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LOW MOMENTS OF DIRICHLET SERIES

HUGH L. MONTGOMERY AND GÉRALD TENENBAUM

ABSTRACT. We determine the maximum possible size of the q^{th} moment of a Dirichlet series, for $1 \leq q \leq 2$.

1. INTRODUCTION

In order to bound the mean value of multiplicative functions, Halász [5] introduced a majorant principle which (after a little refining) asserts that if $\lambda_1, \lambda_2, \dots$ are real numbers, if $|a_n| \leq A_n$ for all n , and $\sum_{n \geq 1} A_n < \infty$, then

$$(1.1) \quad \int_{-T}^T \left| \sum_{n \geq 1} a_n e^{i\lambda_n t} \right|^2 dt \leq 3 \int_{-T}^T \left| \sum_{n \geq 1} A_n e^{i\lambda_n t} \right|^2 dt.$$

For a proof of the principle in this form, see Montgomery [8, §7.3]. In Halász's theory, one needs bounds for integrals of the shape

$$(1.2) \quad I(q) = \int_{-1}^1 \left| \sum_p \frac{a_p \log p}{p^{\sigma+it}} \right|^q dt, \quad J(q) = \int_{-1}^1 \left| \sum_{n \geq 1} \frac{b_n}{n^{\sigma+it}} \right|^q dt,$$

when $|a_p| \leq 1$ for all p , and $|b_n| \leq 1$ for all n . From (1.1) it is immediate that

$$I(2) \leq 3 \int_{-1}^1 \left| \sum_{n \geq 2} \frac{\Lambda(n)}{n^{\sigma+it}} \right|^2 dt = 3 \int_{-1}^1 \left| \frac{\zeta'}{\zeta}(\sigma+it) \right|^2 dt \ll \int_{-1}^1 \frac{dt}{|\sigma+it-1|^2} \ll \frac{1}{\sigma-1}$$

uniformly for $1 < \sigma \leq 2$. Similarly, $J(2) \ll 1/(\sigma-1)$ for σ in this range. By applying the majorant principle to the squares of these Dirichlet series we find that $I(4) \ll (\sigma-1)^{-3}$ and $J(4) \ll (\sigma-1)^{-3}$. Hence by Hölder's inequality,

$$(1.3) \quad I(q) \ll (\sigma-1)^{1-q}, \quad J(q) \ll (\sigma-1)^{1-q} \quad (2 \leq q \leq 4),$$

$$(1.4) \quad I(q) \ll (\sigma-1)^{-q/2}, \quad J(q) \ll (\sigma-1)^{-q/2} \quad (1 \leq q \leq 2)$$

uniformly for $1 < \sigma \leq 2$. The estimate (1.3) is best possible, as we see by taking $a_p = 1$ for all p and $b_n = 1$ for all n . For purposes of Halász's theory, it would be helpful if (1.3) would hold also when $1 < q \leq 2$. However, we construct examples that show that the weaker estimate (1.4) is best possible.

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Theorem 1.1. *Let $I(q)$ and $J(q)$ be defined as in (1.2). There exist numbers a_p with $a_p = \pm 1$ for all p , and b_n with $b_n = \pm 1$ for all n , such that*

$$I(q) \asymp (\sigma - 1)^{-q/2}, \quad J(q) \asymp (\sigma - 1)^{-q/2}$$

uniformly for $1 < \sigma \leq 2$, $1 \leq q \leq 2$.

This is analogous to the situation for Fourier series. For example, if $|b_n| \leq 1$ for $-N \leq n \leq N$ and $e(\vartheta) = e^{2\pi i\vartheta}$, then

$$(1.5) \quad \int_0^1 \left| \sum_{|n| \leq N} b_n e(nx) \right|^q dx \ll N^{q-1}$$

uniformly for $2 \leq q \leq 4$, and

$$(1.6) \quad \int_0^1 \left| \sum_{|n| \leq N} b_n e(nx) \right|^q dx \ll N^{q/2}$$

for $1 \leq q \leq 2$, but there exists a choice of the b_n with $b_n = \pm 1$ for all n such that

$$(1.7) \quad \int_0^1 \left| \sum_{|n| \leq N} b_n e(nx) \right|^q dx \asymp N^{q/2}$$

uniformly for $1 \leq q \leq 2$. Indeed, we use such b_n in our construction.

Antecedents of Halász's majorant principle (1.1) are found in Wiener & Wintner [15] and in Erdős & Fuchs [3]. Logan [6] showed that the constant 3 in (1.1) is best-possible.

2. LEMMAS

We begin with a generalization of a result of H. S. Shapiro [12].

Let the sequence $\{r_n\}_{n=0}^{\infty}$ be defined by the relations $r_0 = 1$, $r_{2n} = r_n$ and $r_{2n+1} = (-1)^n r_n$. The sequence $\{r_n\}_{n=0}^{\infty}$ is the classical Rudin–Shapiro sequence. Suppose that the binary expansion of n is $n = \sum_{j \geq 0} e_j(n) 2^j$ where $e_j(n) = 0$ or 1 . A well-known alternative definition is

$$r_n = (-1)^{H(n)}, \quad \text{where } H(n) := \sum_{j \geq 0} e_j(n) e_{j+1}(n) \quad (n \geq 0).$$

Let $p_m(z)$, $q_m(z)$ denote polynomials defined recursively by the relations $p_0(z) = 1$, $q_0(z) = 1$ and

$$(2.1) \quad \begin{aligned} p_{m+1}(z) &= p_m(z) + z^{2^m} q_m(z) \\ q_{m+1}(z) &= p_m(z) - z^{2^m} q_m(z). \end{aligned}$$

One can easily check that

$$p_m(z) = \sum_{0 \leq n \leq 2^m - 1} r_n z^n \quad (m \geq 0).$$

The Rudin–Shapiro sequence may be generalized by the so-called paper-folding twist. This amounts to introducing a sequence $\{\varepsilon_m\}_{m=0}^{\infty} \in \{\pm 1\}^{\mathbb{N}}$ and replacing (2.1) by

$$(2.2) \quad \begin{aligned} p_{m+1}(z) &= p_m(z) + \varepsilon_m z^{2^m} q_m(z) \\ q_{m+1}(z) &= p_m(z) - \varepsilon_m z^{2^m} q_m(z). \end{aligned}$$

We then obtain

$$p_m(z) = \sum_{0 \leq n \leq 2^m - 1} c_n z^n \quad (m = 0, 1, \dots)$$

with

$$(2.3) \quad c_n = r_n \prod_{j \geq 0} \varepsilon_j^{e_j(n)} \quad (n \geq 0),$$

a formula for which we did not find a reference in the literature and which B. Saffari kindly pointed out to us.

Lemma 2.1. *Let the sequence c_m be defined as above. Put*

$$(2.4) \quad P_M(\vartheta) = \sum_{0 \leq m < M} c_m e(m\vartheta).$$

Then

$$|P_M(\vartheta)| \leq (2 + \sqrt{2})\sqrt{M}$$

for all positive integers M and all real ϑ .

Shapiro proved this in the case $c_m = r_m$ ($m \geq 0$) but never published his work. The coefficients r_m were independently discovered by Golay [4]. Rudin [11] published an account of Shapiro's argument in the case $M = 2^k$, but obtained an inferior constant in the general case. The above lemma is proved in [7, théorème 2].

We note in passing that it follows from the proof of theorem 2 of [7] that, given an arbitrary sequence $\{\eta_j\}_{j=0}^\infty \in \{\pm 1\}^\mathbb{N}$, a generalized Rudin–Shapiro sequence may alternatively be written as

$$(2.5) \quad c_m = (-1)^{v_m}$$

where v_m equals 0 or 1 according to whether $\sum_{j \geq 0} \eta_j |e_j(m) - e_{j+1}(m)|$ belongs to $\{0, 1\}$ or to $\{2, 3\}$ modulo 4. In this setting, we recover r_m by selecting $\eta_j = (-1)^j$ ($j \geq 0$). Also, this easily enables retrieving (2.3).

Lemma 2.2. *For $|z| < 1$, let $f(z) = \sum_{m \geq 0} c_m z^m$ where c_m is defined as in (2.3). Then*

$$|f(re(\vartheta))| \leq \frac{2 + \sqrt{2}}{\sqrt{1-r}}.$$

Numerical studies suggest that, at least in the case $c_j = r_j$ ($j \geq 0$),

$$(2.6) \quad \max_{\vartheta} |f(re(\vartheta))| = f(r).$$

Moreover, it is easy to show that, in the above circumstance, $f(r)\sqrt{1-r}$ does not tend to a limit as $r \rightarrow 1^-$. Indeed, it is proved in Brillhart, Erdős and Morton [1] that

$$(2.7) \quad \sum_{m < n} r_m = \sqrt{n} G\left(\frac{\log n}{\log 4}\right) \quad (n \geq 0)$$

where G is 1-periodic and continuous. Moreover, Dumont and Thomas [2] showed that G is nowhere differentiable and Tenenbaum [13] obtained the oscillation result

$$G(x+h) - G(x) = \Omega(\sqrt{h}) \quad (h \geq 0)$$

for any given real number x .

By partial summation it is readily derived from (2.7) that $f(r)\sqrt{1-r}$ oscillates as $r \rightarrow 1-$: indeed, as $y \rightarrow \infty$,

$$(2.8) \quad 2^y f(\exp(-4^{-y}))$$

tends to a 1-periodic, nowhere differentiable function of y .

It is noteworthy that

$$(2.9) \quad \begin{aligned} f(z) &= f(z^2) + f(-z^2), \\ f(-z) &= f(z^2) - f(-z^2). \end{aligned}$$

Kumiko Nishioka [10] showed that $f(z)$ and $f(-z)$ are algebraically independent, and then used these recurrences and Mahler's method to show that if α is algebraic with $0 < |\alpha| < 1$, then $f(\alpha)$ and $f(-\alpha)$ are algebraically independent.

Proof. Clearly

$$\frac{f(\operatorname{re}(\vartheta))}{1-r} = \sum_{m \geq 0} P_{m+1}(\vartheta) r^m.$$

Hence by Lemma 2.1 and the triangle inequality it follows that

$$(2.10) \quad \left| \frac{f(\operatorname{re}(\vartheta))}{1-r} \right| \leq (2 + \sqrt{2}) \sum_{m \geq 0} \sqrt{m+1} r^m.$$

But

$$(2.11) \quad \sqrt{m+1} \leq \binom{m+1/2}{m}$$

for all non-negative integers m . Hence the right hand side of (2.10) is

$$\leq (2 + \sqrt{2}) \sum_{m \geq 0} \binom{m+1/2}{m} r^m = \frac{2 + \sqrt{2}}{(1-r)^{3/2}},$$

which gives the stated result.

To prove (2.11), let $a_m = \binom{m+1/2}{m} / \sqrt{m+1}$. To show that $a_m \geq 1$, it suffices to note that $a_0 = 1$, and to show that the a_m are increasing. As to this latter point, we observe that

$$\frac{a_m}{a_{m-1}} = \frac{m+1/2}{\sqrt{m(m+1)}} = \frac{2m+1}{\sqrt{(2m+1)^2 - 1}} > 1.$$

□

Lemma 2.3. *Let f be defined as in Lemma 2.2. For each r , $0 < r < 1$, there is a measurable set $A_r \subseteq \mathbb{T}$ with Lebesgue measure $\lambda(A_r) \geq 1/50$, such that*

$$(2.12) \quad |f(\operatorname{re}(\vartheta))| \geq \frac{1}{2\sqrt{1-r}}$$

for all $\vartheta \in A_r$.

Proof. Let $B_r = \mathbb{T} \setminus A_r$ be the complementary set of those ϑ on which $|f|$ is small; precisely $|f(\operatorname{re}(\vartheta))| < 1/(2\sqrt{1-r})$. By Parseval's identity we know that

$$(2.13) \quad \int_0^1 |f(\operatorname{re}(\vartheta))|^2 d\vartheta = \sum_{m \geq 0} r^{2m} = \frac{1}{1-r^2} > \frac{1}{2(1-r)}.$$

By Lemma 2.2, the left hand side above equals

$$\int_{A_r} |f(re(\vartheta))|^2 d\vartheta + \int_{B_r} |f(re(\vartheta))|^2 d\vartheta \leq \frac{(2 + \sqrt{2})^2}{1-r} \lambda(A_r) + \frac{1 - \lambda(A_r)}{4(1-r)}.$$

On combining this with (2.13), we find that

$$\frac{1}{4} \leq ((2 + \sqrt{2})^2 - \frac{1}{4}) \lambda(A_r),$$

which gives the stated result. \square

Lemma 2.4. *Write $s = \sigma + it$. Then*

$$\sum_{n \leq x} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'}{\zeta}(s) + \frac{x^{1-s}}{1-s} + O\left(\frac{x^{1-\sigma}}{\exp(\sqrt{\log x})}\right)$$

for $x \geq 2$, $1 < \sigma \leq 2$, $-1 \leq t \leq 1$.

This is included in equation (III.5.72) of [14] and is proved by Perron's summation formula (see Montgomery–Vaughan [9, Theorem 5.2] or Tenenbaum [14, corollary II.2.4]) appealing to the classical zero-free region and estimates for $\zeta'(s)/\zeta(s)$ in the zero-free region.

Lemma 2.5. *For $x \geq 2$, $1 < \sigma \leq 2$, and $-1 \leq t \leq 1$, we have*

$$\sum_{n \leq x} \frac{1}{n^s} = \zeta(s) + \frac{x^{1-s}}{1-s} + O\left(\frac{1}{x^\sigma}\right).$$

This is immediate by partial summation; see Montgomery–Vaughan [9, Theorem 1.12] or Tenenbaum [14, theorem II.3.5] for the details.

3. PROOF OF THE THEOREM

In view of the upper bounds of (1.4), it suffices to establish lower bounds. For $I(q)$ we let c_m be defined as in (2.3), and take $a_p := c_m$ for $e^{\pi m} < p < e^{\pi(m+1)}$. Then, for $s = \sigma + it$, $1 < \sigma \leq 2$, $|t| \leq 1$,

$$\begin{aligned} \sum_p \frac{a_p \log p}{p^s} &= \sum_{m \geq 0} c_m \sum_{e^{\pi m} < p < e^{\pi(m+1)}} \frac{\log p}{p^s} \\ (3.1) \quad &= \sum_{m \geq 0} c_m \left(\sum_{e^{\pi m} < n < e^{\pi(m+1)}} \frac{\Lambda(n)}{n^s} + O(e^{\pi m(1-2\sigma)}) \right). \end{aligned}$$

By Lemma 2.4 this is

$$= \sum_{m \geq 0} c_m \left(\frac{e^{\pi(m+1)(1-s)}}{1-s} - \frac{e^{\pi m(1-s)}}{1-s} \right) + O\left(\sum_{m \geq 0} \frac{e^{\pi m(1-\sigma)}}{\exp(c\sqrt{m})} \right) + O(1).$$

Here the first error term is also $O(1)$, uniformly for $\sigma \geq 1$. The main term is

$$(3.2) \quad F(s) f(e^{\pi(1-s)})$$

where f is defined in Lemma 2.2, and

$$(3.3) \quad F(s) = \frac{e^{\pi(1-s)} - 1}{1-s}.$$

The zeros of this entire function are the numbers $1+2im$ where m runs over non-zero integers. Thus $|F(s)|$ is bounded away from 0 uniformly on the rectangle $1 \leq \sigma \leq 2$,

$-1 \leq t \leq 1$. With a little computation one can in fact show that the minimum of $|F(s)|$ in this rectangle is $|F(2 \pm i)| \approx 0.73766$. By Lemma 2.3 it follows that if σ is fixed, $1 < \sigma \leq 2$, then

$$\left| \sum_p \frac{a_p \log p}{p^{\sigma+it}} \right| \gg \frac{1}{\sqrt{1 - e^{\pi(\sigma-1)}}} \asymp \frac{1}{\sqrt{\sigma-1}}$$

when $t/2 \in A_{e^{\pi(\sigma-1)}}$, i.e. on a subset of $-1 \leq t \leq 1$ of measure $> 1/25$. Hence $I(q) \gg (\sigma-1)^{-q/2}$.

The proof for $J(q)$ is the same, except that now the passage from $\log p$ to $\Lambda(n)$ in (3.1) is unnecessary, and instead of Lemma 2.4 we use Lemma 2.5, in which the error term is smaller.

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