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Benot Daniel, Pablo Mira

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Benoît Daniel[a] and Pablo Mira[b]

[a] Université Paris 12, Département de Mathématiques, UFR des Sciences et Technologies, 61 avenue du Général de Gaulle, 94010 Créteil cedex, France
e-mail: daniel@univ-paris12.fr

[b] Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, E-30203 Cartagena, Murcia, Spain.
e-mail: pablo.mira@upct.es

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Abstract

We study the classification of immersed constant mean curvature (CMC) spheres in the homogeneous Riemannian 3-manifold Sol_3, i.e., the only Thurston 3-dimensional geometry where this problem remains open. Our main result states that, for every $H > 1/\sqrt{3}$, there exists a unique (up to left translations) immersed CMC H sphere S_H in Sol_3 (Hopf-type theorem). Moreover, this sphere S_H is embedded, and is therefore the unique (up to left translations) compact embedded CMC H surface in Sol_3 (Alexandrov-type theorem). The uniqueness parts of these results are also obtained for all real numbers H such that there exists a solution of the isoperimetric problem with mean curvature H.

1 Introduction

Two fundamental results in the theory of compact constant mean curvature (CMC) surfaces are the Hopf and Alexandrov theorems. The first one [12] states that round spheres are the unique immersed CMC spheres in Euclidean space $\mathbb{R}^3$; the proof relies on the existence of a holomorphic quadratical differential, the so-called Hopf differential.

The second one [4] states that round spheres are the unique compact embedded CMC surfaces in Euclidean space $\mathbb{R}^3$; the proof is based on the so-called Alexandrov reflection technique, and uses the maximum principle. Hopf's theorem can be generalized immediately to hyperbolic space $\mathbb{H}^3$ and the sphere $\mathbb{S}^3$, and Alexandrov's theorem to $\mathbb{H}^3$ and a hemisphere of $\mathbb{S}^3$.

An important problem from several viewpoints is to generalize the Hopf and Alexandrov theorems to more general ambient spaces - for instance, isoperimetric regions in a Riemannian 3-manifold are bounded by compact embedded CMC surfaces. In this sense, among all possible choices of ambient spaces, the simply connected homogeneous 3-manifolds are placed in a privileged position. Indeed, they are the most symmetric Riemannian 3-manifolds other than the spaces of constant curvature, and are tightly linked to Thurston's 3-dimensional geometries. Moreover, the global study of CMC surfaces in these homogeneous spaces in currently a topic of great activity.

Hopf's theorem was extended by Abresch and Rosenberg [1, 2] to all simply connected homogeneous 3-manifolds with a 4-dimensional isometry group, i.e., $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, the Heisenberg group Nil_3, the universal cover of $\mathrm{PSL}_2(\mathbb{R})$ and the Berger spheres: any immersed CMC sphere in any of these spaces is a standard rotational sphere. To do this, they proved the existence of a holomorphic quadratic differential, which is a linear combination of the Hopf differential and of a term coming from a certain ambient Killing field. Once there, the proof is similar to Hopf's: such a differential must vanish on a sphere, and this implies that the sphere is rotational.

On the other hand, Alexandrov's theorem extends readily to $\mathbb{H}^2 \times \mathbb{R}$ and a hemisphere of $\mathbb{S}^2$ times $\mathbb{R}$ in the following way: any compact embedded CMC surface is a standard rotational sphere (see for instance [13]). The key property of these ambient manifolds is that there exist reflections with respect to vertical planes, and this makes the Alexandrov reflection technique work. In contrast, the Alexandrov problem in Nil_3, the universal cover of $\mathrm{PSL}_2(\mathbb{R})$ and the Berger hemispheres is still open, since there are no reflections in these manifolds.

The purpose of this paper is to investigate the Hopf problem in the simply connected homogeneous Lie group Sol_3, i.e., the only Thurston 3-dimensional geometry where this problem remains open. The topic is a natural and widely commented extension of the Abresch-Rosenberg theorem, but in this Sol_3 setting there are substantial difficulties that do not appear in other homogeneous spaces.

One of these difficulties is that Sol_3 has an isometry group only of dimension 3, and has no rotations. Hence, there are no known explicit CMC spheres, since, contrarily to other homogeneous 3-manifolds, we cannot reduce the problem of finding CMC spheres to solving an ordinary differential equation (there are no compact one-parameter subgroups of ambient isometries). Let us also observe that geodesic spheres are not CMC [19]. Moreover, even the existence of a CMC sphere for a specific mean curvature $H \in \mathbb{R}$ needs to be settled (although the existence of isoperimetric CMC spheres is known). Other basic difficulty is that the Abresch-Rosenberg quadratic differential does not exist in Sol_3 (more precisely, Abresch and Rosenberg claimed that there is no holomorphic quadratic differential of a certain form for CMC surfaces in Sol_3 [2]).

As regards the Alexandrov problem in Sol_3, a key fact is that Sol_3 admits two foliations by totally geodesic surfaces such that reflections with respect to the leaves are isometries; this ensures that a compact embedded CMC surface is topologically a sphere (see [10]). Hence the problem of classifying compact embedded CMC surfaces is solved as soon as the Hopf problem is solved and embeddedness of the examples is studied.

We now state the main theorems of this paper. We will generally assume without loss of generality that $H \geqslant 0$, by changing orientation if necessary. We also refer to Section 5.1 for the basic definitions regarding stability, index and the Jacobi operator.

Theorem 1.1 *Let $H > 1/\sqrt{3}$. Then:*

i) There exists an embedded CMC H sphere S_H in Sol_3.

ii) Any immersed CMC H sphere in Sol_3 differs from S_H at most by a left translation.

iii) Any compact embedded CMC H surface in Sol_3 differs from S_H at most by a left translation.

Moreover, these canonical spheres S_H constitute a real analytic family, they all have index one and two reflection planes, and their Gauss maps are global diffeomorphisms into $\mathbb{S}^2$.

As explained before, the Alexandrov-type uniqueness *iii*) follows from the Hopf-type uniqueness *ii*), by using the standard Alexandrov reflection technique with respect to the two canonical foliations of Sol_3 by totally geodesic surfaces.

We actually have the following more general uniqueness theorem.

Theorem 1.2 *Let $H > 0$ such that there exists some immersed CMC H sphere Σ_H in Sol_3 verifying* one *of the properties (a)-(d), where actually $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$:*

(a) It is a solution to the isoperimetric problem in Sol_3.

(b) It is a (weakly) stable surface.

(c) It has index one.

(d) Its Gauss map is a (global) diffeomorphism into $\mathbb{S}^2$.

Then Σ_H is embedded and unique up to left translations in Sol_3 among immersed CMC H spheres (Hopf-type theorem) and among compact embedded CMC H surfaces (Alexandrov-type theorem).

Let us remark that solutions of the isoperimetric problem in Sol_3 are embedded CMC spheres. Hence, we can deduce from results of Pittet [24] that the infimum of the set of $H > 0$ such that there exists a CMC H sphere satisfying (a) is 0 (see Section 2.2). We will additionally prove that for all $H > 1/\sqrt{3}$ there exists a CMC H sphere satisfying (c), which gives Theorem 1.1.

We outline the proof of the theorems. We first study the Gauss map of CMC H immersions into Sol_3. We prove in Section 3 that the Gauss map is nowhere antiholomorphic and satisfies a certain second order elliptic equation. Conversely we obtain a Weierstrass-type representation formula that allows to recover a CMC H immersion from a nowhere antiholomorphic solution of this elliptic equation.

To prove uniqueness of CMC H spheres, the main idea will be to ensure the existence of a quadratic differential $Q\mathrm{d}z^2$ that satisfies the so-called *Cauchy-Riemann inequality* (a property weaker than holomorphicity, introduced by Alencar, do Carmo and Tribuzy [3]) for all CMC H immersions. This will be the purpose of Section 4. The main obstacle is that it seems very difficult and maybe impossible to obtain such a differential (or even just a CMC sphere) *explicitly*. We are able to prove the existence of this differential provided there exists a CMC H sphere whose Gauss map G is a (global) diffeomorphism of $\mathbb{S}^2$ (our differential $Q\mathrm{d}z^2$ will be defined using this G).

The next step is to study the existence of CMC spheres whose Gauss map is a diffeomorphism. This is done in Section 5. We first prove that the Gauss map of an isoperimetric sphere, and more generally of an index one CMC sphere, is a diffeomorphism. For this purpose we use a nodal domain argument. We also prove that a CMC sphere whose Gauss map is a diffeomorphism is embedded.

Then we deform an isoperimetric sphere with large mean curvature by the implicit function theorem. More generally we prove that we can deform index one CMC spheres, and that the property of having index one is preserved by this deformation. In this way we prove that there exists an index one CMC H sphere for all $H > 1/\sqrt{3}$. To do this we need a bound on the second fundamental form and a bound on the diameter of the spheres. This diameter estimate is a consequence of a theorem of Rosenberg [28] and relies on a stability argument; however, this estimate only holds for $H > 1/\sqrt{3}$. This will complete the proof.

We conjecture that Theorem 1.1 should hold for every $H > 0$ (see Section 6). Also, in this last section we will explain how our results give information on the symmetries of solutions to the isoperimetric problem in Sol_3.

A remarkable novelty of our approach to this problem is that we obtain a Hopf-type theorem for a class of surfaces without knowing explicitly beforehand the spheres for which uniqueness is aimed, or at least some key property of them (e.g. that they are rotational). This suggests that the ideas of our approach may be suitable for proving Hopf-type theorems in many other theories (see Remark 4.8).

In this sense, let us mention that Sol_3 belongs to a 2-parameter family of homogeneous 3-manifolds, which also includes $\mathbb{R}^3$, $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{R}$. The manifolds in this family generically possess, as Sol_3, an isometry group of dimension 3. However, they are "less symmetric" than Sol_3, in the sense that their isometry group only has 4 connected components, whereas the isometry group of Sol_3 has 8 connected components. Also, contrarily to Sol_3, these manifolds do not have compact quotients (this is the reason why they are not Thurston geometries, see for instance [6]). We believe that it is very possible that the techniques developed in this paper also work in these manifolds, since we do not use these additional properties of Sol_3.

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2 The Lie group Sol_3

We will study surfaces immersed in the homogeneous Riemannian 3-manifold Sol_3, i.e., the least symmetric of the eight canonical Thurston 3-geometries. This preliminary section is intended to explain the basic geometric elements of this ambient space, and the consequences for CMC spheres of some known results.

2.1 Isometries, connection and foliations

The space Sol_3 can be viewed as $\mathbb{R}^3$ endowed with the Riemannian metric

$$\langle , \rangle = e^{2x_3}\mathrm{d}x_1^2 + e^{-2x_3}\mathrm{d}x_2^2 + \mathrm{d}x_3^2,$$

where (x_1, x_2, x_3) are canonical coordinates of $\mathbb{R}^3$. It is important to observe that Sol_3 has a Lie group structure with respect to which the above metric is left-invariant. The group structure is given by the multiplication

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + e^{-x_3}\, y_1, x_2 + e^{x_3}\, y_2, x_3 + y_3).$$

The isometry group of Sol_3 is easy to understand; it has dimension 3 and the connected component of the identity is generated by the following three families of isometries:

$$(x_1, x_2, x_3) \mapsto (x_1 + c, x_2, x_3), \quad (x_1, x_2, x_3) \mapsto (x_1, x_2 + c, x_3),$$
$$(x_1, x_2, x_3) \mapsto (e^{-c}x_1, e^{c}x_2, x_3 + c).$$

Obviously, these isometries are just *left translations* in Sol_3 with respect to the Lie group structure above, i.e., left multiplications by elements in Sol_3. On the contrary, right translations are not isometries.

The corresponding Killing fields associated to these families of isometries are

$$F_1 = \frac{\partial}{\partial x_1}, \quad F_2 = \frac{\partial}{\partial x_2}, \quad F_3 = -x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}.$$

They are right-invariant.

Another key property of Sol_3 is that it admits reflections. Indeed, Euclidean reflections in the (x_1, x_2, x_3) coordinates with respect to the planes $x_1 = \text{const.}$ and $x_2 = \text{const.}$ are orientation-reversing isometries of Sol_3. A very important consequence of this is that we can use the Alexandrov reflection technique in the x_1 and x_2 directions, as we will explain in Section 2.2.

More specifically, the isotropy group of the origin $(0, 0, 0)$ is isomorphic to the dihedral group D_4 and is generated by the following two isometries:

$$\sigma : (x_1, x_2, x_3) \mapsto (x_2, -x_1, -x_3), \quad \tau : (x_1, x_2, x_3) \mapsto (-x_1, x_2, x_3). \tag{2.1}$$

These two isometries are orientation-reversing; σ has order 4 and τ order 2. Observe that the reflection with respect to the plane $x_2 = 0$ is given by $\sigma^2\tau$.

An important role in Sol_3 is played by the left-invariant orthonormal frame (E_1, E_2, E_3) defined by

$$E_1 = e^{-x_3}\frac{\partial}{\partial x_1}, \quad E_2 = e^{x_3}\frac{\partial}{\partial x_2}, \quad E_3 = \frac{\partial}{\partial x_3}.$$

We call it the *canonical frame*. The coordinates with respect to the frame (E_1, E_2, E_3) of a vector at a point $x = (x_1, x_2, x_3) \in \mathrm{Sol}_3$ will be denoted into brackets; then we have

$$a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\frac{\partial}{\partial x_3} = \begin{bmatrix} e^{x_3}a_1 \\ e^{-x_3}a_2 \\ a_3 \end{bmatrix}. \tag{2.2}$$

The expression of the Riemannian connection $\widehat{\nabla}$ of Sol_3 with respect to the canonical frame is the following:

$$\begin{array}{lll} \widehat{\nabla}_{E_1}E_1 = -E_3, & \widehat{\nabla}_{E_2}E_1 = 0, & \widehat{\nabla}_{E_3}E_1 = 0, \\ \widehat{\nabla}_{E_1}E_2 = 0, & \widehat{\nabla}_{E_2}E_2 = E_3, & \widehat{\nabla}_{E_3}E_2 = 0, \\ \widehat{\nabla}_{E_1}E_3 = E_1, & \widehat{\nabla}_{E_2}E_3 = -E_2, & \widehat{\nabla}_{E_3}E_3 = 0. \end{array} \tag{2.3}$$

From there, we see that the sectional curvatures of the planes (E_2, E_3), (E_1, E_3) and (E_1, E_2) are -1, -1 and 1 respectively, and that the Ricci curvature of Sol_3 is given, with respect to the canonical frame (E_1, E_2, E_3), by

$$\mathrm{Ric} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

In particular, Sol_3 has constant scalar curvature -2.

One of the nicest features of Sol_3 is the existence of three canonical foliations with good geometric properties. First, we have the foliations

$$\mathcal{F}_1 \equiv \{x_1 = \text{constant}\}, \qquad \mathcal{F}_2 \equiv \{x_2 = \text{constant}\},$$

which are orthogonal to the Killing fields F_1, F_2, respectively.

It is immediate from the expression of the metric in Sol_3 that the leaves of this foliation are isometric to the hyperbolic plane $\mathbb{H}^2$. For instance, for a leaf S of $\mathcal{F}_1$, the coordinates $(x_2, x_3) \in \mathbb{R}^2$ are horospherical coordinates of $\mathbb{H}^2$, i.e.

$$\langle,\rangle|_S = e^{-2x_3}\mathrm{d}x_2^2 + \mathrm{d}x_3^2, \qquad (x_2, x_3) \in \mathbb{R}^2.$$

For us, the main property of these foliations is that Euclidean reflections across a leaf of $\mathcal{F}_1, \mathcal{F}_2$ are orientation-reversing isometries of Sol_3. Also, these leaves are the only totally geodesic surfaces in Sol_3 [31].

The third canonical foliation of Sol_3 is $\mathcal{G} \equiv \{x_3 = \text{constant}\}$. Its leaves are isometric to $\mathbb{R}^2$ and are minimal. This foliation $\mathcal{G}$ is less important than $\mathcal{F}_1, \mathcal{F}_2$, since Euclidean reflections with respect to the planes $x_3 =$ constant do not describe isometries in Sol_3 anymore. In addition, $\mathcal{G}$ is no longer orthogonal to a Killing field of Sol_3.

The existence of these foliations by minimal surfaces also implies (by the maximum principle) that there is no compact minimal surface in Sol_3.

Let us also observe that the map $\mathrm{Sol}_3 \to \mathbb{R}, (x_1, x_2, x_3) \mapsto x_3$ is a Riemannian fibration. This means that the coordinate x_3 has a geometric meaning (whereas x_1 and x_2 do not have it).

For more details we refer to [32]. Some papers, such as [32] and [14], have a different convention for the metric of Sol_3.

2.2 Alexandrov reflection and the isoperimetric problem

Let us now focus on the Alexandrov reflection technique. Let S be an embedded compact CMC surface in Sol_3. Then, applying the Alexandrov technique in the x_1 direction, we obtain that S is, up to a left translation, a symmetric bigraph in the x_1 direction over some domain U in the plane $x_1 = 0$. But then we can apply the Alexandrov reflection technique in the x_2 direction. This implies that, up to a left translation, U can be written as

$$U = \{(x_2, x_3) \in \mathbb{R}^2; x_3 \in I, x_2 \in [-f(x_3), f(x_3)]\}$$

for some continuous function f and some set I, which is necessarily an interval (otherwise S would not be connected). Then U is topologically a disk, and S is topologically a sphere. Hence we have the following fundamental result (see the concluding remarks in [10]).

Fact 2.1 *A compact embedded CMC surface in* Sol_3 *is a sphere.*

In particular, all solutions of the isoperimetric problem in Sol_3 are spheres.

Some important information about the mean curvatures of the solutions of the isoperimetric problem in Sol_3 can be deduced from results of Pittet [24].

The identity component of the isometry group of Sol_3 is Sol_3 itself, acting by left multiplication. It is a solvable Lie group, hence amenable (an amenable Lie group is a compact extension of a solvable Lie group). It is unimodular: its Haar measure $\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$ is biinvariant. It has exponential growth, i.e., the volume of a geodesic ball of radius r increases exponentially with r (see for example [8]). Consequently, by Theorem 2.1 in [24], there exist positive constants c_1 and c_2 such that the isoperimetric profile $I(v)$ of Sol_3 satisfies

$$\frac{c_1 v}{\ln v} \leqslant I(v) \leqslant \frac{c_2 v}{\ln v} \tag{2.4}$$

for all v large enough (let us recall that the isoperimetric profile $I(v)$ is defined as the infimum of the areas of compact surfaces enclosing a volume v).

It is well-known that I admits left and right derivatives $I'_-(v)$ and $I'_+(v)$ at every $v \in (0, +\infty)$, and there exist isoperimetric surfaces of mean curvatures $I'_-(v)/2$ and $I'_+(v)/2$ respectively (see for instance [27]; here the mean curvature is computed for the unit normal pointing into the compact domain bounded by the surface, i.e., the mean curvature is positive because of the maximum principle used with respect to a minimal surface $x_3 = \text{constant}$). Also by (2.4) the numbers $I'_-(v)$ and $I'_+(v)$ cannot be bounded from below by a positive constant. From this and the fact that isoperimetric surfaces are spheres (since they are embedded), we get the following result.

Fact 2.2 *Let $\mathcal{J}$ be the set of real numbers $H > 0$ such that there exists an isoperimetric sphere of mean curvature H. Then*

$$\inf \mathcal{J} = 0.$$

Even though we will not use it, it is worth pointing out the following consequence about *entire graphs* (a surface is said to be an entire graph with respect to a non-zero Killing field F of Sol_3 if it is transverse to F and intersects every orbit of F at exactly one point).

Corollary 2.3 *Any entire CMC graph with respect to a non-zero Killing field of* Sol_3 *must be minimal ($H = 0$).*

Proof: Let G be an entire CMC H graph with respect to a non-zero Killing field F with $H \neq 0$. By Fact 2.2, there exists an isoperimetric sphere S whose mean curvature H_0 satisfies $H_0 \leqslant |H|$.

Let $(\varphi_c)_{c\in\mathbb{R}}$ be the one-parameter group of isometries generated by F. Then $(\varphi_c(G))_{c\in\mathbb{R}}$ is a foliation of Sol_3. Let $c_0 = \max\{c \in \mathbb{R}; \varphi_c(G) \cap S \neq \emptyset\}$ and $c_1 = \min\{c \in \mathbb{R}; \varphi_c(G) \cap S \neq \emptyset\}$. Then at $c = c_0$ or at $c = c_1$ the sphere S is tangent to $\varphi_c(G)$ and is situated in the mean convex side of $\varphi_c(G)$; this contradicts the maximum principle since $H_0 \leqslant |H|$.

□

The same proof also shows that no entire graph can have a mean curvature function bounded from below by a positive constant. In contrast, there exists various entire minimal graphs in Sol_3. Indeed, for instance one can easily check that a graph $x_1 = f(x_2, x_3)$ is minimal if and only if

$$(e^{2x_3} f_{x_3}^2 + 1) f_{x_2x_2} - 2e^{2x_3} f_{x_2} f_{x_3} f_{x_2x_3} + (e^{-2x_3} + e^{2x_3} f_{x_2}^2) f_{x_3x_3} - (e^{2x_3} f_{x_2}^2 - e^{-2x_3}) f_{x_3} = 0,$$

and so the following equations define entire F_1-graphs:

$$x_1 = ax_2 + b, \quad x_1 = ae^{-x_3}, \quad x_1 = ax_2e^{-x_3}, \quad x_1 = x_2e^{-2x_3}.$$

3 The Gauss map

In this section we will expose the basic equations for immersed surfaces in Sol_3, putting special emphasis on the geometry of the Gauss map associated to the surface in terms of the Lie group structure of Sol_3.

So, let us consider an immersed oriented surface in Sol_3, that will be seen as a conformal immersion $X:\Sigma\to\mathrm{Sol}_3$ of a Riemann surface Σ. We shall denote by $N:\Sigma\to T\mathrm{Sol}_3$ its unit normal.

If we fix a conformal coordinate $z=u+iv$ in Σ, then we have

$$\langle X_z, X_{\bar z}\rangle = \frac{\lambda}{2} > 0, \qquad \langle X_z, X_z\rangle = 0,$$

where λ is the conformal factor of the metric with respecto to z. Moreover, we will denote the coordinates of X_z and N with respect to the canonical frame (E_1,E_2,E_3) by

$$X_z = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}, \quad N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}.$$

The usual *Hopf differential* of X, i.e., the $(2,0)$ part of its complexified second fundamental form, is defined as

$$P\mathrm{d}z^2 = \langle N, \widehat{\nabla}_{X_z} X_z\rangle \mathrm{d}z^2.$$

From the definitions, we have the basic algebraic relations

$$\begin{cases} |A_1|^2+|A_2|^2+|A_3|^2 = \dfrac{\lambda}{2}, \\ A_1^2+A_2^2+A_3^2 = 0, \\ N_1^2+N_2^2+N_3^2 = 1, \\ A_1N_1+A_2N_2+A_3N_3 = 0. \end{cases} \tag{3.1}$$

A classical computation proves that the Gauss-Weingarten equations of the immersion read as

$$\begin{cases} \widehat{\nabla}_{X_z} X_z = \dfrac{\lambda_z}{\lambda} X_z + PN, \\ \widehat{\nabla}_{X_{\bar z}} X_z = \dfrac{\lambda H}{2} N, \\ \widehat{\nabla}_{X_{\bar z}} X_{\bar z} = \dfrac{\lambda_{\bar z}}{\lambda} X_{\bar z} + \bar P N, \end{cases} \qquad \begin{cases} \widehat{\nabla}_{X_z} N = -HX_z - \dfrac{2P}{\lambda} X_{\bar z}, \\ \widehat{\nabla}_{X_{\bar z}} N = -\dfrac{2\bar P}{\lambda} X_z - HX_{\bar z}. \end{cases} \tag{3.2}$$

Using (2.3) in these equations we get

$$\begin{cases} A_{1z} = \dfrac{\lambda_z}{\lambda} A_1 + PN_1 - A_1A_3, \\ A_{2z} = \dfrac{\lambda_z}{\lambda} A_2 + PN_2 + A_2A_3, \\ A_{3z} = \dfrac{\lambda_z}{\lambda} A_3 + PN_3 + A_1^2 - A_2^2, \end{cases} \qquad \begin{cases} A_{1\bar z} = \dfrac{\lambda H}{2} N_1 - \bar A_1 A_3, \\ A_{2\bar z} = \dfrac{\lambda H}{2} N_2 + \bar A_2 A_3, \\ A_{3\bar z} = \dfrac{\lambda H}{2} N_3 + |A_1|^2 - |A_2|^2, \end{cases} \tag{3.3}$$

$$\begin{cases} N_{1z} = -HA_1 - \dfrac{2P}{\lambda}\bar{A}_1 - A_1N_3, \\ N_{2z} = -HA_2 - \dfrac{2P}{\lambda}\bar{A}_2 + A_2N_3, \\ N_{3z} = -HA_3 - \dfrac{2P}{\lambda}\bar{A}_3 + A_1N_1 - A_2N_2. \end{cases} \tag{3.4}$$

Moreover, the fact that $X_z \times X_{\bar{z}} = i\frac{\lambda}{2}N$ implies that

$$N_1 = -\frac{2i}{\lambda}(A_2\bar{A}_3 - A_3\bar{A}_2), \quad N_2 = -\frac{2i}{\lambda}(A_3\bar{A}_1 - A_1\bar{A}_3), \quad N_3 = -\frac{2i}{\lambda}(A_1\bar{A}_2 - A_2\bar{A}_1). \tag{3.5}$$

Once here, let us define the *Gauss map* associated to the surface. We first set

$$\widehat{N} = (N_1, N_2, N_3) : \Sigma \to \mathbb{S}^2 \subset \mathbb{R}^3.$$

In other words, for each $z \in \Sigma$, $\widehat{N}(z)$ is just the vector in the Lie algebra of Sol_3 (identified with $\mathbb{R}^3$ by means of the canonical frame) that corresponds to $N(z)$ when we apply a left translation in Sol_3 taking $X(z)$ to the origin.

Definition 3.1 Given an immersed oriented surface $X : \Sigma \to \mathrm{Sol}_3$, the *Gauss map* of X is the map

$$g := \varphi \circ \widehat{N} : \Sigma \to \bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\},$$

where φ is the stereographic projection with respect to the southern pole, i.e.

$$g = \frac{N_1 + iN_2}{1 + N_3}.$$

Here, N_1, N_2, N_3 are the coordinates of the unit normal of X with respect to the canonical frame (E_1, E_2, E_3).

Equivalently, we have

$$N = \frac{1}{1 + |g|^2}\begin{bmatrix} 2\,\mathrm{Re}\, g \\ 2\,\mathrm{Im}\, g \\ 1 - |g|^2 \end{bmatrix}. \tag{3.6}$$

Remark 3.2 The Gauss map obviously remains invariant when we apply a left translation to the surface. On the other hand, if we apply the orientation-preserving isometry of Sol_3

$$\sigma\tau : (x_1, x_2, x_3) \mapsto (x_2, x_1, -x_3)$$

then the Gauss map g of the surface changes to $g^* := i/g$.

Definition 3.3 Given an immersed oriented surface $X : \Sigma \to \mathrm{Sol}_3$, let us denote

$$\eta := 2\langle E_3, X_z\rangle = 2A_3.$$

A straightforward computation from (3.1) proves that

$$A_1 = -\frac{1-\bar{g}^2}{4\bar{g}}\eta, \quad A_2 = i\frac{1+\bar{g}^2}{4\bar{g}}\eta, \quad A_3 = \frac{\eta}{2} \tag{3.7}$$

and thereby

$$\lambda = (1+|g|^2)^2 \frac{|\eta|^2}{4|g|^2}. \tag{3.8}$$

Then the right system in (3.3) becomes

$$(\bar{g}^2+1)\frac{\eta \bar{g}_{\bar{z}}}{4\bar{g}^2} + \frac{1}{4}\left(\bar{g} - \frac{1}{\bar{g}}\right)\eta_{\bar{z}} = (1+|g|^2)\frac{|\eta|^2}{4|g|^2} H\,\mathrm{Re}(g) + |\eta|^2 \frac{1-g^2}{8g}, \tag{3.9}$$

$$i(\bar{g}^2-1)\frac{\eta \bar{g}_{\bar{z}}}{4\bar{g}^2} + \frac{i}{4}\left(\bar{g} + \frac{1}{\bar{g}}\right)\eta_{\bar{z}} = (1+|g|^2)\frac{|\eta|^2}{4|g|^2} H\,\mathrm{Im}(g) - i|\eta|^2 \frac{1+g^2}{8g}, \tag{3.10}$$

$$\frac{1}{2}\eta_{\bar{z}} = \frac{|\eta|^2}{8|g|^2}((1-|g|^4)H - g^2 - \bar{g}^2). \tag{3.11}$$

Reporting (3.11) into (3.9) + i (3.10) gives

$$\frac{g\bar{g}_{\bar{z}}}{2\bar{\eta}} = \frac{H}{8}(1+|g|^2)^2 + \frac{1}{8}(\bar{g}^2 - g^2),$$

i.e.

$$\eta = \frac{4\bar{g}g_z}{R(g)} \qquad \text{where} \qquad R(q) = H(1+|q|^2)^2 + q^2 - \bar{q}^2. \tag{3.12}$$

Once here, we are ready to state the main result of this section.

Theorem 3.4 *Let $X : \Sigma \to \mathrm{Sol}_3$ be a CMC H surface with Gauss map $g : \Sigma \to \bar{\mathbb{C}}$. Then, g is nowhere antiholomorphic, i.e., $g_z \neq 0$ at every point for any local conformal parameter z on Σ, and g verifies the second order elliptic equation*

$$g_{z\bar{z}} = A(g)g_z g_{\bar{z}} + B(g)g_z \bar{g}_{\bar{z}}, \tag{3.13}$$

where, by definition,

$$A(q) = \frac{R_q}{R} = \frac{2H(1+|q|^2)\bar{q} + 2q}{R(q)}, \quad B(q) = \frac{R_{\bar{q}}}{R} - \frac{\bar{R}_{\bar{q}}}{\bar{R}} = -\frac{4H(1+|q|^2)(\bar{q}+q^3)}{|R(q)|^2}, \tag{3.14}$$

$$R(q) = H(1+|q|^2)^2 + q^2 - \bar{q}^2.$$

Moreover, the immersion $X = (x_1, x_2, x_3) : \Sigma \to \mathrm{Sol}_3$ can be recovered in terms of the Gauss map g by means of the representation formula

$$(x_1)_z = e^{-x_3}\frac{(\bar{g}^2-1)g_z}{R(g)}, \quad (x_2)_z = ie^{x_3}\frac{(\bar{g}^2+1)g_z}{R(g)}, \quad (x_3)_z = \frac{2\bar{g}g_z}{R(g)}. \tag{3.15}$$

Conversely, if a map $g : \Sigma \to \bar{\mathbb{C}}$ from a simply connected Riemann surface Σ verifies (3.13) *for $H \neq 0$ and the coefficients A, B given in* (3.14)*, and if g is nowhere antiholomorphic, then the map $X : \Sigma \to \mathrm{Sol}_3$ given by the representation formula* (3.15) *defines a CMC H surface in Sol_3 whose Gauss map is g.*

Remark 3.5 It is immediate that (3.13) is invariant by changes of conformal parameter, i.e., g is a solution to (3.13) if and only if $g\circ\psi$ is a solution to (3.13), where ψ is a locally injective meromorphic function. This shows that we can work with (3.13) at $z=\infty$, just by making the change $z\mapsto 1/z$.

In addition, a direct computation shows that if g is a solution to (3.13), then

$$g^* := \frac{i}{g}$$

is also a solution to (3.13) (see Remark 3.2 for the geometric meaning of this duality). Again, this gives a way to work with (3.13) at points where $g=\infty$.

Remark 3.6 If g is a nowhere antiholomorphic solution to (3.13), inducing a CMC H immersion X, then a direct computation shows that ig and $1/g$ are also solutions to (3.13) with H replaced by $-H$, and they induce the CMC $-H$ immersion $\sigma\circ X$ and $\tau\circ X$. Also, the map $z\mapsto 1/\bar{g}(\bar{z})$ is also solution to (3.13) with H replaced by $-H$, and induces the same surface as g but with the opposite orientation.

Proof: The fact that g is nowhere antiholomorphic follows from (3.8) and (3.12).

Formula (3.13) follows from the constancy of H, just by computing the expression $\eta_{\bar{z}}/\eta$ from (3.12) and reporting it then in (3.11). Besides, the representation formula (3.15) follows directly from (3.7) and (3.12), taking into account the relation (2.2).

To prove the converse, we start with a simply connected domain $\mathcal{U}\subset\Sigma$ on which $g\neq\infty$ at every point. Then, using (3.13), we have

$$\frac{\partial}{\partial\bar{z}}\left(\frac{2\bar{g}g_z}{R(g)}\right)=\frac{2|g_z|^2}{|R(g)|^2}\left(H(1-|g|^4)-(g^2+\bar{g}^2)\right)\in\mathbb{R}.$$

Hence, there exists $x_3:\mathcal{U}\subset\Sigma\to\mathbb{R}$ with

$$(x_3)_z=\frac{2\bar{g}g_z}{R(g)}. \tag{3.16}$$

We define now in terms of x_3 the map $\mathcal{F}:\mathcal{U}\to\mathbb{C}^3$ given by

$$\mathcal{F}=\left(e^{-x_3}\frac{(\bar{g}^2-1)g_z}{R(g)},\, i\,e^{x_3}\frac{(\bar{g}^2+1)g_z}{R(g)},\,\frac{2\bar{g}g_z}{R(g)}\right). \tag{3.17}$$

Again using (3.13) it can be checked that $\mathcal{F}_{\bar{z}}\in\mathbb{R}$ at every point. So, there exists $X=(x_1,x_2,x_3):\mathcal{U}\to\mathbb{R}^3\equiv\mathrm{Sol}_3$ with $X_z=\mathcal{F}$. This indicates that the representation formula (3.15) gives, indeed, a well defined map $X:\mathcal{U}\subset\Sigma\to\mathrm{Sol}_3$. Clearly, X is defined up to left translation in Sol_3, due to the integration constants involved in (3.15).

Now, let us assume that $\mathcal{U}\subset\Sigma$ is a simply connected domain on which g may take the value ∞ at some points, but not the value 0. Let us define $g^*=i/g$. By Remark 3.5, g^* is a solution to (3.13) not taking the value ∞ on $\mathcal{U}$, and so we already have proved the existence of a map $X^*=(x_1^*,x_2^*,x_3^*):\mathcal{U}\to\mathrm{Sol}_3$ with

$$X_z^*=\mathcal{F}^*=\left(e^{-x_3}\frac{(\bar{g^*}^2-1)g_z^*}{R(g^*)},\, i\,e^{x_3}\frac{(\bar{g^*}^2+1)g_z^*}{R(g^*)},\,\frac{2\bar{g^*}g_z^*}{R(g^*)}\right).$$

Hence, since $g^* = i/g$, a direct computation shows that $X := (x_2^*, x_1^*, -x_3^*)$ satisfies $X_z = \mathcal{F}$ for $\mathcal{F}$ as in (3.17). Integrating $\mathcal{F}$ again shows that the representation formula (3.15) is well defined also when $g = \infty$ at some points. Finally, as Σ is simply connected, (3.15) can be defined globally on Σ, thus giving rise to a map $X : \Sigma \to \mathrm{Sol}_3$ which is unique up to left translations.

Once here, we need to check that X is a conformal CMC H immersion with Gauss map g. In this sense, it is clear from (3.17) and (2.2) that

$$X_z = [A_1, A_2, A_3],$$

where the A_k's are given by (3.7) and (3.12). Consequently, $\langle X_z, X_z\rangle = 0$ and

$$\langle X_z, X_{\bar z}\rangle = \frac{\lambda}{2} = \frac{2(1+|g|^2)^2}{|R(g)|^2}\,|g_z|^2, \tag{3.18}$$

which is non-zero since g is nowhere antiholomorphic. Thus, X is a conformal immersion.

Besides, denoting by $N = (-2i/\lambda)X_z \times X_{\bar z}$ the unit normal of X, we easily see from (3.5) that (3.6) holds. So, g is indeed the Gauss map of the surface X.

At last, let H_X denote the mean curvature function of X. Putting together (3.7) and (3.12) we see that H_X is given (for any immersed surface in Sol_3) by

$$H_X = \frac{2\bar g g_z}{(1+|g|^2)^2 A_3} - \frac{g^2 - \bar g^2}{(1+|g|^2)^2}.$$

In our case, as (3.7) and (3.12) hold for H, this tells directly that $H = H_X$, as wished. This completes the proof.

□

Remark 3.7 Using (3.4), (3.3), (3.5), (3.7) and (3.12), we obtain after a computation that the Hopf differential $P\,\mathrm{d}z^2$ of the surface is given by

$$\begin{aligned} P &= -A_1 N_{1z} - A_2 N_{2z} - A_3 N_{3z} + N_3(A_2^2 - A_1^2) + A_3(A_1N_1 - A_2N_2) \\ &= \frac{2g_z \bar g_z}{R(g)} - \frac{2(1-\bar g^4)g_z^2}{R(g)^2}. \end{aligned}$$

Let us now make some brief comments regarding the special case of minimal surfaces in Sol_3, i.e. the case $H = 0$. In that situation it is immediate from (3.14) that $B(g) = 0$, and thus the Gauss map equation (3.13) simplifies to

$$g_{z\bar z} = \frac{R_g^*}{R^*}\, g_z g_{\bar z}, \qquad R^*(q) := q^2 - \bar q^2. \tag{3.19}$$

Now, it is immediate that this is the harmonic map equation for maps $g : \Sigma \to (\bar{\mathbb{C}}, \mathrm{d}\sigma^2)$, where $\mathrm{d}\sigma^2$ is the singular metric on $\bar{\mathbb{C}}$ given by

$$\mathrm{d}\sigma^2 = \frac{|\mathrm{d}w|^2}{|w^2 - \bar w^2|}.$$

This result was first obtained in [14]. It is somehow parallel to the case of minimal surfaces in the Heisenberg space Nil_3. Indeed, it is a result of the first author [9] that the Gauss map of a minimal local graph in Nil_3 (with respect to the canonical Riemannian fibration $\mathrm{Nil}_3 \to \mathbb{R}^2$) is a harmonic map into the unit disk $\mathbb{D}$ endowed with the Poincaré metric.

In addition, it is remarkable that *every minimal surface in* Sol_3 *has an associated holomorphic quadratic differential.* Indeed, it follows from (3.19) that the quadratic differential

$$Q^* \mathrm{d}z^2 := \frac{1}{R^*(g)} g_z \bar{g}_z \, \mathrm{d}z^2$$

is holomorphic (whenever it is well defined) on any minimal surface in Sol_3.

Remark 3.8 Theorem 3.4 has been obtained by analyzing the structure equations of immersed CMC surfaces in Sol_3. Alternatively, it could also have been obtained by working with the Dirac equations obtained by Berdinsky and Taimanov in [5].

Example 3.9 Let us compute the CMC surfaces in Sol_3 that are invariant by translations in the x_2 direction. Then we have to look for a real-valued Gauss map g, and we may choose without loss of generality a conformal parameter $z = u + iv$ such that g only depends on u. Then (3.13) becomes

$$g_{uu} = (A(g) + B(g)) g_u^2 = \frac{2Hg(1+g^2) - 2g}{H(1+g^2)^2} g_u^2.$$

From this we get

$$g_u = \beta(1+g^2) \exp\left(\frac{1}{H(1+g^2)}\right) \tag{3.20}$$

for some constant $\beta \in \mathbb{R} \setminus \{0\}$. Up to a multiplication of the conformal parameter by a constant, we may assume that $\beta = 1$.

Conversely, assume that g satisfies $g_v = 0$ and (3.20) with $\beta = 1$. Let us first notice that such a real-valued function is defined on a domain $(u_0, u_1) \times \mathbb{R}$ where u_0 and u_1 are finite, since

$$\int \frac{1}{1+x^2} \exp\left(-\frac{1}{H(1+x^2)}\right) \mathrm{d}x$$

is bounded. On the other hand, the function $\widetilde{g} := 1/g$ satisfies

$$\widetilde{g}_u = -(1+\widetilde{g}^2) \exp\left(\frac{\widetilde{g}^2}{H(1+\widetilde{g}^2)}\right).$$

Hence by allowing the value ∞ we can define a function $g : \mathbb{C} \to \mathbb{R} \cup \{\infty\}$, which will be periodic in u. Then, (3.12) gives

$$A_3 = \frac{g g_u}{H(1+g^2)^2},$$

and so, up to a translation,

$$x_3 = -\frac{1}{H(1+g^2)} + \frac{1}{2H}.$$

Then we see from (3.7) and (3.20) that

$$A_1 = \frac{(g^2-1)g_u}{2H(1+g^2)^2} = \frac{g^2-1}{2H(1+g^2)}e^{-x_3+1/(2H)}, \quad A_2 = \frac{ig_u}{2H(1+g^2)} = \frac{i}{2H}e^{-x_3+1/(2H)},$$

so, up to a translation in the x_2 direction, we get from (2.2)

$$x_1 = \int e^{-2x_3+1/(2H)} \frac{g^2-1}{H(1+g^2)} \mathrm{d}u, \quad x_2 = -\frac{e^{1/(2H)}}{H} v.$$

We now define a new variable t by

$$\cos t = \frac{g^2-1}{1+g^2}, \quad \sin t = \frac{2g}{1+g^2}.$$

Then from (3.20) we have

$$t_u = -2\exp\left(\frac{1}{H(1+g^2)}\right),$$

and hence we get that the immersion $X = (x_1, x_2, x_3)$ is given with respect to these (t, v) coordinates by

$$x_1(t,v) = -\frac{1}{2H}\int e^{-(\cos t)/(2H)} \cos t \,\mathrm{d}t, \quad x_2(t,v) = -\frac{e^{1/(2H)}}{H} v, \quad x_3(t,v) = \frac{\cos t}{2H}.$$

This surface is complete, simply connected and not embedded since

$$\int_{-\pi/2}^{3\pi/2} e^{-(\cos t)/(2H)} \cos t \,\mathrm{d}t = \int_{-\pi/2}^{\pi/2} -2\sinh\left(\frac{\cos t}{2H}\right) \cos t \,\mathrm{d}t < 0.$$

Also, it is conformally equivalent to the complex plane $\mathbb{C}$. The profile curve is drawn in Figure 1.

4 Uniqueness of immersed CMC spheres

The uniqueness problem for a class of immersed spheres is usually approached by seeking a holomorphic quadratic differential $Q\,\mathrm{d}z^2$ for the class of surfaces under study. Once this is done, this holomorphic differential will vanish on spheres, what provides the key step for proving uniqueness.

As a matter of fact, and as already pointed out by Hopf, is not necessary that $Q\,\mathrm{d}z^2$ be holomorphic. Indeed, it suffices to find a complex quadratic differential $Q\,\mathrm{d}z^2$ whose

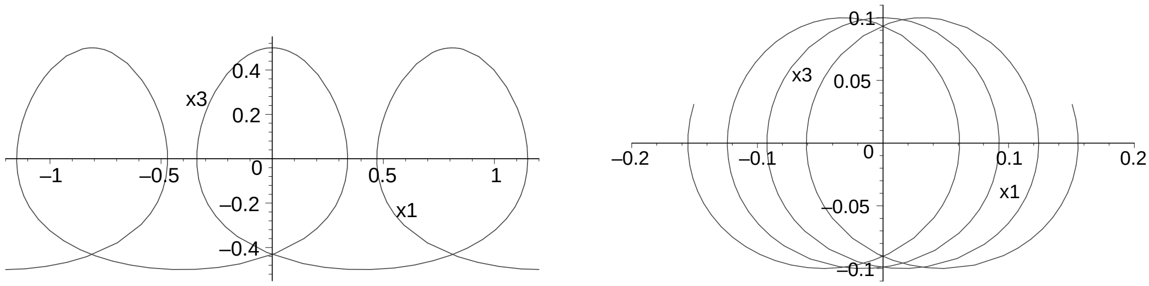

Figure 1: The profile curve in the (x_1, x_3)-plane when $H = 1$ (left) and $H = 5$ (right).

zeros (when not identically zero) are isolated and of negative index. This condition implies again, by the Poincaré-Hopf theorem, that $Q\,\mathrm{d}z^2$ will vanish on topological spheres.

By means of this idea and a careful local analysis, Alencar, do Carmo and Tribuzy recently obtained in [3] the following result:

Theorem 4.1 *Let $Q\,\mathrm{d}z^2$ denote a complex quadratic differential on $\bar{\mathbb{C}}$. Assume that around every point z_0 of $\bar{\mathbb{C}}$ we have*

$$|Q_{\bar{z}}|^2 \leqslant f\,|Q|^2, \tag{4.1}$$

where f is a continuous non-negative real function around z_0, and z is a local conformal parameter. Then $Q \equiv 0$ on $\bar{\mathbb{C}}$.

Our next objective is to seek a quadratic differential $Q\,\mathrm{d}z^2$ satisfying the so-called *Cauchy-Riemann inequality* (4.1) for every CMC H surface in Sol_3. Our main result in this section will be that such a quadratic differential exists, provided there also exists an immersed CMC H sphere in Sol_3 whose Gauss map is a global diffeomorphism of $\bar{\mathbb{C}} \equiv \mathbb{S}^2$. The existence of such a sphere will be discussed in Section 5.

We will make a constructive approach to this result, in order to clarify its nature. To start, we consider a very general quadratic differential $Q\,\mathrm{d}z^2$ on a CMC H surface $X : \Sigma \to \mathrm{Sol}_3$, defined in terms of its Gauss map $g : \Sigma \to \bar{\mathbb{C}}$ as

$$Q = L(g)g_z^2 + M(g)g_z\bar{g}_z, \tag{4.2}$$

where

$$L: \begin{array}{l} \bar{\mathbb{C}} \to \mathbb{C} \\ q \mapsto L(q), \end{array} \qquad M: \begin{array}{l} \bar{\mathbb{C}} \to \mathbb{C} \\ q \mapsto M(q) \end{array}$$

are to be determined. It is immediate that $Q\,\mathrm{d}z^2$ is invariant by conformal changes of coordinates in Σ, so it gives a well defined quadratic differential, at least at points where $g \neq \infty$. In order to ensure that Q is well defined at points where $g = \infty$, we use the conformal chart $w = i/q$ on $\bar{\mathbb{C}}$, the Riemann sphere on which g takes its values. From there, we get the following restrictions, which imply that Q is well defined at every point:

$$q^4 L(q) \text{ is smooth and bounded around } q = \infty, \tag{4.3}$$

and

$$|q|^4 M(q) \text{ is smooth and bounded around } q = \infty. \tag{4.4}$$

Then, using (3.13) we get from (4.2)

$$\begin{aligned} Q_{\bar{z}} &= g_{\bar{z}} g_z^2 (L_q + 2LA) + g_z |g_z|^2 (L_{\bar{q}} + 2LB + M\bar{B}) \\ &\quad + g_z |g_{\bar{z}}|^2 (M_q + MA) + \bar{g}_z |g_z|^2 (M_{\bar{q}} + MB + M\bar{A}). \end{aligned} \tag{4.5}$$

We first notice that this expression simplifies for some specific choice for M. Indeed, let us define $M : \mathbb{C} \to \mathbb{C}$ by

$$M(q) = \frac{1}{R(q)} = \frac{1}{H(1+|q|^2)^2 + q^2 - \bar{q}^2}. \tag{4.6}$$

A computation shows that

$$M\left(\frac{i}{q}\right) = |q|^4 M(q), \tag{4.7}$$

what implies that M can be extended to a map $M : \bar{\mathbb{C}} \to \mathbb{C}$ satisfying (4.4) by setting

$$M(\infty) = 0.$$

Moreover, the following formulas are a direct consequence of the definition of the coefficients A, B:

$$M_q = -MA, \quad M_{\bar{q}} = -M(\bar{A} + B). \tag{4.8}$$

Thereby, if in the expression (4.2) we choose M as in (4.6), then (4.5) simplifies to

$$Q_{\bar{z}} = g_z^2 \left\{ (L_q + 2LA) g_{\bar{z}} + (L_{\bar{q}} + 2LB + M\bar{B}) \bar{g}_{\bar{z}} \right\}. \tag{4.9}$$

From here, we have the following result.

Lemma 4.2 *Let $L : \bar{\mathbb{C}} \to \mathbb{C}$ be a global solution of the differential equation*

$$(L_q + 2LA)\bar{L} = (L_{\bar{q}} + 2LB + M\bar{B})\bar{M}, \tag{4.10}$$

where A, B, M are given by (3.14) *and* (4.6), *and which satisfies the condition* (4.3). *Then, for any immersed CMC H surface $X : \Sigma \to \mathrm{Sol}_3$ with Gauss map $g : \Sigma \to \bar{\mathbb{C}}$, the quadratic differential $Q(g)\mathrm{d}z^2$ on Σ given by*

$$Q(g) = L(g) g_z^2 + M(g) g_z \bar{g}_z$$

satisfies the Cauchy-Riemann inequality (4.1) *at every point of Σ.*

Proof: Let $g : \Sigma \to \bar{\mathbb{C}}$ denote the Gauss map of a CMC H surface in Sol_3, and define

$$\alpha = \frac{L_q + 2LA}{\bar{M}}. \tag{4.11}$$

This function is well defined on $\mathbb{C}$, since M does not vanish on $\mathbb{C}$. We need to study $\alpha(q)$ in a neighbourhood of $q = \infty$. For this we shall use that, since L verifies the growth condition (4.3), we have

$$L(q) = \frac{1}{q^4}\,\varphi\left(\frac{i}{q}\right) \tag{4.12}$$

where φ is a smooth bounded function in a neighbourhood of 0. Using (4.12) together with (4.7) and the easily verified relation

$$A\left(\frac{i}{q}\right) = iq^2 A(q) - 2iq,$$

we get from (4.11) that

$$\alpha(q) = -\frac{i\bar{q}^2}{q^4}\bar{R}\left(\frac{i}{q}\right)\left(\varphi_q\left(\frac{i}{q}\right) + 2\varphi\left(\frac{i}{q}\right)A\left(\frac{i}{q}\right)\right) = \frac{\bar{q}^2}{q^4}\psi\left(\frac{i}{q}\right) \tag{4.13}$$

where ψ is bounded and continuous in a neighbourhood of 0.

Once here, observe that by (4.9) and (4.10) we have

$$Q_{\bar{z}} = \alpha(g)\overline{(L(g)g_z + M(g)\bar{g}_z)}\,g_z^2,$$

and so

$$|Q_{\bar{z}}| = \beta\,|Q|, \qquad \text{with} \qquad \beta(z) = |\alpha(g(z))g_z(z)|.$$

This function β is continuous except, possibly, at points $z \in \Sigma$ where $g(z) = \infty$. But in the neighbourhood of such a point, by (4.13) we have

$$\beta = \left|\psi\left(\frac{i}{g}\right)\frac{\partial}{\partial z}\left(\frac{i}{g}\right)\right|.$$

Consequently β is continuous at every $z \in \Sigma$. Thus, Q verifies the Cauchy-Riemann inequality (4.1), as claimed.

□

Once here, it comes clear from Theorem 4.1 and Lemma 4.2 that, in order to have a quadratic differential that vanishes on CMC H spheres in Sol_3, we just need a global solution L to (4.10) that satisfies the growth condition (4.3). Next, we prove that such a solution exists *provided* we know beforehand the existence of a CMC H sphere whose Gauss map is a global diffeomorphism of $\bar{\mathbb{C}}$.

Proposition 4.3 *Let $H \neq 0$. Assume that there exists a CMC H sphere $S_H \equiv \bar{\mathbb{C}}$ whose Gauss map $G : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a diffeomorphism. Then there exists a solution $L : \bar{\mathbb{C}} \to \mathbb{C}$ to* (4.10) *that satisfies the condition at infinity* (4.3).

Consequently, we can define, associated to every smooth map $g : \Sigma \to \bar{\mathbb{C}}$ from a Riemann surface Σ, the quadratic differential $Q(g)\mathrm{d}z^2$ where

$$Q(g) = L(g)g_z^2 + M(g)g_z\bar{g}_z,$$

and then $Q(g)\mathrm{d}z^2$ satisfies the Cauchy-Riemann inequality (4.1) *if $g : \Sigma \to \bar{\mathbb{C}}$ is the Gauss map of any CMC H immersion $X : \Sigma \to \mathrm{Sol}_3$ from a Riemann surface Σ.*

Proof: We view S_H as a conformal immersion from $\bar{\mathbb{C}}$ into Sol_3, whose Gauss map $G : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a global diffeomorphism, and $G_z \neq 0$ at every point. Then, we can define a function $L : \mathbb{C} \to \mathbb{C}$ by

$$L(G(z)) = -\frac{M(G(z))\bar{G}_z(z)}{G_z(z)}. \tag{4.14}$$

It is important to remark that this function L has been chosen so that

$$L(G)G_z^2 + M(G)G_z\bar{G}_z = 0 \tag{4.15}$$

identically on $\bar{\mathbb{C}}$ (we prove next that L is well defined at ∞).

Let us analyze the behaviour of $L(q)$ at $q = \infty$. We set $G = i/\Gamma$ in the neighbourhood of the point $z_0 \in \bar{\mathbb{C}}$ where $G(z_0) = \infty$. Then by (4.7) we can write

$$L(G(z)) = \Gamma(z)^4 M(\Gamma(z))\frac{\bar{\Gamma}_z(z)}{\Gamma_z(z)}.$$

We know that Γ_z does not vanish (indeed, Γ is the Gauss map of the image of S by the isometry $\sigma\tau$, see Remark 3.5), $M(0) = 1/H$ and $\bar{\Gamma}_z$ has a finite limit when $z \to z_0$, so we can write

$$L(q) = \frac{1}{q^4}\varphi\left(\frac{i}{q}\right)$$

where φ is bounded in a neighbourhood of 0. Hence we can define $L(q)$ on the whole Riemann sphere $\bar{\mathbb{C}}$, and moreover the condition (4.3) is satisfied.

We want to prove next that $L(q)$ verifies (4.10). For that, we divide (4.15) by G_z and then differentiate with respect to $\bar{z}$. As G verifies the Gauss map equation (3.13), $G_z \neq 0$ and M satisfies (4.8), we obtain the relation

$$(L_q + 2LA)G_{\bar{z}} + (L_{\bar{q}} + 2LB + M\bar{B})\bar{G}_{\bar{z}} = 0$$

(where $L_q + 2LA$ and $L_{\bar{q}} + 2LB + M\bar{B}$ are evaluated at the point $G(z)$). So, using again (4.15) we have

$$(L_q + 2LA)\bar{L} = (L_{\bar{q}} + 2LB + M\bar{B})\bar{M}$$

at every point $q = G(z)$. Since G is a global diffeomorphism, this equation holds at every $q \in \bar{\mathbb{C}}$.

Thus, we have proved the existence of a solution $L : \bar{\mathbb{C}} \to \mathbb{C}$ to (4.10) that satisfies the condition at infinity (4.3), so by Lemma 4.2 this concludes the proof.

□

Observe that by (4.15) it holds

$$Q(G) = 0.$$

We will now show that, in these conditions, this CMC H sphere S_H is unique (up to left translations) among immersed CMC H spheres in Sol_3. For that, we shall use the following auxiliary results.

Lemma 4.4 *We have*

$$\left|\frac{L(q)}{M(q)}\right| < 1$$

for all $q \in \bar{\mathbb{C}}$.

Proof: When $q \in \mathbb{C}$, this is a consequence of (4.14) and the fact that G is an orientation-preserving diffeomorphism. When $q = \infty$, we use Remark 3.5.

□

Lemma 4.5 *Let* $g : \Sigma \to \mathbb{C}$ *be a nowhere antiholomorphic smooth map from a Riemann surface* Σ *such that* $Q(g) = 0$. *Then* g *is a local diffeomorphism.*

Proof: Since $Q(g) = 0$, we have $\bar{g}_z/g_z = -L(g)/M(g)$ and so $|g_z| > |\bar{g}_z|$ by Lemma 4.4.

□

The following lemma states that solutions of the equation $Q(g) = 0$ are locally unique up to a conformal change of parameter.

Lemma 4.6 *Let* $g_1 : \Sigma_1 \to \mathbb{C}$ *and* $g_2 : \Sigma_2 \to \mathbb{C}$ *be two nowhere antiholomorphic smooth maps from Riemann surfaces* Σ_1, Σ_2. *Assume that* $Q(g_1) = 0$ *and* $Q(g_2) = 0$ *(consequently,* g_1 *and* g_2 *are local diffeomorphisms by Lemma 4.5). Assume that there exists an open set* $U \subset \Sigma_2$ *and a diffeomorphism* $\varphi : U \to \varphi(U) \subset \Sigma_1$ *such that* $g_2 = g_1 \circ \varphi$ *on* U. *Then* φ *is holomorphic.*

Proof: Since g_{1z} and g_{2z} do not vanish, the fact that $Q(g_1) = 0$ and $Q(g_2) = 0$ implies that

$$0 = L(g_1)g_{1z} + M(g_1)\bar{g}_{1z}, \tag{4.16}$$

$$0 = L(g_2)g_{2z} + M(g_2)\bar{g}_{2z}. \tag{4.17}$$

We have

$$g_{2z} = (g_{1z} \circ \varphi)\varphi_z + (g_{1\bar{z}} \circ \varphi)\bar{\varphi}_z,$$

$$\bar{g}_{2z} = (\bar{g}_{1z} \circ \varphi)\varphi_z + (\bar{g}_{1\bar{z}} \circ \varphi)\bar{\varphi}_z.$$

Consequently, using (4.17) and then (4.16) at the point $\varphi(z)$, we obtain

$$(L(g_2)(g_{1\bar{z}} \circ \varphi) + M(g_2)(\bar{g}_{1\bar{z}} \circ \varphi))\bar{\varphi}_z = 0.$$

Conjugating and applying again (4.16) at the point $\varphi(z)$ gives

$$\frac{g_{1z} \circ \varphi}{M(g_2)}(|M(g_2)|^2 - |L(g_2)|^2)\varphi_{\bar{z}} = 0.$$

We have $g_{1z} \neq 0$ and $|M(g_2)| > |L(g_2)|$ by Lemma 4.4. Hence we conclude that $\varphi_{\bar{z}} = 0$.

□

Theorem 4.7 *Let $H \neq 0$. Assume that there exists an immersed CMC H sphere S_H in* Sol_3 *whose Gauss map is a global diffeomorphism. Then, any other immersed CMC H sphere Σ in* Sol_3 *differs from S_H at most by a left translation.*

Proof: We view S_H as a conformal immersion from $\bar{\mathbb{C}}$ into Sol_3, whose Gauss map $G : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a global diffeomorphism, and $G_z \neq 0$ at every point.

By Proposition 4.3 and Theorem 4.1 we can conclude that if Σ is another CMC H sphere in Sol_3, and $g : \Sigma \equiv \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ denotes its Gauss map, then the well-defined quadratic differential $Q(g)$ vanishes identically on Σ.

Nonetheless, we also have $Q(G) = 0$. So, as G is a global diffeomorphism, we have from Lemma 4.6 (and real analyticity) that $g = G \circ \psi$ where $\psi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a conformal automorphism of $\bar{\mathbb{C}}$. Thus, by conformally reparametrizing Σ if necessary we have two conformal CMC H immersions $X_1, X_2 : \bar{\mathbb{C}} \to \mathrm{Sol}_3$ with the same Gauss map G. From Theorem 3.4, this implies that X_1 and X_2, i.e. Σ and S_H, coincide up to a left translation, as desired.

□

Remark 4.8 Our proof of Theorem 4.7 has two key ideas. One is to prove that, associated to a sphere S_H whose Gauss map is a global diffeomorphism, we can construct a geometric differential $Q\,\mathrm{d}z^2$ for arbitrary surfaces in Sol_3 such that:

(*a*) $Q\,\mathrm{d}z^2$ vanishes identically on S_H (this is the meaning of (4.14) and (4.15)), and

(*b*) $Q\,\mathrm{d}z^2$ vanishes identically *only* on S_H (this is Lemma 4.6).

This idea is very general, it does not use any differential equation, and could also work in many other contexts. As a matter of fact, in this general strategy the role of the Gauss map g could be played by some other geometric mapping into $\bar{\mathbb{C}}$ that determines the surface uniquely.

The second key idea is to prove, using the Gauss map equation, that this quadratic differential $Q\,\mathrm{d}z^2$ must actually vanish identically on any sphere, as otherwise it would only have isolated zeros of negative index, thus contradicting the Poincaré-Hopf theorem. This is also a rather general strategy, that actually goes back to Hopf.

It is hence our impression that the underlying ideas in the proof of Theorem 4.7 actually provide a new and flexible way of proving Hopf-type theorems.

Corollary 4.9 *Let S be a CMC sphere whose Gauss map is a diffeomorphism. Then there exists a point $p \in \mathrm{Sol}_3$, which we will call the* center *of S, such that S is globally invariant by all isometries of* Sol_3 *fixing p.*

In particular, there exists two constants d_1 and d_2 such that S is invariant by reflections with respect to the planes $x_1 = d_1$ and $x_2 = d_2$.

Proof: Let H be the mean curvature of S. By Theorem 4.7, S is the unique CMC H sphere up to left translations. In particular, there exists a left translation

$$T : (x_1, x_2, x_3) \mapsto (e^{-c_3}(x_1 + c_1), e^{c_3}(x_2 + c_2), x_3 + c_3)$$

such that $T\sigma(S) = S$, where $\sigma : (x_1, x_2, x_3) \mapsto (x_2, -x_1, -x_3)$. The isometry $T\sigma$ fixes the point

$$p = \left(\frac{e^{-c_3} c_1 + c_2}{2}, \frac{e^{c_3} c_2 - c_1}{2}, \frac{c_3}{2} \right).$$

In the same way, there exists a left translation

$$T' : (x_1, x_2, x_3) \mapsto (e^{-c_3'}(x_1 + c_1'), e^{c_3'}(x_2 + c_2'), x_3 + c_3')$$

such that $T'\tau(S) = S$, where $\tau : (x_1, x_2, x_3) \mapsto (-x_1, x_2, x_3)$. We have $T'\tau(x_1, x_2, x_3) = (e^{-c_3'}(-x_1 + c_1'), e^{c_3'}(x_2 + c_2'), x_3 + c_3')$, so since S is compact we have $c_3' = 0$ and $c_2' = 0$. Now we have $(T\sigma T'\tau)^2(S) = S$, and

$$(T\sigma T'\tau)^2(x_1, x_2, x_3) = (x_1 - c_1' + c_2 + e^{-c_3} c_1, x_2 + c_1 + e^{c_3}(c_2 - c_1'), x_3),$$

so again since S is compact we get $c_1' = c_2 + e^{-c_3} c_1$. Consequently, $T'\tau$ fixes p.

So $T\sigma$ and $T'\tau$ generate the isotropy group of p, which finishes the proof.

□

Remark 4.10 Consider an equation of the form

$$g_{z\bar{z}} = A(g) g_z g_{\bar{z}} + B(g) g_z \bar{g}_{\bar{z}},$$

where $A : \bar{\mathbb{C}} \to \mathbb{C}$ and $B : \bar{\mathbb{C}} \to \mathbb{C}$ are given functions satisfying a certain growth condition at ∞, so that the change of function $g \to i/g$ induces an equation of the same form. Then using the techniques above we can prove that it admits at most a unique (up to conformal change of parameter) solution $g : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ if the partial differential equations (4.8) and (4.10) admit solutions $L, M : \bar{\mathbb{C}} \to \mathbb{C}$ satisfying a certain growth condition at ∞ and such that $|L/M| < 1$.

5 Existence and properties of CMC spheres

The object of this section is to prove existence of index one CMC spheres and some of their properties. For this purpose we will use stability and nodal domain arguments. We will first recall some well known results of this context, and their application to CMC surfaces in Sol_3. A good reference is [7] (see also [11, 30, 29]).

5.1 Preliminaries on the stability operator

Let S be a compact CMC surface in Sol_3. Its *stability operator* (or *Jacobi operator*) is defined by

$$\mathcal{L} = \Delta + ||\mathcal{B}||^2 + \mathrm{Ric}(N)$$

where Δ is the Laplacian with respect to the induced metric, $\mathcal{B}$ the second fundamental form of S, N its unit normal vector field and Ric the Ricci curvature of Sol_3. The stability operator is also the linearized operator of the mean curvature functional.

The operator $-\mathcal{L}$ admits a sequence

$$\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$$

of eigenvalues such that $\lim_{k\to+\infty} \lambda_k = +\infty$. We will call λ_k the k-th eigenvalue. Moreover the corresponding eigenspaces are orthogonal. The number of negative eigenvalues is called the *index* of S and will be denoted by $\mathrm{Ind}(S)$. To the operator $-\mathcal{L}$ is associated the quadratic form $\mathcal{Q}$ defined by

$$\mathcal{Q}(f,f) = -\int_S f\mathcal{L}f = \int_S ||\operatorname{grad} f||^2 - (||\mathcal{B}||^2 + \mathrm{Ric}(N))f^2. \tag{5.1}$$

The quadratic form $\mathcal{Q}$ acts naturally on the Sobolev space $\mathrm{H}^1(S)$.

A *Jacobi function* is a function f on S such that $\mathcal{L}f = 0$. If F is a Killing field in Sol_3, then $\langle N, F\rangle$ is a Jacobi function on S. Since S is compact, it is not invariant by a one-parameter group of isometries; consequently the Jacobi fields F_1, F_2 and F_3 induce three linearly independent Jacobi functions. This implies that 0 is an eigenvalue of $\mathcal{L}$ with multiplicity at least 3. Hence it cannot be the first eigenvalue, and so $\lambda_1 < 0$. Consequently, $\mathrm{Ind}(S) \geqslant 1$, and $\mathrm{Ind}(S) = 1$ if and only if $\lambda_2 = 0$.

We say that S is *stable* (or *weakly stable*) if

$$\mathcal{Q}(f,f) = -\int_S f\mathcal{L}f \geqslant 0$$

for any smooth function f on S satisfying

$$\int_S f = 0.$$

If S is stable, then $\mathrm{Ind}(S) = 1$ (indeed, if $\lambda_2 < 0$, then there exists a linear combination f of the first two eigenfunctions such that $-\int_S f\mathcal{L}f < 0$ and $\int_S f = 0$, which contradicts stability). Another well known property in this context is that solutions to the isoperimetric problem are stable compact embedded CMC surfaces (hence stable spheres in Sol_3).

We state the following well known theorems:

- Courant's nodal domain theorem: any eigenfunction of λ_2 has at most 2 nodal domains (Proposition 1.1 in [7]),
- if S is a sphere, then the dimension of the eigenspace of λ_2 is at most 3 (Theorem 3.4 in [7]).

The results in [7] are stated for the Laplacian but can be extended for an operator of the form $\Delta + V$ where V is a function (see for instance [29] for a proof of Courant's nodal domain theorem; then the other results are deduced using topological arguments and the local behaviour of solutions of $\Delta f + Vf = 0$). Also the convention for the numbering of the eigenfunctions in [7] differs from ours: what is called the k-th eigenvalue in [7] is what we call the $(k+1)$-th eigenvalue.

In particular, if S is an index one compact CMC surface in Sol_3, then any Jacobi function on S admits at most two nodal domains (see also Proposition 1 in [26]).

5.2 The Gauss map of index one CMC spheres

Proposition 5.1 *Let S be an index one CMC sphere. Then its Gauss map $g : S \to \bar{\mathbb{C}}$ is an orientation-preserving diffeomorphism.*

Proof: Since g is a map from a sphere to a sphere, it suffices to prove that its Jacobian $J(g) = |g_z|^2 - |g_{\bar z}|^2$ is positive everywhere.

Let $F = a_1F_1 + a_2F_2 + a_3F_3$ be a non-zero Killing field. Then

$$f = \langle F, N\rangle = (a_1 - a_3x_1)e^{x_3}N_1 + (a_2 + a_3x_2)e^{-x_3}N_2 + a_3N_3 \tag{5.2}$$

is a Jacobi function. Since $\mathrm{Ind}(S) = 1$, the function f has at most two nodal domains, so since S is a sphere, f cannot have a singular point on the nodal curve (see also Theorem 3.2 in [7]). Consequently, if $f(p) = 0$ for some p, then $f_z(p) \neq 0$.

We have

$$\begin{aligned} f_z &= (a_1 - a_3x_1)e^{x_3}N_{1z} + (a_2 + a_3x_2)e^{-x_3}N_{2z} + a_3N_{3z} \\ &\quad +(a_1 - a_3x_1)e^{x_3}x_{3z}N_1 - a_3e^{x_3}x_{1z}N_1 \\ &\quad -(a_2 + a_3x_2)e^{-x_3}x_{3z}N_2 + a_3e^{-x_3}x_{2z}N_2. \end{aligned}$$

Setting

$$b = (a_1 - a_3x_1)e^{x_3} + i(a_2 + a_3x_2)e^{-x_3}, \tag{5.3}$$

we get

$$\begin{aligned} f_z &= \mathrm{Re}\, b\frac{\partial}{\partial z}\frac{g+\bar g}{1+|g|^2} + \mathrm{Im}\, b\frac{\partial}{\partial z}\frac{i(\bar g - g)}{1+|g|^2} + a_3\frac{\partial}{\partial z}\frac{1-|g|^2}{1+|g|^2} \\ &\quad +\frac{bg+\bar b\bar g}{1+|g|^2}A_3 - a_3A_1N_1 + a_3A_2N_2 \\ &= \frac{\bar b g_z + b\bar g_z}{1+|g|^2} - \frac{\bar b g + b\bar g}{(1+|g|^2)^2}(\bar g g_z + g\bar g_z) - \frac{2a_3}{(1+|g|^2)^2}(\bar g g_z + g\bar g_z) \\ &\quad +\frac{bg+\bar b\bar g}{1+|g|^2}\frac{\eta}{2} + a_3\frac{\eta}{2\bar g}\frac{g-\bar g^3}{1+|g|^2}. \end{aligned} \tag{5.4}$$

Let $z_0 \in S$. Assume first that $|g(z_0)| \neq 1$. We can also assume that $g(z_0) \neq \infty$, since $g(z_0) = \infty$ can be dealt with in the same way using the isometry $\sigma\tau$ (see Remark 3.2). Then $f(z_0) = 0$ if and only if

$$a_3 = -\frac{\bar b g + b\bar g}{1-|g|^2}$$

at z_0. If this condition is satisfied, then, at the point z_0, using (3.12) we get

$$\begin{aligned}(1+|g|^2)^2 f_z &= (1+|g|^2)(\bar{b}g_z + b\bar{g}_z) - (\bar{b}g + b\bar{g})(\bar{g}g_z + g\bar{g}_z) + 2\frac{\bar{b}g+b\bar{g}}{1-|g|^2}(\bar{g}g_z + g\bar{g}_z) \\ &\quad +(1+|g|^2)(bg+\bar{b}\bar{g})\frac{2\bar{g}g_z}{R(g)} - \frac{\bar{b}g+b\bar{g}}{1-|g|^2}(1+|g|^2)\frac{2g_z}{R(g)}(g-\bar{g}^3) \\ &= (1+|g|^2)(\bar{b}g_z + b\bar{g}_z) + \frac{1+|g|^2}{1-|g|^2}(\bar{b}g+b\bar{g})(\bar{g}g_z+g\bar{g}_z) \\ &\quad +\frac{2g_z}{R(g)}\frac{1+|g|^2}{1-|g|^2}(\bar{b}+b\bar{g}^2)(\bar{g}^2-g^2),\end{aligned}$$

and so

$$(1-|g|^4)f_z = C_1 b + C_2\bar{b}$$

with

$$\begin{aligned}C_1 &:= \left(1+\frac{2(\bar{g}^2-g^2)}{R(g)}\right)\bar{g}^2 g_z + \bar{g}_z = \frac{\overline{R(g)}}{R(g)}\bar{g}^2 g_z + \bar{g}_z, \\ C_2 &:= \left(1+\frac{2(\bar{g}^2-g^2)}{R(g)}\right) g_z + g^2\bar{g}_z = \frac{\overline{R(g)}}{R(g)} g_z + g^2\bar{g}_z.\end{aligned}$$

From this we get $C_1 b + C_2\bar{b} \neq 0$. This holds for all non-zero Killing fields F such that $\langle F, N\rangle$ vanishes at z_0, i.e., for all $b \in \mathbb{C}\setminus\{0\}$. This means that the real-linear equation $C_1 b + C_2\bar{b} = 0$ in $(b,\bar{b})$ has $b=0$ as unique solution, so we get $|C_1|^2 \neq |C_2|^2$, which is equivalent to

$$|g_z|^2 \neq |\bar{g}_z|^2.$$

Assume now that $|g(z_0)| = 1$ at some $z_0 \in S$ where $f(z_0) = 0$. Then $N_3(z_0) = 0$ and from (5.2), (5.3) we have (at z_0) $\mathrm{Re}(b\bar{g}) = 0$, that is,

$$b = i\mu g$$

for some $\mu \in \mathbb{R}$. Hence, using (3.12) and the fact that $|g(z_0)| = 1$, equation (5.4) simplifies (at z_0) to

$$f_z = -\frac{(a_3+i\mu)\bar{g}}{2}g_z - \frac{(a_3-i\mu)g}{2}\bar{g}_z + \frac{(a_3+i\mu)(g-\bar{g}^3)}{R(g)}g_z,$$

and after some computations to

$$f_z = -g_z(a_3+i\mu)\frac{\overline{R(g)}}{R(g)}\frac{\bar{g}}{2} - \bar{g}_z(a_3-i\mu)\frac{g}{2}.$$

So, as $f_z(z_0) \neq 0$, this quantity must be non-zero for all non-zero Killing fields F such that $\langle F, N\rangle$ vanishes at z_0, i.e., for all $a_3 + i\mu \in \mathbb{C}\setminus\{0\}$. Again, this means that $|g_z|(z_0) \neq |\bar{g}_z|(z_0)$.

We have proved that $|g_z|^2 \neq |\bar{g}_z|^2$ everywhere, i.e., that g is a diffeomorphism. To prove that g preserves orientation, it suffices to prove that $|g_z|^2 > |\bar{g}_z|^2$ at some point. At a point of S where x_3 has an extremum, we have $g = 0$ or $g = \infty$, so since g is a diffeomorphism, x_3 admits exactly one extremum where $g = 0$, at a point that we will denote z_0.

We have $(x_3)_z = \eta/2 = 2\bar{g}g_z/R(g)$. Hence, at the point z_0, using that $g(z_0) = 0$ and $R(0) = H$, we get

$$(x_3)_{zz} = \frac{2g_z\bar{g}_z}{H}, \quad (x_3)_{z\bar{z}} = \frac{2|g_z|^2}{H}, \quad (x_3)_{\bar{z}\bar{z}} = \frac{2g_{\bar{z}}\bar{g}_{\bar{z}}}{H}.$$

Since x_3 has an extremum at z_0, the Hessian of x_3 at z_0 has a non-negative determinant, i.e., $4(x_3)_{z\bar{z}}^2 - 4(x_3)_{zz}(x_3)_{\bar{z}\bar{z}} \geqslant 0$. Since since $g_z \neq 0$, we obtain that $|g_z|^2 \geqslant |\bar{g}_z|^2$ at z_0, which concludes the proof.

□

5.3 Bounds on the second fundamental form and the diameter

Proposition 5.2 *Let $H \neq 0$ and let S be a CMC H sphere whose Gauss map is an orientation-preserving diffeomorphism. Then its second fundamental form $\mathcal{B}$ satisfies*

$$||\mathcal{B}||^2 < 4H^2 + 4|H| + 2. \tag{5.5}$$

Proof: By a classical computation we have

$$\det \mathcal{B} = H^2 - \frac{4|P|^2}{\lambda^2}$$

where $P\mathrm{d}z^2$ is the Hopf differential, and so

$$||\mathcal{B}||^2 = 4H^2 - 2\det \mathcal{B} = 2H^2 + \frac{8|P|^2}{\lambda^2}.$$

By (3.8), (3.12) and Remark 3.7 we have

$$\frac{|P|}{\lambda} = \frac{1}{2(1+|g|^2)^2}\left|R(g)\frac{\bar{g}_z}{g_z} - 1 + \bar{g}^4\right|.$$

Since g is an orientation-preserving diffeomorphism we have $|\bar{g}_z| < |g_z|$ and so

$$\frac{|P|}{\lambda} < \frac{|R(g)| + 1 + |g|^4}{2(1+|g|^2)^2} \leqslant \frac{|H|+1}{2},$$

which gives the result (notice that the first inequality is strict since $R(g)$ does not vanish when $H \neq 0$).

□

The following diameter estimate when $H > 1/\sqrt{3}$ is a consequence of a theorem of Rosenberg [28]. We also refer to [11] for some arguments used in the proof.

Lemma 5.3 *Let $H > 1/\sqrt{3}$ and S be an index one CMC H sphere. Then its intrinsic diameter is less than or equal to $8\pi/\sqrt{3(3H^2-1)}$.*

Proof: We recall that a domain $U \subset S$ is said to be *strongly stable* if $-\int_U f\mathcal{L}f \geqslant 0$ for all smooth functions f on U that vanish on ∂U.

Fix $p_0 \in S$ and for $r > 0$ let B_r denote the geodesic ball of radius r centered at p_0. Let

$$\rho := \sup\{r > 0; B_r \text{ is strongly stable}\}.$$

We have $\rho < +\infty$ since S is not strongly stable.

Let $\varepsilon > 0$ and $U = S \setminus B_{\rho+\varepsilon}$. We claim that U is strongly stable (when $U \neq \emptyset$).

Indeed, assume that U is not strongly stable. Then the first eigenvalue $\lambda_1(U)$ of $-\mathcal{L}$ (for the Dirichlet problem with zero boundary condition) on U is negative; let f_1 be an eigenfunction on U for $\lambda_1(U)$, extended by 0 on $B_{\rho+\varepsilon}$. In the same way, $B_{\rho+\varepsilon}$ is not strongly stable and so $\lambda_1(B_{\rho+\varepsilon}) < 0$; let f_2 be an eigenfunction on $B_{\rho+\varepsilon}$ for $\lambda_1(B_{\rho+\varepsilon})$, extended by 0 on U. Then f_1 and f_2 have supports that overlap on a set of measure 0, so they are orthogonal for $\mathcal{Q}$; then $\mathcal{Q}$ is negative definite on the space spanned by f_1 and f_2, which has dimension 2. This contradicts the fact that $\mathrm{Ind}(S) = 1$. This proves the claim.

Since Sol_3 has scalar curvature -2 and since $H > 1/\sqrt{3}$, Theorem 1 in [28] states that if Ω is a strongly stable domain in S, then for all $p \in \Omega$ we have

$$\mathrm{dist}_S(p, \partial\Omega) \leqslant \frac{2\pi}{\sqrt{3(3H^2-1)}} \tag{5.6}$$

where dist_S denotes the intrinsic distance on S. (Observe that in [28] the scalar curvature is defined as one half of the trace of the Ricci tensor, whereas in our convention it is defined as the trace of the Ricci tensor.) Applying (5.6) for p_0 in B_ρ we get

$$\rho \leqslant \frac{2\pi}{\sqrt{3(3H^2-1)}}.$$

Let $p \in S$. We claim that

$$\mathrm{dist}_S(p, p_0) \leqslant \frac{4\pi}{\sqrt{3(3H^2-1)}} + \varepsilon.$$

If $p \in B_{\rho+\varepsilon}$, this is clear. So assume that $p \in U$ (in the case where $U \neq \emptyset$). Then applying (5.6) for the strongly stable domain U we get

$$\mathrm{dist}_S(p, \partial U) \leqslant \frac{2\pi}{\sqrt{3(3H^2-1)}}.$$

But $\partial U = \partial B_{\rho+\varepsilon}$, so

$$\mathrm{dist}_S(p, p_0) \leqslant \frac{2\pi}{\sqrt{3(3H^2-1)}} + \rho + \varepsilon.$$

This finishes proving the claim.

Since this holds for all $\varepsilon > 0$, the lemma is proved.

□

5.4 Embeddedness

Proposition 5.4 *Let S be an immersed CMC sphere whose Gauss map is a diffeomorphism. Then S is embedded, and consequently it is a symmetric bigraph in the x_1 and x_2 directions.*

Proof: We view S as a conformal immersion $X : S \equiv \bar{\mathbb{C}} \to \mathrm{Sol}_3$. By Corollary 4.9, up to a left translation, $X(S)$ is symmetric with respect to the planes $\Pi_1 = \{x_1 = 0\}$ and $\Pi_2 = \{x_2 = 0\}$. It will be important to keep in mind that the frame $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ is orthogonal.

Let $N = [N_1, N_2, N_3] : S \to T\mathrm{Sol}_3$ be the unit normal vector field of X and $g : S \to \bar{\mathbb{C}}$ the Gauss map of S. We first notice that at a critical point of x_3 we have $N = \pm E_3$, i.e., $g = 0$ or $g = \infty$. Hence, since g is a diffeomorphism, x_3 admits exactly one minimum at a point $p \in S$, one maximum at a point $q \in S$ and no other critical point. Since $X(S)$ is symmetric with respect to the planes Π_1 and Π_2, we necessarily have $X(p) \in \Pi_1$, $X(q) \in \Pi_1$, $X(p) \in \Pi_2$, $X(q) \in \Pi_2$.

Let γ be the set of the points of S where N is orthogonal to E_1. Then $\gamma = \{z \in S; g(z) \in i\mathbb{R} \cup \{\infty\}\}$. Since g is a diffeomorphism, γ is a closed embedded curve in S.

Let $z_0 \in \gamma$. Since $X(S)$ is invariant by the symmetry τ with respect to Π_1, there exists a point $z_1 \in S$ such that $X(z_1) = \tau(X(z_0))$ and $N(z_1) = [-N_1(z_0), N_2(z_0), N_3(z_0)]$. But $z_0 \in \gamma$ so $N_1(z_0) = 0$ and so $g(z_1) = g(z_0)$. Since g is a diffeomorphism we have $z_1 = z_0$ and so $X(z_0) \in \Pi_1$. This proves that $X(\gamma) \subset \Pi_1$.

We claim that $X(\gamma)$ is embedded and is a bigraph in the x_2 direction.

We have $p, q \in \gamma$ and, as explained before, $X(p), X(q) \in \Pi_2$. The set $\gamma \setminus \{p, q\}$ consists of the following two disjoint curves:

$$\gamma^+ := \{z \in S; g(z) = ir, r \in (0, +\infty)\}, \quad \gamma^- := \{z \in S; g(z) = ir, r \in (-\infty, 0)\}.$$

Then the curve $X(\gamma^+)$ is transverse to $\frac{\partial}{\partial x_2}$ since N is never parallel to E_3 on γ^+. Consequently, $X(\gamma^+)$ can be written as a graph $\{(x_2, x_3); x_3 \in I, x_2 = f(x_3)\}$ where $I = (x_3(p), x_3(q))$ and $f : I \to \mathbb{R}$ is a smooth function such that $\lim_{t \to x_3(p)} f(t) = \lim_{t \to x_3(q)} f(t) = 0$. By symmetry, $X(\gamma^-)$ is the graph $\{(x_2, x_3); x_3 \in I, x_2 = -f(x_3)\}$.

We now prove that $f > 0$ or $f < 0$. Without loss of generality, we may assume that a maximum of x_2 on γ is attained at a point of γ^+, and consequently a minimum of x_2 on γ is attained at a point of γ^-. But at a critical point of x_2 on γ we have $N = \pm E_2$, i.e., $g = \pm i$. Since g is a diffeomorphism, x_2 admits exactly two critical point on γ, so exactly one on γ^+, which is the abovementioned maximum. This implies that $f > 0$.

This finishes proving the claim.

Consequently, $X(\gamma)$ is embedded and bounds a domain U in Π_1; this domain U is topologically a disk. Let

$$S^+ := \{z \in S; g(z) \in \mathbb{C}, \mathrm{Re}(g(z)) > 0\}, \quad S^- := \{z \in S; g(z) \in \mathbb{C}, \mathrm{Re}(g(z)) < 0\}.$$

Then S is the disjoint union of S^+, S^- and γ. Also, $X(S^+)$ is transverse to $\frac{\partial}{\partial x_1}$ since N is never orthogonal to E_1 on S^+, and is bounded by $X(\gamma) = \partial U$. Consequently, $X(S^+)$ can be written as a graph $\{(x_1, x_2, x_3); (x_2, x_3) \in U, x_1 = h(x_2, x_3)\}$ where $h : U \to \mathbb{R}$ is a smooth function. By symmetry, $X(\gamma^-)$ is the graph $\{(x_1, x_2, x_3); (x_2, x_3) \in U, x_1 = -h(x_2, x_3)\}$. By the same argument as above, we prove that $h > 0$ or $h < 0$ (otherwise x_1 would admit at least 3 critical points). So $X(S)$ is embedded.

□

5.5 Deformations of CMC spheres

Lemma 5.5 *Let Σ be an oriented embedded compact surface (not necessarily CMC) in* Sol_3*. Let N be its unit normal vector and $\mathcal{H}$ its mean curvature function. Then, for all Killing field F of* Sol_3*, it holds*

$$\int_\Sigma \langle F, N\rangle = 0, \qquad \int_\Sigma \mathcal{H}\langle F, N\rangle = 0. \tag{5.7}$$

Proof: These formulas are well-known. We include a proof for the convenience of the reader.

Since Killing fields have divergence zero, the first formula follows from Stokes' formula applied on the compact region bounded by Σ.

Let $J : \mathrm{T}\Sigma \to \mathrm{T}\Sigma$ be the rotation of angle $\pi/2$. Denote by $\widehat{\nabla}$, ∇ and $\mathcal{B}$ the Riemannian connection of Sol_3, the Riemannian connection of Σ and the second fundamental form of Σ respectively. We define on Σ the 1-form ω by

$$\forall v \in \mathrm{T}\Sigma, \quad \omega(v) = \langle F, Jv\rangle.$$

Let (e_1, e_2) be a local direct orthonormal frame on Σ (hence $Je_1 = e_2$ and $Je_2 = -e_1$). Then we have

$$\begin{aligned}
\mathrm{d}\omega(e_1, e_2) &= e_1 \cdot \omega(e_2) - e_2 \cdot \omega(e_1) - \omega([e_1, e_2]) \\
&= -e_1 \cdot \langle F, e_1\rangle - e_2 \cdot \langle F, e_2\rangle - \langle F, J(\nabla_{e_1} e_2 - \nabla_{e_2} e_1)\rangle \\
&= -\langle F, \widehat{\nabla}_{e_1} e_1\rangle - \langle F, \widehat{\nabla}_{e_2} e_2\rangle + \langle F, \nabla_{e_1} e_1\rangle + \langle F, \nabla_{e_1} e_1\rangle \\
&= -\langle F, \mathcal{B}(e_1, e_1)N + \mathcal{B}(e_2, e_2)N\rangle \\
&= -2\mathcal{H}\langle F, N\rangle.
\end{aligned}$$

Here we used the fact that J commutes with ∇ and the fact that $\langle \widehat{\nabla}_v F, v\rangle = 0$ for all v since F is a Killing field. Then Stokes' formula yields

$$0 = \int_\Sigma \mathrm{d}\omega = -2\int_\Sigma \mathcal{H}\langle F, N\rangle.$$

□

In the next propostion we will deform index one CMC spheres using the implicit function theorem and the fact that all the Jacobi functions on such a sphere come from ambient Killing fields.

Proposition 5.6 *Let S be an index one CMC H sphere. Then there exist $\varepsilon > 0$ and a real analytic family $(S_H)_{H\in(H_0-\varepsilon,H_0+\varepsilon)}$ of spheres such that $S_{H_0} = S$ and S_H has CMC H for all $H \in (H_0-\varepsilon, H_0+\varepsilon)$. Also, if $H \in (H_0-\varepsilon, H_0+\varepsilon)$ and $\widetilde{S}$ is a CMC H sphere close enough to S, then $\widetilde{S} = S_H$ (up to a left translation).*

Moreover, the spheres S_H, $H \in (H_0-\varepsilon, H_0+\varepsilon)$, have index one.

Proof: By Propositions 5.1 and 5.4, the Gauss map of S is a diffeomorphism and S is embedded. We view S as an embedding $X : S \to \mathrm{Sol}_3$ and we will identify S and $X(S)$.

Let us start by proving the existence of the family (S_H) by means of a classical deformation technique (see for instance [22, 17] for details).

Let $\alpha > 0$. For a function $\varphi \in \mathrm{C}^{2,\alpha}(S)$, we define the normal variation $X_\varphi : S \to \mathrm{Sol}_3$ by

$$X_\varphi(s) = \exp_{X(s)}(\varphi(s)N(s)),$$

where exp denotes the exponential map for the Riemannian metric on Sol_3 and N the unit normal vector field of S. There exists some neighbourhood Ω of 0 in $\mathrm{C}^{2,\alpha}(S)$ such that X_φ is an embedding for all $\varphi \in \Omega$; then let $\mathcal{H}(\varphi)$ be the mean curvature of X_φ. Hence we have defined a map

$$\mathcal{H} : \Omega \to \mathrm{C}^{0,\alpha}(S),$$

and the differential of $\mathcal{H}$ at $0 \in \mathrm{C}^{2,\alpha}(S)$ is one half the stability operator of S:

$$\mathrm{d}_0\mathcal{H} = \frac{1}{2}\mathcal{L} : \mathrm{C}^{2,\alpha}(S) \to \mathrm{C}^{0,\alpha}(S).$$

If $E \subset \mathrm{C}^{0,\alpha}(S)$, we will let $E^\perp$ denote the orthogonal of E in $\mathrm{C}^{0,\alpha}(S)$ for the $\mathrm{L}^2(S)$ scalar product. As explained in Section 5.1, all Jacobi functions on S come from ambient Killing fields, i.e., $\ker\mathcal{L}$ has dimension 3 and is generated by the functions $f_k := \langle F_k, N\rangle$, $k = 1, 2, 3$, where (F_1, F_2, F_3) is the basis of Killing fields defined in Section 2. We also have $\operatorname{Im}\mathcal{L} = (\ker\mathcal{L})^\perp$.

We now consider the map

$$\begin{aligned} \mathcal{K} :\ & \Omega \times \mathbb{R}^3 \to \mathrm{C}^{0,\alpha}(S) \\ & (\varphi, a_1, a_2, a_3) \mapsto \mathcal{H}(\varphi) - a_1\langle F_1, N_\varphi\rangle - a_2\langle F_2, N_\varphi\rangle - a_3\langle F_3, N_\varphi\rangle \end{aligned}$$

where N_φ denotes the unit normal vector field of X_φ. Then we have

$$\begin{aligned} \mathrm{d}_{(0,0,0,0)}\mathcal{K} = \Phi :\ & \mathrm{C}^{2,\alpha}(S) \times \mathbb{R}^3 \to \mathrm{C}^{0,\alpha}(S) \\ & (u, b_1, b_2, b_3) \mapsto \tfrac{1}{2}\mathcal{L}u - b_1 f_1 - b_2 f_2 - b_3 f_3. \end{aligned}$$

We first claim that Φ is surjective. Let $v \in \mathrm{C}^{0,\alpha}(S)$. If $v \in \ker\mathcal{L}$, then there exists (b_1, b_2, b_3) such that $v = \Phi(0, b_1, b_2, b_3)$. If $v \in (\ker\mathcal{L})^\perp$, then there exists $u \in \mathrm{C}^{2,\alpha}(S)$ such that $\mathcal{L}u = 2v$, and so $v = \Phi(u, 0, 0, 0)$. This proves the claim.

Also, since $\operatorname{Im}\mathcal{L} = (\ker\mathcal{L})^{\perp}$, we have $\ker\Phi = \ker\mathcal{L} \times \{(0,0,0)\}$, so $\ker\Phi$ has dimension 3.

We can now apply the implicit function theorem (see for instance [18]): there exist $\varepsilon > 0$, a real analytic family $(\varphi_H)_{H\in(H_0-\varepsilon,H_0+\varepsilon)}$ of functions and real analytic functions $a_1, a_2, a_3 : (H_0-\varepsilon, H_0+\varepsilon) \to \mathbb{R}$ such that, for all $H \in (H_0-\varepsilon, H_0+\varepsilon)$,

$$\mathcal{K}(\varphi_H, a_1(H), a_2(H), a_3(H)) = H$$

and $\mathcal{K}^{-1}(H)$ is locally a manifold of dimension 3.

Now, applying (5.7) to $S_H := X_{\varphi_H}(S)$, we get $a_1(H) = a_2(H) = a_3(H) = 0$, and so $\mathcal{H}(\varphi_H) = H$. This gives the existence of the family (S_H) as announced. Also, in a neighbourhood of S_H the set of CMC H spheres is a 3-dimensional manifold, so it is composed uniquely of images of S_H by left translations. This gives the announced uniqueness result.

Let us show next that all spheres in this deformation have index one. Assume that there is some $H_1 \in (H_0-\varepsilon, H_0+\varepsilon)$ such that $\operatorname{Ind}(S_H) \neq 1$. Without loss of generality, we can assume that $H_1 < H_0$. Let

$$\widehat{H} = \inf\{H > H_1; \operatorname{Ind}(S_H) = 1\}.$$

The functions $\lambda_k(S_H)$ and their eigenspaces are continuous with respect to H [15, 16]. Consequently, $\lambda_2(S_{\widehat{H}}) = 0$, $H_1 < \widehat{H}$, $\lambda_2(S_H) < 0$ for all $H \in [H_1, \widehat{H})$ and there exists a continuous family of functions

$$(f_H)_{H\in[H_1,\widehat{H}]}$$

such that f_H is an eigenfunction of $\lambda_2(S_H)$, $\int_{S_H} f_H^2 = 1$ and f_H is orthogonal to the Jacobi functions on S_H coming from ambient Killing fields. This implies in particular that $\lambda_2(S_{\widehat{H}})$ has multiplicity at least 4, which contradicts Theorem 3.4 in [7].

□

Proposition 5.7 *There exists a real analytic family*

$$(S_H)_{H>1/\sqrt{3}}$$

of spheres such that, for all $H > 1/\sqrt{3}$, S_H has CMC H and $\operatorname{Ind}(S_H) = 1$.

Proof: Let S be a solution of the isoperimetric problem that has CMC H_0 with $H_0 > 1/\sqrt{3}$ (such a solution exists since there exist solutions with arbitrary large mean curvature). Then S is a compact embedded CMC surface, hence it is a sphere by the Alexandrov reflection argument. Moreover, S is stable and so has index one. Then by Proposition 5.6 there exists a local deformation $(S_H)_{H\in(H_0-\varepsilon,H_0+\varepsilon)}$ for some $\varepsilon > 0$.

Let (H^-, H^+) be the largest interval to which we can extend the deformation (S_H). By the same argument as that of the proof of Proposition 5.6, all the spheres S_H have index one. Consequently, by Propositions 5.1 and 5.4, they are embedded and symmetric bigraphs in the x_1 and x_2 directions.

We first notice that $H^- \geqslant 0$ since there is no compact minimal surface in Sol_3. We will prove that $H^- \leqslant 1/\sqrt{3}$ and $H^+ = +\infty$.

Assume that $H^- > 1/\sqrt{3}$. Let (H_n) be a sequence converging to H^- and such that $H_n > H^-$ for all n. Then by (5.5) the second fundamental forms of the spheres S_{H_n} are uniformly bounded. Also, by Lemma 5.6 the diameters of the spheres S_{H_n} are bounded; thus the areas of the spheres S_{H_n} are bounded (by the Rauch comparison theorem), so in particular the spheres S_{H_n} satisfy local uniform area bounds (see for instance [25]). Without loss of generality we will assume that all spheres S_{H_n} contain a certain fixed point. Consequently, by standard arguments [23, 20] there exists a subsequence, which will also be denoted (H_n), such that the spheres S_{H_n} converge to a properly weakly embedded CMC H^- surface S, and S must be compact since the diameters of the spheres S_{H_n} are bounded. (The convergence is the convergence in the C^k topology, for every $k \in \mathbb{N}$, as local graphs over disks in the tangent planes with radius independent of the point.) Since S is a limit of bigraphs in the x_1 and x_2 directions, it is a sphere (and the convergence is with multiplicity one). Finally, S has index one since the spheres S_{H_n} have index one (having index one is equivalent to $\lambda_2 = 0$).

Thus we can apply Proposition 5.6 to S: we obtain the existence of a unique deformation $(\widetilde{S}_H)_{H \in (H^- - \eta, H^- + \eta)}$ for some $\eta > 0$. Since the spheres S_{H_n} converge to S, there exists some integer m such that $S_{H_m} = \widetilde{S}_{H_m}$, and so we have $S_H = \widetilde{S}_H$ for H in some neighbourhood of H_m (here the equalities between spheres are always up to a left translation). This means that we can analytically extend the family (S_H) for $H \leqslant H^-$ by setting $S_H := \widetilde{S}_H$ for all $H \in (H^- - \eta, H^-]$. This contradicts the fact that (H^-, H^+) is the largest interval for which the S_H are defined.

Hence we have proved that $H^- \leqslant 1/\sqrt{3}$. The proof that $H^+ = +\infty$ is similar (indeed, if H^+ is finite, then we have the bound on the second fundamental form).

□

Remark 5.8 Even if S_{H_0} is stable, it is not clear whether the S_H are stable or not. For instance, in $\mathbb{S}^2 \times \mathbb{R}$, Souam [30] proved that among the one parameter family of rotational CMC spheres, the ones with mean curvature less than some constant are unstable.

6 Conclusion and final remarks

We can now summarize and conclude the proofs of the main theorems.

Proof of Theorems 1.1 and 1.2: In Theorem 1.2, the implications $(a) \Rightarrow (b) \Rightarrow (c)$ are explained in Section 5.1 and the implication $(c) \Rightarrow (d)$ is Proposition 5.1. The embeddedness of Σ_H is Proposition 5.4. The uniqueness among immersed spheres is Theorem 4.7, and the uniqueness among compact embedded surfaces is then obtained thanks to the Alexandrov reflection technique. Finally, Theorem 1.1 is a corollary of Theorem 1.2 and Proposition 5.7.

□

Let us now focus on some consequences for the isoperimetric problem in Sol_3. First, index one spheres constitute a finite or countable number of real analytic families $(S_H)_{H\in(H^-,H^+)}$ parametrized by mean curvature belonging to pairwise disjoint intervals. Hence the solutions to the isoperimetric problem belong to these families. Moreover, in such a family, if $A(H)$ and $V(H)$ denote respectively the area of S_H and the volume enclosed by S_H, then $A'(H) = 2HV'(H)$ and the sphere S_{H_0} is stable if and only if $V'(H_0) \leqslant 0$ (this was proved in [30] for $\mathbb{S}^2 \times \mathbb{R}$ but the proof readily extends to Sol_3).

Proposition 6.1 *The following statements hold in* Sol_3*.*

(a) *Every solution to the isoperimetric problem is globally invariant by* all *isometries fixing some point.*

(b) *Two different solutions (up to left translations) to the isoperimetric problem cannot have the same mean curvature.*

Proof: Item (a) is an immediate consequence of Corollary 4.9 and Proposition 5.1. Item (b) is a particular case of Theorem 1.2.

□

However, we do not know if, for a given volume, there can exist several solutions (hence with different mean curvatures). Item (a) states that the solutions to the isoperimetric problem are as symmetric as possible; in [21] this was proved for small volumes in a general compact manifold.

Remark 6.2 One can prove using Propostion 5.6 that for a given volume there exists a finite number of solutions to the isoperimetric problem. Using moreover the above characterization of stable spheres, one can prove that the isoperimetric profile is concave.

The value $1/\sqrt{3}$ in Theorem 1.1 does not seem optimal and does not seem to have a geometric signification, as we explain next.

In $\mathbb{H}^3$, the value 1 is a special value for mean curvature, in the sense that CMC H surfaces in $\mathbb{H}^3$ have very different behaviours if $|H| > 1$, $|H| = 1$ or $|H| < 1$ (for instance, compact CMC H surfaces in $\mathbb{H}^3$ exist if and only if $|H| > 1$). In the same way, $1/2$ is a special value for mean curvature in $\mathbb{H}^2 \times \mathbb{R}$.

On the contrary, in Sol_3 there exist CMC H spheres with $H \neq 0$ and $|H|$ arbitrarily small. Also, in no equation regarding conformal immersions is the fact that $H > 1/\sqrt{3}$ important. This number $1/\sqrt{3}$ only appeared in the Diameter Lemma 5.3, which relies on a stability argument: the fact that $H > 1/\sqrt{3}$ implies that some quantity in the Jacobi operator is necessarily positive. However it is only a *sufficient* condition. The same kind of problems have appeared previously in related questions: for instance, it is proved in [30] that a stable compact CMC H surface is a sphere if $H > 1/\sqrt{2}$, but this value of H is conjectured not to be optimal.

This is why it is natural to propose the following conjecture.

Conjecture 6.3 *For every $H > 0$ there exists an embedded CMC H sphere S_H, which is the unique (up to left translations) immersed CMC H sphere and the unique (up to left translations) embedded compact CMC H surface. Moreover, S_H is a solution to the isoperimetric problem, and the family $(S_H)_{H>0}$ is real analytic.*

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