \subseteq X \times Y$ is given by $R_x = \{y \in Y \mid x \mathrel{R} y\}$. The *set of unicity* of R is $\{x \in X \mid |R_x| = 1\}$.

Theorem 1.5 (Lusin)**.** *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is Borel. Then the set of unicity of R is co-analytic.*

Proof. See, for example, [Kec95, Theorem 18.11]. ⊠

A *graph* on a set X is an irreflexive, symmetric set $G \subseteq X \times X$. Such a graph is *locally countable* if its vertical sections are countable. An *edge N-coloring* of G is a map $c \colon G \to N$ with $\forall (x, y) \in G \; c(x, y) = c(y, x)$ and $\forall (x, y), (x, z) \in G \; (y \neq z \implies c(x, y) \neq c(x, z))$.

Theorem 1.6 (Feldman-Moore)**.** *Suppose that X is a Polish space and G is a locally countable Borel graph on X. Then there is a Borel edge $\mathbb{N}$-coloring of G.*

Proof. This follows from the proof of [FM77, Theorem 1]. ⊠

We say that a Borel equivalence relation is *smooth* if it is Borel reducible to equality on $2^{\mathbb{N}}$. An *embedding* is an injective reduction.

Theorem 1.7 (Harrington-Kechris-Louveau)**.** *Suppose that X is a Polish space and E is a Borel equivalence relation on X. Then exactly one of the following holds:*

(1) *The relation E is smooth.*

(2) *There is a continuous embedding of* $\mathbb{E}_0$ *into* E.

Proof. See [HKL90, Theorem 1.1]. ⊠

A *partial transversal* of an equivalence relation E on X is a set $B \subseteq X$ intersecting every equivalence class of E in at most one point. A *transversal* of an equivalence relation E on X is a set $B \subseteq X$ intersecting every equivalence class of E in exactly one point.

Following the standard abuse of language, we say that an equivalence relation is *countable* if all of its equivalence classes are countable.

Proposition 1.8. *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then E is smooth if and only if X is the union of countably many Borel partial transversals of E.*

Proof. This is a straightforward consequence of Theorem 1.4. ⊠

Again following the standard abuse of language, we say that an equivalence relation is *finite* if all of its equivalence classes are finite.

Proposition 1.9. *Suppose that X is a Polish space and E is a finite Borel equivalence relation on X. Then E is smooth.*

Proof. This is also a straightforward consequence of Theorem 1.4. ⊠

We say that a Borel equivalence relation is *hyperfinite* if it is the union of an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite Borel subequivalence relations. By [DJK94, Theorem 1], a countable Borel equivalence relation is hyperfinite if and only if it is Borel reducible to $\mathbb{E}_0$.

Proposition 1.10 (Dougherty-Jackson-Kechris)**.** *Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations on X and Y, E is Borel reducible to F, and F is hyperfinite. Then E is hyperfinite.*

Proof. See [DJK94, Proposition 5.2]. ⊠

We say that a Borel equivalence relation is *hypersmooth* if it is the union of an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of smooth Borel subequivalence relations. By [KL97, Propositions 1.1 and 1.3], a Borel equivalence relation is hypersmooth if and only if it is Borel reducible to $\mathbb{E}_1$.

Theorem 1.11 (Dougherty-Jackson-Kechris)**.** *Suppose that X is a Polish space and E is a Borel equivalence relation on X. Then E is hyperfinite if and only if E is both countable and hypersmooth.*

Proof. See, for example, [DJK94, Theorem 5.1]. ⊠

A property P of subsets of Y is $\mathbf{\Pi^1_1}$-*on*-$\mathbf{\Sigma^1_1}$ if whenever X is a Polish space and $R \subseteq X \times Y$ is an analytic set, the corresponding set $\{x \in X \mid R_x \text{ satisfies } P\}$ is co-analytic.

Theorem 1.12. *Suppose that X is a Polish space, Φ is a $\mathbf{\Pi^1_1}$-on-$\mathbf{\Sigma^1_1}$ property of subsets of X, and $A \subseteq X$ is an analytic set on which Φ holds. Then there is a Borel set $B \supseteq A$ on which Φ holds.*

Proof. See, for example, [Kec95, Theorem 35.10]. ⊠

A *path* through a graph G is a sequence $(x_i)_{i \leq n}$ with the property that $\forall i < n \; x_i \; G \; x_{i+1}$. Such a path is a *cycle* if $n > 2$, $(x_i)_{i<n}$ is injective, and $x_0 = x_n$. A graph is *acyclic* if it has no cycles.

The equivalence relation *generated* by a graph G on a set X is the smallest equivalence relation E_G on X containing it. A *graphing* of an equivalence relation is a graph generating it. We say that a Borel equivalence relation is *treeable* if it has an acyclic Borel graphing.

Theorem 1.13 (Hjorth)**.** *Suppose that X is a Polish space and E is a treeable Borel equivalence relation on X. Then the following are equivalent:*

(1) *The relation E is essentially countable.*
(2) *There is a Borel set $B \subseteq X$ whose intersection with each equivalence class of E is countable and non-empty.*

Proof. See [Hjo08, Theorem 6]. ⊠

Theorem 1.14. *Suppose that X is a Polish space and E is a Borel equivalence relation on X essentially generated by a Borel subgraph of an acyclic compact graph. Then exactly one of the following holds:*

(1) *The relation E is essentially countable.*
(2) *There is a continuous embedding of $\mathbb{E}_1$ into E.*

Proof. See [CLM14, Theorem 6.3 and Proposition 6.4]. ⊠

A topological group is *Polish* if it is Polish as a topological space.

Theorem 1.15 (Hjorth-Kechris-Louveau)**.** *The family of orbit equivalence relations induced by Borel actions of Polish groups on Polish spaces has cofinal essential complexity.*

Proof. See [HKL98, Theorem 4.1]. ⊠

Theorem 1.16 (Kechris-Louveau)**.** *Suppose that X is a Polish space and E is the orbit equivalence relation induced by a Borel action of a Polish group on a Polish space. Then $\mathbb{E}_1$ is not Borel reducible to E.*

Proof. See [KL97, Theorem 4.2]. ⊠

2. Partition stratifications

Suppose that X is a Polish space, E is a Borel equivalence relation on X, and G is a Borel graph on X. We use G_E to denote the graph on X/E given by

$$G_E = \{(C, D) \in (X/E) \times (X/E) \mid C \neq D \text{ and } (C \times D) \cap G \neq \emptyset\}.$$

We say that G has *faithful cycles* over E if among all G-paths $(x_i)_{i \leq k}$, only G-cycles have the property that $([x_i]_E)_{i \leq k}$ is a G_E-cycle.

A *partition stratification* of G is a sequence of the form $(E_n, G_n)_{n \in \mathbb{N}}$, where $(E_n)_{n \in \mathbb{N}}$ is a decreasing sequence of equivalence relations on X, each of which having only countably many classes, whose intersection is the diagonal, $(G_n)_{n \in \mathbb{N}}$ is an increasing sequence of Borel graphs whose union is G, and each G_n has faithful cycles on E_n.

Proposition 2.1. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and G is a Borel graphing of E. Then E is hyperfinite if and only if there is a partition stratification of G.*

Proof. Suppose first that E is hyperfinite, and fix an increasing sequence $(F_n)_{n \in \mathbb{N}}$ of finite Borel subequivalence relations whose union is E. As Propositions 1.8 and 1.9 ensure that spaces underlying finite Borel equivalence relations are countable unions of Borel partial transversals, it follows that there is a decreasing sequence $(E_n)_{n \in \mathbb{N}}$ of Borel equivalence relations such that every E_n has only countably many classes, each of which is a partial transversal of F_n of diameter at most $1/n$. For each $n \in \mathbb{N}$, set $G_n = F_n \cap G$, and note that if $(x_i)_{i \leq k}$ is a G_n-path for which $([x_i]_{E_n})_{i \leq k}$ is a $(G_n)_{E_n}$-cycle, then $x_0 \ (E_n \cap F_n) \ x_k$, so $x_0 = x_k$, thus $(x_i)_{i \leq k}$ is a G_n-cycle. It follows that each G_n has faithful cycles on E_n, so $(E_n, G_n)_{n \in \mathbb{N}}$ is a partition stratification of G.

Conversely, suppose that $(E_n, G_n)_{n \in \mathbb{N}}$ is a partition stratification of G. By Theorem 1.6, there is a Borel edge $\mathbb{N}$-coloring c of G. Let H_n denote the subgraph of G_n consisting of all $(x, y) \in G_n \setminus E_n$ for which $c(x, y)$ is minimal both among natural numbers of the form $c(x', y)$ where $x \ E_n \ x' \ G \ y$, and those of the form $c(x, y')$ where $x \ G \ y' \ E_n \ y$. Then $(H_n)_{n \in \mathbb{N}}$ is an increasing sequence of Borel graphs whose union is G. By Theorem 1.11, to see that E_G is hyperfinite, we need only check that the relations $F_n = E_{H_n}$ are smooth. By Proposition 1.8, it is sufficient to show that for all $n \in \mathbb{N}$, every equivalence class of E_n is a partial transversal of F_n.

Suppose, towards a contradiction, that $k \in \mathbb{N}$ is least for which there is an injective H_n-path $(x_i)_{i \leq k}$ beginning and ending at distinct E_n-related points. The definition of partition stratification ensures that $([x_i]_{E_n})_{i \leq k}$ is not a $(G_n)_{E_n}$-cycle, so there exists $0 < i < k$ for

which $x_{i-1}\ E_n\ x_{i+1}$. Set $x = x_{i-1}$, $y = x_i$, and $x' = x_{i+1}$. Then $c(x, y) \neq c(x', y)$, so the definition of H_n ensures that $\neg x\ H_n\ y$ or $\neg x'\ H_n\ y$, the desired contradiction. ⊠

We say that properties P and Q of Borel equivalence relations *coincide* below a given Borel equivalence relation F if the family of Borel equivalence relations on Polish spaces which are Borel reducible to F and satisfy P is the same as the family of Borel equivalence relations on Polish spaces which are Borel reducible to F and satisfy Q.

Proposition 2.2. *Suppose that X is a Polish space, E is a Borel equivalence relation on X, and G is a Borel graphing of E which admits a partition stratification. Then countability and hyperfiniteness coincide below E.*

Proof. Fix a partition stratification $(E_n, G_n)_{n\in\mathbb{N}}$ of G.

Lemma 2.3. *There are only countably many injective G-paths between any two points.*

Proof. Suppose, towards a contradiction, that $k \in \mathbb{N}$ is the least natural number for which there exist $x, y \in X$ between which there are uncountably many injective G-paths $(z_i)_{i\leq k}$ from x to y. Then for $n \in \mathbb{N}$ sufficiently large, there are uncountably many injective G_n-paths $(z_i)_{i\leq k}$ from x to y with the further property that $([z_i]_{E_n})_{i\leq k}$ is an injective $(G_n)_{E_n}$-path. Fix such an injective G_n-path $(z_i)_{i\leq k}$ from x to y for which there are uncountably many injective G_n-paths $(z'_i)_{i\leq k}$ from x to y inducing the same injective $(G_n)_{E_n}$-path as $(z_i)_{i\leq k}$. The minimality of k then ensures that there are uncountably many injective G_n-paths $(z'_i)_{i\leq k}$ from x to y inducing the same injective $(G_n)_{E_n}$-path as $(z_i)_{i\leq k}$ but avoiding $\{z_i \mid 0 < i < k\}$. Then for $n' \in \mathbb{N}$ sufficiently large, there are uncountably many injective G_n-paths $(z'_i)_{i\leq k}$ from x to y inducing the same injective $(G_n)_{E_n}$-path as $(z_i)_{i\leq k}$ but for which the corresponding injective $(G_n)_{E_{n'}}$-path avoids $\{[z_i]_{E_{n'}} \mid 0 < i < k\}$. It follows that there is such an injective G_n-path $(z'_i)_{i\leq k}$ from x to y for which there are uncountably many such injective G_n-paths $(z''_i)_{i\leq k}$ inducing the same injective $(G_n)_{E_{n'}}$-path as $(z'_i)_{i\leq k}$. By one more appeal to the minimality of k, there are uncountably many such injective G_n-paths $(z''_i)_{i\leq k}$ inducing the same injective $(G_n)_{E_{n'}}$-path as $(z'_i)_{i\leq k}$ but avoiding $\{z'_i \mid 0 < i < k\}$. Fix any such injective G_n-path $(z''_i)_{i\leq k}$, and observe that $(z'_1, \ldots, z'_k = z_k, \ldots, z_0 = z''_0, z''_1)$ is an injective G_n-path inducing a $(G_n)_{E_{n'}}$-cycle, contradicting the fact that $G_{n'}$ has faithful cycles on $E_{n'}$. ⊠

By Theorem 1.4 and Proposition 1.10, it is sufficient to show that E is hyperfinite on every Borel set $B \subseteq X$ on which it is countable.

Towards this end, let C denote the *convex closure* of B with respect to G, that is, the set of points lying along an injective G-path between two points of B. As Theorem 1.4 and Lemma 2.3 ensure that C is also a Borel set on which E is countable, Proposition 2.1 implies that E is hyperfinite on C, thus hyperfinite on B. ⊠

In the treeable case, we can say even more.

Proposition 2.4. *Suppose that X is a Polish space, E is a treeable Borel equivalence relation on X, and countability and hyperfiniteness coincide below E. Then essential countability and essential hyperfiniteness also coincide below E.*

Proof. Suppose that Y is a Polish space and F is an essentially countable Borel equivalence relation on Y which admits a Borel reduction $\phi\colon Y \to X$ to E. Fix a Polish space Y' and a countable Borel equivalence relation F' on Y' for which there is a Borel reduction $\psi\colon Y \to Y'$ of F to F'. Then the set $R_0 = \{(x, \psi(y)) \mid \phi(y) = x\}$ *induces a partial injection* of X/E into Y'/F', in the sense that $x_1 \mathrel{E} x_2 \iff y_1' \mathrel{F'} y_2'$, for all $(x_1, y_1'), (x_2, y_2') \in R_0$.

The *product* of the equivalence relations E and F' is the relation on $X \times Y'$ given by $(x_1, y_1') \mathrel{(E \times F')} (x_2, y_2') \iff (x_1 \mathrel{E} x_2 \text{ and } y_1' \mathrel{F'} y_2')$.

Lemma 2.5. *There is an $(E \times F')$-invariant Borel set $R \supseteq R_0$ inducing a partial injection of X/E into Y'/F'.*

Proof. As the property of inducing a partial injection of X/E into Y'/F' is $\mathbf{\Pi^1_1}$-on-$\mathbf{\Sigma^1_1}$ and closed under $(E \times F')$-saturation, by repeatedly applying Theorem 1.12, we obtain Borel sets $R_{n+1} \supseteq [R_n]_{E \times F'}$ inducing Borel partial injections of X/E into Y'/F'. Define $R = \bigcup_{n \in \mathbb{N}} R_n$. ⊠

As F' is countable, Theorem 1.4 ensures that the set $C = \mathrm{proj}_X(R)$ is Borel, and that there is a Borel uniformization $\pi\colon C \to Y'$ of R. As any such function is necessarily a reduction of E to F' on C, it follows that E is essentially countable on C. An application of Theorem 1.13 then yields a Borel set $D \subseteq C$, whose E-saturation is C, on which E is countable. As countability and hyperfiniteness coincide below E, it follows that E is hyperfinite on D, and one more application of Theorem 1.4 yields a Borel reduction of $E \restriction C$ to $E \restriction D$, so E is essentially hyperfinite on C, thus F is essentially hyperfinite. ⊠

3. Ideals

We say that a family $\mathcal{K}$ of subsets of $\mathbb{N} \times \mathbb{N}$ is *determined by cardinalities on vertical sections* if $A \in \mathcal{K} \iff B \in \mathcal{K}$, whenever $\forall n \in \mathbb{N}\ |A_n| = |B_n|$.

For each family $\mathscr{N} \subseteq \mathcal{P}(\mathbb{N})$ of subsets of the natural numbers, we use $\mathrm{cl}(\mathscr{N})$ to denote the closure of $\mathscr{N}$ under finite unions, and we define $\mathcal{K}_{\mathscr{N}} = \bigcup_{N \in \mathrm{cl}(\mathscr{N})} \mathcal{K}_N$, where

$$\mathcal{K}_N = \{A \subseteq \mathbb{N} \times \mathbb{N} \mid \forall n \in \mathbb{N}\ (|A_n| = \aleph_0 \implies n \in N)\}.$$

Note that every such family is both determined by cardinalities on vertical sections and an *ideal*, in the sense that it is closed under containment and finite unions.

Proposition 3.1. *Suppose that $\mathscr{N} \subseteq \mathcal{P}(\mathbb{N})$ is a family of pairwise almost disjoint infinite subsets of $\mathbb{N}$. Then there is a continuous function $\pi \colon \mathscr{N} \to \mathcal{P}(\mathbb{N} \times \mathbb{N})$ Wadge reducing $\mathscr{M}$ to $\mathcal{K}_{\mathscr{M}}$, for all $\mathscr{M} \subseteq \mathscr{N}$.*

Proof. Define $\pi(N) = N \times \mathbb{N}$. Given $\mathscr{M} \subseteq \mathscr{N}$, observe first that if $M \in \mathscr{M}$, then $\pi(M) \in \mathcal{K}_M \subseteq \mathcal{K}_{\mathscr{M}}$. Conversely, if $N \in \mathscr{N}$ and $\pi(N) \in \mathcal{K}_{\mathscr{M}}$, then there exist $n \in \mathbb{N}$ and $M_1, \ldots, M_n \in \mathscr{M}$ such that $\pi(N) \in \mathcal{K}_{M_1 \cup \cdots \cup M_n}$, in which case $N \subseteq M_1 \cup \cdots \cup M_n$. As $\mathscr{N}$ consists of pairwise almost disjoint infinite sets, it follows that $N = M_i$, for some $1 \leq i \leq n$, thus $N \in \mathscr{M}$. ⊠

A weak converse to this result is provided by the following.

Proposition 3.2. *Suppose that $\mathscr{N} \subseteq \mathcal{P}(\mathbb{N})$ is a Borel family of pairwise almost disjoint infinite subsets of $\mathbb{N}$. Then $\mathcal{K}_{\mathscr{N}}$ is also Borel.*

Proof. Clearly $\mathcal{K}_{\mathscr{N}}$ is analytic, so by Theorem 1.3, it is sufficient to show that it is co-analytic. But this follows from Theorem 1.5 and the fact that a set A is in $\mathcal{K}_{\mathscr{N}}$ if and only if there exist $k \in \mathbb{N}$ and a unique subfamily $\mathscr{F}$ of $\mathscr{N}$ of cardinality k for which there is a finite subset F of a set in $\mathrm{cl}(\mathscr{N})$ such that $\forall n \in \mathbb{N}\ (|A_n| = \aleph_0 \implies n \in F \cup \bigcup \mathscr{F})$. ⊠

4. Trees

Suppose that $t_{i,n} \in 2^n$, for all $i < 2$ and $n \in \mathbb{N}$. Associated with $(t_{i,n})_{i<2, n \in \mathbb{N}}$ are the graphs T_n on 2^n obtained recursively by letting T_{n+1} be the union of the graph $\{(s \frown (i), t \frown (i)) \mid i < 2 \text{ and } (s,t) \in T_n\}$ with the singleton edge $\{(t_{i,n} \frown (i), t_{1-i,n} \frown (1-i)) \mid i < 2\}$, as well as the set $T = \{(\emptyset, \emptyset)\} \cup \bigcup_{n \in \mathbb{N}} T_n$.

A straightforward induction shows that each T_n is a tree on 2^n. It follows that if the set T is closed under initial segments, then the set $[T] = \{(x,y) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} \mid \forall n \in \mathbb{N}\ (x \restriction n, y \restriction n) \in T_n\}$ of *branches* through T is an acyclic compact graph admitting a partition stratification, thus all of its Borel subgraphs admit partition stratifications as well. As equivalence relations induced by acyclic Borel graphs are themselves Borel (by Theorems 1.3 and 1.5), Propositions 2.2 and 2.4

therefore imply that essential countability and essential hyperfiniteness coincide below equivalence relations generated by such subgraphs.

The *support* of a sequence $c \in 2^{\mathbb{N}}$ is given by $\mathrm{supp}(c) = c^{-1}(1)$.

Proposition 4.1. *Suppose that $t_{i,n} \in 2^n$ for all $i < 2$ and $n \in \mathbb{N}$, the corresponding set T is closed under initial segments, $(a, b), (c, d) \in [T]$, and $\{a, b\} \neq \{c, d\}$. Then $(\mathrm{supp}(a) \triangle \mathrm{supp}(b)) \cap (\mathrm{supp}(c) \triangle \mathrm{supp}(d))$ is finite.*

Proof. For each $n \in \mathbb{N}$, let T_n denote the tree on 2^n associated with $(t_{i,n})_{i<2,n\in\mathbb{N}}$, and note that if $n \in \mathrm{supp}(a) \triangle \mathrm{supp}(b)$, then the pair of restrictions $(a \restriction (n+1), b \restriction (n+1))$ is in T_{n+1}, from which it follows that $\{a \restriction (n+1), b \restriction (n+1)\} = \{t_{i,n} \frown (i) \mid i < 2\}$, and therefore that $\{a \restriction n, b \restriction n\} = \{t_{i,n} \mid i < 2\}$. In particular, if $n \in \mathbb{N}$ is sufficiently large that $\{a \restriction n, b \restriction n\} \neq \{c \restriction n, d \restriction n\}$, then n is not in $(\mathrm{supp}(a) \triangle \mathrm{supp}(b)) \cap (\mathrm{supp}(c) \triangle \mathrm{supp}(d))$. ⊠

Given a family $\mathcal{I}$ of subsets of $\mathbb{N}$, let $E_{\mathcal{I}}$ denote the binary relation on $2^{\mathbb{N}}$ given by $c \; E_{\mathcal{I}} \; d \iff \mathrm{supp}(c) \triangle \mathrm{supp}(d) \in \mathcal{I}$.

Proposition 4.2. *Suppose that $t_{i,n} \in 2^n$ for all $i < 2$ and $n \in \mathbb{N}$, the corresponding set T is closed under initial segments, and $\mathcal{I}$ is an ideal on $\mathbb{N}$ containing all finite subsets of $\mathbb{N}$. Then $E_{\mathcal{I}} \cap [T]$ is a graphing of $E_{\mathcal{I}} \cap E_{[T]}$.*

Proof. It is sufficient to show that if $n \in \mathbb{N}$, $(c_i)_{i\leq n}$ is an injective $[T]$-path, and $c_0 \; E_{\mathcal{I}} \; c_n$, then $(c_i)_{i\leq n}$ is an $(E_{\mathcal{I}} \cap [T])$-path. Towards this end, appeal to Proposition 4.1 repeatedly to obtain a finite set $F \subseteq \mathbb{N}$ containing $(\mathrm{supp}(c_i) \triangle \mathrm{supp}(c_{i+1})) \cap (\mathrm{supp}(c_j) \triangle \mathrm{supp}(c_{j+1}))$, for all $i < j < n$. Then $\mathrm{supp}(c_0) \triangle \mathrm{supp}(c_n)$ and $\bigcup_{i<n} \mathrm{supp}(c_i) \triangle \mathrm{supp}(c_{i+1})$ agree off of F, so $\mathrm{supp}(c_i) \triangle \mathrm{supp}(c_{i+1}) \subseteq F \cup (\mathrm{supp}(c_0) \triangle \mathrm{supp}(c_n))$, and is therefore in $\mathcal{I}$, for all $i < n$. ⊠

5. Density

Suppose that $t_{i,n} \in 2^n$, for all $i < 2$ and $n \in \mathbb{N}$, and the corresponding set T is closed under initial segments. Then for each $n \in \mathbb{N}$, there exist $k \in \{0, \ldots, n\}$ and $s \in 2^{n-k}$ with $t_{i,n+1} = t_{i,k} \frown (i) \frown s$. Conversely, if $k_n \in \{0, \ldots, n\}$ and $s_n \in 2^{n-k_n}$ for all $n \in \mathbb{N}$, then the set T associated with $(t_{i,n})_{i<2,n\in\mathbb{N}}$, where $t_{i,0} = \emptyset$ and $t_{i,n+1} = t_{i,k_n} \frown (i) \frown s_n$ for $i < 2$ and $n \in \mathbb{N}$, is closed under initial segments. We say that $(k_n, s_n)_{n\in\mathbb{N}}$ is *suitable* if $k_n \in \{0, \ldots, n\}$ and $s_n \in 2^{n-k_n}$, for all $n \in \mathbb{N}$.

Fix an injective enumeration $(i_n, j_n)_{n\in\mathbb{N}}$ of $\mathbb{N} \times \mathbb{N}$ with $i_0 = 0$, and let e denote its inverse. We say that $(k_n, s_n)_{n\in\mathbb{N}}$ is *dense* (with respect to

our fixed enumeration) if for all $i, k \in \mathbb{N}$ and all $s \in 2^{<\mathbb{N}}$, there exists $n \in \mathbb{N}$ such that $i = i_{n+1}$, $k = k_n$, and $s \sqsubseteq s_n$.

The *push-forward* of a family $\mathcal{K}$ of subsets of $\mathbb{N} \times \mathbb{N}$ through e is given by $e_*\mathcal{K} = \{e(A) \mid A \in \mathcal{K}\}$.

Proposition 5.1. *Suppose that $(k_n, s_n)_{n\in\mathbb{N}}$ is dense and suitable, T is the corresponding set, $C \subseteq 2^{\mathbb{N}}$ is comeager, $\mathcal{K}$ is a family of subsets of $\mathbb{N} \times \mathbb{N}$ which is determined by cardinalities on vertical sections and invariant under finite alterations of the leftmost column, and $\mathcal{I} = e_*\mathcal{K}$. Then there is a Wadge reduction $\pi\colon \mathcal{P}(\mathbb{N}) \to [T] \restriction C$ of $\mathcal{I}$ to $E_{\mathcal{I}}$.*

Proof. Fix dense open sets $U_n \subseteq 2^{\mathbb{N}}$ whose intersection is contained in C, and let $t_{i,n}$ denote the sequences associated with $(k_n, s_n)_{n\in\mathbb{N}}$. We will recursively construct natural numbers $\ell_n > 0$, in addition to natural numbers $n_u < \ell_n$ and sequences $t_u \in 2^{\ell_n - n_u - 1}$ for $u \in 2^n$, from which we define $\pi_{i,n}\colon 2^n \to 2^{\ell_n}$ by $\pi_{i,n}(u) = t_{i,n_u} \frown (i) \frown t_u$, for $i < 2$ and $u \in 2^n$. We will ensure that the following conditions hold:

(1) $\forall i, j < 2 \forall u \in 2^n\ \pi_{i,n}(u) \sqsubset \pi_{i,n+1}(u \frown (j))$.
(2) $\forall i < 2 \forall u \in 2^{n+1}\ \mathcal{N}_{\pi_{i,n+1}(u)} \subseteq U_n$.
(3) $\forall u \in 2^n\ (\pi_{0,n}(u), \pi_{1,n}(u)) \in T$.
(4) $\forall u \in 2^n\ i_n = i_{n_{u\frown(1)}}$.
(5) $\forall u \in 2^n\ \mathrm{supp}(\pi_{0,n+1}(u \frown (0))) \triangle \mathrm{supp}(\pi_{1,n+1}(u \frown (0)))$
$= \mathrm{supp}(\pi_{0,n}(u)) \triangle \mathrm{supp}(\pi_{1,n}(u))$.
(6) $\forall u \in 2^n\ \mathrm{supp}(\pi_{0,n+1}(u \frown (1))) \triangle \mathrm{supp}(\pi_{1,n+1}(u \frown (1)))$
$= (\mathrm{supp}(\pi_{0,n}(u)) \triangle \mathrm{supp}(\pi_{1,n}(u))) \cup \{n_{u\frown(1)}\}$.

We begin by setting $\ell_0 = 1$, $n_\emptyset = 0$, and $t_\emptyset = \emptyset$.

Suppose now that $n \in \mathbb{N}$ and we have already found ℓ_n, n_u, and t_u, for $u \in 2^n$. For each $u \in 2^n$, set $n_{u\frown(0)} = n_u$, and fix $t'_u \in 2^{<\mathbb{N}}$ with the property that $\mathcal{N}_{\pi_{i,n}(u)\frown t'_u} \subseteq U_n$ for all $i < 2$. By density, there exists $n_{u\frown(1)} > 0$ with $i_n = i_{n_{u\frown(1)}}$, $n_u = k_{n_{u\frown(1)}-1}$, and $t_u \frown t'_u \sqsubseteq s_{n_{u\frown(1)}-1}$. Define $\ell_{n+1} = \max_{u\in 2^n} n_{u\frown(1)} + 1$, and for each $u \in 2^n$, fix an extension $t_{u\frown(0)} \in 2^{\ell_{n+1} - n_u - 1}$ of $t_u \frown t'_u$, as well as $t_{u\frown(1)} \in 2^{\ell_{n+1} - n_{u\frown(1)} - 1}$.

By condition (1), the function $\pi\colon \mathcal{P}(\mathbb{N}) \to 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ which is given by $\pi(A) = (\pi_0(\chi_A), \pi_1(\chi_A))$, where $\pi_i(c) = \bigcup_{n\in\mathbb{N}} \pi_{i,n}(c \restriction n)$ for all $i < 2$, is well-defined and continuous. Condition (2) then ensures that $\pi_i(2^{\mathbb{N}}) \subseteq C$ for all $i < 2$, so $\pi(\mathcal{P}(\mathbb{N})) \subseteq C \times C$, thus condition (3) implies that $\pi(\mathcal{P}(\mathbb{N})) \subseteq [T] \restriction C$. And conditions (4) - (6) ensure that for all $N \subseteq \mathbb{N}$, the cardinalities of all but the leftmost vertical sections of $e^{-1}(\mathrm{supp}(\pi_0(\chi_N)) \triangle \mathrm{supp}(\pi_1(\chi_N)))$ and $e^{-1}(N)$ agree, whereas the cardinalities of their leftmost vertical sections differ by at most one, thus $N \in \mathcal{I}$ if and only if $\pi(N) \in E_{\mathcal{I}}$. ⊠

As a consequence, we obtain the following.

Theorem 5.2. *The family of treeable Borel equivalence relations below which essential hyperfiniteness and the inexistence of a continuous embedding of* $\mathbb{E}_1$ *coincide has cofinal essential complexity.*

Proof. By appealing to Proposition 1.1, we obtain a continuous injection $\pi\colon 2^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ into a family of pairwise almost disjoint infinite sets. Given a Borel set $B \subseteq 2^{\mathbb{N}}$, set $\mathscr{N} = \pi(B)$. Proposition 3.1 then ensures that B is Wadge reducible to $\mathcal{K}_{\mathscr{N}}$, and Proposition 3.2 implies that the latter is Borel, thus the same holds of the ideal $\mathcal{I} = e_*\mathcal{K}_{\mathscr{N}}$.

Fix a dense suitable sequence $(k_n, s_n)_{n\in\mathbb{N}}$, and let T be the associated set. Proposition 4.2 then ensures that the equivalence relation E on $2^{\mathbb{N}}$ given by $E = E_{\mathcal{I}} \cap E_{[T]}$ is generated by a Borel subgraph of an acyclic compact graph, and since it is clearly Borel, by Theorem 1.14 we need only check that its essential complexity is at least the complexity of B.

Towards this end, suppose that Y is a Polish space, and F is a Borel equivalence relation on Y for which there is a Borel reduction $\phi\colon X \to Y$ of E to F. Proposition 1.2 then yields a comeager Borel set $C \subseteq X$ on which ϕ is continuous. By Proposition 5.1, there is a Wadge reduction $\psi\colon \mathcal{P}(\mathbb{N}) \to [T] \restriction C$ of $\mathcal{I}$ to E, and it follows that $(\phi \times \phi) \circ \psi$ is a Wadge reduction of $\mathcal{I}$ to F, so B is Wadge reducible to F, thus the essential complexity of E is at least the complexity of B. ⊠

6. Anti-basis results

We say that a family $\mathscr{F}$ of Borel equivalence relations on Polish spaces is *unbounded* if for every Borel equivalence relation E on a Polish space, there is a Borel equivalence relation $F \in \mathscr{F}$ which is not Borel reducible to E.

We say that the non-linearity of Borel reducibility is *captured* off of a family $\mathscr{F}$ of Borel equivalence relations if every non-essentially-hyperfinite Borel equivalence relation is incompatible with a Borel equivalence relation outside of $\mathscr{F}$.

Theorem 6.1. *Suppose that* $\mathscr{F}$ *is a class of Borel equivalence relations whose complement contains unboundedly many Borel equivalence relations below which essential hyperfiniteness and the inexistence of a Borel reduction of* $\mathbb{E}_1$ *coincide, as well as unboundedly many Borel equivalence relations to which* $\mathbb{E}_1$ *does not admit a Borel reduction. Then the non-linearity of Borel reducibility is captured off of* $\mathscr{F}$*.*

Proof. Suppose that E is a Borel equivalence relation compatible with every Borel equivalence relation outside of $\mathscr{F}$. Fix a Borel equivalence relation E', outside of $\mathscr{F}$ and not Borel reducible to E, below which essential hyperfiniteness and the inexistence of a Borel reduction of $\mathbb{E}_1$

coincide. In addition, fix a Borel equivalence relation E'', outside of $\mathscr{F}$ and not Borel reducible to E, to which $\mathbb{E}_1$ does not admit a Borel reduction. As E is Borel reducible to E'', it follows that $\mathbb{E}_1$ is not Borel reducible to E. As E is Borel reducible to E', it therefore follows that E is essentially hyperfinite. ⊠

The following corollary strengthens [KL97, Theorem 2].

Theorem 6.2. *Suppose that X is a Polish space and E is a Borel equivalence relation on X. Then the following are equivalent:*

(1) *The relation E is not essentially hyperfinite.*
(2) *There is a Borel equivalence relation on a Polish space which is incompatible with E.*
(3) *The family of Borel equivalence relations on Polish spaces which are incompatible with E under Borel reducibility has cofinal essential complexity.*

Proof. As (3) $\implies$ (2) is trivial and (2) $\implies$ (1) is a consequence of Theorem 1.7, it is sufficient to show (1) $\implies$ (3). Towards this end, note that Theorem 5.2 ensures that the family of Borel equivalence relations on Polish spaces below which essential hyperfiniteness and the inexistence of a Borel reduction of $\mathbb{E}_1$ coincide has cofinal essential complexity, and Theorems 1.15 and 1.16 imply that the family of Borel equivalence relations on Polish spaces to which $\mathbb{E}_1$ does not Borel reduce has cofinal essential complexity. So, by Theorem 6.1, the family of Borel equivalence relations on Polish spaces which are incompatible with E under Borel reducibility has cofinal essential complexity. ⊠

We can now establish our primary result.

Theorem 6.3. *Suppose that $\mathscr{F}$ is a dichotomical class of Borel equivalence relations on Polish spaces of bounded essential complexity. Then every equivalence relation in $\mathscr{F}$ is smooth.*

Proof. Fix a Borel equivalence relation F witnessing that $\mathscr{F}$ is dichotomical. Then F is necessarily essentially hyperfinite, since otherwise Theorem 6.2 would yield a Borel equivalence relation outside of $\mathscr{F}$ and incompatible with F. But Theorem 1.7 then implies that every relation in $\mathscr{F}$ is smooth. ⊠

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