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# Iwahori-Hecke algebras for Kac-Moody groups over local fields

Nicole Bardy-Panse, Stéphane Gaussent and Guy Rousseau

December 18, 2014

## Abstract

We define the Iwahori-Hecke algebra  ${}^I\mathcal{H}$  for an almost split Kac-Moody group  $G$  over a local non-archimedean field. We use the hovel  $\mathcal{S}$  associated to this situation, which is the analogue of the Bruhat-Tits building for a reductive group. The fixer  $K_I$  of some chamber in the standard apartment plays the role of the Iwahori subgroup. We can define  ${}^I\mathcal{H}$  as the algebra of some  $K_I$ -bi-invariant functions on  $G$  with support consisting of a finite union of double classes. As two chambers in the hovel are not always in a same apartment, this support has to be in some large subsemigroup  $G^+$  of  $G$ . In the split case, we prove that the structure constants of  ${}^I\mathcal{H}$  are polynomials in the cardinality of the residue field, with integer coefficients depending on the geometry of the standard apartment. We give a presentation of this algebra  ${}^I\mathcal{H}$ , similar to the Bernstein-Lusztig presentation in the reductive case, and embed it in a greater algebra  ${}^{BL}\mathcal{H}$ , algebraically defined by the Bernstein-Lusztig presentation. Actually, our results apply to abstract “locally finite” hovels, so that we can define the Iwahori-Hecke algebra with unequal parameters.

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## Introduction

Let  $\mathcal{K}$  be a local non-archimedean field, with residue field  $\mathbb{F}_q$ . Suppose  $G$  is a split, simple and simply connected algebraic group over  $\mathcal{K}$  and  $K_0$  an open compact subgroup. The space  $\mathcal{H}_{K_0}$  of complex functions on  $G$ , bi-invariant by  $K_0$  and with compact support, is an algebra for the natural convolution product. Ichiro Satake [Sa63] studied such algebras  $\mathcal{H}_{K_0}$  to define the spherical functions and proved, in particular, that  $\mathcal{H}_{K_0}$  is commutative for a good choice  $K_s$  of

$K_0$ , maximal compact; the corresponding convolution algebra  $\mathcal{H}_{K_s} = {}^s\mathcal{H}(G)$  is now called the spherical Hecke algebra. From the work of Nagayoshi Iwahori and Hideya Matsumoto [IM65], we know that there is an interesting open subgroup  $K_I$  of  $K_s$  (now called Iwahori subgroup) with a Bruhat decomposition  $G = K_I.W.K_I$ , where  $W$  is an infinite Coxeter group. The corresponding convolution algebra  $\mathcal{H}_{K_I} = {}^I\mathcal{H}(G)$  (now called Iwahori-Hecke algebra) may be described as the abstract Hecke algebra associated to this Coxeter group and the parameter  $q$ . There is another presentation of this Hecke algebra, stated by Joseph Bernstein and proved in the most general case by George Lusztig [Lu89]; it emphasizes the role of the translations in  $W$  and uses new relations, now often called the Bernstein-Lusztig relations. We know also now that a good interpretation of these algebras uses the Bruhat-Tits building  $\mathcal{S}$  of  $G$  over  $\mathcal{K}$ , whose structure is explained in [BrT72]:  $K_s$  (resp.  $K_I$ ) is the fixer of a special vertex (resp. a chamber) in this building and these Hecke algebras may be defined using only this building, see *e.g.* [P06].

Kac-Moody groups are interesting generalizations of semisimple groups and it is natural to try to define the above Hecke algebras in that case. So let  $G$  be a Kac-Moody group over  $\mathcal{K}$ , assumed minimal or "algebraic", *i.e.* as studied by Jacques Tits [T87] in the split case and by Bertrand Rémy [Re02] in the almost split case. Unfortunately there is, up to now, no good topology on  $G$  and no good compact subgroup, so the "convolution product" has to be defined only by other means. Alexander Braverman and David Kazhdan [BrK11] succeeded in defining geometrically such a spherical Hecke algebra, when  $G$  is split and untwisted affine, see also the survey [BrK14] by the same authors. We were able, in [GR14], to generalize their construction to any Kac-Moody group over  $\mathcal{K}$ . In [BrKP14], Alexander Braverman, David Kazhdan and Manish Patnaik construct the spherical Hecke algebra and the Iwahori-Hecke algebra by algebraic computations (including results of [Ga95] and [BrGKP14]) in the Kac-Moody group, still assumed split and untwisted affine (and even simply laced for some results). They are convolution algebras of functions on  $G$  bi-invariant under a group  $K_s$  or  $K_I$  ( $\subset K_s$ ), but there are two new features: the support has to be in a subsemigroup  $G^+$  of  $G$  and the description of the Iwahori-Hecke algebra has to use Bernstein-Lusztig type relations (as  $W$  is no longer a Coxeter group).

We shall generalize these results on the Iwahori-Hecke algebra to the case of any Kac-Moody group. As in [GR14], our idea is to define this Iwahori-Hecke algebra using the hovel associated to the almost split Kac-Moody group  $G$  that we built in [GR08], [Ro12] and [Ro13]. This hovel  $\mathcal{S}$  is a set with an action of  $G$  and a covering by subsets called apartments. They are in one-to-one correspondence with the maximal split subtori, hence permuted transitively by  $G$ . Each apartment  $A$  is a finite dimensional real affine space and its stabilizer  $N$  in  $G$  acts on it via a generalized affine Weyl group  $W = W^v \rtimes Y$  (where  $Y \subset \overrightarrow{A}$  is a discrete subgroup of translations) which stabilizes a set  $\mathcal{M}$  of affine hyperplanes called walls. So,  $\mathcal{S}$  looks much like the Bruhat-Tits building of a reductive group, but  $\mathcal{M}$  is not a locally finite system of hyperplanes (as the root system  $\Phi$  is infinite) and two points in  $\mathcal{S}$  are not always in a same apartment (this is why  $\mathcal{S}$  is called a hovel). There is on  $\mathcal{S}$  a  $G$ -invariant preorder  $\leq$  which induces on each apartment  $A$  the preorder given by the Tits cone  $\mathcal{T} \subset \overrightarrow{A}$ .

Now, we consider the fixer  $K_I$  in  $G$  of some (local) chamber  $C_0^+$  in a chosen standard apartment  $\mathbb{A}$ ; it is our Iwahori subgroup. Fix a ring  $R$ . The Iwahori-Hecke algebra  ${}^I\mathcal{H}_R$  is the space of some  $K_I$ -bi-invariant functions on  $G$  with values in  $R$ . In other words, it is the space  ${}^I\mathcal{H}_R^\mathcal{S}$  of some  $G$ -invariant functions on  $\mathcal{C}_0^+ \times \mathcal{C}_0^+$ , where  $\mathcal{C}_0^+ = G/K_I$  is the orbit of  $C_0^+$  in the set  $\mathcal{C}$  of chambers of  $\mathcal{S}$ . The convolution product is easy to guess

from this point of view:  $(\varphi * \psi)(C_x, C_y) = \sum_{C_z \in \mathcal{C}_0^+} \varphi(C_x, C_z)\psi(C_z, C_y)$  (if this sum means something). As two chambers in  $\mathcal{S}$  are not always in a same apartment (*i.e.* the Bruhat-Iwahori decomposition fails:  $G \neq K_I.N.K_I$ ), we have to consider pairs  $(C_x, C_y) \in \mathcal{C}_0^+ \times_{\leq} \mathcal{C}_0^+$ , *i.e.*  $C_x$  (resp.  $C_y$ )  $\in \mathcal{C}_0^+$  has  $x$  (resp.  $y$ ) for vertex and  $x \leq y$  (this implies that  $C_x, C_y$  are in a same apartment). For  ${}^I\mathcal{H}_R$ , this means that the support of  $\varphi \in \mathcal{H}_R$  has to be in  $K_I \backslash G^+ / K_I$  where  $G^+ = \{g \in G \mid 0 \leq g.0\}$  is a semigroup; we suppose moreover this support finite. In addition,  $K_I \backslash G^+ / K_I$  is in bijective correspondence with the subsemigroup  $W^+ = W^v \rtimes Y^+$  of  $W$  (where  $Y^+ = Y \cap \mathcal{T}$ ).

With this definition we are able to prove that  ${}^I\mathcal{H}_R$  is really an algebra, which generalizes the known Iwahori-Hecke algebras in the semi-simple case (see §2).

The structure constants of  ${}^I\mathcal{H}_R$  are the non-negative integers  $a_{\mathbf{w}, \mathbf{v}}^{\mathbf{u}}$  (for  $\mathbf{w}, \mathbf{v}, \mathbf{u} \in W^+$ ) such that  $T_{\mathbf{w}} * T_{\mathbf{v}} = \sum_{\mathbf{u} \in W^+} a_{\mathbf{w}, \mathbf{v}}^{\mathbf{u}} T_{\mathbf{u}}$  (finite sum), where  $T_{\mathbf{w}}$  is the characteristic function of  $K_I.\mathbf{w}.K_I$ . Each chamber in  $\mathcal{S}$  has only a finite number of adjacent chambers along a given panel. These numbers are called parameters of  $\mathcal{S}$  and they form a finite set  $\mathcal{Q}$ . In the split case, there is only one parameter  $q$ : the number of elements of the residue field of  $\mathcal{K}$ . We conjecture that each  $a_{\mathbf{w}, \mathbf{v}}^{\mathbf{u}}$  is a polynomial in these parameters with integral coefficients depending only on the geometry of the model apartment  $\mathbb{A}$  and on  $W$ . We prove this only partially: this is true if  $G$  is split or if we replace "polynomial" by "Laurent polynomial" (*cf.* 6.7); this is also true for  $\mathbf{w}, \mathbf{v}$  "generic" (*cf.* 3.8). Actually in this generic case, we give, in section 3, an explicit formula for  $a_{\mathbf{w}, \mathbf{v}}^{\mathbf{u}}$ .

If the parameters in  $\mathcal{Q}$  are invertible in the ring  $R$ , we are able, in section 4, to deduce from the geometry of  $\mathcal{S}$  a set of generators and some relations in  ${}^I\mathcal{H}_R$ . The family  $(T_{\lambda} * T_w)_{\lambda \in Y^+, w \in W^v}$  is an  $R$ -basis of  ${}^I\mathcal{H}_R$  and  $\sum_{w \in W^v} R.T_w$  is the abstract Hecke algebra  $\mathcal{H}_R(W^v)$  associated to the Coxeter group  $W^v$ , generated by the  $T_i = T_{r_i}$  (where the  $r_i$  are the fundamental reflections in  $W^v$ ). So  ${}^I\mathcal{H}_R$  is a free right  $\mathcal{H}_R(W^v)$ -module. We get also some commuting relations between the  $T_{\lambda}$  and the  $T_w$ , including some relations of Bernstein-Lusztig type (Theorem 4.8).

From all these relations, we deduce algebraically in section 5 that there is a new basis  $(X^{\lambda} * T_w)_{\lambda \in Y^+, w \in W^v}$  of  ${}^I\mathcal{H}_R$ , associated to some new elements  $X^{\lambda} \in {}^I\mathcal{H}_R$ , satisfying  $X^{\lambda} = T_{\lambda}$  for  $\lambda \in Y^{++} = Y \cap C_f^v$  (where  $C_f^v$  is the fundamental Weyl chamber) and  $X^{\lambda} * X^{\mu} = X^{\lambda+\mu} = X^{\mu} * X^{\lambda}$  for  $\lambda, \mu \in Y^+$ . As, for any  $\lambda \in Y^+$ , there is  $\mu \in Y^{++}$  with  $\lambda + \mu \in Y^{++}$ , these  $X^{\lambda}$  are some quotients of elements  $T_{\mu}$ . The Bernstein-Lusztig type relations may be translated for this new basis. When  $R$  contains sufficiently high roots of the parameters in  $\mathcal{Q}$  (*e.g.* if  $R \supset \mathbb{R}$ ), we may replace the  $T_w$  and  $X^{\lambda}$  by some  $R^*$ -multiples  $H_w$  and  $Z^{\lambda}$ . We get a new basis  $(Z^{\lambda} * H_w)_{\lambda \in Y^+, w \in W^v}$  of  ${}^I\mathcal{H}_R$ , satisfying a set of relations very close to the Bernstein-Lusztig presentation in the semi-simple case (*cf.* 5.7).

In section 6, we define algebraically the Bernstein-Lusztig-Hecke algebra  ${}^{BL}\mathcal{H}_{R_1}$ : it is a free module with basis written  $(Z^{\lambda} H_w)_{\lambda \in Y^+, w \in W^v}$  over the algebra  $R_1 = \mathbb{Z}[(\sigma_i^{\pm 1}, \sigma_i'^{\pm 1})_{i \in I}]$ , where  $\sigma_i, \sigma_i'$  are indeterminates (with some identifications); its product  $*$  is given by the same relations as above for the  $Z^{\lambda} * H_w$ , one just extends  $\lambda \in Y^+$  to  $\lambda \in Y$  and replace  $\sqrt{q_i}, \sqrt{q_i'}$  by  $\sigma_i, \sigma_i'$ . We prove then that, up to a change of scalars,  ${}^I\mathcal{H}_R$  may be identified to a subalgebra of  ${}^{BL}\mathcal{H}_{R_1}$ . This last algebra may be considered as a ring of quotients of the Iwahori-Hecke algebra.

Actually, this article is written in a more general framework (explained in §1): we ask  $\mathcal{S}$  to be an abstract ordered affine hovel (as defined in [Ro11]) and  $G$  to be a strongly transitive group of (positive, type-preserving) automorphisms.

# 1 General framework

## 1.1 Vectorial data

We consider a quadruple  $(V, W^v, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$  where  $V$  is a finite dimensional real vector space,  $W^v$  a subgroup of  $GL(V)$  (the vectorial Weyl group),  $I$  a finite set,  $(\alpha_i^\vee)_{i \in I}$  a family in  $V$  and  $(\alpha_i)_{i \in I}$  a free family in the dual  $V^*$ . We ask these data to satisfy the conditions of [Ro11, 1.1]. In particular, the formula  $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$  defines a linear involution in  $V$  which is an element in  $W^v$  and  $(W^v, \{r_i \mid i \in I\})$  is a Coxeter system.

To be more concrete, we consider the Kac-Moody case of [l.c. ; 1.2]: the matrix  $\mathbb{M} = (\alpha_j(\alpha_i^\vee))_{i,j \in I}$  is a generalized Cartan matrix. Then  $W^v$  is the Weyl group of the corresponding Kac-Moody Lie algebra  $\mathfrak{g}_{\mathbb{M}}$  and the associated real root system is

$$\Phi = \{w(\alpha_i) \mid w \in W^v, i \in I\} \subset Q = \bigoplus_{i \in I} \mathbb{Z}.\alpha_i.$$

We set  $\Phi^\pm = \Phi \cap Q^\pm$  where  $Q^\pm = \pm(\bigoplus_{i \in I} (\mathbb{Z}_{\geq 0}).\alpha_i)$  and  $Q^\vee = (\bigoplus_{i \in I} \mathbb{Z}.\alpha_i^\vee)$ ,  $Q_\pm^\vee = \pm(\bigoplus_{i \in I} (\mathbb{Z}_{\geq 0}).\alpha_i^\vee)$ . We have  $\Phi = \Phi^+ \cup \Phi^-$  and, for  $\alpha = w(\alpha_i) \in \Phi$ ,  $r_\alpha = w.r_i.w^{-1}$  and  $r_\alpha(v) = v - \alpha(v)\alpha^\vee$ , where the coroot  $\alpha^\vee = w(\alpha_i^\vee)$  depends only on  $\alpha$ .

The set  $\Phi$  is an (abstract, reduced) real root system in the sense of [MoP89], [MoP95] or [Ba96]. We shall sometimes also use the set  $\Delta = \Phi \cup \Delta_{im}^+ \cup \Delta_{im}^-$  of all roots (with  $-\Delta_{im}^- = \Delta_{im}^+ \subset Q^+$ ,  $W^v$ -stable) defined in [Ka90]. It is an (abstract, reduced) root system in the sense of [Ba96].

The *fundamental positive chamber* is  $C_f^v = \{v \in V \mid \alpha_i(v) > 0, \forall i \in I\}$ . Its closure  $\overline{C_f^v}$  is the disjoint union of the vectorial faces  $F^v(J) = \{v \in V \mid \alpha_i(v) = 0, \forall i \in J, \alpha_i(v) > 0, \forall i \in I \setminus J\}$  for  $J \subset I$ . The positive (resp. negative) vectorial faces are the sets  $w.F^v(J)$  (resp.  $-w.F^v(J)$ ) for  $w \in W^v$  and  $J \subset I$ . The support of such a face is the vector space it generates. The set  $J$  or the face  $w.F^v(J)$  or an element of this face is called *spherical* if the group  $W^v(J)$  generated by  $\{r_i \mid i \in J\}$  is finite. An element of a vectorial chamber  $\pm w.C_f^v$  is called *regular*.

The *Tits cone*  $\mathcal{T}$  is the (disjoint) union of the positive vectorial faces. It is a  $W^v$ -stable convex cone in  $V$ .

## 1.2 The model apartment

As in [Ro11, 1.4] the model apartment  $\mathbb{A}$  is  $V$  considered as an affine space and endowed with a family  $\mathcal{M}$  of walls. These walls are affine hyperplanes directed by  $\text{Ker}(\alpha)$  for  $\alpha \in \Phi$ .

We ask this apartment to be **semi-discrete** and the origin 0 to be **special**. This means that these walls are the hyperplanes defined as follows:

$$M(\alpha, k) = \{v \in V \mid \alpha(v) + k = 0\} \quad \text{for } \alpha \in \Phi \text{ and } k \in \Lambda_\alpha$$

(with  $\Lambda_\alpha = k_\alpha.\mathbb{Z}$  a non trivial discrete subgroup of  $\mathbb{R}$ ). Using the Lemma 1.3 in [GR14] (*i.e.* replacing  $\Phi$  by another system  $\Phi_1$ ) we may (and shall) assume that  $\Lambda_\alpha = \mathbb{Z}, \forall \alpha \in \Phi$ .

For  $\alpha = w(\alpha_i) \in \Phi$ ,  $k \in \mathbb{Z}$  and  $M = M(\alpha, k)$ , the reflection  $r_{\alpha, k} = r_M$  with respect to  $M$  is the affine involution of  $\mathbb{A}$  with fixed points the wall  $M$  and associated linear involution  $r_\alpha$ . The affine Weyl group  $W^a$  is the group generated by the reflections  $r_M$  for  $M \in \mathcal{M}$ ; we assume that  $W^a$  stabilizes  $\mathcal{M}$ .

An automorphism of  $\mathbb{A}$  is an affine bijection  $\varphi : \mathbb{A} \rightarrow \mathbb{A}$  stabilizing the set of pairs  $(M, \alpha^\vee)$  of a wall  $M$  and the coroot associated with  $\alpha \in \Phi$  such that  $M = M(\alpha, k)$ ,  $k \in \mathbb{Z}$ . The group  $\text{Aut}(\mathbb{A})$  of these automorphisms contains  $W^a$  and normalizes it.

For  $\alpha \in \Phi$  and  $k \in \mathbb{R}$ ,  $D(\alpha, k) = \{v \in V \mid \alpha(v) + k \geq 0\}$  is an half-space, it is called an *half-apartment* if  $k \in \mathbb{Z}$ . We write  $D(\alpha, \infty) = \mathbb{A}$ .

The Tits cone  $\mathcal{T}$  and its interior  $\mathcal{T}^o$  are convex and  $W^v$ -stable cones, therefore, we can define two  $W^v$ -invariant preorder relations on  $\mathbb{A}$ :

$$x \leq y \Leftrightarrow y - x \in \mathcal{T}; \quad x \overset{o}{<} y \Leftrightarrow y - x \in \mathcal{T}^o.$$

If  $W^v$  has no fixed point in  $V \setminus \{0\}$  and no finite factor, then they are orders; but they are not in general.

### 1.3 Faces, sectors, chimneys...

The faces in  $\mathbb{A}$  are associated to the above systems of walls and half-apartments. As in [BrT72], they are no longer subsets of  $\mathbb{A}$ , but filters of subsets of  $\mathbb{A}$ . For the definition of that notion and its properties, we refer to [BrT72] or [GR08].

If  $F$  is a subset of  $\mathbb{A}$  containing an element  $x$  in its closure, the germ of  $F$  in  $x$  is the filter  $\text{germ}_x(F)$  consisting of all subsets of  $\mathbb{A}$  which are intersections of  $F$  and neighbourhoods of  $x$ . In particular, if  $x \neq y \in E$ , we denote the germ in  $x$  of the segment  $[x, y]$  (resp. of the interval  $]x, y]$ ) by  $[x, y)$  (resp.  $]x, y)$ ).

Given  $F$  a filter of subsets of  $\mathbb{A}$ , its *enclosure*  $cl_{\mathbb{A}}(F)$  (resp. *closure*  $\overline{F}$ ) is the filter made of the subsets of  $\mathbb{A}$  containing an element of  $F$  of the shape  $\cap_{\alpha \in \Delta} D(\alpha, k_{\alpha})$ , where  $k_{\alpha} \in \mathbb{Z} \cup \{\infty\}$  (resp. containing the closure  $\overline{S}$  of some  $S \in F$ ).

We shall actually here speak only of local faces, and sometimes forget the word local.

A *local face*  $F$  in the apartment  $\mathbb{A}$  is associated to a point  $x \in \mathbb{A}$  (its vertex) and a vectorial face  $F^v$  in  $V$  (its direction), it is  $F = F^{\ell}(x, F^v) = \text{germ}_x(x + F^v)$ . Its closure is  $\overline{F}^{\ell}(x, F^v) = \text{germ}_x(x + \overline{F^v})$

There is an order on the local faces: the assertions “ $F$  is a face of  $F'$ ”, “ $F'$  covers  $F$ ” and “ $F \leq F'$ ” are by definition equivalent to  $F \subset \overline{F}'$ . The dimension of a local face  $F$  is the smallest dimension of an affine space generated by some  $S \in F$ . The (unique) such affine space  $E$  of minimal dimension is the support of  $F$ ; if  $F = F^{\ell}(x, F^v)$ ,  $\text{supp}(F) = x + \text{supp}(F^v)$ . A local face  $F = F^{\ell}(x, F^v)$  is spherical if the direction of its support meets the open Tits cone (*i.e.* when  $F^v$  is spherical), then its pointwise stabilizer  $W_F$  in  $W^a$  is finite.

Any point  $x \in \mathbb{A}$  is contained in a unique face  $F(x, V_0) \subset cl_{\mathbb{A}}(\{x\})$  which is minimal of positive and negative direction (but seldom spherical). For any local face  $F^{\ell} = F^{\ell}(x, F^v)$ , there is a unique face  $F$  (as defined in [Ro11]) containing  $F^{\ell}$ . Then  $\overline{F}^{\ell} \subset \overline{F} = cl_{\mathbb{A}}(F^{\ell}) = cl_{\mathbb{A}}(F)$  is also the enclosure of any interval-germ  $]x, y) = \text{germ}_x([x, y])$  included in  $F^{\ell}$ .

A *local chamber* is a maximal local face, *i.e.* a local face  $F^{\ell}(x, \pm w.C_f^v)$  for  $x \in \mathbb{A}$  and  $w \in W^v$ . The *fundamental local chamber* is  $C_0^+ = \text{germ}_0(C_f^v)$ .

A *(local) panel* is a spherical local face maximal among local faces which are not chambers, or, equivalently, a spherical face of dimension  $n - 1$ . Its support is a wall.

A *sector* in  $\mathbb{A}$  is a  $V$ -translate  $\mathfrak{s} = x + C^v$  of a vectorial chamber  $C^v = \pm w.C_f^v$  ( $w \in W^v$ ),  $x$  is its *base point* and  $C^v$  its *direction*. Two sectors have the same direction if, and only if, they are conjugate by  $V$ -translation, and if, and only if, their intersection contains another sector.

The *sector-germ* of a sector  $\mathfrak{s} = x + C^v$  in  $\mathbb{A}$  is the filter  $\mathfrak{S}$  of subsets of  $\mathbb{A}$  consisting of the sets containing a  $V$ -translate of  $\mathfrak{s}$ , it is well determined by the direction  $C^v$ . So, the

set of translation classes of sectors in  $\mathbb{A}$ , the set of vectorial chambers in  $V$  and the set of sector-germs in  $\mathbb{A}$  are in canonical bijection. We denote the sector-germ associated to the negative fundamental vectorial chamber  $-C_f^v$  by  $\mathfrak{S}_{-\infty}$ .

A *sector-face* in  $\mathbb{A}$  is a  $V$ -translate  $\mathfrak{f} = x + F^v$  of a vectorial face  $F^v = \pm w.F^v(J)$ . The sector-face-germ of  $\mathfrak{f}$  is the filter  $\mathfrak{F}$  of subsets containing a translate  $\mathfrak{f}'$  of  $\mathfrak{f}$  by an element of  $F^v$  (i.e.  $\mathfrak{f}' \subset \mathfrak{f}$ ). If  $F^v$  is spherical, then  $\mathfrak{f}$  and  $\mathfrak{F}$  are also called spherical. The sign of  $\mathfrak{f}$  and  $\mathfrak{F}$  is the sign of  $F^v$ .

A *chimney* in  $\mathbb{A}$  is associated to a face  $F = F(x, F_0^v)$ , called its basis, and to a vectorial face  $F^v$ , its direction, it is the filter

$$\mathfrak{r}(F, F^v) = cl_{\mathbb{A}}(F + F^v).$$

A chimney  $\mathfrak{r} = \mathfrak{r}(F, F^v)$  is *splayed* if  $F^v$  is spherical, it is *solid* if its support (as a filter, i.e. the smallest affine subspace containing  $\mathfrak{r}$ ) has a finite pointwise stabilizer in  $W^v$ . A splayed chimney is therefore solid. The enclosure of a sector-face  $\mathfrak{f} = x + F^v$  is a chimney.

A ray  $\delta$  with origin in  $x$  and containing  $y \neq x$  (or the interval  $]x, y]$ , the segment  $[x, y]$ ) is called *preordered* if  $x \leq y$  or  $y \leq x$  and *generic* if  $x \overset{o}{<} y$  or  $y \overset{o}{<} x$ . With these new notions, a chimney can be defined as the enclosure of a preordered ray and a preordered segment-germ sharing the same origin. The chimney is splayed if, and only if, the ray is generic.

## 1.4 The hovel

In this section, we recall the definition and some properties of an ordered affine hovel given by Guy Rousseau in [Ro11].

**1)** An apartment of type  $\mathbb{A}$  is a set  $A$  endowed with a set  $Isom^W(\mathbb{A}, A)$  of bijections (called Weyl-isomorphisms) such that, if  $f_0 \in Isom^W(\mathbb{A}, A)$ , then  $f \in Isom^W(\mathbb{A}, A)$  if, and only if, there exists  $w \in W^a$  satisfying  $f = f_0 \circ w$ . An isomorphism (resp. a Weyl-isomorphism) between two apartments  $\varphi : A \rightarrow A'$  is a bijection such that, for any  $f \in Isom^W(\mathbb{A}, A)$ ,  $f' \in Isom^W(\mathbb{A}, A')$ ,  $f'^{-1} \circ \varphi \circ f \in Aut(\mathbb{A})$  (resp.  $\in W^a$ ). As the filters in  $\mathbb{A}$  defined in 1.3 above (e.g. local faces, sectors, walls,..) are permuted by  $Aut(\mathbb{A})$ , they are well defined in any apartment of type  $\mathbb{A}$  and exchanged by any isomorphism.

**Definition.** An ordered affine hovel of type  $\mathbb{A}$  is a set  $\mathcal{S}$  endowed with a covering  $\mathcal{A}$  of subsets called apartments such that:

- (MA1) any  $A \in \mathcal{A}$  admits a structure of an apartment of type  $\mathbb{A}$ ;
- (MA2) if  $F$  is a point, a germ of a preordered interval, a generic ray or a solid chimney in an apartment  $A$  and if  $A'$  is another apartment containing  $F$ , then  $A \cap A'$  contains the enclosure  $cl_A(F)$  of  $F$  and there exists a Weyl-isomorphism from  $A$  onto  $A'$  fixing  $cl_A(F)$ ;
- (MA3) if  $\mathfrak{R}$  is the germ of a splayed chimney and if  $F$  is a face or a germ of a solid chimney, then there exists an apartment that contains  $\mathfrak{R}$  and  $F$ ;
- (MA4) if two apartments  $A, A'$  contain  $\mathfrak{R}$  and  $F$  as in (MA3), then their intersection contains  $cl_A(\mathfrak{R} \cup F)$  and there exists a Weyl-isomorphism from  $A$  onto  $A'$  fixing  $cl_A(\mathfrak{R} \cup F)$ ;
- (MAO) if  $x, y$  are two points contained in two apartments  $A$  and  $A'$ , and if  $x \leq_A y$  then the two line segments  $[x, y]_A$  and  $[x, y]_{A'}$  are equal.

We ask here  $\mathcal{S}$  to be thick of **finite thickness**: the number of local chambers containing a given (local) panel has to be finite  $\geq 3$ . This number is the same for any panel in a given wall  $M$  [Ro11, 2.9]; we denote it by  $1 + q_M$ .

An automorphism (resp. a Weyl-automorphism) of  $\mathcal{S}$  is a bijection  $\varphi : \mathcal{S} \rightarrow \mathcal{S}$  such that  $A \in \mathcal{A} \iff \varphi(A) \in \mathcal{A}$  and then  $\varphi|_A : A \rightarrow \varphi(A)$  is an isomorphism (resp. a Weyl-isomorphism).

**2)** For  $x \in \mathcal{S}$ , the set  $\mathcal{T}_x^+ \mathcal{S}$  (resp.  $\mathcal{T}_x^- \mathcal{S}$ ) of segment germs  $[x, y)$  for  $y > x$  (resp.  $y < x$ ) may be considered as a building, the positive (resp. negative) tangent building. The corresponding faces are the local faces of positive (resp. negative) direction and vertex  $x$ . The associated Weyl group is  $W^v$ . If the  $W$ -distance (calculated in  $\mathcal{T}_x^\pm \mathcal{S}$ ) of two local chambers is  $d^W(C_x, C'_x) = w \in W^v$ , to a reduced decomposition  $w = r_{i_1} \cdots r_{i_n}$  corresponds a unique minimal gallery from  $C_x$  to  $C'_x$  of type  $(i_1, \dots, i_n)$ : we shall say, abusively, that this gallery is of type  $w$ .

The buildings  $\mathcal{T}_x^+ \mathcal{S}$  and  $\mathcal{T}_x^- \mathcal{S}$  are actually twinned.

**Lemma.** [Ro11, 2.9] *Let  $D$  be a half-apartment in  $\mathcal{S}$  and  $M = \partial D$  its wall (i.e. its boundary). One considers a panel  $F$  in  $M$  and a local chamber  $C$  in  $\mathcal{S}$  covering  $F$ . Then there is an apartment containing  $D$  and  $C$ .*

**3)** We assume that  $\mathcal{S}$  has a strongly transitive group of automorphisms  $G$  (i.e. all isomorphisms involved in the above axioms are induced by elements of  $G$ , cf. [Ro13, 4.10]). We choose in  $\mathcal{S}$  a fundamental apartment which we identify with  $\mathbb{A}$ . As  $G$  is strongly transitive, the apartments of  $\mathcal{S}$  are the sets  $g.\mathbb{A}$  for  $g \in G$ . The stabilizer  $N$  of  $\mathbb{A}$  in  $G$  induces a group  $W = \nu(N)$  of affine automorphisms of  $\mathbb{A}$  which permutes the walls, local faces, sectors, sector-faces... and contains the affine Weyl group  $W^a = W^v \ltimes Q^\vee$  [Ro13, 4.13.1].

We denote the stabilizer of  $0 \in \mathbb{A}$  in  $G$  by  $K$  and the stabilizer (or fixer) of  $C_0^+$  by  $K_I$ ; this group  $K_I$  is the *Iwahori subgroup*.

Note that there is only a finite number of constants  $q_M$  as above: as  $q_{wM} = q_M, \forall w \in \nu(N)$  and  $w.M(\alpha, k) = M(w(\alpha), k), \forall w \in W^v$ , we may suppose  $M = M(\alpha_i, k)$  with  $i \in I$  and  $k \in \mathbb{Z}$ . Now  $\alpha_i^\vee \in Q^\vee$ ; as  $\alpha_i(\alpha_i^\vee) = 2$  the translation by  $\alpha_i^\vee$  permutes the walls  $M = M(\alpha_i, k)$  (for  $k \in \mathbb{Z}$ ) with two orbits. So,  $Q^\vee \subset W^a$  has at most two orbits in the set of the constants  $q_{M(\alpha_i, k)}$ : one containing the  $q_i = q_{M(\alpha_i, 0)}$  and the other containing the  $q'_i = q_{M(\alpha_i, \pm 1)}$ . Hence, the number of (possibly) different  $q_M$  is at most  $2 \cdot |I|$ . We denote by  $\mathcal{Q} = \{q_i, q'_i \mid i \in I\}$  this set of parameters.

If  $\alpha_i(\alpha_j^\vee)$  is odd for some  $j \in I$ , the translation by  $\alpha_j^\vee$  exchanges the two walls  $M(\alpha_i, 0)$  and  $M(\alpha_i, -\alpha_i(\alpha_j^\vee))$ ; so  $q_i = q'_i$ . If  $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$ , one knows that the element  $r_i r_j r_i$  of  $W^v(\{i, j\})$  exchanges  $\alpha_i$  and  $-\alpha_j$ , so  $q_i = q'_i = q_j = q'_j$ .

Actually many of the following results (in sections 2, 3) are true without assuming the existence of  $G$ : we have only to assume that the parameters  $q_M$  satisfy the above conditions.

**4)** We ask  $W = \nu(N)$  to be **positive** and **type-preserving** for its action on the vectorial faces. This means that the associated linear map  $\vec{w}$  of any  $w \in \nu(N)$  is in  $W^v$ . As  $\nu(N)$  contains  $W^a$  and stabilizes  $\mathcal{M}$ , we have  $W = \nu(N) = W^v \ltimes Y$ , where  $W^v$  fixes the origin  $0$  of  $\mathbb{A}$  and  $Y$  is a group of translations such that:  $Q^\vee \subset Y \subset P^\vee = \{v \in V \mid \alpha(v) \in \mathbb{Z}, \forall \alpha \in \Phi\}$ . An element  $\mathbf{w} \in W$  will often be written  $\mathbf{w} = \lambda.w$ , with  $\lambda \in Y$  and  $w \in W^v$ .

As above we see that  $q_i = q'_i$  when  $\alpha_i(Y) = \mathbb{Z}$  i.e.  $\alpha_i(Y)$  contains an odd integer.

We ask  $Y$  to be **discrete** in  $V$ . This is clearly satisfied if  $\Phi$  generates  $V^*$  i.e.  $(\alpha_i)_{i \in I}$  is a basis of  $V^*$ .



**5) Examples.** The main examples of all the above situation are provided by the hovels of almost split Kac-Moody groups over fields complete for a discrete valuation and with a finite residue field, see [Ro12], [Ch10], [Ch11] or [Ro13]. Some details in the split case can be found in [GR14].

**6) Remark.** a) In the following, we sometimes refer to [GR08] that deals with split Kac-Moody groups and residue fields containing  $\mathbb{C}$ . But the cited results are easily generalizable to our present framework, using the above references.

b) All isomorphisms in [Ro11] are Weyl-isomorphisms, and, when  $G$  is strongly transitive, all isomorphisms constructed in *l.c.* are induced by an element of  $G$ .

## 1.5 Type 0 vertices

The elements of  $Y$ , through the identification  $Y = N.0$ , are called *vertices of type 0* in  $\mathbb{A}$ ; they are special vertices. We note  $Y^+ = Y \cap \mathcal{T}$  and  $Y^{++} = Y \cap \overline{C_f^v}$ . The type 0 vertices in  $\mathcal{S}$  are the points on the orbit  $\mathcal{S}_0$  of 0 by  $G$ . This set  $\mathcal{S}_0$  is often called the affine Grassmannian as it is equal to  $G/K$ .

In general,  $G$  is not equal to  $KYK = KNK$  [GR08, 6.10] *i.e.*  $\mathcal{S}_0 \neq KY$ .

We know that  $\mathcal{S}$  is endowed with a  $G$ -invariant preorder  $\leq$  which induces the known one on  $\mathbb{A}$  and satisfies  $x \leq y \implies \exists A \in \mathcal{A}$  with  $x, y \in A$  [Ro11, 5.9]. We set  $\mathcal{S}^+ = \{x \in \mathcal{S} \mid 0 \leq x\}$ ,  $\mathcal{S}_0^+ = \mathcal{S}_0 \cap \mathcal{S}^+$  and  $G^+ = \{g \in G \mid 0 \leq g.0\}$ ; so  $\mathcal{S}_0^+ = G^+.0 = G^+/K$ . As  $\leq$  is a  $G$ -invariant preorder,  $G^+$  is a semigroup.

If  $x \in \mathcal{S}_0^+$  there is an apartment  $A$  containing 0 and  $x$  (by definition of  $\leq$ ) and all apartments containing 0 are conjugated to  $\mathbb{A}$  by  $K$  (axiom (MA2)); so  $x \in KY^+$  as  $\mathcal{S}_0^+ \cap \mathbb{A} = Y^+$ . But  $\nu(N \cap K) = W^v$  and  $Y^+ = W^v.Y^{++}$  (with uniqueness of the element in  $Y^{++}$ ); so  $\mathcal{S}_0^+ = KY^{++}$ , more precisely  $\mathcal{S}_0^+ = G^+/K$  is the union of the  $KyK/K$  for  $y \in Y^{++}$ . This union is disjoint, for the above construction does not depend on the choice of  $A$  (*cf.* 1.9.a).

Hence, we have proved that the map  $Y^{++} \rightarrow K \backslash G^+/K$  is one-to-one and onto.

## 1.6 Vectorial distance and $Q^\vee$ -order

For  $x \in \mathcal{T}$ , we denote by  $x^{++}$  the unique element in  $\overline{C_f^v}$  conjugated by  $W^v$  to  $x$ .

Let  $\mathcal{S} \times_{\leq} \mathcal{S} = \{(x, y) \in \mathcal{S} \times \mathcal{S} \mid x \leq y\}$  be the set of increasing pairs in  $\mathcal{S}$ . Such a pair  $(x, y)$  is always in a same apartment  $g.\mathbb{A}$ ; so  $(g^{-1}).y - (g^{-1}).x \in \mathcal{T}$  and we define the *vectorial distance*  $d^v(x, y) \in \overline{C_f^v}$  by  $d^v(x, y) = ((g^{-1}).y - (g^{-1}).x)^{++}$ . It does not depend on the choices we made (by 1.9.a below).

For  $(x, y) \in \mathcal{S}_0 \times_{\leq} \mathcal{S}_0 = \{(x, y) \in \mathcal{S}_0 \times \mathcal{S}_0 \mid x \leq y\}$ , the vectorial distance  $d^v(x, y)$  takes values in  $Y^{++}$ . Actually, as  $\mathcal{S}_0 = G.0$ ,  $K$  is the stabilizer of 0 and  $\mathcal{S}_0^+ = KY^{++}$  (with uniqueness of the element in  $Y^{++}$ ), the map  $d^v$  induces a bijection between the set  $\mathcal{S}_0 \times_{\leq} \mathcal{S}_0/G$  of  $G$ -orbits in  $\mathcal{S}_0 \times_{\leq} \mathcal{S}_0$  and  $Y^{++}$ .

Any  $g \in G^+$  is in  $K.d^v(0, g.0).K$ .

For  $x, y \in \mathbb{A}$ , we say that  $x \leq_{Q^\vee} y$  (resp.  $x \leq_{Q_{\mathbb{R}}^\vee} y$ ) when  $y - x \in Q_+^\vee$  (resp.  $y - x \in Q_{\mathbb{R}^+}^\vee = \sum_{i \in I} \mathbb{R}_{\geq 0} \cdot \alpha_i^\vee$ ). We get thus a preorder which is an order at least when  $(\alpha_i^\vee)_{i \in I}$  is free or  $\mathbb{R}_+$ -free (*i.e.*  $\sum a_i \alpha_i^\vee = 0, a_i \geq 0 \implies a_i = 0, \forall i$ ).

## 1.7 Paths

We consider piecewise linear continuous paths  $\pi : [0, 1] \rightarrow \mathbb{A}$  such that each (existing) tangent vector  $\pi'(t)$  belongs to an orbit  $W^v \cdot \lambda$  for some  $\lambda \in \overline{C_f^v}$ . Such a path is called a  $\lambda$ -path; it is increasing with respect to the preorder relation  $\leq$  on  $\mathbb{A}$ .

For any  $t \neq 0$  (resp.  $t \neq 1$ ), we let  $\pi'_-(t)$  (resp.  $\pi'_+(t)$ ) denote the derivative of  $\pi$  at  $t$  from the left (resp. from the right). Further, we define  $w_\pm(t) \in W^v$  to be the smallest element in its  $(W^v)_\lambda$ -class such that  $\pi'_\pm(t) = w_\pm(t) \cdot \lambda$  (where  $(W^v)_\lambda$  is the stabilizer in  $W^v$  of  $\lambda$ ).

Hecke paths of shape  $\lambda$  (with respect to the sector germ  $\mathfrak{S}_{-\infty} = \text{germ}_\infty(-C_f^v)$ ) are  $\lambda$ -paths satisfying some further precise conditions, see [KM08, 3.27] or [GR14, 1.8]. For us their interest will appear just below in 1.8.

But to give a formula for the structure constants of the forthcoming Iwahori-Hecke algebra, we will need need slight different Hecke paths whose definition is detailed in Section 3.3.

## 1.8 Retractions onto $Y^+$

For all  $x \in \mathcal{I}^+$  there is an apartment containing  $x$  and  $C_0^- = \text{germ}_0(-C_f^v)$  [Ro11, 5.1] and this apartment is conjugated to  $\mathbb{A}$  by an element of  $K$  fixing  $C_0^-$  (axiom (MA2)). So, by the usual arguments and [l.c., 5.5] (i.e. 1.10.a) we can define the retraction  $\rho_{C_0^-}$  of  $\mathcal{I}^+$  into  $\mathbb{A}$  with center  $C_0^-$ ; its image is  $\rho_{C_0^-}(\mathcal{I}^+) = \mathcal{T} = \mathcal{I}^+ \cap \mathbb{A}$  and  $\rho_{C_0^-}(\mathcal{I}_0^+) = Y^+$ .

There is also the retraction  $\rho_{-\infty}$  of  $\mathcal{I}$  onto  $\mathbb{A}$  with center the sector-germ  $\mathfrak{S}_{-\infty}$ , defined using axioms (MA3), (MA4), cf. [GR08, 4.4].

More generally, we may define a retraction  $\rho$  of  $\mathcal{I}$  (resp. the subset  $\mathcal{I}_{\geq z} = \{y \in \mathcal{I} \mid y \geq z\}$ ) onto an apartment  $A$  with center any sector germ (resp. any local chamber of negative direction with vertex  $z$ ) of  $A$ . For any such retraction  $\rho$ , the image of any segment  $[x, y]$  with  $(x, y) \in \mathcal{I} \times_{\leq} \mathcal{I}$  and  $d^v(x, y) = \lambda \in \overline{C_f^v}$  (resp. and moreover  $x, y \in \mathcal{I}_{\geq z}$ ) is a  $\lambda$ -path [GR08, 4.4]. In particular,  $\rho(x) \leq \rho(y)$ .

Actually, the image by  $\rho_{-\infty}$  of any segment  $[x, y]$  with  $(x, y) \in \mathcal{I} \times_{\leq} \mathcal{I}$  and  $d^v(x, y) = \lambda \in Y^{++}$  is a Hecke path of shape  $\lambda$  with respect to  $\mathfrak{S}_{-\infty}$  [GR08, th. 6.2], and we have:

**Lemma.** a) For  $\lambda \in Y^{++}$  and  $w \in W^v$ ,  $w \cdot \lambda \in \lambda - Q_+^v$ , i.e.  $w \cdot \lambda \leq_{Q^v} \lambda$ .

b) Let  $\pi$  be a Hecke path of shape  $\lambda \in Y^{++}$  with respect to  $\mathfrak{S}_{-\infty}$ , from  $y_0 \in Y$  to  $y_1 \in Y$ . Then, for  $0 \leq t < t' < 1$ ,  $\lambda = \pi'_+(t)^{++} = \pi'_-(t')^{++}$ ,  $\pi'_+(t) \leq_{Q^v} \pi'_-(t') \leq_{Q^v} \pi'_+(t') \leq_{Q^v} \pi'_-(1)$ ,  $\pi'_+(0) \leq_{Q^v} \lambda$ ,  $\pi'_+(0) \leq_{Q_{\mathbb{R}}^v} (y_1 - y_0) \leq_{Q_{\mathbb{R}}^v} \pi'_-(1) \leq_{Q^v} \lambda$  and  $y_1 - y_0 \leq_{Q^v} \lambda$ .

Moreover  $y_1 - y_0$  is in the convex hull  $\text{conv}(W^v \cdot \lambda)$  of all  $w \cdot \lambda$  for  $w \in W^v$ , more precisely in the convex hull  $\text{conv}(W^v \cdot \lambda, \geq \pi'_+(0))$  of all  $w' \cdot \lambda$  for  $w' \in W^v$ ,  $w' \leq w$ , where  $w$  is the element with minimal length such that  $\pi'_+(0) = w \cdot \lambda$ .

c) If moreover  $(\alpha_i^v)_{i \in I}$  is free, we may replace above  $\leq_{Q_{\mathbb{R}}^v}$  by  $\leq_{Q^v}$ .

d) If  $x \leq z \leq y$  in  $\mathcal{I}_0$ , then  $d^v(x, y) \leq_{Q^v} d^v(x, z) + d^v(z, y)$ .

**N.B.** In the following, we always assume  $(\alpha_i^v)_{i \in I}$  free.

*Proof.* Everything is in [GR14, 2.4], except the second paragraph of b). Actually we see in l.c. that  $y_1 - y_0$  is the integral of the locally constant vector-valued function  $\pi'_+(t) = w_+(t) \cdot \lambda$ , where  $w_+(t)$  is decreasing for the Bruhat order [GR14, 5.4], hence the result.  $\square$

## 1.9 Chambers of type 0

Let  $\mathcal{C}_0^\pm$  be the set of all local chambers with vertices of type 0 and positive or negative direction. An element of vertex  $x \in \mathcal{I}_0$  in this set (resp. its direction) will often be written  $C_x$  (resp.  $C_x^v$ ). We consider  $\mathcal{C}_0^+ \times_{\leq} \mathcal{C}_0^+ = \{(C_x, C_y) \in \mathcal{C}_0^+ \times \mathcal{C}_0^+ \mid x \leq y\}$ . We sometimes write  $C_x \leq C_y$  when  $x \leq y$ .

**Proposition.** [Ro11, 5.4 and 5.1] *Let  $x, y \in \mathcal{I}$  with  $x \leq y$ . We consider two local faces  $F_x, F_y$  with respective vertices  $x, y$ .*

a)  *$\{x, y\}$  is included in an apartment and two such apartments  $A, A'$  are isomorphic by a Weyl-isomorphism in  $G$ , fixing  $cl_A(\{x, y\}) = cl_{A'}(\{x, y\}) \supset [x, y]$ .*

b) *There is an apartment containing  $F_x$  and  $F_y$ , if we assume moreover  $x \overset{\circ}{<} y$  or  $x = y$  when  $F_x$  and  $F_y$  are respectively of positive and negative direction.*

**Consequences.** 1) We define  $W^+ = W^v \times Y^+$  which is a subsemigroup of  $W$ .

If  $C_x \in \mathcal{C}_0^+$ , we know by b) above, that there is an apartment  $A$  containing  $C_0^+$  and  $C_x$ . But all apartments containing  $C_0^+$  are conjugated to  $\mathbb{A}$  by  $K_I$  (Axiom (MA2)), so there is  $k \in K_I$  with  $k^{-1}.C_x \subset \mathbb{A}$ . Now the vertex  $k^{-1}.x$  of  $k^{-1}.C_x$  satisfies  $k^{-1}.x \geq 0$ , so there is  $\mathbf{w} \in W^+$  such that  $k^{-1}.C_x = \mathbf{w}.C_0^+$ .

When  $g \in G^+$ ,  $g.C_0^+$  is in  $\mathcal{C}_0^+$  and there are  $k \in K_I$ ,  $\mathbf{w} \in W^+$  with  $g.C_0^+ = k.\mathbf{w}.C_0^+$ , i.e.  $g \in K_I.W^+.K_I$ . We have proved the Bruhat decomposition  $G^+ = K_I.W^+.K_I$ .

2) Let  $x \in \mathcal{I}_0$ ,  $C_y \in \mathcal{C}_0^+$  with  $x \leq y$ ,  $x \neq y$ . We consider an apartment  $A$  containing  $x$  and  $C_y$  (by b) above) and write  $C_y = F(y, C_y^v)$  in  $A$ . For  $y' \in y + C_y^v$  sufficiently near to  $y$ ,  $\alpha(y' - x) \neq 0$  for any root  $\alpha$ . So  $]x, y']$  is in a unique local chamber  $\pi_x(C_y)$  of vertex  $x$ ; this chamber satisfies  $]x, y] \subset \overline{\pi_x(C_y)} \subset cl_A(\{x, y'\})$  and does not depend of the choice of  $y'$ . Moreover, if we change  $A$  to  $A'$ , we may suppose  $y' \in A \cap A'$  and  $]x, y']$ ,  $cl_A(\{x, y'\})$ ,  $\pi_x(C_y)$  are the same in  $A'$ . The local chamber  $\pi_x(C_y)$  is well determined by  $x$  and  $C_y$ , it is the projection of  $C_y$  in  $\mathcal{T}_x^+ \mathcal{I}$ .

The same things may be done when changing  $+$  to  $-$  and  $\leq$  to  $\geq$ . But, in the above situation, if  $C_x \in \mathcal{C}_0^+$ , we have to assume  $x \overset{\circ}{<} y$  to define the analogous  $\pi_y(C_x) \in \mathcal{C}_0^+$ .

**Proposition 1.10.** *In the situation of Proposition 1.9,*

a) *If  $x \overset{\circ}{<} y$  or  $F_x$  and  $F_y$  are respectively of negative and positive direction, any two apartments  $A, A'$  containing  $F_x$  and  $F_y$  are isomorphic by a Weyl-isomorphism in  $G$  fixing the convex hull of  $F_x$  and  $F_y$  (in  $A$  or  $A'$ ).*

b) *If  $x = y$  and the directions of  $F_x, F_y$  have the same sign, any two apartments  $A, A'$  containing  $F_x$  and  $F_y$  are isomorphic by a Weyl-isomorphism in  $G$ ,  $\varphi : A \rightarrow A'$ , fixing  $F_x$  and  $F_y$ . If moreover  $F_x$  is a local chamber, any minimal gallery from  $F_x$  to  $F_y$  is fixed by  $\varphi$  (and in  $A \cap A'$ ).*

c) *If  $F_x$  and  $F_y$  are of positive directions and  $F_y$  is spherical, any two apartments  $A, A'$  containing  $F_x$  and  $F_y$  are isomorphic by a Weyl-isomorphism in  $G$  fixing  $F_x$  and  $F_y$ .*

**Remark.** The conclusion in c) above is less precise than in a) or in 1.9.a. We may actually improve it when the hovel is associated to a very good family of parahorics, as defined in [Ro13] and already used in [GR08]. Then, using the notion of half good fixers, we may assume that the isomorphism in c) above fixes some kind of enclosure of  $F_x$  and  $F_y$  (containing the convex hull). This particular case includes the case of an almost split Kac-Moody group over a local field.

*Proof.* The assertion a) (resp. b)) is Proposition 5.5 (resp. 5.2) of [Ro11]. To prove c) we improve a little the proof of 5.5 in *l.c.* and use a classical trick to assume that, either  $F_x$  or  $F_y$  is a local chamber. We assume now that  $F_x = C_x$  is a local chamber; the other case is analogous.

We consider an element  $\Omega_x$  (resp.  $\Omega_y$ ) of the filter  $C_x$  (resp.  $F_y$ ) contained in  $A \cap A'$ . We have  $x \in \overline{\Omega_x}$ ,  $y \in \overline{\Omega_y}$  and one may suppose  $\Omega_x$  open and  $\Omega_y$  open in the support of  $F_y$ . There is an isomorphism  $\varphi : A \rightarrow A'$  fixing  $\Omega_x$ . Let  $y' \in \Omega_y$ , we want to prove that  $\varphi(y') = y'$ . As  $F_y$  is spherical,  $x \leq y \overset{\circ}{<} y'$ , hence  $x \overset{\circ}{<} y'$ . So  $x' \leq y'$  for any  $x' \in \Omega_x$  ( $\Omega_x$  sufficiently small). Moreover  $[x', y'] \cap \Omega_x$  is an open neighbourhood of  $x'$  in  $[x', y']$ . By the following lemma, we get  $\varphi(y') = y'$ .  $\square$

**Lemma.** *Let us consider two apartments  $A, A'$  in  $\mathcal{F}$ , a subset  $\Omega \subset A \cap A'$ , a point  $z \in A \cap A'$  and an isomorphism  $\varphi : A \rightarrow A'$  fixing (pointwise)  $\Omega$ . We assume that there is  $z' \in \Omega$  with  $z' \leq z$  or  $z' \geq z$  and  $[z', z] \cap \Omega$  open in  $[z', z]$ ; then  $\varphi(z) = z$ .*

**N.B.** This lemma tells, in particular, that any isomorphism  $\varphi : A \rightarrow A'$  fixing a local facet  $F \subset A \cap A'$  fixes  $\overline{F}$ .

*Proof.*  $\varphi|_{[z', z]}$  is an affine bijection of  $[z', z]$  onto its image in  $A'$ , which is the identity in a neighbourhood of  $z'$ . But 1.9.a tells that  $[z', z] \subset A \cap A'$  and the identity of  $[z', z]$  is an affine bijection (for the affine structures induced by  $A$  and  $A'$ ). Hence  $\varphi|_{[z', z]}$  coincides with this affine bijection; in particular  $\varphi(z) = z$ .  $\square$

### 1.11 $W$ -distance

Let  $(C_x, C_y) \in \mathcal{C}_0^+ \times_{\leq} \mathcal{C}_0^+$ , there is an apartment  $A$  containing  $C_x$  and  $C_y$ . We identify  $(\mathbb{A}, \mathcal{C}_0^+)$  with  $(A, C_x)$  *i.e.* we consider the unique  $f \in \text{Isom}(\mathbb{A}, A)$  such that  $f(C_0^+) = C_x$ . Then  $f^{-1}(y) \geq 0$  and there is  $\mathbf{w} \in W^+$  such that  $f^{-1}(C_y) = \mathbf{w} \cdot C_0^+$ . By 1.10.c,  $\mathbf{w}$  does not depend on the choice of  $A$ .

We define the  $W$ -distance between the two local chambers  $C_x$  and  $C_y$  to be this unique element:  $d^W(C_x, C_y) = \mathbf{w} \in W^+$ . If  $\mathbf{w} = \lambda \cdot w$ , with  $\lambda \in Y^+$  and  $w \in W^v$ , we write also  $d^W(C_x, y) = \lambda$ . As  $\leq$  is  $G$ -invariant, the  $W$ -distance is also  $G$ -invariant. When  $x = y$ , this definition coincides with that in 1.4.2.

If  $C_x, C_y, C_z \in \mathcal{C}_0^+$ , with  $x \leq y \leq z$ , are in a same apartment, we have the Chasles relation:  $d^W(C_x, C_z) = d^W(C_x, C_y) \cdot d^W(C_y, C_z)$ .

When  $C_x = C_0^+$  and  $C_y = g \cdot C_0^+$  (with  $g \in G^+$ ),  $d^W(C_x, C_y)$  is the only  $\mathbf{w} \in W^+$  such that  $g \in K_I \cdot \mathbf{w} \cdot K_I$ . We have proved the uniqueness in Bruhat decomposition:  $G^+ = \coprod_{\mathbf{w} \in W^+} K_I \cdot \mathbf{w} \cdot K_I$ .

The  $W$ -distance classifies the orbits of  $K_I$  on  $\{C_y \in \mathcal{C}_0^+ \mid y \geq 0\}$ , hence also the orbits of  $G$  on  $\mathcal{C}_0^+ \times_{\leq} \mathcal{C}_0^+$ .

## 2 Iwahori-Hecke Algebras

Throughout this Section, we assume that  $(\alpha_i^\vee)_{i \in I}$  is free and we consider any ring  $R$ . To each  $\mathbf{w} \in W^+$ , we associate a function  $T_{\mathbf{w}}$  from  $\mathcal{C}_0^+ \times_{\leq} \mathcal{C}_0^+$  to  $R$  defined by

$$T_{\mathbf{w}}(C, C') = \begin{cases} 1 & \text{if } d^W(C, C') = \mathbf{w}, \\ 0 & \text{otherwise.} \end{cases}$$

Now we consider the following free  $R$ -module

$${}^I\mathcal{H}_R^{\mathcal{J}} = \left\{ \varphi = \sum_{\mathbf{w} \in W^+} a_{\mathbf{w}} T_{\mathbf{w}} \mid a_{\mathbf{w}} \in R, a_{\mathbf{w}} = 0 \text{ except for a finite number} \right\},$$

We endow this  $R$ -module with the convolution product:

$$(\varphi * \psi)(C_x, C_y) = \sum_{C_z} \varphi(C_x, C_z) \psi(C_z, C_y).$$

where  $C_z \in \mathcal{C}_0^+$  is such that  $x \leq z \leq y$ . It is clear that this product is associative and  $R$ -bilinear. We prove in the next section that this product is well defined.

## 2.1 Well-definedness of the product

**Lemma 2.2.** *Let  $\mathfrak{S}^- \subset A$  be a sector-germ with negative direction in an apartment  $A$ ,  $\rho_- : \mathcal{J} \rightarrow A$  the corresponding retraction, and  $\mathbf{w} \in W^+$ . Then the set*

$$P = \{d^W(\rho_-(C_x), \rho_-(C_y)) \in W^+ \mid \forall (C_x, C_y) \in \mathcal{C}_0^+ \times_{\leq} \mathcal{C}_0^+, d^W(C_x, C_y) = \mathbf{w}\}$$

is finite and included in a finite subset  $P'$  of  $W^+$  depending only on  $\mathbf{w}$  and on the position of  $C_x$  with respect to  $\mathfrak{S}^-$  (i.e. on the codistance  $w_x \in W^v$  from  $C_x$  to the local chamber  $C_x^-$  in  $x$  of direction  $\mathfrak{S}^-$ ).

Let us write  $\mathbf{w} = \lambda.w$  for  $\lambda \in Y^+$  and  $w \in W^v$ . If we assume  $C_x$  and  $\mathfrak{S}^-$  opposite (i.e.  $w_x = 1$ ), then any  $\mathbf{v} = \mu.v \in P'$  satisfies  $\lambda \leq_{Q^v} \mu \leq_{Q^v} \lambda^{++}$  and  $\mu$  is in  $\text{conv}(W^v.\lambda^{++})$ . More precisely  $\mu$  is in the convex hull  $\text{conv}(W^v.\lambda^{++}, \geq \lambda)$  of all  $w'.\lambda^{++}$  for  $w' \in W^v$ ,  $w' \leq w_\lambda$ , where  $w_\lambda$  is the element with minimal length such that  $\lambda = w_\lambda.\lambda^{++}$ .

If moreover  $\lambda \in Y^{++}$ , then  $\mu = \lambda$  and  $v \leq w$ . In particular, for  $\mathbf{w} = \lambda \in Y^{++}$ ,  $P = \{\mathbf{w}\} = \{\lambda\}$ .

*Proof.* We consider an apartment  $A_1$  containing  $C_x$  and  $C_y$ . We set  $C'_y = C_x + (y - x)$  in  $A_1$ . Identifying  $(\mathbb{A}, C_0^+)$  with  $(A_1, C_x)$  (resp.  $(A_1, C'_y)$ ), we have  $y = x + \lambda$  (resp.  $C_y = wC'_y$ ).

We have to prove that the possibilities for  $\rho_-(C_y)$  vary in a finite set determined by  $\rho_-(C_x)$ ,  $\mathbf{w}$ , and  $w_x$ . We shall prove this by successively showing the same kind of result for  $\rho_-([x, y])$ ,  $\rho_-(y)$  and  $\rho_-(C'_y)$ . Up to isomorphism, one may suppose  $C_x \subset A$ .

a) Let  $w_\lambda \in W^v$  such that  $\lambda^{++} = w_\lambda^{-1}(\lambda)$ . Fixing a reduced decomposition for  $w_\lambda$  gives a minimal gallery between  $C_x$  and  $[x, y]$ . By retraction, we get a gallery with the same type from  $\rho_-(C_x)$  to  $\rho_-([x, y])$ . The possible foldings of this gallery determine the possibilities for  $\rho_-([x, y])$ . More precisely,  $\rho_-([x, y]) = x + w'(\lambda_A^{++})[0, 1)$  for  $w' \leq w_\lambda$  and  $\lambda_A^{++}$  the image in  $A$  of  $\lambda^{++}$  by the identification of  $(\mathbb{A}, C_0^+)$  with  $(A, C_x)$ .

b) We fix now  $\rho_-([x, y])$ . By Lemma 1.8 b),  $\rho_-([x, y])$  is a Hecke path  $\pi$  of shape  $\lambda^{++}$  (with respect to  $\mathfrak{S}^-$ ). Its derivative  $\pi'_+(0)$  is well determined by  $\rho_-([x, y])$ . We identify  $A$  with  $\mathbb{A}$  in such a way that  $\mathfrak{S}^-$  has direction  $-C_f^v$ . Then  $\lambda_A^{++} = w_x(\lambda^{++})$  and  $\pi'_+(0) = w'w_x(\lambda^{++})$ , with  $w'$  as above. By Lemma 1.8 b),  $\pi'_+(0) \leq_{Q^v} \rho_-(y) - \rho_-(x) \leq_{Q^v} \lambda^{++}$ . So there is a finite number of possibilities for  $\rho_-(y)$ .

c) Now we fix  $\rho_-([x, y])$ ,  $\rho_-(y)$  and investigate the possibilities for  $\rho_-(C'_y)$ . Let  $\xi \in Y^{++}$  and in the interior of the fundamental chamber  $C_f^v$ . In the apartment  $A_1$ , with  $(A_1, C_x)$  identified with  $(\mathbb{A}, C_0^+)$ , we consider  $x' = x + \xi$  and  $y' = y + \xi$  (hence  $y' = x' + \lambda$ ).

As in a) and b) above, we get that there is a finite number of possibilities for  $\rho_-(x')$ .

c1) On one side, we may also enlarge the segment  $[x, x']$  to  $[x, x'')$ , where  $x'' = x' + \xi[0, 1]$ . On the other side,  $[x, x']$  can be described as a path  $\pi_1 : [0, 1] \rightarrow A_1$ ,  $\pi_1(t) = x + t\xi$ . The retracted path  $\pi = \rho_-(\pi_1)$  satisfies  $\rho_-(x') - \rho_-(x) \leq_{Q^\vee} \pi'_+(1) \leq_{Q^\vee} \lambda^{++}$ , again by Lemma 1.8. So there is a finite number of possibilities for  $\pi'_+(1)$ , i.e. for  $[x', x'')$ . But there exists (in  $A_1$ ) a minimal gallery of the type of a reduced decomposition of  $w_\lambda$  from the unique local chamber  $(C_x + \xi)$  containing  $[x', x'')$  to  $[x', y']$ . Hence, there exists a gallery of the same type between (a local chamber containing)  $\rho_-([x', x''))$  and  $\rho_-([x', y'])$ . Therefore, there is a finite number of possibilities for  $\rho_-([x', y'])$ .

As in b) above, we deduce that there is a finite number of possibilities for  $\rho_-(y')$ .

c2) The path  $\rho_-([y, y'])$  is a Hecke path of shape  $\xi$  from  $\rho_-(y)$  to  $\rho_-(y')$ . By [GR08] Corollary 5.9, there exists a finite number of such paths. In particular, there is a finite number of possibilities for the segment-germ  $\rho_-([y, y'])$  and for  $\rho_-(C'_y)$ .

d) Next, we fix  $\rho_-(C'_y)$ . Fixing a reduced decomposition for  $w$  gives a minimal gallery between  $C'_y$  and  $C_y$ , hence a gallery of the same type between  $\rho_-(C'_y)$  and  $\rho_-(C_y)$ . So, the number of possible  $\rho_-(C_y)$  is finite and  $d^W(\rho_-(C'_y), \rho_-(C_y)) \leq w$ .

e) Finally, let us consider the case  $w_x = 1$ , then  $\lambda_A^{++} = \lambda^{++}$ . So, in b), we get  $\pi'_+(0) = w'(\lambda^{++})$  with  $w' \leq w_\lambda$ , hence  $\pi'_+(0) \geq_{Q^\vee} w_\lambda(\lambda^{++}) = \lambda$  and  $\lambda \leq_{Q^\vee} \pi'_+(0) \leq_{Q^\vee} \rho_-(y) - \rho_-(x) = \mu \leq_{Q^\vee} \lambda^{++}$ . If moreover  $\lambda$  is in  $Y^{++}$ , then  $\lambda = \lambda^{++}$  and  $\mu = \lambda$ . The Hecke path of shape  $\lambda$   $\rho_-([x, y])$  is the segment  $[\rho_-(x), \rho_-(x) + \lambda]$ . Its dual dimension is 0 (see [GR08] 5.7). By [GR08] 6.3, there is one and only one segment in  $\mathcal{S}$  with end  $y$  that retracts onto this Hecke path: any apartment containing  $y$  and  $\mathfrak{S}^-$  contains  $[x, y]$ . But  $C_x$  is the enclosure of  $x$  and  $C'_y = C_y$  (computation in  $A_1$ ). So, any apartment containing  $\mathfrak{S}^-$  and  $C'_y$  contains  $C_x$ . Therefore, we have  $\lambda = d^W(C_x, C'_y) = d^W(\rho_-(C_x), \rho_-(C'_y))$ .

The end of the proof of the lemma follows then from d) above.  $\square$

**Proposition 2.3.** *Let  $C_x, C_y, C_z \in \mathcal{C}_0^+$  be such that  $x \leq z \leq y$  and  $d^W(C_x, C_z) = \mathbf{w} \in W^+$ ,  $d^W(C_z, C_y) = \mathbf{v} \in W^+$ . Then  $d^W(C_x, C_y)$  varies in a finite subset  $P_{\mathbf{w}, \mathbf{v}}$  of  $W^+$ , depending only on  $\mathbf{w}$  and  $\mathbf{v}$ .*

*Let us write  $\mathbf{w} = \lambda.w$  and  $\mathbf{v} = \mu.v$  for  $\lambda, \mu \in Y^+$  and  $w, v \in W^v$ . If we assume  $\lambda = \lambda^{++}$  and  $w = 1$ , then any  $\mathbf{w}' = \nu.u \in P_{\mathbf{w}, \mathbf{v}}$  satisfies  $\lambda + \mu \leq_{Q^\vee} \nu \leq_{Q^\vee} \lambda + \mu^{++}$  and  $\nu - \lambda \in \text{conv}(W^v.\mu^{++}, \geq \mu) \subset \text{conv}(W^v.\mu^{++})$ .*

*If, moreover,  $\mu = \mu^{++} \in Y^{++}$ , then  $\nu = \lambda + \mu$  and  $u \leq v$ . In particular, for  $\mathbf{w} = \lambda$ ,  $\mathbf{w}' = \mu$  in  $Y^{++}$ ,  $P_{\mathbf{w}, \mathbf{v}} = \{\lambda + \mu\}$ .*

*Proof.* Now we consider any apartment  $A$  containing  $C_x$ , the sector-germ  $\mathfrak{S}^-$  opposite  $C_x$  and the retraction  $\rho_-$  as in Lemma 2.2. Then  $\rho_-(C_x) = C_x$  and  $d^W(C_x, \rho_-(C_z))$  varies in a finite subset  $P_x$  of  $W^+$  depending on  $\mathbf{w}$ , by Lemma 2.2. If  $d^W(C_x, \rho_-(C_z)) = \lambda'.w'$ , then the relative position  $w_z \in W^v$  of  $C_z$  and  $\mathfrak{S}^-$  is equal to  $w'$ . Applying once more Lemma 2.2 to  $C_z$  and  $C_y$ , we get that  $d^W(\rho_-(C_z), \rho_-(C_y))$  varies in a finite subset  $P_{w'}$  of  $W^+$  depending only on  $\mathbf{v}$  and  $w'$ . Finally,  $d^W(C_x, \rho_-(C_y))$  varies in the finite subset  $P_{\mathbf{w}, \mathbf{v}} = \{\mathbf{w}'.\mathbf{v}' \in W^+ \mid \mathbf{w}' = \lambda'.w' \in P_x, \mathbf{v}' \in P_{w'}\}$ . Taking now  $A$  containing  $C_x$  and  $C_y$ , we get  $d^W(C_x, C_y) = d^W(C_x, \rho_-(C_y)) \in P_{\mathbf{w}, \mathbf{v}}$ .

To finish, suppose that  $\lambda = \lambda^{++}$  and  $w = 1$ . By Lemma 2.2,  $P_1 = \{\lambda\}$ , hence  $w' = w_z = 1$ . Applying again Lemma 2.2, any  $\mathbf{v}' = \mu'.v' \in P_{w'}$  satisfies  $\mu \leq_{Q^\vee} \mu' \leq_{Q^\vee} \mu^{++}$ . So any  $\mathbf{w}'' = \nu.u$  in  $P_{\mathbf{w}, \mathbf{v}}$  is equal to  $(\lambda + \mu').v'$  for  $\mu'.v' \in P_{w'} = P_1$ , hence  $\lambda + \mu \leq_{Q^\vee} \nu = \lambda + \mu' \leq_{Q^\vee} \lambda + \mu^{++}$ . If moreover  $\mu \in Y^{++}$ , then  $\nu = \lambda + \mu$  and  $u \leq v$ . The last particular case is now clear.  $\square$

**Proposition 2.4.** *Let us fix two local chambers  $C_x$  and  $C_y$  in  $\mathcal{C}_0^+$  with  $x \leq y$  and  $d^W(C_x, C_y) = \mathbf{u} \in W^+$ . We consider  $\mathbf{w}$  and  $\mathbf{v}$  in  $W^+$ . Then the number  $a_{\mathbf{w}, \mathbf{v}}^{\mathbf{u}}$  of  $C_z \in \mathcal{C}_0^+$  with  $x \leq z \leq y$ ,  $d^W(C_x, C_z) = \mathbf{w}$  and  $d^W(C_z, C_y) = \mathbf{v}$  is finite (i.e. in  $\mathbb{N}$ ).*

*If we set  $\mathbf{w} = \lambda.w$ ,  $\mathbf{v} = \mu.v$  and  $\mathbf{u} = \nu.u$ , with  $\lambda, \mu, \nu \in Y^+$  and  $w, v, u \in W^v$ , then  $a_{\mathbf{w}, \mathbf{v}}^{\mathbf{u}} = a_{\lambda, \mu}^{\nu} \geq 1$  (resp.  $= 1$ ) when  $w = v = u = 1$  and  $\lambda \in Y^{++}$ ,  $\mu \in Y^+$  (resp.  $\lambda, \mu \in Y^{++}$ ),  $\nu = \lambda + \mu$ .*

*Proof.* We have  $d^v(x, z) = \lambda^{++}$  and  $d^v(z, y) = \mu^{++}$ . So, by [GR14], 2.5, the number of possible  $z$  is finite. Hence, we fix  $z$  and count the possible  $C_z$ .

Let  $C'_z$  be the local chamber in  $z$  containing  $[z, y)$  and  $[z, y')$  for  $y'$  in a sufficiently small element of the filter  $C_y$ . By convexity,  $C'_z$  is well determined by  $z$  and  $C_y$ . But in an apartment containing  $C_y, C_z$  (hence also  $C'_z$ ), we see that  $d^W(C'_z, C_z)$  is well determined by  $\mathbf{v}$ . So there is a gallery (of a fixed type) from  $C'_z$  to  $C_z$ , thus the number of possible  $C_z$  is finite.

Assume now that  $\mathbf{w} = \lambda \in Y^{++}$ ,  $\mathbf{v} = \mu \in Y^+$  and  $\mathbf{u} = \lambda + \mu$ . Taking an apartment  $A_1$  containing  $C_x$  and  $C_y$ , it is clear that the local chamber  $C_z$  in  $A_1$  such that  $d^W(C_x, C_z) = \lambda$  satisfies also  $d^W(C_z, C_y) = \mu$  (as  $d^W(C_x, C_y) = \lambda + \mu$ ). So  $a_{\lambda, \mu}^{\lambda + \mu} \geq 1$ . We consider now any  $C_z$  satisfying the conditions, with moreover  $\mu \in Y^{++}$ .

As in Proposition 2.3, we choose  $A$  containing  $C_x$  and  $\mathfrak{S}^-$  opposite  $C_x$ . We saw in Lemma 2.2 e) that any apartment containing  $C_z$  and  $\mathfrak{S}^-$  contains  $C_x$  and  $d^W(C_x, \rho_-(C_z)) = \lambda$ . With the same Lemma applied to  $C_z$  and  $C_y$ , we see that any apartment containing  $C_z$  and  $\mathfrak{S}^-$  contains  $C_y$ . In particular, there is an apartment  $A_1$  containing  $C_x, C_z, C_y$  and  $d^W(C_x, C_z) = \lambda$ ,  $d^W(C_z, C_y) = \mu$ ,  $d^W(C_x, C_y) = \lambda + \mu$ . But  $\lambda, \mu \in Y^{++}$ , so  $C_z$  is in the enclosure of  $C_x$  and  $C_y$ . Therefore,  $C_z$  is unique: any other apartment  $A_2$  containing  $C_x$  and  $C_y$  contains  $x, y$  (with  $x \leq y$ ) and  $x' = x + \xi$ ,  $y' = y + \xi$  (with  $x' \leq y'$ ), for  $\xi \in C_x^v = C_y^v$  small; by 1.9.a,  $A_2$  contains  $z \in cl_{A_1}(\{x, y\})$ ,  $z' = z + \xi \in cl_{A_1}(\{x', y'\})$ , hence also  $C_z \subset cl_{A_1}(\{z, z'\})$ .  $\square$

**Theorem 2.5.** *For any ring  $R$ ,  ${}^I\mathcal{H}_R^{\mathcal{J}}$  is an algebra such that*

$$T_{\mathbf{w}} * T_{\mathbf{v}} = \sum_{\mathbf{u} \in P_{\mathbf{w}, \mathbf{v}}} a_{\mathbf{w}, \mathbf{v}}^{\mathbf{u}} T_{\mathbf{u}}$$

and  $T_{\lambda} * T_{\mu} = T_{\lambda + \mu}$ , for  $\lambda, \mu \in Y^{++}$ .

*Proof.* It is a simple consequence of Propositions 2.3 and 2.4.  $\square$

**Definition 2.6.** The algebra  ${}^I\mathcal{H}_R^{\mathcal{J}}$  is the Iwahori-Hecke algebra associated to  $\mathcal{J}$  with coefficients in  $R$ .

The structure constants  $a_{\mathbf{w}, \mathbf{v}}^{\mathbf{u}}$  are non-negative integers. We conjecture that they are polynomials in the parameters  $q_i, q'_i$  with coefficients in  $\mathbb{Z}$  and that these polynomials depend only on  $\mathbb{A}$  and  $W$ . We prove this in the following section for  $\mathbf{w}, \mathbf{v}$  generic, see the precise hypothesis just below. We get also this conjecture for some  $\mathbb{A}, W$  when all  $q_i, q'_i$  are equal; in the general case we get only that they are Laurent polynomials, see 6.7.

Geometrically, it is possible to get more informations about  $T_{\lambda} * T_{\mu}$  when  $\lambda \in Y^{++}, \mu \in Y^+$ , but we shall get them algebraically (Corollary 5.3).

### 3 Structure constants

In this section, we compute the structure constants  $a_{\mathbf{w}, \mathbf{v}}^{\mathbf{u}}$  of the Iwahori-Hecke algebra  ${}^I\mathcal{H}_R^{\mathcal{J}}$ , assuming that  $\mu$  is regular in  $\mathbf{v} = \mu.v$  and  $\lambda$  spherical in  $\mathbf{w} = \lambda.w$ , here  $\lambda$  and  $\mu$  are in  $Y^+$ . We will adapt some results obtained in the spherical case in [GR14] to our situation.

These structure constants depend on the shape of the standard apartment  $\mathbb{A}$  and on the numbers  $q_M$  of 1.4. Recall that the number of (possibly) different parameters is at most  $2 \cdot |I|$ . We denote by  $\mathcal{Q} = \{q_1, \dots, q_l, q'_1 = q_{l+1}, \dots, q'_l = q_{2l}\}$  this set of parameters.

#### 3.1 Centrifugally folded galleries of chambers

Let  $z$  be a point in the standard apartment  $\mathbb{A}$ . We have twinned buildings  $\mathcal{T}_z^+ \mathcal{J}$  (resp.  $\mathcal{T}_z^- \mathcal{J}$ ). We consider their unrestricted structure, so the associated Weyl group is  $W^v$  and the chambers (resp. closed chambers) are the local chambers  $C = \text{germ}_z(z + C^v)$  (resp. local closed chambers  $\overline{C} = \text{germ}_z(z + \overline{C^v})$ ), where  $C^v$  is a vectorial chamber, cf. [GR08, 4.5] or [Ro11, § 5]. The distances (resp. codistances) between these chambers are written  $d^W$  (resp.  $d^{*W}$ ). To  $\mathbb{A}$  is associated a twin system of apartments  $\mathbb{A}_z = (\mathbb{A}_z^-, \mathbb{A}_z^+)$ .

We choose in  $\mathbb{A}_z^-$  a negative (local) chamber  $C_z^-$  and denote by  $C_z^+$  its opposite in  $\mathbb{A}_z^+$ . We consider the system of positive roots  $\Phi^+$  associated to  $C_z^+$  (i.e.  $\Phi^+ = w.\Phi_f^+$ , if  $\Phi_f^+$  is the system  $\Phi^+$  defined in 1.1 and  $C_z^+ = \text{germ}_z(z + w.C_f^v)$ ). We denote by  $(\alpha_i)_{i \in I}$  the corresponding basis of  $\Phi$  and by  $(r_i)_{i \in I}$  the corresponding generators of  $W^v$ .

Fix a reduced decomposition of an element  $w \in W^v$ ,  $w = r_{i_1} \cdots r_{i_r}$  and let  $\mathbf{i} = (i_1, \dots, i_r)$  be the type of the decomposition. We consider now galleries of (local) chambers  $\mathbf{c} = (C_z^-, C_1, \dots, C_r)$  in the apartment  $\mathbb{A}_z^-$  starting at  $C_z^-$  and of type  $\mathbf{i}$ .

The set of all these galleries is in bijection with the set  $\Gamma(\mathbf{i}) = \{1, r_{i_1}\} \times \cdots \times \{1, r_{i_r}\}$  via the map  $(c_1, \dots, c_r) \mapsto (C_z^-, c_1 C_z^-, \dots, c_1 \cdots c_r C_z^-)$ . Let  $\beta_j = -c_1 \cdots c_j (\alpha_{i_j})$ , then  $\beta_j$  is the root corresponding to the common limit wall  $M_j = M(\beta_j, -\beta_j(z))$  of type  $i_j$  of  $C_{j-1} = c_1 \cdots c_{j-1} C_z^-$  and  $C_j = c_1 \cdots c_j C_z^-$  and satisfying  $\beta_j(C_j) \geq \beta_j(z)$ .

**Definition.** Let  $\Omega$  be a chamber in  $\mathbb{A}_z^+$ . A gallery  $\mathbf{c} = (C_z^-, C_1, \dots, C_r) \in \Gamma(\mathbf{i})$  is said to be centrifugally folded with respect to  $\Omega$  if  $C_j = C_{j-1}$  implies that  $M_j$  is a wall and separates  $\Omega$  from  $C_j = C_{j-1}$ . We denote this set of centrifugally folded galleries by  $\Gamma_{\Omega}^+(\mathbf{i})$ .

#### 3.2 Liftings of galleries

Next, let  $\rho_{\Omega} : \mathcal{J}_z \rightarrow \mathbb{A}_z$  be the retraction centered at  $\Omega$ . To a gallery of chambers  $\mathbf{c} = (C_z^-, C_1, \dots, C_r)$  in  $\Gamma(\mathbf{i})$ , one can associate the set of all galleries of type  $\mathbf{i}$  starting at  $C_z^-$  in  $\mathcal{J}_z^-$  that retract onto  $\mathbf{c}$ , we denote this set by  $\mathcal{C}_{\Omega}(\mathbf{c})$ . We denote the set of minimal galleries (i.e.  $C_{j-1} \neq C_j$ ) in  $\mathcal{C}_{\Omega}(\mathbf{c})$  by  $\mathcal{C}_{\Omega}^m(\mathbf{c})$ . Recall from [GR14] that the set  $\mathcal{C}_{\Omega}^m(\mathbf{c})$  is nonempty if, and only if, the gallery  $\mathbf{c}$  is centrifugally folded with respect to  $\Omega$ . Recall also from loc. cit. that if  $\mathbf{c} \in \Gamma_{\Omega}^+(\mathbf{i})$ , then the number of elements in  $\mathcal{C}_{\Omega}^m(\mathbf{c})$  is:

$$\#\mathcal{C}_{\Omega}^m(\mathbf{c}) = \prod_{j \in J_1} (q_j - 1) \times \prod_{j \in J_2} q_j$$

where  $q_j = q_{M_j} \in \mathcal{Q}$ ,  $J_1 = \{j \in \{1, \dots, r\} \mid c_j = 1\}$  and  $J_2 = \{j \in \{1, \dots, r\} \mid c_j = r_{i_j} \text{ and } M_j \text{ is a wall separating } \Omega \text{ from } C_j\}$  (see [GR14] Corollary 4.5).



### 3.3 Liftings of Hecke paths

The Hecke paths we consider here are slight modifications of those used in [GR14]. Namely, a Hecke path of shape  $\mu^{++}$  with respect to a local positive chamber  $C_x$  in  $\mathbb{A}$ ,  $\pi = [z' = z_0, z_1, \dots, z_{\ell_\pi}, y]$ , is a  $\mu^{++}$ -path in  $\mathbb{A}$ . For all  $z = \pi(t)$ ,  $z \neq z_0 = \pi(0)$ , we ask  $x \overset{o}{<} z$  and then we define a local negative chamber  $C_z^- = \pi_z(C_x)$  by the fact that  $\overline{C_z^-}$  contains  $[z, x)$  and  $[z, x')$  for  $x'$  in a sufficiently small element of the filter  $C_x$ . Then we assume moreover that for all  $k \in \{1, \dots, \ell_\pi\}$ , there exists a  $(W_{z_k}^v, C_{z_k}^-)$ -chain from  $\pi'_-(t_k)$  to  $\pi'_+(t_k)$ , where  $z_k = \pi(t_k)$ . More precisely, this means that, for all  $k \in \{1, \dots, \ell_\pi\}$ , there exist finite sequences  $(\xi_0 = \pi'_-(t), \xi_1, \dots, \xi_s = \pi'_+(t))$  of vectors in  $V$  and  $(\beta_1, \dots, \beta_s)$  of real roots such that, for all  $j = 1, \dots, s$ :

- i)  $r_{\beta_j}(\xi_{j-1}) = \xi_j$ ,
- ii)  $\beta_j(\xi_{j-1}) < 0$ ,
- iii)  $r_{\beta_j} \in W_{\pi(t_k)}^v$  i.e.  $\beta_j(\pi(t_k)) \in \mathbb{Z}$ ,
- iv) each  $\beta_j$  is positive with respect to  $C_x$  i.e.  $\beta_j(z_k - C_x) > 0$ .

The centrifugally folded galleries are related to the lifting of Hecke paths by the following Lemma that we proved in [GR14] 4.6.

Suppose  $z \in \mathbb{A}, x \overset{o}{<} z$ . Let  $\xi$  and  $\eta$  be two segment germs in  $\mathbb{A}_z^+$ . Let  $-\eta$  and  $-\xi$  opposite respectively  $\eta$  and  $\xi$  in  $\mathbb{A}_z^-$ . Let  $\mathbf{i}$  be the type of a minimal gallery between  $C_z^-$  and  $C_{-\xi}$ , where  $C_{-\xi}$  is the negative (local) chamber containing  $-\xi$  such that  $d^W(C_z^-, C_{-\xi})$  is of minimal length. Let  $\mathfrak{Q}$  be a chamber of  $\mathbb{A}_z^+$  containing  $\eta$ . We suppose  $\xi$  and  $\eta$  conjugated by  $W_z^v$ .

**Lemma.** *The following conditions are equivalent:*

- (i) *There exists an opposite  $\zeta$  to  $\eta$  in  $\mathcal{S}_z^-$  such that  $\rho_{\mathbb{A}_z, C_z^-}(\zeta) = -\xi$ .*
- (ii) *There exists a gallery  $\mathbf{c} \in \Gamma_{\mathfrak{Q}}^+(\mathbf{i})$  ending in  $-\eta$ .*
- (iii) *There exists a  $(W_z^v, C_z^-)$ -chain from  $\xi$  to  $\eta$ .*

*Moreover the possible  $\zeta$  are in one-to-one correspondence with the disjoint union of the sets  $C_{\mathfrak{Q}}^m(\mathbf{c})$  for  $\mathbf{c}$  in the set  $\Gamma_{\mathfrak{Q}}^+(\mathbf{i}, -\eta)$  of galleries in  $\Gamma_{\mathfrak{Q}}^+(\mathbf{i})$  ending in  $-\eta$ .*

For an Hecke path as above and for  $k \in \{1, \dots, \ell_\pi\}$ , we define the segment germs  $\eta_k = \pi_+(t_k) = \pi(t_k) + \pi'_+(t_k).[0, 1)$  and  $-\xi_k = \pi_-(t_k) = \pi(t_k) - \pi'_-(t_k).[0, 1)$ . As above  $\mathbf{i}_k$  is the type of a minimal gallery between  $C_{z_k}^-$  and  $C_{-\xi_k}$ , where  $C_{-\xi_k}$  is the negative (local) chamber such that  $-\xi_k \in \overline{C_{-\xi_k}}$  and  $d^W(C_{z_k}^-, C_{-\xi_k})$  is of minimal length. Let  $\mathfrak{Q}_k$  be a fixed chamber in  $\mathbb{A}_{z_k}^+$  containing  $\eta_k$  and  $\Gamma_{\mathfrak{Q}_k}^+(\mathbf{i}_k, -\eta_k)$  be the set of all the galleries  $(C_{z_k}^-, C_1, \dots, C_r)$  of type  $\mathbf{i}_k$  in  $\mathbb{A}_{z_k}^-$ , centrifugally folded with respect to  $\mathfrak{Q}_k$  and with  $-\eta_k \in \overline{C_r}$ .

Let us denote the retraction  $\rho_{\mathbb{A}, C_x} : \mathcal{S}_{\geq x} \rightarrow \mathbb{A}$  simply by  $\rho$  and recall that  $y = \pi(1)$ . Let  $S_{C_x}(\pi, y)$  be the set of all segments  $[z, y]$  such that  $\rho([z, y]) = \pi$ , in particular,  $\rho(z) = z'$ . The following two theorems are proved in the same way as Theorem 4.8 and Theorem 4.12 of [GR14].

**Theorem 3.4.**  $S_{C_x}(\pi, y)$  is non empty if, and only if,  $\pi$  is a Hecke path with respect to  $C_x$ . Then, we have a bijection

$$S_{C_x}(\pi, y) \simeq \prod_{k=1}^{\ell_\pi} \prod_{\mathbf{c} \in \Gamma_{\Omega_k}^+(\mathbf{i}_k, -\eta_k)} \mathcal{C}_{\Omega_k}^m(\mathbf{c}).$$

In particular, the number of elements in this set is a polynomial in the numbers  $q \in \mathcal{Q}$  with coefficients in  $\mathbb{Z}$  depending only on  $\mathbb{A}$ .

**Theorem 3.5.** Let  $\lambda, \mu, \nu \in Y^{++}$  with  $\lambda$  spherical. Then, the number  $m_{\lambda, \mu}(\nu)$  of triangles  $[0, z, \nu]$  in  $\mathcal{S}$  with  $d^v(0, z) = \lambda$  and  $d^v(z, \nu) = \mu$  is equal to:

$$m_{\lambda, \mu}(\nu) = \sum_{w \in W^v / (W^v)_\lambda} \sum_{\pi} \prod_{k=1}^{\ell_\pi} \sum_{\mathbf{c} \in \Gamma_{\Omega_k}^+(\mathbf{i}_k, -\eta_k)} \#\mathcal{C}_{\Omega_k}^m(\mathbf{c}) \quad (1)$$

where  $\pi$  runs over the set of Hecke paths of shape  $\mu$  with respect to  $C_x$  from  $w.\lambda$  to  $\nu$  and  $\ell_\pi$ ,  $\Gamma_{\Omega_k}^+(\mathbf{i}_k, -\eta_k)$  and  $\mathcal{C}_{\Omega_k}^m(\mathbf{c})$  are defined as above for each such  $\pi$ .

**Remark.** In theorems 3.4, 3.5 above and in [GR14], it is interesting to precise that, if  $t_{\ell_\pi} = 1$ , i.e.  $z_{\ell_\pi} = y$ , then, in the above formulas,  $-\eta_{\ell_\pi}$  and  $\Omega_{\ell_\pi}$  are not well defined:  $\pi_+(1)$  does not exist. We have to understand that  $\prod_{\mathbf{c} \in \Gamma_{\Omega_{\ell_\pi}}^+(\mathbf{i}_{\ell_\pi}, -\eta_{\ell_\pi})} \mathcal{C}_{\Omega_{\ell_\pi}}^m(\mathbf{c})$  is the set of all minimal galleries of type  $\mathbf{i}_{\ell_\pi}$  starting from  $C_y^-$  (whose cardinality is  $\prod_{j=1}^r q_{i_j}$ , if  $\mathbf{i}_{\ell_\pi} = (i_1, \dots, i_r)$ ).

### 3.6 The formula

Let us fix two local chambers  $C_x$  and  $C_y$  in  $\mathcal{C}_0^+$  with  $x \leq y$  and  $d^W(C_x, C_y) = \mathbf{u} \in W^+$ . We consider  $\mathbf{w}$  and  $\mathbf{v}$  in  $W^+$ . Then we know that the number  $a_{\mathbf{w}, \mathbf{v}}^{\mathbf{u}}$  of  $C_z \in \mathcal{C}_0^+$  with  $x \leq z \leq y$ ,  $d^W(C_x, C_z) = \mathbf{w}$  and  $d^W(C_z, C_y) = \mathbf{v}$  is finite, see Proposition 2.4. We want to use equivalent conditions on the  $W$ -distance between the chambers.

**Lemma.** 1) Assume  $\lambda$  spherical. Let  $C_z^- = \pi_z(C_x)$  be the local chamber in  $z$  containing  $[z, x)$  and  $[z, x')$  for  $x'$  in a sufficiently small element of the filter  $C_x$  and let  $w_\lambda^+$  be the longest element such that  $w_\lambda^+.\lambda \in \overline{C_f^v}$ . Then

$$d^W(C_x, C_z) = \lambda.w \iff \begin{cases} d^W(C_x, z) = \lambda \\ d^{*W}(C_z^-, C_z) = w_\lambda^+w. \end{cases}$$

2) Let  $C_z^+ = \pi_z(C_y)$  be the local chamber in  $z$  containing  $[z, y)$  and  $[z, y')$  for  $y'$  in a sufficiently small element of the filter  $C_y$  and let  $w_\mu$  be the smallest element such that  $\mu^{++} = w_\mu.\mu \in \overline{C_f^v}$ . Then

$$d^W(C_z, C_y) = \mu.v \iff \begin{cases} d^W(C_z, C_z^+) = w_\mu^{-1} \\ d^W(C_z^+, C_y) = \mu^{++}w_\mu v. \end{cases}$$

If we assume moreover  $\mu$  regular, then  $C'_y = \pi_y(C_z)$  (resp.  $C_z^+ = \pi_z(C_y)$ ) is the unique local chamber in  $y$  (resp.  $z$ ) containing  $[y, z)$  (resp.  $[z, y)$ ) and we have :

$$d^W(C_z^+, C_y) = \mu^{++}w_\mu v \iff d^v(z, y) = \mu^{++} \text{ and } d^{*W}(C'_y, C_y) = w_\mu v.$$

*Proof.* 1) By convexity,  $C_z^-$  is in any apartment containing  $C_x$  and  $C_z$ . Let us fix such an apartment  $A$  and identify  $(A, C_x)$  with  $(\mathbb{A}, \text{germ}_0(C_f^v))$ . By definition, we have  $d^W(C_x, z) = d^W(C_x, z + C_x)$ . Then, of course,  $d^W(C_x, z) = \lambda$ . Next as  $\lambda$  is supposed spherical, the stabilizer  $(W^v)_\lambda$  is finite, so  $w_\lambda^+$  is well defined and  $x \stackrel{o}{<} z$ , so  $C_z^-$  is well defined. Moreover,  $d^W(\text{op}_A C_z^-, z + C_x) = w_\lambda^+$  and  $d^W(z + C_x, C_z) = w$ . Therefore, by Chasles, we get  $d^W(\text{op}_A C_z^-, C_z) = w_\lambda^+ w$ , but, by definition,  $d^{*W}(C_z^-, C_z) = d^W(\text{op}_A C_z^-, z + C_z)$ .

2) The first assertion is the Chasles' relation, as  $C_z, C_y, C_z^+$  (and  $C'_y$ ) are in a same apartment  $A'$ . The second comes from the fact that, if  $\mu$  is regular, then  $d^W(C_z^+, C_{zy}^+) = d^v(z, y) \in Y^{++}$ , where  $C_{zy}^+$  opposites  $C'_y$  at  $y$  in  $A'$ . Moreover,  $d^{*W}(C'_y, C_y) = d^W(C_{zy}^+, C_y) \in W^v$  by definition, so we conclude by Chasles.  $\square$

**Theorem 3.7.** *Assume  $\mu$  is regular and  $\lambda$  is spherical. We choose the standard apartment  $\mathbb{A}$  containing  $C_x$  and  $C_y$ . Then*

$$a_{\mathbf{w}, \mathbf{v}}^{\mathbf{u}} = \sum_{\pi, t_{\ell_\pi=1}} \left[ \left( \prod_{k=1}^{\ell_\pi-1} \sum_{\mathbf{c} \in \Gamma_{\Omega_k}^+(\mathbf{i}_k, -\eta_k)} \# \mathcal{C}_{\Omega_k}^m(\mathbf{c}) \right) \left( \sum_{\mathbf{d} \in \Gamma_{C_y}^+(\mathbf{i}_\ell, \tilde{C}_y)} \# \mathcal{C}_{C_y}^m(\mathbf{d}) \right) \left( \sum_{\mathbf{e} \in \Gamma_{C_{z_0}^-}^+(\mathbf{i}, C_{z_0}^-)} \# \mathcal{C}_{C_{z_0}^-}^m(\mathbf{e}) \right) \right] +$$

$$+ \sum_{\pi, t_{\ell_\pi < 1}} \left[ \left( \prod_{k=1}^{\ell_\pi} \sum_{\mathbf{c} \in \Gamma_{\Omega_k}^+(\mathbf{i}_k, -\eta_k)} \# \mathcal{C}_{\Omega_k}^m(\mathbf{c}) \right) \left( \sum_{\mathbf{e} \in \Gamma_{C_{z_0}^-}^+(\mathbf{i}, C_{z_0}^-)} \# \mathcal{C}_{C_{z_0}^-}^m(\mathbf{e}) \right) \right],$$

where the  $\pi$ , in the first sum, runs over the set of all Hecke paths in  $\mathbb{A}$  with respect to  $C_x$  of shape  $\mu^{++}$  from  $x + \lambda = z_0$  to  $x + \nu = y$  such that  $t_{\ell_\pi} = 1$ , whereas in the second sum, the paths have to satisfy  $t_{\ell_\pi} < 1$  and  $d^{*W}(C'_y, C_y) = w_\mu v$ , where  $C_y^- = \pi_y(C_x)$  is the local chamber in  $y$  containing  $[y, x)$  and  $[y, x')$  for  $x'$  in a sufficiently small element of the filter  $C_x$ .

Moreover  $\mathbf{i}$  is a reduced decomposition of  $w_\mu$ ,  $C'_{z_0}$  is the local chamber at  $z_0$  in  $\mathbb{A}$  defined by  $d^{*W}(C_{z_0}^-, C'_{z_0}) = w_\lambda^+ w$ ,  $\mathbf{i}_\ell$  is the type of a minimal gallery from  $C_y^-$  to the local chamber  $C_y^*$  at  $y$  in  $\mathbb{A}$  containing the segment  $\text{germ } \pi_-(y) = y - \pi'_-(1).[0, 1)$  and  $\tilde{C}_y$  is the unique local chamber at  $y$  in  $\mathbb{A}$  such that  $d^{*W}(\tilde{C}_y, C_y) = w_\mu v$ . The rest of the notation is as defined above.

*Proof.* Recall that, to compute the structure constants, we use the retraction  $\rho = \rho_{\mathbb{A}, C_x} : \mathcal{S} \rightarrow \mathbb{A}$ , where  $C_x$  and  $C_y$  are fixed and in  $\mathbb{A}$ . We have  $y = \rho(y) = x + \nu$  and the condition  $d^W(C_x, z) = \lambda$  is equivalent to  $\rho(z) = x + \lambda = z_0$ . We want to prove a formula of the form

$$a_{\mathbf{w}, \mathbf{v}}^{\mathbf{u}} = \sum_{\pi} \left( \text{number of liftings of } \pi \right) \times \left( \text{number of } C_z \right),$$

where  $\pi$  runs over some set of Hecke paths with respect to  $C_x$  of shape  $\mu^{++}$  from  $x + \lambda$  to  $x + \nu$ . It is possible to calculate like that for, in the case of a regular  $\mu^{++}$ ,  $\rho(C_z^+)$  is well determined by  $\pi$ . Hence, the number of  $C_z$  only depends on  $\pi$  and not on the lifting of  $\pi$ .

The local chambers  $C_z$  satisfying  $d^{*W}(C_z^-, C_z) = w_\lambda^+ w$  and  $d^W(C_z, C_z^+) = w_\mu^{-1}$  are at the end of a minimal gallery starting at  $C_z^+$  of type  $\mathbf{i}$  and retracting by  $\rho_{A', C_z^-}$  onto the local chamber  $C'_z$  at  $z$  defined by  $d^{*W}(C_z^-, C'_z) = w_\lambda^+ w$  in a fixed apartment  $A'$  containing  $C_x$  and  $C_z^+$ . So their number is given by the number of minimal galleries starting at  $C_z^+$  of type  $\mathbf{i}$  and retracting on a centrifugally folded gallery  $\mathbf{e}$  of type  $\mathbf{i}$  ending in  $C'_z$ . In other words, their number is given by the cardinality of the set  $\mathcal{C}_{C_z^-}^m(\mathbf{e})$ , for each  $\mathbf{e} \in \Gamma_{C_z^-}^+(\mathbf{i}, C'_z)$ . Using

an isomorphism fixing  $C_x$  and sending  $A'$  to  $\mathbb{A}$ , we may replace in this formula  $z, C_z^-, C'_z$  and  $C_z^+$  by  $z_0, C_{z_0}^-, C'_{z_0}$  and the unique local chamber  $C_{z_0}^+$  in  $\mathbb{A}$  containing the segment germ  $\pi_+(0) = z_0 + \pi'_+(0).[0, 1)$ .

Now, we compute the number of liftings of a Hecke path  $\pi$  starting from the formula in Theorem 3.5 and according to the two conditions  $d^W(C_x, z) = \lambda$  and  $d^W(C_z^+, C_y) = \mu^{++}w_\mu v$ . The first one fixes one element in the set  $W^v/(W^v)_\lambda$ , namely the coset of  $w_\lambda^+$ , *i.e.*  $\pi(0) = x + \lambda$ . The second one is equivalent to the fact that the segment  $[z, y]$  is of type  $\mu^{++}$  and  $d^{*W}(C'_y, C_y) = w_\mu v$ , as we have seen in the Lemma above.

Further, we have

$$t_{\ell_\pi} < 1 \iff \pi_-(y) \in C_y^-,$$

If  $\pi_-(y) \in C_y^-$  then  $\rho(C'_y) = C'_y = C_y^-$ , whence,  $d^{*W}(C_y^-, C_y) = w_\mu v$ . Since we lift the Hecke path into a segment starting with its behaviour at  $y$ , there is nothing more to count.

If  $t_{\ell_\pi} = 1$ , then  $\pi_-(y) \in C_y^* = \rho(C'_y) \neq C_y^-$ . We want to lift the path but with the condition that  $d^{*W}(C'_y, C_y) = w_\mu v$ , which may be translated in  $\rho'(C'_y) = \tilde{C}_y$ , for  $\rho' = \rho_{\mathbb{A}, C_y}$ . Since  $\mu^{++}$  is regular, to find  $[y, z]$  it is enough to find  $C'_y$  *i.e.* to  $\rho'$ -lift  $\tilde{C}_y$ . The  $\rho'$ -liftings are then given by  $\rho'$ -lifting all the centrifugally folded galleries in  $\mathbb{A}$  with respect to  $C_y$  of type  $\mathbf{i}_\ell$  from  $C_y^-$  to  $\tilde{C}_y$  to minimal galleries. Therefore, their number is given by the cardinality of the set  $\mathcal{C}_{C_y}^m(\mathbf{d})$ , for each  $\mathbf{d} \in \Gamma_{C_y}^+(\mathbf{i}_\ell, \tilde{C}_y)$ . The rest of the lifting procedure is the same as in the proof of Theorem 3.5.  $\square$

### 3.8 Consequence

The above explicit formula, together with the formula for  $\sharp\mathcal{C}_\Omega^m(\mathbf{c})$  in 3.2, tell us that the structure constant  $a_{\mathbf{w}, \mathbf{v}}^{\mathbf{u}}$  is a polynomial in the parameters  $q_i, q'_i \in \mathcal{Q}$  with coefficients in  $\mathbb{Z}$  and that this polynomial depends only on  $\mathbb{A}, W, \mathbf{w}, \mathbf{v}$  and  $\mathbf{u}$ . So we have proved the conjecture following Definition 2.6 in this generic case: when  $\lambda$  is spherical and  $\mu$  regular.

## 4 Relations

For short, we write  ${}^I\mathcal{H}_R = {}^I\mathcal{H}_R^{\mathcal{J}}$  and  $T_i = T_{r_i}$  (when  $i \in I$ ).

**Proposition 4.1.** *Let  $\lambda \in Y^+$ ,  $w \in W^v$  and  $i \in I$ , then,*

1)  $T_{\lambda.w} * T_i = T_{\lambda.wr_i}$  *if, and only if, either  $(w(\alpha_i))(\lambda) < 0$  or  $(w(\alpha_i))(\lambda) = 0$  and  $\ell(wr_i) > \ell(w)$ . Otherwise  $T_{\lambda.w} * T_i = (q_i - 1)T_{\lambda.w} + q_i T_{\lambda.wr_i}$ .*

2)  $T_i * T_{\lambda.w} = T_{r_i(\lambda).r_iw}$  *if, and only if, either  $\alpha_i(\lambda) > 0$  or  $\alpha_i(\lambda) = 0$  and  $\ell(r_iw) > \ell(w)$ . Otherwise  $T_i * T_{\lambda.w} = (q_i - 1)T_{\lambda.w} + q_i T_{r_i(\lambda).r_iw}$ .*

*Proof.* We consider local chambers  $C_x, C_z, C_y$  with  $x \leq z \leq y$  and  $d^W(C_x, C_z) = \lambda.w$ ,  $d^W(C_z, C_y) = r_i$ . So there is an apartment  $A$  containing  $C_x, C_z$  and, if we identify  $(A, C_x)$  to  $(\mathbb{A}, C_0^+)$ , we have  $C_z = (\lambda.w)(C_x)$ . Moreover  $y = z$ ,  $C_z \neq C_y$  and  $C_z, C_y$  share a panel  $F_i$  of type  $i$ . We write  $D$  the half apartment of  $A$  containing  $C_x$  and with wall  $\partial D$  containing  $F_i$ .

It is clear that  $C_z \subset D$  if, and only if, either  $(w(\alpha_i))(\lambda) < 0$  or  $(w(\alpha_i))(\lambda) = 0$  and  $\ell(wr_i) > \ell(w)$ . Then, by Lemma 1.4.2, there is an apartment  $A'$  containing  $C_y$  and  $D$ , hence  $C_x, C_z, C_y$ . So  $d^W(C_x, C_y) = \lambda.wr_i$ . The panel  $F_i = F^\ell(z, F_i^v) \subset A$  is a spherical local face, so, for any  $p \in z + F_i^v \subset A$ , we have  $z \overset{o}{<} p$ , hence  $x \overset{o}{<} p$ . By 1.10.a, any apartment  $A''$

containing  $C_x$  and  $F_i$  contains  $C_z$ ; moreover  $C_z$  is well determined by  $F_i$  and  $C_x$ . The number  $a_{\lambda.w,r_i}^{\lambda.wr_i}$  of 2.4 is equal to 1 and we have proved that  $T_{\lambda.w} * T_i = T_{\lambda.wr_i}$ .

If  $C_z$  is not in  $D$ , we write  $C'_z$  the local chamber in  $D$  with panel  $F_i$ ; by the above argument,  $C'_z$  is well determined by  $F_i$  and  $C_x$ , moreover  $d^W(C_x, C'_z) = \lambda.wr_i$ . There are two cases for  $C_y$ : either  $C_y = C'_z$  or not. If  $C_y = C'_z$ , then  $d^W(C_x, C_y) = \lambda.wr_i$  and, if  $C_x, C_y$  are given, there are  $q_i$  possibilities for  $C_z$  (all local chambers covering  $F_i$  and different from  $C'_z$ ):  $a_{\lambda.w,r_i}^{\lambda.wr_i} = q_i$ . If  $C_y \neq C'_z$ , then  $d^W(C_x, C_y) = \lambda.w$  and, if  $C_x, C_y$  are given, there are  $q_i - 1$  possibilities for  $C_z$  (all local chambers covering  $F_i$  and different from  $C'_z, C_y$ ):  $a_{\lambda.w,r_i}^{\lambda.w} = q_i - 1$ .

We have proved 1) and we leave to the reader the similar proof of 2).  $\square$

## 4.2 The subalgebra $\mathcal{H}_R(W^v)$

We consider the  $R$ -submodule  $\mathcal{H}_R(W^v)$  of  ${}^I\mathcal{H}_R$  with basis  $(T_w)_{w \in W^v}$ . It is clearly a subalgebra of  ${}^I\mathcal{H}_R$  (as  $d^W(C_x, C_y) \in W^v \iff x = y$ ). Actually  $\mathcal{H}_R(W^v)$  is the Iwahori-Hecke algebra of the tangent building  $\mathcal{T}_x^+ \mathcal{S}$  for any  $x \in \mathcal{S}$ .

By Proposition 4.1, we have:

- $T_w * T_i = T_{wr_i}$  if  $\ell(wr_i) > \ell(w)$  and  $T_w * T_i = (q_i - 1)T_w + q_i T_{wr_i}$  otherwise.
- $T_i * T_w = T_{r_i w}$  if  $\ell(r_i w) > \ell(w)$  and  $T_i * T_w = (q_i - 1)T_w + q_i T_{r_i w}$  otherwise.

In particular  $T_i^2 = (q_i - 1)T_i + q_i Id$  and, for any reduced decomposition  $w = r_{i_1} \cdots r_{i_n}$ ,  $T_w = T_{i_1} \cdots T_{i_n}$ .

So  $\mathcal{H}_R(W^v)$  is the well known Hecke algebra associated to the Coxeter system  $(W^v, \{r_i \mid i \in I\})$  with (in general unequal) parameters  $(q_i)_{i \in I}$  and coefficients in the ring  $R$ . It is generated, as an  $R$ -algebra, by the  $T_i$ , for  $i \in I$ .

Suppose each  $q_i$  invertible in  $R$ , then, as well known,  $T_i^{-1} = q_i^{-1}(T_i - (q_i - 1)Id) \in \mathcal{H}_R(W^v)$  is the inverse of  $T_i$ . In particular any  $T_w$  is invertible: for any reduced decomposition  $w = r_{i_1} \cdots r_{i_n}$ ,  $T_w^{-1} = T_{i_n}^{-1} \cdots T_{i_1}^{-1}$ .

**Remark.** If  $q_i$  is invertible, it is easy to see from Proposition 4.1 that, either  $T_{\lambda.wr_i} = T_{\lambda.w} * T_i$  or  $T_{\lambda.wr_i} = T_{\lambda.w} * T_i^{-1}$  and, either  $T_{r_i(\lambda).r_i w} = T_i * T_{\lambda.w}$  or  $T_{r_i(\lambda).r_i w} = T_i^{-1} * T_{\lambda.w}$ .

**Corollary 4.3.** *Suppose each  $q_i$  invertible in  $R$  and consider  $\lambda \in Y^+$ . We may write  $\lambda = w.\lambda^{++}$ , with  $w \in W^v$ . Then  $T_\lambda = T_w * T_{\lambda^{++}} * T_w^{-1}$ .*

*Proof.* We consider a reduced decomposition  $w = r_{i_n} \cdots r_{i_1}$  and argue by induction on  $n$ . So, for  $w' = r_{i_{n-1}} \cdots r_{i_1}$  and  $\lambda' = w'.\lambda^{++}$ , we have  $T_{\lambda'} = T_{w'} * T_{\lambda^{++}} * T_{w'}^{-1}$ . We consider  $T_w * T_{\lambda^{++}} * T_w^{-1} = T_{i_n} * T_{\lambda'} * T_{i_n}^{-1}$ . But  $\ell(r_{i_n} w') > \ell(w')$  and  $\lambda^{++} \in Y^{++} \subset \overline{C_f^v}$ , so  $\alpha_{i_n}(w'.\lambda^{++}) \geq 0$ , i.e.  $\alpha_{i_n}(\lambda') \geq 0$ . We get  $T_{i_n} * T_{\lambda'} = T_{r_{i_n}(\lambda').r_{i_n}}$  by 4.1.2 and then  $T_{i_n} * T_{\lambda'} * T_{i_n}^{-1} = T_{r_{i_n}(\lambda')} = T_\lambda$  by 4.1.1 (and the above remark).  $\square$

**Corollary 4.4.** *Let  $\lambda \in Y^+$  and  $w, w' \in W^v$ , then we may write*

$$T_{\lambda.w'} * T_w = \sum_{w'' \leq w} a_{\lambda.w',w}^{\lambda.w'w''} T_{\lambda.w'w''}$$

where each  $a_{\lambda.w',w}^{\lambda.w'w''}$  is a polynomial in the  $q_i$  with coefficients in  $\mathbb{Z}$  and, when  $w' = 1$ ,  $a_{\lambda,w}^{\lambda.w} > 0$  is a primitive monomial. This polynomial  $a_{\lambda.w',w}^{\lambda.w'w''}$  depends only on  $\mathbb{A}$  and on  $W$ .

*Proof.* We write  $w = r_{i_1} \cdots r_{i_n}$  and argue by induction on  $n$ . The result is then clear from Proposition 4.1.1. We get actually that  $a_{\lambda,w}^{\lambda.w}$  is the product of some of the  $q_{i_j}$  ( $1 \leq j \leq n$ ).  $\square$

#### 4.5 The Iwahori-Hecke algebra as a right $\mathcal{H}_R(W^v)$ -module

We assume here that each  $q_i$  is invertible in  $R$ .

Given  $\lambda \in Y^+$ , we see from Corollary 4.4 that  $\{T_\lambda * T_w \mid w \in W^v\}$  and  $\{T_{\lambda.w} \mid w \in W^v\}$  are two bases of the same  $R$ -module. The base-change matrix is triangular with respect to the Bruhat order on  $W^v$  and the coefficients are Laurent polynomials in the  $q_i$ , with coefficients in  $\mathbb{Z}$  (primitive Laurent monomials on the diagonal). These polynomials depend only on  $\mathbb{A}$  and  $W$ .

As  $\{T_{\lambda.w} \mid \lambda \in Y^+, w \in W^v\}$  (resp.  $\{T_w \mid w \in W^v\}$ ) is a  $R$ -basis of  ${}^I\mathcal{H}_R$  (resp.  $\mathcal{H}_R(W^v)$ ), this means in particular that  ${}^I\mathcal{H}_R$  is a free right  $\mathcal{H}_R(W^v)$ -module with basis  $\{T_\lambda \mid \lambda \in Y^+\}$ .

In particular the  $R$ -algebra  ${}^I\mathcal{H}_R$  is generated by the  $T_i$  (for  $i \in I$ ) and the  $T_\lambda$  (for  $\lambda \in Y^+$ ) and even by the  $T_i$  (for  $i \in I$ ) and the  $T_\lambda$  (for  $\lambda \in Y^{++}$ ), as we see from Corollary 4.3.

**Lemma 4.6.** *Let  $C_1, C_2 \in \mathcal{C}_0^+$ , with vertices  $x_1, x_2$  be such that  $d^W(C_1, C_2) = \lambda \in Y^{++}$ . We consider  $i \in I$ ,  $F_1^i$  (resp.  $F_2^i$ ) the panel of type  $i$  of  $C_1$  (resp.  $C_2$ ). In an apartment  $A_1$  (resp.  $A_2$ ) containing  $C_1$  (resp.  $C_2$ ), we consider the sector panel  $\mathfrak{f}_1^-$  (resp.  $\mathfrak{f}_2^+$ ) with base point  $x_1$  (resp.  $x_2$ ) and direction opposite the direction of  $F_1^i$  (resp. equal to the direction of  $F_2^i$ ).*

*Then there is an apartment  $A$  containing  $\mathfrak{f}_1^-, \mathfrak{f}_2^+, C_1, C_2$  and, in this apartment  $A$ , the directions of  $\mathfrak{f}_1^-$  and  $\mathfrak{f}_2^+, F_2^i$  and  $\mathfrak{f}_1^-$  (resp.  $F_1^i$  and  $\mathfrak{f}_2^+$ ) are opposite (resp. equal).*

*Proof.* We choose  $\lambda_i \in F^v(\{i\}) \cap Y \subset Y^{++}$ . We write  $\mathfrak{F}_j^\pm$  the germ of  $\mathfrak{f}_j^\pm$  and  $F_j^{\pm v}$  its direction in  $A_j$ . In  $A_1$  (resp.  $A_2$ ) we consider the splayed chimney  $\mathfrak{r}_1^- = \mathfrak{r}(C_1, F_1^{-v})$  (resp.  $\mathfrak{r}_2^+ = \mathfrak{r}(C_2, F_2^{+v})$ ) containing  $\mathfrak{f}_1^-$  (resp.  $\mathfrak{f}_2^+$ ) and, for  $n \in \mathbb{N}$ , the chamber of type 0  $C_1(-n) = C_1 - n\lambda_i \subset \mathfrak{r}_1^-$  (resp.  $C_2(+n) = C_2 + n\lambda_i \subset \mathfrak{r}_2^+$ ); actually we identify  $(\mathbb{A}, \mathcal{C}_0^+)$  with  $(A_1, C_1)$  (resp.  $(A_2, C_2)$ ) to consider  $\lambda_i$  in  $\overline{A}_1$  (resp.  $\overline{A}_2$ ).

Then  $d^W(C_1(-n), C_1) = d^W(C_2, C_2(+n)) = n\lambda_i \in Y^{++}$  and  $d^W(C_1, C_2) = \lambda \in Y^{++}$ . By (MA3) there is an apartment  $A$  containing the germs  $\mathfrak{R}_1^-$  and  $\mathfrak{R}_2^+$  of  $\mathfrak{r}_1^-$  and  $\mathfrak{r}_2^+$ , hence  $C_1(-n)$  and  $C_2(+n)$  for  $n$  great. By Proposition 2.3 and the last paragraph of the proof of 2.4,  $d^W(C_1(-n), C_2(+n)) = \lambda + 2n\lambda_i \in Y^{++}$  and  $A$  contains  $C_1, C_2$ . By (MA4)  $A$  contains also  $\mathfrak{f}_1^- \subset \mathfrak{r}_1^- \subset cl_{A_1}(C_1, \mathfrak{R}_1^-)$  and  $\mathfrak{f}_2^+ \subset \mathfrak{r}_2^+ \subset cl_{A_2}(C_2, \mathfrak{R}_2^+)$ . So all assertions of the Lemma are satisfied.  $\square$

**Proposition 4.7.** *Let  $C_1, C_2, C_3, C_4 \in \mathcal{C}_0^+$  be such that  $d^W(C_1, C_2) = \lambda \in Y^{++}$ ,  $d^W(C_2, C_3) = r_i$  and  $d^W(C_3, C_4) = \mu \in Y^{++}$ . Then there is a direction of wall  $M_i^\infty$  (cf. [Ro11, § 4] or [GR14, 5.5]), chosen according to  $C_1, C_2$  (but independently from  $C_3, C_4$ ), such that  $C_1, C_2, C_3, C_4$  are in the extended tree  $\mathcal{S}(M_i^\infty)$ .*

*Proof.* We write  $x_1, x_2 = x_3, x_4$  the vertices of  $C_1, C_2, C_3, C_4$  and  $F_1^i, F_2^i = F_3^i, F_4^i$  their panels of type  $i$ . We choose  $\mathfrak{f}_1^-$  associated to  $C_1$  and  $F_1^i$  in an apartment  $A_1$  (resp.  $\mathfrak{f}_4^+$  associated to  $C_4$  and  $F_4^i$  in an apartment  $A_4$ ), as in Lemma 4.6. By this Lemma, using  $C_1$  and  $C_2$ , the direction of  $\mathfrak{f}_1^-$  is opposite that of  $F_2^i = F_3^i$  in some apartment  $A_2$  and, using  $C_3$  and  $C_4$ , the direction of  $\mathfrak{f}_4^+$  is the same as that of  $F_2^i = F_3^i$  in some apartment  $A_3$ . In  $A_3$  (resp.  $A_2$ ) we consider the sector face  $\mathfrak{f}_3^+$  (resp.  $\mathfrak{f}_2^-$ ) with base point  $x_2 = x_3$  and same direction as  $\mathfrak{f}_4^+$  or  $F_2^i = F_3^i$  (resp. same direction as  $\mathfrak{f}_1^-$  and opposite  $F_2^i = F_3^i$ ).

We may use the Lemma for  $C_1, C_2, \mathfrak{f}_1^-, \mathfrak{f}_3^+$ ; so the directions of  $\mathfrak{f}_1^-$  (or  $\mathfrak{f}_2^-$ ) and  $\mathfrak{f}_3^+$  (or  $\mathfrak{f}_4^+$ ) are opposite and  $C_1, C_2$  are in a same apartment  $A_5$  of  $\mathcal{S}(M_i^\infty)$ , if we consider the direction of wall  $M_i^\infty$  associated to the directions of  $\mathfrak{f}_1^-$  and  $\mathfrak{f}_4^+$ . Using now the Lemma for  $C_3, C_4, \mathfrak{f}_2^-, \mathfrak{f}_4^+$ , we see that these filters are in a same apartment  $A_6$  of  $\mathcal{S}(M_i^\infty)$ .  $\square$

**Theorem 4.8.** *Let  $\lambda, \mu \in Y^{++}$  and  $i \in I$ . We write  $N = \inf(\alpha_i(\lambda), \alpha_i(\mu)) \in \mathbb{N}$  and, for  $n \in \mathbb{N}$ ,  $q_i^{*n} = q_i q'_i q'_i \cdots$ , with  $n$  terms in this product.*

a) *If  $N = \alpha_i(\mu) \leq \alpha_i(\lambda)$ , then  $T_\lambda * T_i * T_\mu = T_{\lambda+\mu} * T_i$  for  $N = 0$  and, for  $N > 0$ ,*

$$T_\lambda * T_i * T_\mu = q_i^{*N} T_{\lambda+\mu-N\alpha_i^\vee} * T_i + (q_i^{*N} - q_i^{*(N-1)}) T_{\lambda+\mu-(N-1)\alpha_i^\vee} + \cdots \\ \cdots + (q_i^{*2} - q_i) T_{\lambda+\mu-\alpha_i^\vee} + (q_i - 1) T_{\lambda+\mu}$$

b) *If  $N = \alpha_i(\lambda) \leq \alpha_i(\mu)$ , then  $T_\lambda * T_i * T_\mu = T_i * T_{\lambda+\mu}$  for  $N = 0$  and, for  $N > 0$ ,*

$$T_\lambda * T_i * T_\mu = q_i^{*N} T_i * T_{\lambda+\mu-N\alpha_i^\vee} + (q_i^{*N} - q_i^{*(N-1)}) T_{\lambda+\mu-(N-1)\alpha_i^\vee} + \cdots \\ \cdots + (q_i^{*2} - q_i) T_{\lambda+\mu-\alpha_i^\vee} + (q_i - 1) T_{\lambda+\mu}$$

**Remarks.** 1) The case b) is less interesting for us, as we try to express any element in the basis of 4.5 for  ${}^I\mathcal{H}_R$  considered as a right  $\mathcal{H}_R(W^v)$ -module.

2) In the case a) we have  $\mu - N\alpha_i^\vee = r_i(\mu)$  and  $\lambda + \mu - N\alpha_i^\vee \in Y^{++}$  (as  $\alpha_i(\lambda + \mu - N\alpha_i^\vee) = \alpha_i(\lambda) - N$  and  $\alpha_j(\lambda + \mu - N\alpha_i^\vee) \geq \alpha_j(\lambda) + \alpha_j(\mu)$  for  $j \neq i$ ). So all  $\nu$  such that  $T_\nu$  appears on the right of the formula are in the  $\alpha_i^\vee$ -chain between  $\lambda + \mu$  and  $\lambda + r_i(\mu)$ ; in particular they are all in  $Y^{++}$ .

3) We call relation a) or relation b) the Bernstein-Lusztig relation for the  $T_\lambda$ , (BLT) for short. We shall use it essentially when  $\lambda = \mu$ .

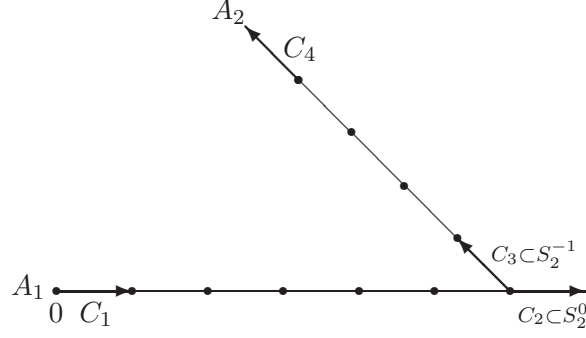
4) When  $\alpha_i(\lambda)$  or  $\alpha_i(\mu)$  is odd, we know that  $q'_i = q_i$ , cf. 1.4.4.

*Proof.* We consider  $C_1, C_2, C_3, C_4$  and  $M_i^\infty$  as in Proposition 4.7. When  $N = 0$  the results come from 4.1. We concentrate on the case  $0 < N = \alpha_i(\mu) \leq \alpha_i(\lambda)$ ; the other case is left to the reader. We have to evaluate  $d^W(C_1, C_4)$  and, given  $C_1, C_4$  satisfying  $d^W(C_1, C_4) = \mathbf{u}$ , to count the number of possible  $C_2, C_3$ . By Proposition 4.7 everything is in the extended tree  $\mathcal{S}(M_i^\infty)$ , which is semi-homogeneous with thicknesses  $1 + q_i, 1 + q'_i$ . By Proposition 4.1.2,  $C_3$  is well determined by  $C_2, C_4$  and lies in any apartment containing  $C_2, C_4$ ; moreover  $d^W(C_2, C_4) = r_i(\mu).r_i$ .

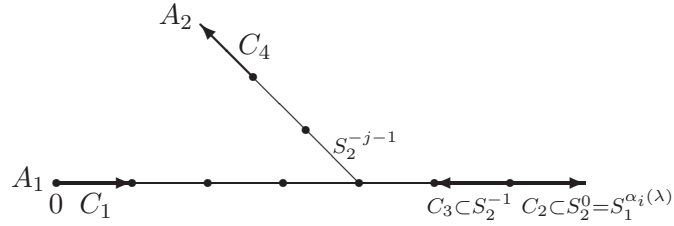
We consider an apartment  $A_1$  (resp.  $A_2$ ) of  $\mathcal{S}(M_i^\infty)$  containing  $C_1$  and  $C_2$  (resp.  $C_2$  and  $C_4$ , hence also  $C_3$ ). We identify  $(A_1, C_1)$  and  $(A_2, C_2)$  with  $(\mathbb{A}, C_0^+)$ ; we consider the retraction  $\rho_1$  (resp.  $\rho_2$ ) of  $\mathcal{S}(M_i^\infty)$  onto  $A_1$  (resp.  $A_2$ ) with center  $C_1$  (resp.  $C_2$ ). The closed chambers in an apartment of  $\mathcal{S}(M_i^\infty)$  are stripes limited by walls of direction  $M_i^\infty$ . In  $A_1 = \mathbb{A}$ , these walls are  $M(\alpha_i, n)$ ,  $n \in \mathbb{Z}$  and we write  $S_1^k$  the stripe  $S_1^k = \{x \mid k \leq \alpha_i(x) \leq k + 1\}$ , in particular  $C_1 \subset S_1^0$  and  $C_2 \subset S_1^{\alpha_i(\lambda)}$ . In  $A_2 = \mathbb{A}$ , we get also stripes  $S_2^k = \{x \mid k \leq \alpha_i(x) \leq k + 1\}$  such that  $C_2 \subset S_2^0 = S_1^{\alpha_i(\lambda)}$ ,  $C_3 \subset S_2^{-1}$  and  $C_4 \subset S_2^{-N-1}$ .

We have  $C_2 = C_1 + \lambda$  in  $A_1$  and  $\rho_2(C_4) = C_3 + r_i(\mu)$  in  $A_2$ . To find  $d^W(C_1, C_4)$  we have to determine the image of  $C_4$  under  $\rho_1$ . It depends actually on the highest number  $j$  such that  $S_2^{-j}$  (hence also  $S_2^0, \dots, S_2^{-j+1}$ ) is in  $A_1$ . A classical result for affine buildings (clear for extended trees and generalized to hovels in [Ro11, 2.9.2]) tells, then, that there is an apartment containing the stripes  $S_2^{-j-1}, \dots, S_2^{-N-1}$  and the half apartment  $\bigcup_{k \leq \alpha_i(\lambda) - j - 1} S_1^k$ .

If  $j = 0$ ,  $S_2^{-1}$  or  $C_3$  is not in  $A_1$ , so  $\rho_1(C_3) = C_2$  and, more generally,  $\rho_1(S_2^{-k}) = S_1^{\alpha_i(\lambda) + k - 1}$ , for  $k \geq 1$ . We get  $\rho_1(C_4) = C_2 + \mu$  and  $d^W(C_1, C_4) = \lambda + \mu$ . When  $C_1$  and  $C_4$  are fixed with this  $W$ -distance, we have to count the number of possible  $C_2$ . But  $C_3 \subset S_2^{-1}$  is in the enclosure of  $C_1 \subset S_1^0$  and  $C_4 \subset S_2^{-N-1}$ : it is well determined by  $C_1$  and  $C_4$ . Now  $C_2$  has to share its panel of type  $i$  with  $C_3$  and to be neither in  $S_2^{-1}$  nor in  $S_1^{\alpha_i(\lambda) - 1}$ ; so there are  $q_i - 1$  possibilities.



If  $1 \leq j \leq N - 1$ , then  $A_1$  contains  $S_2^0 = S_1^{\alpha_i(\lambda)}$ ,  $S_2^{-1} = S_1^{\alpha_i(\lambda)-1}$ ,  $\dots$ ,  $S_2^{-j} = S_1^{\alpha_i(\lambda)-j}$  but not  $S_2^{-j-1}, \dots, S_2^{-N-1}$ . So  $\rho_1(S_2^{-k}) = S_1^{\alpha_i(\lambda)-2j+k}$ , for  $k \geq j$ . The image of the line segment  $[x_2, x_4] = [x_2, x_2 + \mu]$  under  $\rho_1$  is  $\rho_1([x_2, x_4]) = [x_2, x_2 + (j/N)r_i(\mu)] \cup [x_2 + (j/N)r_i(\mu), x_2 + (j/N)r_i(\mu) + ((N-j)/N)\mu]$ . As  $N = \alpha_i(\mu)$  and  $r_i(\mu) = \mu - N\alpha_i^\vee$ , this means that  $\rho_1(C_4) = C_2 + \mu - j\alpha_i^\vee$ . When  $C_1$  and  $C_4$  are fixed with this  $W$ -distance, we have to count the number of possible  $C_2$ . As  $S_1^0, \dots, S_1^{\alpha_i(\lambda)-j-1}, S_2^{-j-1}, \dots, S_2^{-N-1}$  are well determined by  $C_1, C_4$ , we have to count the possibilities for  $(S_1^{\alpha_i(\lambda)-j}, \dots, S_1^{\alpha_i(\lambda)})$ . As above there are  $q_i - 1$  possibilities for  $S_1^{\alpha_i(\lambda)-j}$  (or  $q'_i - 1$  if  $j$  is odd) and then  $q'_i$  (or  $q_i$ ) possibilities for  $S_1^{\alpha_i(\lambda)-j+1}$ , etc. . Finally the total number of possibilities is  $(q_i - 1)q'_i q_i q'_i \dots$  or  $(q'_i - 1)q_i q'_i q_i \dots$  (according to  $j$  being even or odd) with  $j + 1$  terms in the product. The last factor is necessarily  $q_i$ , so this total number is  $(q_i^{*j+1} - q_i^{*j})$ .



It is convenient to look at the cases  $j = N$  or  $j = N + 1$  simultaneously. This means that  $S_2^{-N} = S_1^{\alpha_i(\lambda)-N}$  is in  $A_1$ ; in particular the panel  $F_4^i$  of type  $i$  of  $C_4$  is in  $A_1$ , in the wall  $\{x \mid \alpha_i(x) = \alpha_i(\lambda) - N\}$ . More precisely  $F_4^i$  is the panel of type  $i$  of  $C'_4 = C_1 + \lambda + r_i(\mu) \subset A_1$ . This means that  $(T_{\lambda+r_i(\mu)} * T_i)(C_1, C_4) \geq 1$ . Conversely if  $C_1, C_4$  are fixed satisfying this condition, we can find  $C_2, C_3$  with the required  $W$ -distances. We have now to count the number of possibilities for  $C_2, C_3$  i.e. for  $C_2$  or for  $(S_1^{\alpha_i(\lambda)-N}, \dots, S_1^{\alpha_i(\lambda)})$ . The number of possibilities for  $S_1^{\alpha_i(\lambda)-N}$  is exactly  $(T_{\lambda+r_i(\mu)} * T_i)(C_1, C_4)$ . Then the number of possibilities



for  $S_1^{\alpha_i(\lambda)-N+1}, \dots, S_1^{\alpha_i(\lambda)}$  is alternatively  $q_i$  or  $q'_i$ . Finally the total number of possibilities for  $C_2$  is  $q_i^{*N} (T_{\lambda+r_i(\mu)} * T_i)(C_1, C_4)$  (as, when  $N$  is odd,  $q_i = q'_i$ ).  $\square$

## 5 New basis

We suppose  $\mathbb{Z} \subset R$  and each  $q_i, q'_i$  invertible in  $R$ . By previous calculations,  ${}^I\mathcal{H}_R$  is a free right  $\mathcal{H}_R(W^v)$ -module with basis  $\{T_\lambda \mid \lambda \in Y^+\}$ . We say that  $T_\lambda * H$  with  $\lambda \in Y^{++}$  and  $H \in \mathcal{H}_R(W^v)$  is of degree  $\lambda$ .

For  $i \in I$  and  $\Omega$  a subset of  $V$ , we write  $c(i)(\Omega)$  the convex hull of  $\Omega \cup r_i(\Omega)$ . For  $(i_1, i_2, \dots, i_h) \in I^h$  and  $(\lambda_0, \lambda_1, \dots, \lambda_h) \in (Y^{++})^{h+1}$ , we define  $D(i_h)(\lambda_{h-1}, \lambda_h) = \lambda_{h-1} + c(i_h)(\lambda_h)$  and, by induction for  $k$  from  $h-1$  to  $1$ ,  $D(i_k, \dots, i_h)(\lambda_{k-1}, \lambda_k, \dots, \lambda_h) = \lambda_{k-1} + c(i_k)(D(i_{k+1}, \dots, i_h)(\lambda_k, \lambda_{k+1}, \dots, \lambda_h))$  (of course  $c(i_h)(\lambda_h) = c(i_h)(\{\lambda_h\})$ ).

**Lemma 5.1.** *With notation as above,*

- a) if  $\lambda'_{h-1} \in D(i_h)(\lambda_{h-1}, \lambda_h)$ , then  
 $D(i_k, \dots, i_{h-2}, i_{h-1})(\lambda_{k-1}, \lambda_k, \dots, \lambda_{h-2}, \lambda'_{h-1}) \subset D(i_k, \dots, i_{h-1}, i_h)(\lambda_{k-1}, \lambda_k, \dots, \lambda_{h-1}, \lambda_h)$ ;
- b) if  $r_{i_1} r_{i_2} \cdots r_{i_h}$  is a reduced word in  $W^v$  and  $\lambda \in D(i_1, \dots, i_h)(\lambda_0, \lambda_1, \dots, \lambda_h)$ , then  
 $\lambda_0 + r_{i_1}(\lambda_1) + r_{i_1} r_{i_2}(\lambda_2) + \cdots + r_{i_1} r_{i_2} \cdots r_{i_h}(\lambda_h) \leq_{Q_{\mathbb{R}}^v} \lambda$ .

**Remark.** If the expression  $r_{i_1} r_{i_2} \cdots r_{i_h}$  is reduced, we get  $D(i_1, \dots, i_h)(0, 0, \dots, 0, \lambda_h) = \text{conv}(\{w(\lambda_h) \mid w \leq_B r_{i_1} r_{i_2} \cdots r_{i_h}\})$  where  $\leq_B$  denotes the Bruhat order.

*Proof.* The proof of a) is easy.

b) We have  $D(i_1, \dots, i_h)(\lambda_0, \lambda_1, \dots, \lambda_h) \subset \lambda_0 + c(i_1)(\lambda_1) + c(i_1, i_2)(\lambda_2) + \cdots + c(i_1, i_2, \dots, i_h)(\lambda_h)$ , with  $c(i_1, i_2, \dots, i_k)(\lambda_k) = c(i_1)(c(i_2)(\dots(c(i_k)(\lambda_k))\dots)) = \text{conv}(\{w(\lambda_k) \mid w \leq_B r_{i_1} r_{i_2} \cdots r_{i_k}\})$  where  $0 \leq k \leq h$  and  $\leq_B$  denotes the Bruhat order. For  $w \leq_B r_{i_1} r_{i_2} \cdots r_{i_k}$ , there is a sequence  $w = w_0, w_1, \dots, w_r = r_{i_1} r_{i_2} \cdots r_{i_k}$  such that, for each  $1 \leq i < r$ , there is a reduced decomposition  $w_{i+1} = r_{j_1} r_{j_2} \cdots r_{j_{p-1}} r_{j_p} r_{j_{p+1}} \cdots r_{j_q}$  with  $w_i = r_{j_1} r_{j_2} \cdots r_{j_{p-1}} r_{j_p} \cdots r_{j_q}$ . Then  $w_i(\lambda_k) = w_{i+1}(\lambda_k) + \alpha_{j_p}(r_{j_{p+1}} \cdots r_{j_q}(\lambda_k)) r_{j_1} r_{j_2} \cdots r_{j_{p-1}}(\alpha_{j_p}^v)$  and  $Q_+^v$  contains the term  $(r_{j_q} \cdots r_{j_{p+1}}(\alpha_{j_p}^v))(\lambda_k) r_{j_1} r_{j_2} \cdots r_{j_{p-1}}(\alpha_{j_p}^v)$  by minimality of the expressions  $r_{j_1} r_{j_2} \cdots r_{j_{p-1}} r_{j_p}$  and  $r_{j_q} \cdots r_{j_{p+1}} r_{j_p}$ . So we get by induction that  $w(\lambda_k) \geq_{Q^v} r_{i_1} r_{i_2} \cdots r_{i_k}(\lambda_k)$  and  $w(\mu) \geq_{Q_{\mathbb{R}}^v} r_{i_1} r_{i_2} \cdots r_{i_k}(\lambda_k)$  for any  $\mu \in c(i_1, \dots, i_k)(\lambda_k)$ . The expected result is now clear.  $\square$

**Proposition 5.2.** *For any expression  $H_k = T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * T_{i_2} * \cdots * T_{\lambda_{k-1}} * T_{i_k} * T_{\lambda_k} * H$  with  $\lambda_i \in Y^{++}$ ,  $H \in \mathcal{H}_{\mathbb{Z}}(W^v)$  and  $\mu \in Y^{++}$  sufficiently great, the product  $T_\mu * H_k$  may be written as a  $R$ -linear combination of elements  $T_\nu * H_\nu$  with  $\nu \in \mu + D(i_1, \dots, i_k)(\lambda_0, \lambda_1, \dots, \lambda_k)$  and  $H_\nu \in \mathcal{H}_R(W^v)$ .*

*Moreover, if  $r_{i_1} r_{i_2} \cdots r_{i_k}$  is a reduced word and  $\nu_0 = \mu + \lambda_0 + r_{i_1}(\lambda_1) + r_{i_1} r_{i_2}(\lambda_2) + \cdots + r_{i_1} r_{i_2} \cdots r_{i_k}(\lambda_k)$ , then  $H_{\nu_0} \in R^* T_{i_1} * T_{i_2} * \cdots * T_{i_k} * H$  and more precisely the constant in  $R^*$  is a primitive monomial in the  $q_i, q'_i$ . Further,  $H_{\nu_0}$  is the only  $H_\nu$  in  $R T_{i_1} * T_{i_2} * \cdots * T_{i_k} * H$ .*

**N.B.** So one may write  $T_\mu * H_k = \sum_{\nu, w} a_{\nu, w} T_\nu * T_w$ , with  $a_{\nu, w} \in R$ ,  $\nu$  running in  $\mu + D(i_1, \dots, i_k)(\lambda_0, \lambda_1, \dots, \lambda_k)$  and  $w$  in  $W$ . Moreover we get from the following proof, that each  $a_{\nu, w}$  is a Laurent polynomial in the parameters  $q_i, q'_i$ , with coefficients in  $\mathbb{Z}$ ; these polynomials depend only on the expression  $H_k$ , on  $\mathbb{A}$  and on  $W$ .

*Proof.* The proof is easy in a special case (I).

(I). We say that the expression of  $H_k$  is **normalizable of length  $k$**  when it satisfies the following properties:

(i)  $\lambda_{k-1} - \lambda_k \in Y^{++}$ ,

(ii) For all  $h$  from  $k$  to  $2$ ,  $\lambda_{h-2} - D(i_h, \dots, i_k)(\lambda_{h-1}, \lambda_h, \dots, \lambda_k) \in \overline{C_f^v}$ .

For such an expression, we write  $D(H_k) = D(i_1, \dots, i_k)(\lambda_0, \lambda_1, \dots, \lambda_k)$ .

We will then prove that  $T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * T_{i_2} * \dots * T_{\lambda_{k-1}} * T_{i_k} * T_{\lambda_k} * H$  is a  $\mathbb{Z}[q_i, q'_i]$ -linear combination of normalizable elements  $H'_{k-1}$  of length  $k-1$  such that  $D(H'_{k-1}) \subset D(H_k)$ .

Using the fact  $\lambda_{k-1} - \lambda_k \in Y^{++}$  and Theorem 4.8, or (BLT) we have :

$$(E) \quad H_k = q_{i_k}^{*(\alpha_{i_k}(\lambda_k))} T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * \dots * T_{i_{k-1}} * T_{\lambda_{k-1}}^{*(\alpha_{i_k}(\lambda_k))} * (T_{i_k} * H) \\ + \sum_{h=0}^{\alpha_{i_k}(\lambda_k)-1} (q_i^{*(h+1)} - q_i^{*(h)}) T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * \dots * T_{i_{k-1}} * T_{\lambda_{k-1}}^{*(h)} * H$$

with  $\lambda_{k-1}^{(h)} = \lambda_{k-1} + \lambda_k - h\alpha_{i_k}^\vee$ , in particular  $\lambda_{k-1}^{(\alpha_{i_k}(\lambda_k))} = \lambda_{k-1} + r_{i_k}(\lambda_k)$ . Let us consider, for each  $0 \leq h \leq \alpha_{i_k}(\lambda_k)$ ,  $\lambda'_i = \lambda_i$  for  $i \leq k-2$  and  $\lambda'_{k-1} = \lambda_{k-1}^{(h)}$ , then  $(\lambda'_0, \dots, \lambda'_{k-1})$  satisfies  $\lambda'_{k-2} - \lambda'_{k-1} \in Y^{++}$  (by (ii) for  $h=k$  and  $\lambda'_{k-1} \in D(i_k)(\lambda_{k-1}, \lambda_k)$ ), and, for all  $h$  from  $k-1$  to  $2$ ,  $\lambda'_{h-2} - D(i_h, \dots, i_{k-1})(\lambda'_{h-1}, \dots, \lambda'_{k-1}) \in \overline{C_f^v}$ . This last result comes from (ii)  $\lambda'_{h-2} - D(i_h, \dots, i_k)(\lambda_{h-1}, \lambda_h, \dots, \lambda_k) \in \overline{C_f^v}$  and the inclusion  $D(i_h, \dots, i_{k-1})(\lambda'_{h-1}, \lambda'_h, \dots, \lambda'_{k-1}) \subset D(i_h, \dots, i_k)(\lambda_{h-1}, \lambda_h, \dots, \lambda_k)$  (by Lemma 5.1 a)). We have  $T_{i_k} * H \in \mathcal{H}_R(W^v)$ , so every term of the right hand side of (E) is a normalizable element  $H'_{k-1}$  of length  $k-1$  with  $D(H'_{k-1}) \subset D(H_k)$ .

By induction on each term, after  $k$  steps, we obtain  $H_k$  as a  $\mathbb{Z}[q_i, q'_i]$ -linear combination of  $T_\nu * H_\nu$  with  $\nu \in D(H_k)$  and  $H_\nu \in \mathcal{H}_R(W^v)$ .

Moreover, if the decomposition  $r_{i_1} r_{i_2} \dots r_{i_k}$  is reduced, we take  $\nu_0 = \lambda_0 + r_{i_1}(\lambda_1) + r_{i_1} r_{i_2}(\lambda_2) + \dots + r_{i_1} r_{i_2} \dots r_{i_k}(\lambda_k)$  and look more carefully at the decomposition (E). For  $0 \leq h < \alpha_{i_k}(\lambda_k)$ , we have  $\nu_0 \notin D(T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * \dots * T_{\lambda_{k-1}}^{*(h)} * H) \subset D(H_k)$  by Lemma 5.1b). Indeed, if  $\lambda \in D(T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * \dots * T_{\lambda_{k-1}}^{*(h)} * H)$ , then, by minimality of  $r_{i_1} r_{i_2} \dots r_{i_k}$ ,

$\nu_0 \leq_{Q^v} \nu_0^{(h)} \leq_{Q^v} \lambda$  with  $\nu_0^{(h)} = \lambda_0 + r_{i_1}(\lambda_1) + r_{i_1} r_{i_2}(\lambda_2) + \dots + r_{i_1} r_{i_2} \dots r_{k-1}(\lambda_{i_{k-1}}^{(h)}) \neq \nu_0$ . So the unique term of degree  $\nu_0$  of the final decomposition comes from the term of first kind (i.e. obtained like the first term of the right hand side of (E)) in every step of the reduction and is also the only term containing all the  $T_{i_j}$ . And so, we prove that, in front of the term  $T_{\nu_0} * T_{i_1} * T_{i_2} * \dots * T_{i_k} * H$  obtained for  $\nu_0$  the constant is equal to the primitive monomial  $C = q_{i_k}^{*(\alpha_{i_k}(\lambda_k))} q_{i_{k-1}}^{*(\alpha_{i_k}(\lambda_{k-1} + r_{i_k}(\lambda_k))} \dots q_{i_1}^{*(\alpha_{i_1}(\lambda_1 + r_{i_2}(\lambda_2) + \dots + r_{i_2} \dots r_{i_k}(\lambda_k))}$ .

Let us consider now the general case and first prove the following result (II).

(II) If  $H_k = T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * T_{i_2} * \dots * T_{\lambda_{k-1}} * T_{i_k} * T_{\lambda_k} * H$  with  $\lambda_i \in Y^{++}$ ,  $H \in \mathcal{H}_R(W^v)$ , we can choose  $\mu_0 \in Y^{++}$  such that  $T_{\mu_0} * H_k$  can be written as a  $R$ -linear combination of normalizable expressions  $H'_k$  of length  $\leq k$  and with  $D(H'_k) \subset \mu_0 + D(i_1, \dots, i_k)(\lambda_0, \lambda_1, \dots, \lambda_k)$ .

We prove this result for  $H_{k-h} = T_{\lambda_h} * T_{i_{h+1}} * T_{\lambda_{h+1}} * \dots * T_{\lambda_{k-1}} * T_{i_k} * T_{\lambda_k} * H$  by decreasing induction on  $0 \leq h \leq k-1$ . For  $h = k-1$ , we have  $H_1 = T_{\lambda_{k-1}} * T_{i_k} * T_{\lambda_k} * H$ . Choose  $\mu_{k-1} = \lambda_k$ , then  $T_{\mu_{k-1}} * H_1$  is normalizable of length 1 and  $D(T_{\mu_{k-1}} * H_1) \subset \mu_{k-1} + D(i_k)(\lambda_{k-1}, \lambda_k)$ .

Let  $0 \leq h \leq k-2$  and suppose that we can choose  $\mu_{h+1} \in Y^{++}$  such that  $T_{\mu_{h+1}} * H_{k-(h+1)} = T_{\mu_{h+1}} * T_{\lambda_{h+1}} * T_{i_{h+2}} * \dots * T_{i_k} * T_{\lambda_k} * H$  can be written as a  $R$ -linear combination of normalizable expressions  $H'_{k-(h+1)}$  of length  $\leq k - (h+1)$  and with  $D(H'_{k-(h+1)}) \subset \mu_{h+1} +$

$D(i_{h+2}, \dots, i_k)(\lambda_{h+1}, \dots, \lambda_k)$ . Let us write these normalizable expressions  $H'_{k-(h+1)} = T_{\lambda'_0} * T_{i'_1} * T_{\lambda'_1} * T_{i'_2} * \dots * T_{i'_{k'}} * T_{\lambda'_{k'}} * H'$ , where  $k' \leq k - (h + 1)$  and  $(\lambda'_0, \dots, \lambda'_{k'})$  satisfies (i) and (ii). Consider  $\mu_h^{min} \in Y^{++}$  such that  $\mu_h^{min} - D(i'_1, \dots, i'_{k'})(\lambda'_0, \lambda'_1, \dots, \lambda'_{k'}) \subset \overline{C_f^v}$  for all these expressions. We take  $\mu_h = \mu_h^{min} + 2\mu_{h+1} + r_{i_{h+1}}(\mu_{h+1})$ , then  $T_{\mu_h} * H_{k-h} = T_{\mu_h} * T_{\lambda_h} * T_{i_{h+1}} * H_{k-(h+1)} = T_{\mu_h^{min} + \lambda_h + \mu_{h+1}} * T_{\mu_{h+1} + r_{i_{h+1}}(\mu_{h+1})} * T_{i_{h+1}} * H_{k-(h+1)}$ .

By (BLT), we have:

$$(E') \quad \begin{aligned} & q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\mu_{h+1}))} T_{\mu_h} * H_{k-h} = T_{\mu_h^{min} + \lambda_h + 2\mu_{h+1}} * T_{i_{h+1}} * T_{\mu_{h+1}} * H_{k-(h+1)} \\ & - \sum_{j=0}^{\alpha_{i_{h+1}}(\mu_{h+1})-1} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)}) T_{\lambda_h + \mu_h^{min} + 2\mu_{h+1} - j\alpha_{i_{h+1}}^\vee} * T_{\mu_{h+1}} * H_{k-(h+1)}. \end{aligned}$$

The choice of  $\mu_h^{min}$  and the hypothesis on  $T_{\mu_{h+1}} * H_{k-(h+1)}$  allow us to say that we have written  $T_{\mu_h} * H_{k-h}$  as a  $R$ -linear combination of normalizable expressions  $H'_{k-h}$  of length  $\leq k - h$  with  $D(H'_{k-h}) \subset \mu_h^{min} + 2\mu_{h+1} + D(i_{h+1}, \dots, i_k)(\lambda_h, \lambda_{h+1} + \mu_{h+1}, \dots, \lambda_k)$  for the first term and  $D(H'_{k-h}) \subset \mu_h^{min} + 2\mu_{h+1} - j\alpha_{i_{h+1}}^\vee + D(i_{h+1}, \dots, i_k)(\lambda_h, \lambda_{h+1} + \mu_{h+1}, \dots, \lambda_k)$  for the others. We need to be more precise to prove  $D(H'_{k-h}) \subset \mu_h + D(i_{h+1}, \dots, i_k)(\lambda_h, \dots, \lambda_k)$ .

By the part I) of this proof and the hypothesis on  $T_{\mu_{h+1}} * H_{k-(h+1)}$  we know that this element can be written  $\sum_{\Lambda} c_{\Lambda} T_{\Lambda} * H^{\Lambda}$  with  $\Lambda = \mu_{h+1} + \Lambda'$  where  $\Lambda' \in D(i_{h+2}, \dots, i_k)(\lambda_{h+1}, \dots, \lambda_k)$   $c_{\Lambda} \in R$ ,  $H^{\Lambda} \in \mathcal{H}_R(W^v)$ . The first term of the second member of (E') becomes :

$$T_{\mu_h^{min} + \lambda_h + 2\mu_{h+1}} * T_{i_{h+1}} * \left( \sum_{\Lambda} c_{\Lambda} T_{\Lambda} * H^{\Lambda} \right) = T_{\lambda_h + 2\mu_{h+1}} * \left( \sum_{\Lambda} c_{\Lambda} T_{\mu_h^{min}} * T_{i_{h+1}} * T_{\Lambda} * H^{\Lambda} \right).$$

By the condition on  $\mu_h^{min}$  and (BLT), we write it

$$\begin{aligned} & T_{\lambda_h + 2\mu_{h+1}} * \left( \sum_{\Lambda} c_{\Lambda} \left( q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\Lambda))} T_{\mu_h^{min} + r_{i_{h+1}}(\Lambda)} * T_{i_{h+1}} * H^{\Lambda} \right) \right) \\ & + T_{\lambda_h + 2\mu_{h+1}} * \left( \sum_{\Lambda} c_{\Lambda} \left( \sum_{j=0}^{\alpha_{i_{h+1}}(\Lambda)-1} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)}) T_{\mu_h^{min} + \Lambda - j\alpha_{i_{h+1}}^\vee} * H^{\Lambda} \right) \right). \end{aligned}$$

The first term of this sum will be  $\sum_{\Lambda} c_{\Lambda} q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\Lambda))} T_{\lambda_h + 2\mu_{h+1} + \mu_h^{min} + r_{i_{h+1}}(\Lambda)} * T_{i_{h+1}} * H^{\Lambda}$  and  $\lambda_h + 2\mu_{h+1} + \mu_h^{min} + r_{i_{h+1}}(\Lambda) = \lambda_h + 2\mu_{h+1} + \mu_h^{min} + r_{i_{h+1}}(\mu_{h+1}) + r_{i_{h+1}}(\Lambda') = \lambda_h + \mu_h + r_{i_{h+1}}(\Lambda')$  is an element of  $\lambda_h + \mu_h + r_{i_{h+1}}(D(i_{h+2}, \dots, i_k)(\lambda_{h+1}, \dots, \lambda_k))$  which is included, as expected, in  $\mu_h + D(i_{h+1}, i_{h+2}, \dots, i_k)(\lambda_h, \lambda_{h+1}, \dots, \lambda_k)$ .

The second term is  $\sum_{\Lambda} c_{\Lambda} \left( \sum_{j=0}^{\alpha_{i_{h+1}}(\Lambda)-1} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)}) T_{\lambda_h + 2\mu_{h+1} + \mu_h^{min} + \Lambda - j\alpha_{i_{h+1}}^\vee} * H^{\Lambda} \right)$ . And we see that in fact (E') becomes :

$$\begin{aligned}
(E'') \quad q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\mu_{h+1}))} T_{\mu_h} * H_{k-h} &= \sum_{\Lambda} c_{\Lambda} q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\Lambda))} T_{\lambda_h + \mu_h + r_{i_{h+1}}(\Lambda')} * T_{i_{h+1}} * H^{\Lambda} \\
&+ \sum_{\Lambda} c_{\Lambda} \sum_{j=0}^{\alpha_{i_{h+1}}(\Lambda)-1} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)}) T_{\lambda_h + 2\mu_{h+1} + \mu_h^{\min} + \Lambda - j\alpha_{i_{h+1}}^{\vee}} * H^{\Lambda} \\
&- \sum_{\Lambda} c_{\Lambda} \sum_{j=0}^{\alpha_{i_{h+1}}(\mu_{h+1})-1} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)}) T_{\lambda_h + \mu_h^{\min} + 2\mu_{h+1} - j\alpha_{i_{h+1}}^{\vee}} * T_{\Lambda} * H^{\Lambda} \quad . \\
&= \sum_{\Lambda} c_{\Lambda} q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\Lambda))} T_{\lambda_h + \mu_h + r_{i_{h+1}}(\Lambda')} * T_{i_{h+1}} * H^{\Lambda} \\
&+ \sum_{\Lambda} c_{\Lambda} \varepsilon_{\Lambda} \sum_j (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)}) T_{\lambda_h + 2\mu_{h+1} + \mu_h^{\min} + \Lambda - j\alpha_{i_{h+1}}^{\vee}} * H^{\Lambda}
\end{aligned}$$

where  $\alpha_{i_{h+1}}(\mu_{h+1}) \leq j \leq \alpha_{i_{h+1}}(\Lambda) - 1$  and  $\varepsilon_{\Lambda} = 1$  if  $\alpha_{i_{h+1}}(\mu_{h+1}) \leq \alpha_{i_{h+1}}(\Lambda)$  (i.e.  $\alpha_{i_{h+1}}(\Lambda') \geq 0$ ) and  $\alpha_{i_{h+1}}(\Lambda) \leq j \leq \alpha_{i_{h+1}}(\mu_{h+1}) - 1$  and  $\varepsilon_{\Lambda} = -1$  if  $\alpha_{i_{h+1}}(\mu_{h+1}) \geq \alpha_{i_{h+1}}(\Lambda)$  (i.e.  $\alpha_{i_{h+1}}(\Lambda') \leq 0$ ). For these values of  $j$ , by using  $\Lambda - j\alpha_{i_{h+1}}^{\vee} = r_{i_{h+1}}(\mu_{h+1}) + j'\alpha_{i_{h+1}}^{\vee} + \Lambda'$  with  $j' = \alpha_{i_{h+1}}(\mu_{h+1}) - j$ , we have  $\lambda_h + 2\mu_{h+1} + \mu_h^{\min} + \Lambda - j\alpha_{i_{h+1}}^{\vee} = \lambda_h + \mu_h + j'\alpha_{i_{h+1}}^{\vee} + \Lambda'$ . If  $\alpha_{i_{h+1}}(\mu_{h+1}) \leq \alpha_{i_{h+1}}(\Lambda)$ ,  $\alpha_{i_{h+1}}(\mu_{h+1}) - \alpha_{i_{h+1}}(\Lambda) + 1 \leq j' \leq 0$  that is  $-\alpha_{i_{h+1}}(\Lambda') + 1 \leq j' \leq 0$ . If  $\alpha_{i_{h+1}}(\mu_{h+1}) \geq \alpha_{i_{h+1}}(\Lambda)$ , then  $\alpha_{i_{h+1}}(\mu_{h+1}) - \alpha_{i_{h+1}}(\Lambda) \geq j' \geq 1$  that is  $-\alpha_{i_{h+1}}(\Lambda') \geq j' \geq 1$ . In all cases,  $j'\alpha_{i_{h+1}}^{\vee} + \Lambda'$  is between  $\Lambda'$  and  $r_{i_{h+1}}(\Lambda')$  and so, as expected,  $\lambda_h + 2\mu_{h+1} + \mu_h^{\min} + \Lambda - j\alpha_{i_{h+1}}^{\vee} \in \mu_h + D(i_{h+1}, i_{h+2}, \dots, i_k)(\lambda_h, \lambda_{h+1}, \dots, \lambda_k)$ .

So we have proved that  $T_{\mu_0} * H_k$  can be written as a  $R$ -linear combination of normalizable expressions  $H'_k$  of length  $\leq k$  and with  $D(H'_k) \subset \mu_0 + D(i_1, \dots, i_k)(\lambda_0, \lambda_1, \dots, \lambda_k)$ . By the I) of the proof we can write it as a  $R$ -linear combination of elements  $T_{\nu} * H_{\nu}$  with  $\nu \in \mu_0 + D(i_1, \dots, i_k)(\lambda_0, \lambda_1, \dots, \lambda_k)$  and  $H_{\nu} \in \mathcal{H}_R(W^{\nu})$ .

Like in I) we can say, if moreover the decomposition  $r_{i_1} r_{i_2} \cdots r_{i_k}$  is reduced, that only the term  $\sum_{\Lambda} c_{\Lambda} q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\Lambda))} T_{\lambda_h + 2\mu_{h+1} + \mu_h^{\min} + r_{i_{h+1}}(\Lambda')} * T_{i_{h+1}} * H^{\Lambda}$  (which contains  $T_{i_{h+1}}$ ) in  $(E'')$  can give us a term of lowest degree  $\mu_h + \lambda_h + r_{i_{h+1}}(\lambda_{h+1}) + \cdots + r_{i_{h+1}} \cdots r_{i_k}(\lambda_k)$ . More precisely, the term of lowest degree comes from the term with  $\Lambda_0 = \mu_{h+1} + \lambda_{h+1} + r_{i_{h+2}}(\lambda_{h+2}) + \cdots + r_{i_{h+2}} \cdots r_{i_k}(\lambda_k)$  for which we have  $\alpha_{i_{k+1}}(\Lambda_0) \geq \alpha_{i_{k+1}}(\mu_{h+1})$ . So, it's easy to see by induction that the coefficient of that term is a primitive monomial in the  $q_i, q'_i$ .  $\square$

**Corollary 5.3.** a) For  $\lambda \in Y^+$  and  $\mu \in Y^{++}$  sufficiently great, we have

$$T_{\mu} * T_{\lambda} = \sum_{\lambda \leq_{Q^{\vee}} \nu \leq_{Q^{\vee}} \lambda^{++}} T_{\mu+\nu} * H^{\nu} \text{ with } H^{\nu} \in \mathcal{H}_R(W^{\nu}).$$

b) More precisely, if  $H^{\nu} \neq 0$  then  $\mu + \nu \in Y^{++}$  and  $\nu$  is in the convex hull  $\text{conv}(W^{\nu}.\lambda^{++})$  of  $W^{\nu}.\lambda^{++}$  or better in the convex hull  $\text{conv}(W^{\nu}.\lambda^{++}, \geq \lambda)$  of all  $w'.\lambda^{++}$  for  $w' \leq_B w_{\lambda}$  (with  $w_{\lambda}$  the smallest element of  $W^{\nu}$  such that  $\lambda = w_{\lambda}.\lambda^{++}$ ).

c) For  $\nu = \lambda$ ,  $H^{\lambda}$  is a strictly positive integer  $a_{\lambda}$  which may be written as a primitive monomial in  $q_i, q'_i$ ,  $i \in I$  (depending only on  $\mathbb{A}$ ).

d) In a) above, we may write  $H^{\nu} = \sum_{w \in W^{\nu}} a_{\mu, \lambda}^{\nu, w} T_w$  and, then each  $a_{\mu, \lambda}^{\nu, w}$  is a Laurent polynomial in the parameters  $q_i, q'_i$  with coefficients in  $\mathbb{Z}$ , depending only on  $\mathbb{A}$  and  $W$ .

*Proof.* Only the result c) is new (cf. Propositions 2.3 and 2.4), and we already saw that the constant term in  $H^{\lambda}$  is in  $\mathbb{Z}_{>0}$ . We have to prove that  $H^{\lambda} \in \mathcal{H}_R(W^{\nu})$  is actually a constant (for  $\mu$  sufficiently great). Write  $\lambda = w_{\lambda}(\lambda^{++})$  (with  $w_{\lambda}$  minimal in  $W^{\nu}$  for this property), choose a minimal decomposition  $w_{\lambda} = r_{i_1} r_{i_2} \cdots r_{i_k}$ , by corollary 4.3 we have  $T_{\lambda} = T_{i_1} * T_{i_2} * \cdots * T_{i_k} * T_{\lambda^{++}} * T_{i_k}^{-1} * \cdots * T_{i_1}^{-1}$ . Then, by Proposition 5.2, for  $\mu$  great,

$T_\mu * T_\lambda$  may be written as a  $R$ -linear combination of elements  $T_{\mu+\nu} * (H_1^\nu * T_{i_k}^{-1} * \cdots * T_{i_1}^{-1})$  with  $\nu \in D(i_1, \dots, i_k)(0, \dots, 0, \lambda^{++})$  and  $H_1^\nu \in \mathcal{H}_R(W^\nu)$  with term of lowest degree  $\nu_0 = \lambda$ . Moreover  $H^\lambda = H_1^\lambda * T_{i_k}^{-1} * \cdots * T_{i_1}^{-1}$  is a primitive monomial in the  $q_i, q'_i$ .

To prove d), we remark that  $T_{i_k}^{-1} * \cdots * T_{i_1}^{-1}$  may be written  $\sum_{w \in W^v} a_w T_w$  with  $a_w \in \mathbb{Z}[(q_i^{\pm 1})_{i \in I}]$  and we apply 5.2 with  $H = T_w$ .  $\square$

**Corollary 5.4.** *In  ${}^I\mathcal{H}_R$ , for  $\mu \in Y^{++}$  the left multiplication by  $T_\mu$  is injective.*

*Proof.* As  $T_{\mu_1+\mu_2} = T_{\mu_1} * T_{\mu_2}$  for  $\mu_1, \mu_2 \in Y^{++}$ , we may assume  $\mu$  sufficiently great. Let  $H \in {}^I\mathcal{H}_R \setminus \{0\}$ . We may write  $H = \sum_{j \in J} T_{\lambda_j} * H^j$  with  $\lambda_j \in Y^+$  and  $0 \neq H^j \in \mathcal{H}_R(W^\nu)$ . We choose  $\lambda_{j_0}$  minimal among the  $\lambda_j$  for  $\leq_{Q^\vee}$ . Then  $T_\mu * H = \sum_{j \in J} \sum_{\mu+\lambda_j \leq_{Q^\vee} \nu_j} T_{\nu_j} * H^{\nu_j, j} * H^j$ . Hence  $\nu_{j_0} = \mu + \lambda_{j_0}$  is minimal for  $\leq_{Q^\vee}$  and  $H^{\nu_{j_0}, j_0}$  is a monomial in  $q_i, q'_i$ ; so  $H^{\nu_{j_0}, j_0} * H^{j_0} \neq 0$  and  $T_\mu * H \neq 0$ .  $\square$

**Theorem 5.5.** 1) *For any  $\lambda \in Y^+$ , there is a unique  $X^\lambda \in {}^I\mathcal{H}_R$  such that: for all  $\mu \in Y^{++}$  with  $\lambda + \mu \in Y^{++}$ , we have  $T_\mu * X^\lambda = T_{\lambda+\mu}$ .*

2) *More precisely,*

$$X^\lambda = b_\lambda T_\lambda + \sum_{\nu} T_\nu * H^\nu,$$

where  $H^\nu \in \mathcal{H}_R(W^\nu)$ ,  $\nu \in \text{conv}(W^v \cdot \lambda^{++}, \geq \lambda) \setminus \{\lambda\}$  and  $b_\lambda$  is a primitive monomial in  $q_i^{-1}, q'_i^{-1}$ .

3) *For  $\lambda \in Y^{++}$ , we have  $X^\lambda = T_\lambda$ . For  $\lambda, \lambda' \in Y^+$ ,  $X^\lambda * X^{\lambda'} = X^{\lambda+\lambda'} = X^{\lambda'} * X^\lambda$ .*

**Remarks. :**

a) We have two bases for the free right  $\mathcal{H}_R(W^v)$ -module  ${}^I\mathcal{H}_R$ ,  $\{T_\lambda \mid \lambda \in Y^+\}$  and  $\{X^\lambda \mid \lambda \in Y^+\}$ . The change of bases matrix is triangular (for the order  $\geq_{Q^\vee}$ ) with diagonal coefficients primitive monomials in  $q_i^{-1}, q'_i^{-1}$ . From 5.3.d we get that all coefficients of this matrix are Laurent polynomials in the parameters  $q_i, q'_i$ , with coefficients in  $\mathbb{Z}$ , depending only on  $\mathbb{A}$  and on  $W$ .

b) By 1) above and Corollary 5.4, it is clear that the left multiplication by  $X^\lambda$  is injective, for any  $\lambda \in Y^+$ .

*Proof.* By Corollary 5.4, the uniqueness is clear and 3) follows from the relation  $T_\lambda * T_\mu = T_{\lambda+\mu}$  of the Theorem 2.5. We have just to prove 1) and 2) for a  $\mu \in Y^{++}$  (chosen sufficiently great).

We argue by induction on the height  $ht(\lambda^{++} - \lambda)$  of  $\lambda^{++} - \lambda$  with respect to the free family  $(\alpha_i^\vee)$  in  $Q^\vee$ . When the height is 0,  $\lambda = \lambda^{++}$  and  $X^\lambda = T_\lambda$ . By Corollary 5.3, we write  $T_\mu * T_\lambda = a_\lambda T_{\mu+\lambda} + \sum_{\lambda \leq_{Q^\vee} \nu \leq_{Q^\vee} \lambda^{++}; \lambda \neq \nu} T_{\mu+\nu} * H^\nu$  with  $H^\nu \in \mathcal{H}_R(W^\nu)$  and  $\nu \in \text{conv}(W^v \cdot \lambda^{++})$

hence  $\nu^{++} \in \text{conv}(W^v \cdot \lambda^{++})$  (in particular  $\nu^{++} \leq_{Q^\vee} \lambda^{++}$ ). *cf.* Lemme 1.8.a).

So  $ht(\nu^{++} - \nu) < ht(\lambda^{++} - \lambda)$ . By induction and for  $\mu$  sufficiently great, we can consider the element  $X^\nu$  such that  $T_{\mu+\nu} = T_\mu * X^\nu$ . We can write it  $X^\nu = \sum_{\nu \leq_{Q^\vee} \nu' \leq_{Q^\vee} \nu^{++}} T_{\nu'} * H^{\nu', \nu}$

$$\begin{aligned} \text{and we may take } X^\lambda &= a_\lambda^{-1} T_\lambda - \left( \sum_{\lambda \leq_{Q^\vee} \nu \leq_{Q^\vee} \lambda^{++}; \lambda \neq \nu} X^\nu * H^\nu \right) \\ &= a_\lambda^{-1} T_\lambda - \left( \sum_{\lambda \leq_{Q^\vee} \nu \leq_{Q^\vee} \lambda^{++}; \lambda \neq \nu} \left( \sum_{\nu \leq_{Q^\vee} \nu' \leq_{Q^\vee} \nu^{++}} T_{\nu'} * H^{\nu', \nu} \right) * H^\nu \right). \end{aligned}$$

$\square$

**Proposition 5.6.** For  $\lambda \in Y^+$  and  $i \in I$  we have the following relations :

$$\begin{aligned} a) \text{ If } \alpha_i(\lambda) \geq 0, \text{ then } T_i * X^\lambda &= q_i^{*(\alpha_i(\lambda))} X^{r_i(\lambda)} * T_i + \sum_{h=0}^{\alpha_i(\lambda)-1} (q_i^{*(h+1)} - q_i^{*(h)}) X^{\lambda-h\alpha_i^\vee}. \\ b) \text{ If } \alpha_i(\lambda) < 0, \text{ then } T_i * X^\lambda &= \frac{1}{q_i^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda)} * T_i - \frac{1}{q_i^{*(-\alpha_i(\lambda))}} \sum_{h=\alpha_i(\lambda)}^{-1} \left( q_i^{*(-\alpha_i(\lambda)+h+1)} - \right. \\ &\left. q_i^{*(-\alpha_i(\lambda)+h)} \right) X^{\lambda-h\alpha_i^\vee} \end{aligned}$$

**N.B.** These relations are the Bernstein-Lusztig relations for the  $X^\lambda$ , (BLX) for short.

*Proof.* If  $\lambda \in Y^{++}$ , by Proposition 4.8 a) , we know that  $X^\lambda * T_i * X^\lambda = X^{\lambda+\lambda} * T_i$  when  $\alpha_i(\lambda) = 0$  and, when  $\alpha_i(\lambda) > 0$ ,  $X^\lambda * T_i * X^\lambda = q_i^{*\alpha_i(\lambda)} X^{\lambda+r_i(\lambda)} * T_i + (q_i^{*(\alpha_i(\lambda))} - q_i^{*(\alpha_i(\lambda)-1)}) X^{\lambda+\lambda-(\alpha_i(\lambda)-1)\alpha_i^\vee} + \dots + (q_i^{*2} - q_i) X^{\lambda+\lambda-\alpha_i^\vee} + (q_i - 1) X^{\lambda+\lambda}$ , so we have the result.

In the general case,  $\lambda \in Y^+$ , we write  $\lambda = \mu - \nu$  with  $\mu, \nu$  chosen in  $Y^{++}$ . By Theorem 5.5,  $X^\nu * X^\lambda = X^\mu$ . From (BLX) for  $X^\mu$  and  $X^\nu$ , we have :

$$T_i * X^\mu = q_i^{*(\alpha_i(\lambda+\nu))} X^{r_i(\lambda+\nu)} * T_i + \sum_{h=0}^{\alpha_i(\lambda+\nu)-1} (q_i^{*(h+1)} - q_i^{*(h)}) X^{\nu+\lambda-h\alpha_i^\vee}$$

which can also be written

$$\begin{aligned} T_i * X^{\nu+\lambda} &= (T_i * X^\nu) * X^\lambda = (q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * T_i + \sum_{h=0}^{\alpha_i(\nu)-1} (q_i^{*(h+1)} - q_i^{*(h)}) X^{\nu-h\alpha_i^\vee}) * X^\lambda \\ &= q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * T_i * X^\lambda + \sum_{h=0}^{\alpha_i(\nu)-1} (q_i^{*(h+1)} - q_i^{*(h)}) X^{\nu+\lambda-h\alpha_i^\vee}. \end{aligned}$$

If  $\alpha_i(\lambda) \geq 0$ , we obtain

$$q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * T_i * X^\lambda = q_i^{*(\alpha_i(\lambda+\nu))} X^{r_i(\mu)} * T_i + \sum_{h=\alpha_i(\nu)}^{\alpha_i(\lambda+\nu)-1} (q_i^{*(h+1)} - q_i^{*(h)}) X^{\nu+\lambda-h\alpha_i^\vee}.$$

We take  $h' = h - \alpha_i(\nu)$ , then  $X^{\nu+\lambda-h\alpha_i^\vee} = X^{\nu-\alpha_i(\nu)\alpha_i^\vee+\lambda-h'\alpha_i^\vee} = X^{r_i(\nu)+\lambda-h'\alpha_i^\vee}$  and  $q_i^{*(\alpha_i(\nu)+h')} = q_i^{*\alpha_i(\nu)} q_i^{*h'}$  (by  $q_i = q'_i$  if  $\alpha_i(\nu)$  is odd, and an easy calculation if  $\alpha_i(\nu)$  is even). So,

$$q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * T_i * X^\lambda = q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} (q_i^{*(\alpha_i(\lambda))} X^{r_i(\lambda)} * T_i + \sum_{h'=0}^{\alpha_i(\lambda)-1} (q_i^{*(h'+1)} - q_i^{*(h')}) X^{\lambda-h'\alpha_i^\vee}).$$

If  $\alpha_i(\lambda) < 0$ , we obtain

$$q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * T_i * X^\lambda = q_i^{*(\alpha_i(\lambda+\nu))} X^{r_i(\lambda+\nu)} * T_i - \sum_{h=\alpha_i(\lambda+\nu)}^{\alpha_i(\nu)-1} (q_i^{*(h+1)} - q_i^{*(h)}) X^{\nu+\lambda-h\alpha_i^\vee}.$$

We have  $q_i^{*(\alpha_i(\nu))} = q_i^{*(-\alpha_i(\lambda))} q_i^{*(\alpha_i(\lambda+\nu))}$  by an easy calculus if  $\alpha_i(\nu)$  and  $\alpha_i(\lambda)$  are even and because  $q_i = q'_i$  whenever  $\alpha_i(\nu)$  or  $\alpha_i(\lambda)$  is odd. So,

$X^{r_i(\nu)} * T_i * X^\lambda = \frac{1}{q_i^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda+\nu)} * T_i - \frac{1}{q_i^{*(\alpha_i(\nu))}} \sum_{h=\alpha_i(\lambda+\nu)}^{\alpha_i(\nu)-1} (q_i^{*(h+1)} - q_i^{*(h)}) X^{\nu+\lambda-h\alpha_i^\vee}$  and we

have (because of the injectivity of the left multiplication by  $X^{r_i(\nu)}$ ):

$$\begin{aligned} T_i * X^\lambda &= \frac{1}{q_i^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda)} * T_i - \frac{1}{q_i^{*(\alpha_i(\nu))}} \sum_{h=\alpha_i(\lambda+\nu)}^{\alpha_i(\nu)-1} (q_i^{*(h+1)} - q_i^{*(h)}) X^{\lambda+(\alpha_i(\nu)-h)\alpha_i^\vee} \\ &= \frac{1}{q_i^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda)} * T_i - \frac{1}{q_i^{*(\alpha_i(\nu))} q_i^{*(-\alpha_i(\lambda))}} \sum_{h=\alpha_i(\lambda)}^{-1} (q_i^{*(\alpha_i(\nu)-\alpha_i(\lambda)+h+1)} - q_i^{*(\alpha_i(\nu)-\alpha_i(\lambda)+h)}) X^{\lambda-h\alpha_i^\vee} \\ &= \frac{1}{q_i^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda)} * T_i - \frac{1}{q_i^{*(-\alpha_i(\lambda))}} \sum_{h=\alpha_i(\lambda)}^{-1} (q_i^{*(-\alpha_i(\lambda)+h+1)} - q_i^{*(-\alpha_i(\lambda)+h)}) X^{\lambda-h\alpha_i^\vee} \end{aligned}$$

□

## 5.7 The classical Bernstein-Lusztig relation

The module  $\delta : Q^\vee \rightarrow R$  is defined by  $\delta(\sum_{i \in I} a_i \alpha_i^\vee) = \prod_{i \in I} (q_i q_i')^{a_i}$  [GR14, 5.3.2]. After replacing eventually  $R$  by a bigger ring  $R'$  containing some square roots  $\sqrt{q_i}, \sqrt{q_i'}$  of  $q_i, q_i'$  (with  $\sqrt{q_i} = \sqrt{q_i'}$ , if  $q_i = q_i'$ ), we assume moreover that there exists an homomorphism  $\delta^{1/2} : Y \rightarrow R^*$ , such that  $\delta(\lambda) = \delta^{1/2}(\lambda)^2$  for any  $\lambda \in Q^\vee$  and  $\delta^{1/2}(\alpha_i^\vee) = \sqrt{q_i} \cdot \sqrt{q_i'}$ . In particular  $\sqrt{q_i}^{\pm 1}$  and  $\sqrt{q_i'}^{\pm 1}$  are well defined in  $R^*$ . In the common example where  $R = \mathbb{R}$  or  $\mathbb{C}$ , these expressions are chosen to be the classical ones:  $\delta^{1/2}(Y) \subset \mathbb{R}_+^*$ .

We define  $H_i = (\sqrt{q_i})^{-1} T_i$  and  $Z^\lambda = \delta^{-1/2}(\lambda) X^\lambda$  for  $\lambda \in Y^+$ . When  $w = r_{i_1} \cdots r_{i_n}$  is a reduced decomposition, we set  $H_w = H_{i_1} * \cdots * H_{i_n}$ ; this does not depend of the chosen decomposition of  $w$ .

We may translate the relations (BLX) for these elements.

**Proposition.** *For  $\lambda \in Y^{++}$ , we have the following relation:*

$$H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i + \sum_{k=0}^{\lfloor \frac{\alpha_i(\lambda)-1}{2} \rfloor} (\sqrt{q_i} - \sqrt{q_i}^{-1}) Z^{\lambda-(2k)\alpha_i^\vee} + \sum_{k=0}^{\lfloor \frac{\alpha_i(\lambda)}{2} \rfloor - 1} (\sqrt{q_i'} - \sqrt{q_i'}^{-1}) Z^{\lambda-(2k+1)\alpha_i^\vee}$$

**Remarks.** 1) This is the Bernstein-Lusztig relation for the  $Z^\lambda$ , (BLZ) for short.

2) In the following section, we shall consider an algebra containing  ${}^I\mathcal{H}_R$  and, for any  $i \in I$ , an element  $Z^{-\alpha_i^\vee}$  satisfying  $Z^{\lambda-h\alpha_i^\vee} = Z^\lambda (Z^{-\alpha_i^\vee})^h$  for  $h \in \mathbb{N}$ ,  $\lambda, \lambda - h\alpha_i^\vee \in Y^+$ . In such an algebra the relation (BLZ) may be rewritten (using that  $\sqrt{q_i} = \sqrt{q_i'}$  if  $\alpha_i(\lambda)$  is odd) as the classical Bernstein-Lusztig relation (BL):

$$H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i + (\sqrt{q_i} - \sqrt{q_i}^{-1}) \frac{Z^\lambda - Z^{r_i(\lambda)}}{1 - Z^{-2\alpha_i^\vee}} + (\sqrt{q_i'} - \sqrt{q_i'}^{-1}) \frac{Z^{\lambda-\alpha_i^\vee} - Z^{r_i(\lambda)-\alpha_i^\vee}}{1 - Z^{-2\alpha_i^\vee}}$$

*i.e.*  $H_i * Z^\lambda - Z^{r_i(\lambda)} * H_i = b(\sqrt{q_i}, \sqrt{q_i'}; Z^{-\alpha_i^\vee})(Z^\lambda - Z^{r_i(\lambda)})$  where  $b(t, u; z) = \frac{t - t^{-1} - (u - u^{-1})z}{1 - z^2}$ .

This is the same relation as in [Ma03, 4.2], up to the order; see below in 3).

3) Actually this relation (BLZ) is still true when  $\lambda \in Y^+$  and  $\alpha_i(\lambda) \geq 0$  (same proof as below). If  $\alpha_i(\lambda) < 0$ , we leave to the reader the proof of the following relation:

$$T_i * Z^\lambda = Z^{r_i(\lambda)} * T_i - \left( \sum_{h \text{ even}, h=2}^{-\alpha_i(\lambda)} (q_i - 1) Z^{\lambda+h\alpha_i^\vee} + \sum_{h \text{ odd}, h=1}^{-\alpha_i(\lambda)} \left( \sqrt{q_i \cdot q_i'} - \frac{\sqrt{q_i \cdot q_i'}}{q_i'} \right) Z^{\lambda+h\alpha_i^\vee} \right)$$

In the situation of 2) above, it may be rewritten:

$$\begin{aligned} H_i * Z^\lambda - Z^{r_i(\lambda)} * H_i &= (\sqrt{q_i} - \sqrt{q_i}^{-1}) \frac{Z^\lambda - Z^{r_i(\lambda)}}{1 - Z^{-2\alpha_i^\vee}} + (\sqrt{q'_i} - \sqrt{q'_i}^{-1}) \frac{Z^{\lambda - \alpha_i^\vee} - Z^{r_i(\lambda) - \alpha_i^\vee}}{1 - Z^{-2\alpha_i^\vee}} \\ &= b(\sqrt{q_i}, \sqrt{q'_i}; Z^{-\alpha_i^\vee})(Z^\lambda - Z^{r_i(\lambda)}) \end{aligned}$$

It is the same relation (BLZ) as above. Moreover, it's easy to see in the first equality that  $H_i * Z^\lambda - Z^{r_i(\lambda)} * H_i = Z^\lambda * H_i - H_i * Z^{r_i(\lambda)}$ . Actually we shall see in section 6 that this same relation is true for any  $\lambda \in Y$  in a greater algebra containing elements  $Z^\lambda$  for  $\lambda \in Y$ .

*Proof.* From  $Z^\lambda = \delta^{-1/2}(\lambda) X^\lambda$  and  $\delta^{1/2}(\alpha_i^\vee) = \sqrt{q_i \cdot q'_i}$ , we get

$$\begin{aligned} Z^{\lambda - h\alpha_i^\vee} &= \delta^{-1/2}(\lambda - h\alpha_i^\vee) X^{\lambda - h\alpha_i^\vee} \\ &= \delta^{-1/2}(\lambda) (\delta^{1/2}(\alpha_i^\vee))^h X^{\lambda - h\alpha_i^\vee} \\ &= \delta^{-1/2}(\lambda) (\sqrt{q_i \cdot q'_i})^h X^{\lambda - h\alpha_i^\vee} \end{aligned}$$

By  $\alpha_i(\lambda) \geq 0$  and (BLX) we have

$$T_i * Z^\lambda = q_i^{*(\alpha_i(\lambda))} (\sqrt{q_i \cdot q'_i})^{-\alpha_i(\lambda)} Z^{r_i(\lambda)} * T_i + \sum_{h=0}^{\alpha_i(\lambda)-1} (q_i^{*(h+1)} - q_i^{*h}) (\sqrt{q_i \cdot q'_i})^{(-h)} Z^{\lambda - h\alpha_i^\vee}.$$

Moreover  $q_i^{*h} = q_i q'_i q_i \cdots$  with  $h$  terms in the product so  $q_i^{*h} = (\sqrt{q_i \cdot q'_i})^h$  if  $h$  is even and  $q_i^{*h} = q_i (\sqrt{q_i \cdot q'_i})^{(h-1)}$  if  $h$  is odd. So, if  $\alpha_i(\lambda)$  is even, we have  $T_i * Z^\lambda = Z^{r_i(\lambda)} * T_i + \sum_{k=0}^{\frac{\alpha_i(\lambda)-2}{2}} (q_i - 1) Z^{\lambda - (2k)\alpha_i^\vee} + \sum_{k=0}^{\frac{\alpha_i(\lambda)-2}{2}} (q_i q'_i - q_i) (\sqrt{q_i q'_i})^{-1} Z^{\lambda - (2k+1)\alpha_i^\vee}$ . If  $\alpha_i(\lambda)$  is odd, we have

$q_i = q'_i$ , and we obtain  $T_i * Z^\lambda = Z^{r_i(\lambda)} * T_i + \sum_{h=0}^{\alpha_i(\lambda)-1} (q_i - 1) Z^{\lambda - h\alpha_i^\vee}$ . In both cases, by  $H_i = (\sqrt{q_i})^{-1} T_i$ , we get:

$$H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i + \sum_{k=0}^{\lfloor \frac{\alpha_i(\lambda)-1}{2} \rfloor} (\sqrt{q_i} - \sqrt{q_i}^{-1}) Z^{\lambda - (2k)\alpha_i^\vee} + \sum_{k=0}^{\lfloor \frac{\alpha_i(\lambda)}{2} \rfloor - 1} (\sqrt{q'_i} - \sqrt{q'_i}^{-1}) Z^{\lambda - (2k+1)\alpha_i^\vee}.$$

□

## 6 Bernstein-Lusztig-Hecke Algebras

The aim of this section is to define, in a formal way, an associative algebra  ${}^{BL}\mathcal{H}_R$ , called the Bernstein-Lusztig-Hecke algebra. This construction by generators and relations is motivated by the results obtained in the previous section (in particular 5.6) and we will be able next to identify  ${}^I\mathcal{H}_R$  and a subalgebra of  ${}^{BL}\mathcal{H}_R$  (up to some hypotheses on  $R$ ).

We use the same notations as before, even if the objects are somewhat different. This choice will be justified by the identification obtained at the end of this section.

### 6.1 The module ${}^{BL}\mathcal{H}_{R_1}$

We consider now the ring  $R_1 = \mathbb{Z}[(\sigma_i^{\pm 1}, \sigma'_i)^{\pm 1}]_{i \in I}$  where the indeterminates  $\sigma_i, \sigma'_i$  satisfy the following relations (as  $q_i$  and  $q'_i$  in 1.5 3) and 4) because in the further identification,  $\sigma_i, \sigma'_i$  will play the role of  $\sqrt{q_i}$  and  $\sqrt{q'_i}$ ).

If  $\alpha_i(Y) = \mathbb{Z}$ , then  $\sigma_i = \sigma'_i$ .

If  $r_i$  and  $r_j$  are conjugated (i.e. if  $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$ ), then  $\sigma_i = \sigma_j = \sigma'_i = \sigma'_j$ .

We denote by  ${}^{BL}\mathcal{H}_{R_1}$  the free  $R_1$ -module with basis  $(Z^\lambda H_w)_{\lambda \in Y, w \in W^v}$ . For short, we write  $H_i = H_{r_i}, H_w = Z^0 H_w$  and  $Z^\lambda = Z^\lambda H_1$ .



**Theorem 6.2.** *There exists a unique multiplication  $*$  on  ${}^{BL}\mathcal{H}_{R_1}$  which makes it an associative unitary  $R_1$ -algebra with unity  $H_1$  and satisfies the following conditions:*

- (1)  $\forall \lambda \in Y \quad \forall w \in W^v \quad Z^\lambda * H_w = Z^\lambda H_w,$
- (2)  $\forall i \in I \quad \forall w \in W^v \quad H_i * H_w = H_{r_i w}$  if  $\ell(r_i w) > \ell(w)$   
 $= (\sigma_i - \sigma_i^{-1})H_w + H_{r_i w}$  if  $\ell(r_i w) < \ell(w),$
- (3)  $\forall \lambda \in Y \quad \forall \mu \in Y \quad Z^\lambda * Z^\mu = Z^{\lambda+\mu},$
- (4)  $\forall \lambda \in Y \quad \forall i \in I \quad H_i * Z^\lambda - Z^{r_i(\lambda)} * H_i = b(\sigma_i, \sigma'_i; Z^{-\alpha_i^\vee})(Z^\lambda - Z^{r_i(\lambda)});$  where  
 $b(t, u; z) = \frac{(t-t^{-1})-(u-u^{-1})z}{1-z^2}.$

**Remarks 6.3.** 1) It is already known (see *e.g.* [Hu90, Th. 7.1] or [Bo68, IV § 2 exer. 23]) that the free submodule with basis  $(H_w)_{w \in W^v}$  can be equipped, in a unique way, with a multiplication  $*$  that satisfies (2) and gives it a structure of an associative unitary algebra called the "Hecke algebra of the group  $W^v$  over  $R_1$ " and denoted by  $\mathcal{H}_{R_1}(W^v)$ .

2) The submodule  $\mathcal{H}_{R_1}(Y)$  with basis  $(Z^\lambda)_{\lambda \in Y}$  will be a commutative subalgebra.

3) When all  $\sigma_i, \sigma'_i$  are equal, the existence of this algebra  ${}^{BL}\mathcal{H}$  is stated in [GaG95] and justified by an action on some Grothendieck group.

4) This  $R_1$ -algebra depends only on  $\mathbb{A}$  and  $Y$  (*i.e.*  $\mathbb{A}$  and  $W$ ). We call it the Bernstein-Lusztig-Hecke algebra over  $R_1$  (associated to  $\mathbb{A}$  and  $W$ ).

## 6.4 Proof of Theorem 6.2

1) The uniqueness of the multiplication  $*$  is clear: by associativity and distributivity, we have only to identify  $H_w * Z^\mu$ . If  $w = r_{i_1} r_{i_2} \cdots r_{i_n}$  is a reduced decomposition, then, by (2), (4) and remark 1),  $H_w * Z^\mu = H_{i_1} * (H_{i_2} * (\cdots * (H_{i_n} * Z^\mu) \cdots))$  has to be a well defined linear combination of terms  $Z^\nu H_u : H_w * Z^\mu = \sum_k a_k Z^{\nu_k} H_{u_k}$  with  $a_k \in R_1, \nu_k \in Y, u_k \in W^v$ .

2) Construction of  $*$ . We define  $H_w * Z^\mu$  as above and we have to prove that it does not depend of the reduced decomposition  $w = r_{i_1} r_{i_2} \cdots r_{i_n}$ .

a) We define  $L_i \in \text{End}_{R_1}({}^{BL}\mathcal{H}_{R_1})$  by :

$$L_i(Z^\mu H_v) = H_i * (Z^\mu H_v) = Z^{r_i(\mu)}(H_i * H_v) + b(\sigma_i, \sigma'_i; Z^{-\alpha_i^\vee})(Z^\mu - Z^{r_i(\mu)})$$

where  $H_i * H_v = H_{r_i w}$  if  $\ell(r_i w) > \ell(w)$  and  $H_i * H_v = (\sigma_i - \sigma_i^{-1})H_w + H_{r_i w}$  if  $\ell(r_i w) < \ell(w)$ .

By Matsumoto's theorem [Bo68, IV 1.5 prop. 5], the expected independence will be a consequence of the braid relations, *i.e.*:

(\*)  $L_i(L_j(L_i(\dots(Z^\lambda H_v)\dots))) = L_j(L_i(L_j(\dots(Z^\lambda H_v)\dots)))$  (with  $m_{i,j}$  factors  $L$  on each side), whenever the order  $m_{i,j}$  of  $r_i r_j$  is finite.

As  $\mathcal{H}_{R_1}(W^v)$  is known to be an algebra, it is enough to prove (\*) for  $v = 1$ . We may also suppose  $\alpha_j(\alpha_i^\vee) \neq 0$  as otherwise  $L_i$  and  $L_j$  commute clearly.

We choose  $i, j \in I$  with  $m_{i,j}$  finite, then  $\pm\alpha_i, \pm\alpha_j$  generate a finite root system  $\Phi_{i,j}$  of rank 2 (or 1 if  $i = j$ ). Moreover,  $Y' = \ker(\alpha_i) \cap \ker(\alpha_j) \cap Y$  is cotorsion free in  $Y$ ; let  $Y''$  be a supplementary module containing  $\alpha_i^\vee$  and  $\alpha_j^\vee$ .  $Y''$  is a lattice (of rank 2 or 1) between the lattices  $Q_{i,j}^\vee$  of coroots and  $P_{i,j}^\vee$  of coweights, associated to  $\Phi_{i,j}$ .

Any  $\lambda \in Y$  may be written  $\lambda = \lambda' + \lambda''$  with  $\lambda' \in Y'$  and  $\lambda'' \in Y''$ . By (4)  $L_i(Z^{\lambda'}) = Z^{\lambda'} H_i$  and  $L_j(Z^{\lambda'}) = Z^{\lambda'} H_j$ . So we have to prove (\*) for  $\lambda = \lambda'' \in Y''$ . We shall do it by comparing with some Macdonald's results.

b) In [Ma03] Macdonald builds affine Hecke algebras  $\mathcal{H}(W(R, L'))$  over  $\mathbb{R}$ , associated to any finite irreducible root system  $R$  and any lattice  $L'$  between the lattices of coroots and coweights; more precisely this algebra is associated to the extended affine Weyl group

$W(R, L') = W(R) \rtimes L'$ . It is defined by generators and relations, but it is proven that it is endowed with a basis  $(Y^\lambda T(w))_{\lambda \in L', w \in W(R)}$  [l.c. ; 4.2.7] and satisfies relations analogous to (1), (2), (3), (4) as above. There are parameters  $(\tau_i)_{i \in I}$  and  $\tau_0$  which are reals (but may be algebraically independent over  $\mathbb{Q}$ , so may be considered as indeterminates) and satisfy  $\tau_i = \tau_j$  if  $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$ . The relation (4) is satisfied with  $\sigma_i = \tau_i$  and  $\sigma'_i = \tau_i$  when  $\alpha_i(L') = \mathbb{Z}$ ,  $\sigma'_i = \tau_0$  when  $\alpha_i(L') = 2\mathbb{Z}$ .

c) In the case  $R = \Phi_{i,j}$ , irreducible,  $L' = Y''$ , we may choose  $\tau_i, \tau_j$  and  $\tau_0$  such that the relations (4) are the same, for us and Macdonald: either  $\alpha_i(\alpha_j^\vee) = -1$  or  $\alpha_j(\alpha_i^\vee) = -1$ , so  $\tau_0 = \sigma'_i$  or  $\tau_0 = \sigma'_j$ . In particular  $R_1$  may be identified to a subring of  $\mathbb{R}$ . The operators  $L_i$  and  $L_j$  of both theories coincide on the elements  $Z^\lambda H_v$  (identified with  $Y^\lambda T(v)$ ) for  $\lambda \in L' = Y''$  and  $v \in \langle r_i, r_j \rangle$ . So (\*) is satisfied as  $\mathcal{H}(W(R, L'))$  is an associative algebra.

d) So, if  $H_w * Z^\mu = \sum_k a_k Z^{\nu_k} H_{u_k}$ , with  $a_k \in R_1, \nu_k \in Y, u_k \in W^v$ , we define  $(Z^\lambda H_w) * (Z^\mu H_v) = \sum_k a_k Z^{\lambda + \nu_k} * (H_{u_k} * H_v)$ . We get a distributive multiplication on  ${}^{BL}\mathcal{H}_{R_1}$  with unit  $H_1$ .

### 3) Associativity

a) Using the associativity in  $\mathcal{H}_{R_1}(Y)$  and  $\mathcal{H}_{R_1}(W^v)$  and the formula 2.d above, it is clear that, for any  $\lambda \in Y, w \in W^v, E_1, E_2 \in {}^{BL}\mathcal{H}_{R_1}$ , we have:

$$(R1) \quad Z^\lambda * (E_1 * E_2) = (Z^\lambda * E_1) * E_2,$$

$$(R2) \quad E_1 * (E_2 * H_w) = (E_1 * E_2) * H_w.$$

We need also to prove (for  $\lambda_1, \lambda_2 \in Y, w, w_1, w_2 \in W^v, E \in {}^{BL}\mathcal{H}_{R_1}$ ),

$$(A) \quad H_w * (Z^{\lambda_1} * Z^{\lambda_2}) = (H_w * Z^{\lambda_1}) * Z^{\lambda_2},$$

$$(B) \quad H_{w_1} * (H_{w_2} * E) = (H_{w_1} * H_{w_2}) * E.$$

Then the general associativity will follow : using (R1), (R2), (A), (B) and the formula 2d for the product, it is not too difficult (and left to the reader) to prove that :

$$\begin{aligned} (Z^{\lambda_1} H_{w_1}) * ((Z^{\lambda_2} H_{w_2}) * (Z^{\lambda_3} H_{w_3})) &= Z^{\lambda_1} * (H_{w_1} * ((Z^{\lambda_2} H_{w_2}) * Z^{\lambda_3})) * H_{w_3} \\ &= Z^{\lambda_1} * ((H_{w_1} * Z^{\lambda_2}) * (H_{w_2} * Z^{\lambda_3})) * H_{w_3} \\ &= Z^{\lambda_1} * ((H_{w_1} * (Z^{\lambda_2} H_{w_2})) * Z^{\lambda_3}) * H_{w_3} \\ &= ((Z^{\lambda_1} H_{w_1}) * (Z^{\lambda_2} H_{w_2})) * (Z^{\lambda_3} H_{w_3}). \end{aligned}$$

b) Proof of (B). This condition is equivalent to the fact that the left multiplication by  $\mathcal{H}_{R_1}(W^v)$  on  ${}^{BL}\mathcal{H}_{R_1}$  is an action. But the associative algebra  $\mathcal{H}_{R_1}(W^v)$  is generated by the  $H_i$  with relations the braid relations and  $H_i^2 = (\sigma_i - \sigma_i^{-1})H_i + H_1$ . As  $L_i$  is the left multiplication by  $H_i$ , we have (B), i.e. an action if and only if these  $L_i$  satisfy the relation (\*) in 2.a and

$$(**) L_i(L_i(Z^\lambda H_v)) = (\sigma_i - \sigma_i^{-1})L_i(Z^\lambda H_v) + Z^\lambda H_v.$$

As in 2b, we reduce the verification of (\*\*) to the case  $v = 1$  and  $\lambda \in Y''$  (associated to  $i = j$ ) i.e.  $\lambda \in Y'' = \mathbb{Q}\alpha_i^\vee \cap Y$ . Then we look at Macdonald's construction of  $\mathcal{H}(W(\{\pm\alpha_i\}, Y''))$  with  $\tau_i = \sigma_i, \tau_0 = \sigma'_i$ . We conclude, as in 2.c that (\*\*) is satisfied.

c) The proof of (A) is by induction on  $\ell(w)$ .

If  $w = r_i$ , we have:

$$\begin{aligned} (H_i * Z^{\lambda_1}) * Z^{\lambda_2} &= (Z^{r_i(\lambda_1)} H_i) * Z^{\lambda_2} + (b(\sigma_i, \sigma'_i; Z^{-\alpha_i^\vee})(Z^{\lambda_1} - Z^{r_i(\lambda_1)})) * Z^{\lambda_2} \\ &= Z^{r_i(\lambda_1)} * (Z^{r_i(\lambda_2)} H_i) + (b(\sigma_i, \sigma'_i; Z^{-\alpha_i^\vee})(Z^{\lambda_2} - Z^{r_i(\lambda_2)})) + b(\sigma_i, \sigma'_i; Z^{-\alpha_i^\vee})(Z^{\lambda_1 + \lambda_2} - Z^{r_i(\lambda_1) + \lambda_2}) \\ &= Z^{r_i(\lambda_1 + \lambda_2)} H_i + b(\sigma_i, \sigma'_i; Z^{-\alpha_i^\vee})(Z^{r_i(\lambda_1) + \lambda_2} - Z^{r_i(\lambda_1) + r_i(\lambda_2)}) + b(\sigma_i, \sigma'_i; Z^{-\alpha_i^\vee})(Z^{\lambda_1 + \lambda_2} - Z^{r_i(\lambda_1) + \lambda_2}) \\ &= Z^{r_i(\lambda_1 + \lambda_2)} H_i + b(\sigma_i, \sigma'_i; Z^{-\alpha_i^\vee})(Z^{\lambda_1 + \lambda_2} - Z^{r_i(\lambda_1) + \lambda_2}) \\ &= H_i * (Z^{\lambda_1} * Z^{\lambda_2}) \end{aligned}$$

If the result is known when  $\ell(w) = n$ . Let us consider  $w = w'r_i$  with  $\ell(w) = n + 1$  and

$\ell(w') = n$ , then

$$\begin{aligned} H_w * (Z^{\lambda_1} * Z^{\lambda_2}) &= H_{w'} * (H_i * Z^{\lambda_1 + \lambda_2}) \text{ (left multiplication by } \mathcal{H}_{R_1}(W^v) \text{ is an action)} \\ &= H_{w'} * ((H_i * Z^{\lambda_1}) * Z^{\lambda_2}) \text{ (case } \ell(w) = 1) \\ &= H_{w'} * ((Z^{r_i(\lambda_1)} H_i) * Z^{\lambda_2} + (b(\sigma_i, \sigma'_i; Z^{-\alpha_i^\vee})(Z^{\lambda_1} - Z^{r_i(\lambda_1)})) * Z^{\lambda_2}). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (H_w * Z^{\lambda_1}) * Z^{\lambda_2} &= (H_{w'} * (H_i * Z^{\lambda_1})) * Z^{\lambda_2} \\ &= (H_{w'} * (Z^{r_i(\lambda_1)} H_i + b(\sigma_i, \sigma'_i; Z^{-\alpha_i^\vee})(Z^{\lambda_1} - Z^{r_i(\lambda_1)})) * Z^{\lambda_2} \\ &= (H_{w'} * (Z^{r_i(\lambda_1)} H_i)) * Z^{\lambda_2} + (H_{w'} * (b(\sigma_i, \sigma'_i; Z^{-\alpha_i^\vee})(Z^{\lambda_1} - Z^{r_i(\lambda_1)}))) * Z^{\lambda_2}. \end{aligned}$$

The second term of the right hand side is a  $R_1$ -linear combination of  $(H_{w'} * Z^{\lambda_1 + k\alpha_i^\vee}) * Z^{\lambda_2}$  and we see by induction that it is the same as  $H_{w'} * ((b(\sigma_i, \sigma'_i; Z^{-\alpha_i^\vee})(Z^{\lambda_1} - Z^{r_i(\lambda_1)})) * Z^{\lambda_2})$  in  $H_w * (Z^{\lambda_1} * Z^{\lambda_2})$ .

In the first term,  $(H_{w'} * (Z^{r_i(\lambda_1)} H_i)) * Z^{\lambda_2} = ((H_{w'} * Z^{r_i(\lambda_1)}) * H_i) * Z^{\lambda_2}$ , we can write  $H_{w'} * Z^{r_i(\lambda_1)} = \sum_k c_k Z^{\lambda_k} H_{w_k}$  and we will use later in the same way  $H_i * Z^{\lambda_2} = \sum_h a_h Z^{\mu_h} H_{v_h}$  with  $c_k, a_h \in R_1, \lambda_k, \mu_h \in Y$  and  $w_k, v_h \in W^v$ . So, we have :

$$\begin{aligned} ((\sum_k c_k Z^{\lambda_k} H_{w_k}) * H_i) * Z^{\lambda_2} &= (\sum_k c_k (Z^{\lambda_k} * (H_{w_k} * H_i))) * Z^{\lambda_2} \quad \text{(by (R2))} \\ &= \sum_k c_k Z^{\lambda_k} * ((H_{w_k} * H_i) * Z^{\lambda_2}) \quad \text{(by formula 2d)} \\ &= \sum_k c_k Z^{\lambda_k} * (H_{w_k} * (H_i * Z^{\lambda_2})) \quad \text{(by (B))} \\ &= \sum_k c_k (Z^{\lambda_k} * H_{w_k}) * (H_i * Z^{\lambda_2}) \quad \text{(by (R1))} \\ &= \sum_k c_k (Z^{\lambda_k} * H_{w_k}) * (\sum_h a_h Z^{\mu_h} H_{v_h}) \\ &= \sum_{k,h} c_k a_h (Z^{\lambda_k} * H_{w_k}) * (Z^{\mu_h} * H_{v_h}) \\ &= \sum_{k,h} c_k a_h ((Z^{\lambda_k} * H_{w_k}) * Z^{\mu_h}) * H_{v_h} \quad \text{(by (R2))} \\ &= \sum_h a_h (((H_{w'} * Z^{r_i(\lambda_1)}) * Z^{\mu_h}) * H_{v_h}). \\ &= \sum_h a_h ((H_{w'} * (Z^{r_i(\lambda_1)} * Z^{\mu_h})) * H_{v_h}) \quad \text{(by induction)} \\ &= \sum_h a_h H_{w'} * ((Z^{r_i(\lambda_1)} * Z^{\mu_h}) * H_{v_h}) \quad \text{(by (R2))} \\ &= H_{w'} * (Z^{r_i(\lambda_1)} * (H_i * Z^{\lambda_2})). \quad \text{(by (R1))} \end{aligned}$$

This corresponds to the term  $H_{w'} * ((Z^{r_i(\lambda_1)} H_i) * Z^{\lambda_2})$  in  $H_w * (Z^{\lambda_1} * Z^{\lambda_2})$  so we obtain the equality when  $\ell(w) = n + 1$ .  $\square$

## 6.5 Change of scalars

1) Let us suppose that we are given a morphism  $\varphi$  from  $R_1$  to a ring  $R$ , then we are able to consider, by extension of scalars,  ${}^{BL}\mathcal{H}_R = R \otimes_{R_1} {}^{BL}\mathcal{H}_{R_1}$  as an  $R$ -associative algebra. The family  $(Z^\lambda H_w)_{\lambda \in Y, w \in W^v}$  is still a basis of the  $R$ -module  ${}^{BL}\mathcal{H}_R$ .

2) In order to consider elements similar to the  $X^\lambda$  of section 4, we are going to define a ring  $R_3$  containing  $R_1$  such that there exists a group homomorphism  $\delta^{1/2} : Y \rightarrow R_3^*$  with  $\delta(\lambda) = \delta^{1/2}(\lambda)^2$  for any  $\lambda \in Q^\vee$  and  $\delta^{1/2}(\alpha_i^\vee) = \sigma_i \cdot \sigma'_i$ .

Since  $Q^\vee$  is a submodule of the free  $\mathbb{Z}$ -module  $Y$ , by the elementary divisor theorem, if we denote  $m$  the biggest elementary divisor, then for any  $\mu \in Y \cap (Q^\vee \otimes_{\mathbb{Z}} \mathbb{R})$ , we have  $m\mu \in Q^\vee$ . Let us consider the ring  $R_3 = \mathbb{Z}[(\tau_i^{\pm 1}, \tau'_i)^{\pm 1}]_{i \in I}$  (with  $\tau_i, \tau'_i$  satisfying conditions similar to those of 6.1) and the identification of  $R_1$  as a subring of  $R_3$  given by  $\tau_i^m = \sigma_i$  and  $\tau'_i{}^m = \sigma'_i$ . Then, for  $\lambda \in Y$  we have  $m\lambda = \sum_{i \in I} a_i \alpha_i^\vee + \lambda_0$  with the  $a_i \in \mathbb{Z}$  and  $\lambda_0 \notin Q^\vee \otimes_{\mathbb{Z}} \mathbb{R}$  and we can define  $\delta^{1/2}(\lambda) = \prod_{i \in I} (\tau_i \tau'_i)^{a_i}$  and obtain a group homomorphism from  $Y$  to  $R_3$ , with the wanted properties.

In  ${}^{BL}\mathcal{H}_{R_3}$ , let us consider  $X^\lambda = \delta^{1/2}(\lambda) Z^\lambda$  for  $\lambda \in Y$  and  $T_i = \sigma_i H_i = (\tau_i)^m H_i$ . It's easy to see that  $T_w = T_{i_1} * T_{i_2} * \dots * T_{i_n}$  is independent of the choice of a reduced decomposition  $r_{i_1} r_{i_2} \dots r_{i_n}$  of  $w$ . It is clear that the family  $(X^\lambda * T_w)_{\lambda \in Y, w \in W^v}$  is a new basis of the  $R_3$ -module  ${}^{BL}\mathcal{H}_{R_3}$ .

**3)** We can give new formulas to define  $*$  in terms of these generators. The relation (4) of the definition of  ${}^{BL}\mathcal{H}_{R_3}$  can be written as previously :

If  $\alpha_i(\lambda) \geq 0$ , then

$$(BLZ+) \quad H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i + \sum_{k \text{ even}, k=0}^{\alpha_i(\lambda)-1} (\sigma_i - \sigma_i^{-1}) Z^{\lambda - k\alpha_i^\vee} + \sum_{k \text{ odd}, k=0}^{\alpha_i(\lambda)-1} (\sigma'_i - \sigma_i'^{-1}) Z^{\lambda - k\alpha_i^\vee},$$

If  $\alpha_i(\lambda) < 0$ , then

$$(BLZ-) \quad H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i - \sum_{k \text{ even}, k=2}^{-\alpha_i(\lambda)} (\sigma_i - \sigma_i^{-1}) Z^{\lambda + k\alpha_i^\vee} - \sum_{k \text{ odd}, k=1}^{-\alpha_i(\lambda)} (\sigma'_i - \sigma_i'^{-1}) Z^{\lambda + k\alpha_i^\vee}.$$

With the same arguments as in 5.7, these relations (after changing the variables and writing  $(\sigma_i^2)^{*n} = \sigma_i^2 \sigma_i'^2 \sigma_i^2 \sigma_i'^2 \dots$  with  $n$  terms in this product) become :

$$(BLX+) \quad \text{If } \alpha_i(\lambda) \geq 0, \text{ then } T_i * X^\lambda = (\sigma_i^2)^{*(\alpha_i(\lambda))} X^{r_i(\lambda)} * T_i + \sum_{h=0}^{\alpha_i(\lambda)-1} ((\sigma_i^2)^{*(h+1)} - (\sigma_i^2)^{*(h)}) X^{\lambda - h\alpha_i^\vee},$$

(BLX-) If  $\alpha_i(\lambda) < 0$ , then,

$$T_i * X^\lambda = \frac{1}{(\sigma_i^2)^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda)} * T_i - \frac{1}{(\sigma_i^2)^{*(-\alpha_i(\lambda))}} \sum_{h=\alpha_i(\lambda)}^{-1} \left( (\sigma_i^2)^{*(-\alpha_i(\lambda)+h+1)} - q_i^{*(-\alpha_i(\lambda)+h)} \right) X^{\lambda - h\alpha_i^\vee}.$$

The other formulas give easily:

$$(2') \quad \forall i \in I \quad \forall w \in W^v \quad T_i * T_w = T_{r_i w} \text{ if } \ell(r_i w) > \ell(w) \\ = (\sigma_i^2 - 1) T_w + \sigma_i^2 T_{r_i w} \text{ if } \ell(r_i w) < \ell(w),$$

$$(3') \quad \forall \lambda \in Y \quad \forall \mu \in Y \quad X^\lambda * X^\mu = X^{\lambda + \mu}.$$

In all these relations, we can see that the coefficients are in the subring  $R_2 = \mathbb{Z}[(\sigma_i^{\pm 2}, \sigma_i'^{\pm 2})_{i \in I}]$  of  $R_1$ . So, if we consider  ${}^{BLX}\mathcal{H}_{R_2}$  the  $R_2$ -submodule with basis  $(X^\lambda * T_w)_{\lambda \in Y, w \in W^v}$ , the multiplication  $*$  gives it a structure of associative unitary algebra over  $R_2$ .

## 6.6 The positive Bernstein-Lusztig-Hecke algebra

If we consider in  ${}^{BLX}\mathcal{H}_{R_2}$ , the submodule with basis  $(X^\lambda * T_w)_{\lambda \in Y^+, w \in W^v}$ , it is stable by multiplication  $*$  (in (BLX+) and (BLX-) if  $\lambda \in Y^+$  all the  $\lambda \pm h\alpha_i^\vee$  written are also in  $Y^+$ ). We denote  ${}^{BL}\mathcal{H}_{R_2}^+$  this  $R_2$ -subalgebra of  ${}^{BLX}\mathcal{H}_{R_2}$ . Actually, we can define such positive Hecke subalgebras inside all algebras in 6.5.

Like before, if we are given a morphism  $\varphi$  from  $R_2$  to a ring  $R$ , we are able to consider, by extension of scalars,  ${}^{BL}\mathcal{H}_R^+ = R \otimes_{R_2} {}^{BL}\mathcal{H}_{R_2}^+$ . Let us consider the ring  $R$  of the section 4 (such that  $\mathbb{Z} \subset R$  and all  $q_i, q'_i$  are invertible in  $R$ ), we can construct a morphism  $\phi$  from  $R_2$  to  $R$  by  $\phi(\sigma_i^2) = q_i$  and  $\phi(\sigma_i'^2) = q'_i$ . So, we obtain an algebra  ${}^{BL}\mathcal{H}_R^+$  with basis  $(X^\lambda * T_w)_{\lambda \in Y^+, w \in W^v}$  and the same relations as in  ${}^I\mathcal{H}_R$ . So :

**Proposition.** *Over  $R$ , the Iwahori-Hecke algebra  ${}^I\mathcal{H}_R$  and the positive Bernstein-Lusztig-Hecke algebra  ${}^{BL}\mathcal{H}_R^+$  are isomorphic.*

**Remark.**  ${}^{BLX}\mathcal{H}_R$  may be considered as a kind of ring of quotients of  ${}^{BL}\mathcal{H}_R^+ \simeq {}^I\mathcal{H}_R$ , as we added, in it, inverses of the  $X^\lambda = T_\lambda$  for  $\lambda \in Y^{++}$ .

## 6.7 Structure constants

From Proposition 6.6, we get that the structure constants of the convolution product  $*$  of  ${}^I\mathcal{H}_R$ , in the basis  $(X^\lambda * T_w)_{\lambda \in Y^+, w \in W^v}$ , are Laurent polynomials in the parameters  $q_i, q'_i$ , with

coefficients in  $\mathbb{Z}$ , depending only on  $\mathbb{A}$  and  $W$ . By Remark 5.5.a, we get the same result for the structure constants in the basis  $(T_\lambda * T_w)_{\lambda \in Y^+, w \in W^v}$  and then still the same result for the structure constants  $a_{\mathbf{w}, \mathbf{v}}^{\mathbf{u}}$  in the basis  $(T_{\mathbf{w}})_{\mathbf{w} \in W^+}$  (by 4.5).

This last result is not as precise as the one expected in the conjecture of § 2. But there is at least one case where we can prove it:

**Remark.** Suppose  $\mathcal{S}$  is the hovel associated to a split Kac-Moody group  $G$  over a local field  $\mathcal{K}$ , cf. [GR14, § 3]. Then all parameters  $q_i, q'_i$  are equal to the cardinality  $q$  of the residue field; moreover we know that each  $a_{\mathbf{w}, \mathbf{v}}^{\mathbf{u}}$  is an integer and a Laurent polynomial in  $q$ , with coefficients in  $\mathbb{Z}$ , depending only on  $\mathbb{A}$  and  $W$ . But, as  $G$  is split, the same thing is true (without changing  $\mathbb{A}$  and  $W$ ) for all unramified extensions of the field  $\mathcal{K}$ , hence for infinitely many  $q$ . So the Laurent polynomial  $a_{\mathbf{w}, \mathbf{v}}^{\mathbf{u}}$  is an integer for infinitely many integral values of the variable  $q$ : it has to be a true polynomial.

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