



HAL
open science

Output contingent securities and efficient investment by firms

Luis H. B. Braido, Victor Filipe Martins da Rocha

► **To cite this version:**

Luis H. B. Braido, Victor Filipe Martins da Rocha. Output contingent securities and efficient investment by firms. 2014. hal-01097363v1

HAL Id: hal-01097363

<https://hal.science/hal-01097363v1>

Preprint submitted on 19 Dec 2014 (v1), last revised 13 Oct 2017 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

OUTPUT CONTINGENT SECURITIES AND EFFICIENT INVESTMENT BY FIRMS¹

Luis H.B. Braido^a and V. Filipe Martins-da-Rocha^b

We analyze competitive financial economies in which firms make risky investments. Unlike the classic Arrow–Debreu framing, firms and agents cannot contract upon the exogenous states of nature underlying production risks. The only available securities are equities and all possible derivatives written on the endogenous aggregate output. It is well-known that this financial structure is rich enough to promote efficient risk sharing across consumers. However, markets are incomplete from the production perspective, and the absence of market prices for each primitive state of nature raises issues on the objective of firms. We show that asset prices combined with competitive beliefs convey sufficient information to compute a competitive shareholder value that leads to efficient investment by firms.

1. INTRODUCTION

Following the work by Léon Walras in the 19th century, the general equilibrium literature focused on understanding how anonymous markets coordinate the production and consumption of goods in competitive economies. In this setting, firms’ productive decisions and agents’ consumption choices are taken independently, and market prices are the only instruments available to coordinate different wishes. Hayek (1945) supported the view that competitive prices have the capacity of aggregating the necessary social knowledge to induce efficiency of self-interested decisions. This idea was rigorously formulated and independently proven by Kenneth J. Arrow, Gerard Debreu, and Lionel W. McKenzie during the 1950’s. They listed conditions for existence of a competitive equilibrium and proved that, in the absence of externalities and other market frictions, competitive markets lead profit-maximizing firms and utility-maximizing agents to achieve a Pareto optimal allocation of resources. The information embedded in market prices is sufficient to promote efficient social coordination across decision makers.

Arrow (1953) and Debreu (1959) extended the general equilibrium analysis to economies in which random states of nature affect production. They showed that the classic results carry over to environments with uncertainty whenever decision makers are able to trade a complete set of contingent claims—each of them promising to deliver goods in the future contingent to the verification of a given state of nature. However, the description

¹This paper has benefited from insightful interactions with Piero Gottardi, Michael Magill, Herakles Polemarchakis, Martine Quinzii, and Paolo Siconolfi. We are also thankful to comments from Yves Balasko, Alessandro Citanna, Jan Eeckhout, John Geanakoplos, Christian Ghiglini, Zhiguo He, Felix Kubler, Karl Shell, Stephen Spear, Jan Werner, and seminar participants at the Cowles Foundation, Exeter Business School, Sao Paulo School of Economics, Université Paris-Dauphine, University of York, Econometric Society Meetings at Malaga and Northwestern Universities, and NSF/NBER/CEME Conference at Indiana University. Financial support from CNPq is gratefully acknowledged.

^aFGV/EPGE

^bCNRS, U. Paris-Dauphine and FGV/EESP

and verification of primitive states are not simple matters, and most securities traded in modern financial markets are contingent on observed output instead of primitive states of nature.

We analyze competitive financial economies in which firms make risky investments and consumers trade firms' equities and securities written on firms' endogenous production. Since our goal is to analyze efficiency of firms' investment decisions, we assume that financial markets allow consumers to insure each other against idiosyncratic risks and make consumption plans contingent on aggregate output. It is well-known that this market structure is sufficient to implement Pareto optimal allocations in exchange economies. We analyze the conditions under which this also holds for production economies.

This topic was introduced by [Magill and Quinzii \(2009, 2010\)](#). These papers develop a concept for computing the shareholder value of large corporations and show that Pareto efficiency does not always follow from shareholder value maximization. This reasoning is applied in [Magill, Quinzii, and Rochet \(2013\)](#) to criticize the conventional view that firms should maximize shareholder value and to support the alternative idea that they must consider the impact of their actions over all stakeholders, including employees and other members of the society.

We argue that profit maximization can still be socially justified as a decision criterion if firms are assumed to behave as perfect competitors. Although output-contingent prices do not a priori convey all the requisite information to coordinate consumption and investment decisions, we show that this issue can be overcome by a proper anticipation of out-of-equilibrium scenarios. In the standard Arrow–Debreu approach, all relevant information for an efficient investment decision is embedded in the state-contingent prices. In the output-contingent framing, however, financial markets do not necessarily distinguish across states of nature that lead to the same equilibrium aggregate output. This distinction is still relevant for firms' decisions. We accommodate the competitive price-taking paradigm by assuming that firms and agents combine market prices with competitive beliefs about how alternative investment plans would affect the conditional expectation of each firm's output given the economy's aggregate production.¹

We start by stressing our main insights through a simple example in which a single firm makes a binary investment decision that affects its probability of success (high output) or failure (low output). Next, in Sections 3 and 4, we describe the primitives of our economy and define the competitive financial equilibrium. Agents in this economy can only trade

¹This competitiveness assumption on firms is consistent with the literature on the objective of corporations under incomplete markets—see for instance [Drèze \(1974\)](#), [Ekern and Wilson \(1974\)](#), [Leland \(1974\)](#), [Ekern \(1975\)](#), [Hart \(1979\)](#), [Grossman and Hart \(1979\)](#), [Makowski \(1983\)](#), and [Bisin, Gottardi, and Ruta \(2014\)](#). Our markets are also incomplete with respect to exogenous uncertainty, and the spanning condition does not hold here (in the sense that investments do affect the individual ability to transfer wealth across primitive states of nature). However, differently from the aforementioned literature, agents share the same marginal rates of substitution for each given equilibrium aggregate output. The main task faced by our firms and agents is to correctly anticipate the conditional expected output associated with different out-of-equilibrium investment plans.

securities written on endogenous production levels. This financial structure is incomplete with respect to the underlying primitive states of nature, but allows consumers to sell their endowment risks and make consumption plans that are contingent on the equilibrium aggregate output. In Section 5, we introduce our view on how firms should compute the net present value of out-of-equilibrium investment plans. We argue that the financial equilibrium in which firms maximize our definition of market value is the counterpart of the Arrow–Debreu concept. In particular, we show that the associated consumption and investment allocation is Pareto optimal. We also compare our competitive notion of shareholder value to the alternative concept, introduced by Magill and Quinzii (2009), where firms implicitly exert a specific form of market power. We use Section 6 to discuss existence of a competitive equilibrium in different contexts.² Concluding remarks appear in Section 7. Appendix A is reserved for technical arguments.

2. PRELIMINARY ILLUSTRATION

We borrow from Magill and Quinzii (2009) the following simple example with two periods $t \in \{0, 1\}$, one good, one firm, and one agent.³ At date 0, the firm undertakes one of two possible investment levels $a \in \{0, 1\}$. This determines the probability over two possible date-1 output levels, namely, $y_L > 0$ or $y_H > y_L$. The transition $a \mapsto Q_H(a)$ represents the probability of producing y_H (success) under investment a . Investment is productive in the sense that $Q_H(1) > Q_H(0)$.

This technology has an alternative representation with three exogenous states of nature ω_H , ω_M and ω_L . Production is high (resp. low) when the exogenous state is ω_H (resp. ω_L), regardless of the investment level. When the primitive state is ω_M , the firm's production is high if, and only if, it has invested $a = 1$. State probabilities are then $P(\omega_H) = Q_H(0)$, $P(\omega_H) + P(\omega_M) = Q_H(1)$ and, hence, $P(\omega_L) = 1 - Q_H(1)$.

At date 0, the agent owns the firm and receives an initial endowment $e_0 > 1$. Preferences are represented by the expected utility

$$u_0(x_0) + \mathbb{E}(u_1(x_1)),$$

where the Bernoulli utility functions u_0 and u_1 are strictly increasing, continuously differentiable, strictly concave and satisfy the Inada condition at zero.⁴ We also assume

²Among other topics, this section analyzes an economy with a success-or-failure technology and a continuum of investment levels for which an Arrow–Debreu equilibrium does not exist, as discussed in Magill, Quinzii, and Rochet (2013). We suggest an alternative approach to deal with the non-existence problem in that context, by assuming there is a continuum of firms whose productive shocks are perfectly correlated. We derive a general existence result for this alternative model and explicitly compute an equilibrium in which, in spite of the continuum of firms, the (average) aggregate output displays stochastic variability.

³We may also interpret this example as an economy with many identical firms and many identical agents, where we restrict attention to symmetric competitive equilibria.

⁴By that we mean $\lim_{x_t \rightarrow 0} u'_t(x_t) = \infty$, for $t = 0, 1$.

that

$$(1) \quad u_0(e_0) - u_0(e_0 - 1) < (u_1(y_H) - u_1(y_L)) [Q_H(1) - Q_H(0)]$$

as for investing to be Pareto optimal.

The financial markets are complete with respect to aggregate uncertainty. In this simple environment (with two possible output levels), this only requires that markets transact the firm's equity and a non-contingent bond.⁵ The entire production is consumed by the single agent at equilibrium. Since there is only one firm, the aggregate output is either $z_H = y_H$ or $z_L = y_L$. For each given investment \bar{a} , the agent's equilibrium marginal rate of substitution is constant across states of nature associated with the same aggregate output. The equilibrium stochastic discount factor is

$$\bar{\chi}(z_s) = \frac{u'_1(y_s)}{u'_0(e_0 - \bar{a})}, \quad \text{for each } s \in \{L, H\}.$$

The equilibrium equity value is

$$\bar{E} = y_H \bar{\chi}(z_H) Q_H(\bar{a}) + y_L \bar{\chi}(z_L) (1 - Q_H(\bar{a})).$$

The value of a large corporation

Recent works by [Magill and Quinzii \(2009, 2010\)](#) consider the case where the firm takes the stochastic discount factor as given but is aware that its investment decision affects the distribution of aggregated variables. In this case, the firm maximizes

$$(2) \quad M(a) \equiv y_H \bar{\chi}(z_H) Q_H(a) + y_L \bar{\chi}(z_L) (1 - Q_H(a)) - a.$$

They show that the Pareto efficient level $\bar{a} = 1$ does not necessarily maximize M .⁶ In our example, this result follows from the fact that the Pareto efficiency condition (1) may hold simultaneously with $M(0) > M(1)$, or equivalently,⁷

$$(3) \quad u'_0(e_0 - 1) > [y_H u'_1(y_H) - y_L u'_1(y_L)] [Q_H(1) - Q_H(0)].$$

The competitive market value

Defining a competitive concept of market value for an economy with output-contingent financial markets is not straightforward. Securities written on the aggregate output do

⁵Our markets are not complete with respect to exogenous uncertainty, as we have three states of nature but only two assets. Moreover, investment plans affect the individual ability to transfer wealth across primitive states.

⁶A recent work by [Magill, Quinzii, and Rochet \(2013\)](#) develops a similar result in an environment with multiple goods and quasi-linear utilities.

⁷Indeed, since u_0 satisfies the Inada condition at zero, we can always take e_0 close enough to 1 to guarantee the validity of condition (3). However, if u_0 and u_1 were linear functions, then condition (1) would be equivalent to $M(1) > M(0)$.

not necessarily distinguish across primitive states of nature. In our illustration, the equilibrium market prices $\bar{\rho}(z)$ for the security contingent on the realization of the output z satisfy the standard equations

$$\bar{\rho}(z_H) = \bar{\chi}(z_H)Q_H(\bar{a}) \quad \text{and} \quad \bar{\rho}(z_L) = \bar{\chi}(z_L)(1 - Q_H(\bar{a})).$$

Agents are aware that the equilibrium prices $\bar{\rho}(z)$ and the theoretical state prices $\bar{p}(\omega)$ are related to each other. However, market prices alone do not convey sufficient information for splitting $\bar{\rho}(z_H)$ and $\bar{\rho}(z_L)$ into $\bar{p}(\omega_H)$, $\bar{p}(\omega_M)$, and $\bar{p}(\omega_L)$. To overcome this issue, we impose an additional layer of rationality and ask firms and agents to rationally anticipate some important equilibrium features when analyzing investment plans out of the equilibrium.

Consider the case in which the equilibrium investment is $\bar{a} = 1$. When analyzing the net present value associated with the out-of-equilibrium investment decision $a = 0$, the agent and the firm are required to take the output-contingent prices $\bar{\rho}$ as given and make the following considerations regarding the firm's production. In equilibrium, date-1 aggregate output is low if, and only if, the primitive state ω_L has occurred. Therefore, under zero investment, a competitive firm shall expect to produce $\tilde{y}(0|z_L) \equiv y_L$ in events with low aggregate output z_L . On the other hand, at equilibrium, there are two possible primitive states consistent with a high date-1 aggregate output, namely, ω_H and ω_M . Under zero investment, the firm should expect to produce y_H if the primitive state is ω_H and to produce y_L if the primitive state is ω_M . Formally, if the firm invested zero, its conditional expected output given a high equilibrium aggregate output z_H is

$$\tilde{y}(0|z_H) \equiv y_H \frac{P(\omega_H)}{P(\omega_H) + P(\omega_M)} + y_L \frac{P(\omega_M)}{P(\omega_H) + P(\omega_M)}.$$

The competitive market value for this firm should then be

$$V(0) \equiv \tilde{y}(0|z_H)\bar{\rho}(z_H) + \tilde{y}(0|z_L)\bar{\rho}(z_L) - 0.$$

By an analogous argument, we can define the conditional expected output under the investment $\bar{a} = 1$ as being $\tilde{y}(1|z_H) \equiv y_H$ and $\tilde{y}(1|z_L) \equiv y_L$. The competitive market value associated with $\bar{a} = 1$ is

$$V(1) \equiv \tilde{y}(1|z_H)\bar{\rho}(z_H) + \tilde{y}(1|z_L)\bar{\rho}(z_L) - 1$$

and coincides with the firm's equilibrium market value $\bar{E} - 1$. It is then simple to verify that the Pareto optimal investment level $\bar{a} = 1$ does in fact maximize the function $a \mapsto V(a)$. We must just notice that

$$V(1) - V(0) = \frac{u'_1(y_H)}{u'_0(e_0 - 1)}(y_H - y_L)P(\omega_M) - 1 > 0,$$

where the last inequality follows from assumption (1) and the fact that $P(\omega_M) = Q_H(1) - Q_H(0)$.

Relating conjectures to Arrow prices

The competitive conditional expectation $\tilde{y}(a|z)$ connects the output-contingent prices $\bar{\rho}(z)$ and the implicit (or shadow) Arrow prices $\bar{p}(\omega)$. To see that, fix the equilibrium investment level $\bar{a} = 1$. If state-contingent claims were allowed, then the Arrow prices would be

$$\bar{p}(\omega_H) = \bar{\rho}(z_H) \frac{P(\omega_H)}{P(\omega_H) + P(\omega_M)},$$

$$\bar{p}(\omega_M) = \bar{\rho}(z_H) \frac{P(\omega_M)}{P(\omega_H) + P(\omega_M)},$$

and

$$\bar{p}(\omega_L) = \bar{\rho}(z_L).$$

This implies

$$V(0) = y_H \bar{p}(\omega_H) + y_L \bar{p}(\omega_M) + y_L \bar{p}(\omega_L) - 0$$

and

$$V(1) = y_H \bar{p}(\omega_H) + y_H \bar{p}(\omega_M) + y_L \bar{p}(\omega_L) - 1.$$

We can then regard our definition of competitive market value as the counterpart (for an environment with output-contingent prices) of the standard Arrow–Debreu market value.

In a similar fashion, we point out that the value of a large corporation defined in Eq. (2) implies the following conjectures:

$$M(0) = y_H \bar{\chi}(y_H) P(\omega_H) + y_L \bar{\chi}(y_L) P(\omega_M) + y_L \bar{\chi}(y_L) P(\omega_L) - 0$$

and

$$M(1) = y_H \bar{\chi}(y_H) P(\omega_H) + y_H \bar{\chi}(y_H) P(\omega_M) + y_L \bar{\chi}(y_L) P(\omega_L) - 1.$$

Under these conjectures, the firm anticipates the effect of its investment over the stochastic discount factor in the state ω_M .

REMARK 2.1 (Competitive conjectures and non-arbitrage pricing) The competitive market value V is mathematically equivalent to the Arrow-Debreu profit function defined for an environment with Arrow prices. It is however important to stress that V is a rational conjecture and cannot be unambiguously derived from non-arbitrage conditions. Markets are incomplete as we have three primitive states of nature ($\omega_H, \omega_M, \omega_L$) and two assets, namely, the firm's equity and a risk-free bond. Under the equilibrium

investment $\bar{a} = 1$, the equity payoff vector is (y_H, y_H, y_L) , while the bond payoff vector is $(1, 1, 1)$. The production vector (y_H, y_L, y_L) associated with the out-of-equilibrium investment $a = 0$ does not belong to the marketed space, in the sense it cannot be generated by the assets payoff vectors (y_H, y_H, y_L) and $(1, 1, 1)$. Therefore, there are infinitely many arbitrage-free state prices, and we cannot rely on non-arbitrage pricing to unambiguously obtain the out-of-equilibrium equity value. Instead, among all possible arbitrage-free valuations, our competitive market value $V(0)$ corresponds to that where market is risk-neutral conditional on the aggregate output. Since the ratio of marginal utilities is the same across states of nature associated with the same aggregate output, this risk-neutral pricing conditional on the aggregate output is natural.

3. GENERAL MODEL

We extend the main ideas presented in our initial illustration to a general environment. Consider an economy with two periods $t \in \{0, 1\}$, a single good, a finite set K of firms and a finite set I of consumers. At the initial date ($t = 0$), each firm k selects an investment level a^k from a set $A^k \subset \mathbb{R}_+$. Making no investment is always a possibility—i.e., $0 \in A^k$, for every k . At date 1, they are exposed to exogenous shocks ω drawn from a probability space (Ω, \mathcal{F}, P) . Events $B \in \mathcal{F}$ represent primitive causes which odds are represented by the exogenous probability $P(B)$. This probability is independent of consumers' and firms' actions.

Technology

The initial investment a^k and the exogenous shock ω determine firm k 's production $y^k = f^k(\omega, a^k)$ at date 1 from a set $Y^k \subset \mathbb{R}_+$. The production possibilities of the economy are represented by the family $f \equiv (f^k)_{k \in K}$ of non-decreasing random production functions

$$f^k(\omega, \cdot) : A^k \mapsto Y^k.$$

We assume that, for each investment a^k , the function $\omega \mapsto f^k(\omega, a^k)$ is measurable and essentially bounded on (Ω, \mathcal{F}, P) .⁸

From this standard production-function framework, we derive the following alternative representation of the productive sector. Define the sets $A \equiv \prod_{k \in K} A^k$ and $Y \equiv \prod_{k \in K} Y^k$, their respective elements $a \equiv (a^k)_{k \in K}$ and $y \equiv (y^k)_{k \in K}$, and the transition probability $a \mapsto Q(a)$ given by

$$Q(B, a) \equiv P(\{f(a) \in B\}),$$

⁸To fix ideas, we can take Ω to be the product space $\prod_{k \in K} (Y^k)^{A^k}$ and \mathcal{F} to be the product $\otimes_{k \in K} \mathcal{B}^k$ of each Borelian σ -algebra \mathcal{B}^k defined by the product topology of the space $(Y^k)^{A^k}$. The support of the probability P is then assumed to be a subset of $\prod_{k \in K} N(A^k, Y^k)$, where $N(A^k, Y^k)$ is the set of non-decreasing functions from A^k to Y^k .

for every Borelian set B of Y .⁹ The investment profile $a \in A$ undertaken at date 0 determines the joint probability $Q(a)$ of firms' random outcomes at date 1.

To represent aggregate production, we define the σ -operator to be

$$\sigma y \equiv \sum_{k \in K} y^k, \quad \text{for all } y \in Y.$$

The random aggregate production is then represented by the function $\omega \mapsto \sigma f(\omega, a)$. We let $Z \equiv \sum_{k \in K} Y^k$ denote the set of all possible aggregate outputs and derive the transition probability $a \mapsto \mu(a)$ by posing

$$\mu(B, a) \equiv P(\{\sigma f(a) \in B\}),$$

for every Borelian set B of Z . We assume that date-1 output is bounded away from zero, in the sense that there exists $\varepsilon > 0$ such that $\mu([\varepsilon, \infty), a) = 1$, for every $a \in A$.

Agents

Each agent i has initial resources consisting of an endowment $e_0^i > 0$ at date 0 and the ownership shares $\delta_k^i \in [0, 1]$ of each firm k , where $\sum_{i \in I} \delta_k^i = 1$. Agents have no initial endowment at date 1, so that all consumption in that period comes from the firms' output. Preferences are represented by a utility function that is separable across time and has the expected utility form for future risky consumption. Let $x_0^i \geq 0$ denote agent i 's consumption at date 0 and γ^i be a probability measure on \mathbb{R}_+ that represents random consumption at date 1. Agent i 's expected utility function is given by

$$u_0(x_0^i) + \int_{\mathbb{R}_+} u_1(x_1^i) \gamma^i(dx_1^i),$$

where u_0 and u_1 are strictly increasing, continuously differentiable and strictly concave functions which map \mathbb{R}_+ into $[-\infty, \infty)$ and satisfy the Inada condition at zero.

REMARK 3.1 The time separability of the expected utility and the cross-agent homogeneity of the Bernoulli utility functions are only assumed to simplify the notation. It is straightforward to extend all results in this paper to the case with heterogeneous Bernoulli utilities $(x_0^i, x_1^i) \mapsto v^i(x_0^i, x_1^i)$.

4. COMPETITIVE EQUILIBRIUM FOR A GIVEN INVESTMENT

For sake of expositional clarity, we first define a competitive equilibrium for our output-contingent environment by taking the investment level of each firm as given. After understanding how agents insure each other, we analyze the problem of how firms choose their investments.

⁹The set $\{f(a) \in B\}$ stands for $\{\omega \in \Omega : f(\omega, a) \in B\}$. Similar notation omitting ω is used throughout the paper.

Different from the traditional Arrow–Debreu model, we do not consider contracts contingent on the realization of the primitive states of nature ω . We assume that the probabilities and the economic consequences of the events in \mathcal{F} are well-understood by firms and agents, but the costs of describing ex-ante each primitive state of nature and enforcing ex-post state-contingent contracts are too large. The only traded contracts are those based on firm’s output. We consider two types of assets: the equity of each firm $k \in K$ traded in positive net supply; and securities in zero net supply representing bonds and all possible output-contingent derivatives.

Uncertainty only derives from production risks. Hence, an efficient allocation of risks among consumers (for a given vector of firms’ investments) only requires that agents trade firms’ equities and claims contingent on date-1 aggregate output. At date 0, for a given investment profile a , each agent i chooses consumption $x_0^i \in \mathbb{R}_+$, new equity holdings $\eta^i \in \mathbb{R}^K$ and a (measurable) contract $\theta^i : Z \rightarrow \mathbb{R}$ contingent on aggregate output such that

$$(4) \quad x_0^i + \int_Z \theta^i(z) \rho(dz) + E \cdot \eta^i \leq e_0^i + (E - a) \cdot \delta^i,$$

where E stands for the vector of equity prices, and ρ is a positive measure on the Borelian sets of Z such that $\rho([0, z])$ represents the date-0 price of the contract delivering one unit of consumption good contingent on the aggregate output being lower than or equal to z . At date 1, contingent on output profile y , agent i consumes

$$(5) \quad x_1^i(y) \equiv \theta^i(\sigma y) + y \cdot \eta^i \geq 0.$$

Each agent i maximizes the expected utility

$$u_0(x_0^i) + \int_Y u_1(x_1^i(y)) Q(dy, a)$$

among all individual plans $(x_0^i, x_1^i, \eta^i, \theta^i)$ satisfying the budget constraints (4) and (5), for $Q(a)$ -almost every y .

A **financial equilibrium** associated with \bar{a} is a list $(\bar{E}, \bar{\rho}, (\bar{x}_0, \bar{x}_1, \bar{\eta}, \bar{\theta}))$, where $(\bar{x}_0, \bar{x}_1, \bar{\eta}, \bar{\theta}) \equiv (\bar{x}_0^i, \bar{x}_1^i, \bar{\eta}^i, \bar{\theta}^i)_{i \in I}$ is a consumption-portfolio allocation such that:

- (i) for every $i \in I$, the plan $(\bar{x}_0^i, \bar{x}_1^i, \bar{\eta}^i, \bar{\theta}^i)$ solves agent i ’s optimization problem given $(\bar{E}, \bar{\rho}, \bar{a})$;
- (ii) the consumption markets clear, i.e.,

$$(6) \quad \sum_{i \in I} (e_0^i - \bar{x}_0^i) = \sum_{k \in K} \bar{a}^k$$

and

$$(7) \quad \sum_{i \in I} \bar{x}_1^i(y) = \sigma y, \quad \text{for } Q(\bar{a})\text{-almost every } y;$$

(iii) the financial markets clear, i.e.,

$$(8) \quad \sum_{i \in I} \bar{\eta}^i = \mathbf{1} \quad \text{and} \quad \sum_{i \in I} \bar{\theta}^i(z) = 0, \quad \text{for } \mu(\bar{a})\text{-almost every } z. \text{ }^{10}$$

An **allocation** $((\bar{x}_0, \bar{x}_1), \bar{a})$ is said to be **feasible** if the consumption markets clear. An allocation $((x_0, x_1), a)$ is said to Pareto dominate the allocation $((\bar{x}_0, \bar{x}_1), \bar{a})$ whenever

$$u_0(x_0^i) + \int_Y u_1(x_1^i(y))Q(dy, a) \geq u_0(\bar{x}_0^i) + \int_Y u_1(\bar{x}_1^i(y))Q(dy, \bar{a}),$$

for every agent i , with strict inequality for at least one agent i . An allocation $((\bar{x}_0, \bar{x}_1), \bar{a})$ is **Pareto optimal** if $((\bar{x}_0, \bar{x}_1), \bar{a})$ is feasible and there is no other feasible allocation $((x_0, x_1), a)$ that Pareto dominates $((\bar{x}_0, \bar{x}_1), \bar{a})$. A consumption allocation (\bar{x}_0, \bar{x}_1) is **Pareto optimal for a given investment** \bar{a} if $((\bar{x}_0, \bar{x}_1), \bar{a})$ is feasible and there is no other feasible allocation $((x_0, x_1), \bar{a})$ that Pareto dominates $((\bar{x}_0, \bar{x}_1), \bar{a})$. The next result shows that our market structure implements a Pareto optimal distribution of resources among consumers for any given investment profile.

PROPOSITION 4.1 Fix a financial equilibrium $(\bar{E}, \bar{\rho}, (\bar{x}_0, \bar{x}_1, \bar{\eta}, \bar{\theta}))$ associated with an arbitrary investment vector \bar{a} . The corresponding consumption allocation (\bar{x}_0, \bar{x}_1) is Pareto optimal given \bar{a} .

PROOF: Assume by way of contradiction that there is a feasible allocation $((x_0, x_1), \bar{a})$ that Pareto dominates $((\bar{x}_0, \bar{x}_1), \bar{a})$. Since the consumption market clearing condition (7) only involves aggregate output and agents are risk-averse, the allocation $((x_0, \tilde{x}_1), \bar{a})$ also Pareto dominates $((\bar{x}_0, \bar{x}_1), \bar{a})$, where

$$\tilde{x}_1^i(y) \equiv c_1^i(\sigma y) \quad \text{with} \quad c_1^i(z) \equiv \int_Y x_1^i(y)Q(dy, \bar{a}|z)$$

where $Q(B, \bar{a}|z) \equiv P(\{f(\bar{a}) \in B | \sigma f(\bar{a}) = z\})$ is the probability that firms' production profile belongs to $B \subset Y$ under the investment vector \bar{a} and conditional on the aggregate production being z . Since the function \tilde{x}_1^i only varies with the aggregated output σy , this consumption level is implemented by the portfolio $(\theta^i, \eta^i) \equiv (c_1^i, 0)$, in the sense that $(\tilde{x}_1^i, \theta^i, \eta^i)$ satisfies Eq. (5). Since the plan $(\bar{x}_0^i, \bar{x}_1^i, \bar{\eta}^i, \bar{\theta}^i)$ solves agent i 's optimization problem given $(\bar{E}, \bar{\rho}, \bar{a})$, we must have

$$(9) \quad x_0^i + \int_Z c_1^i(z)\bar{\rho}(dz) + \bar{E} \cdot 0 \geq e_0^i + (\bar{E} - \bar{a}) \cdot \delta^i,$$

for every agent i , with a strict inequality for at least one i . By non-arbitrage, we know that

$$\int_Z z\bar{\rho}(dz) = \bar{E} \cdot \mathbf{1}.$$

Therefore, by adding inequality (9) over i , we find

$$\sum_{i \in I} (x_0^i - e_0^i) + \bar{a} \cdot \mathbf{1} > 0,$$

which contradicts the date-0 consumption market clearing condition (6). *Q.E.D.*

The equilibrium measure $\bar{\rho}(dz)$ is absolutely continuous with respect to $\mu(dz, \bar{a})$. This is to say that there is a measurable function $\bar{\chi} : Z \rightarrow \mathbb{R}_+$ (called the stochastic discount factor) such that:

$$(10) \quad \bar{\rho}(dz) = \bar{\chi}(z)\mu(dz, \bar{a}).$$

Pareto optimality of the consumption allocation (\bar{x}_0, \bar{x}_1) given \bar{a} implies the individual consumption $\bar{x}_1^i(y)$ to be constant across output vectors y generating the same aggregate output σy . Since date-1 aggregate production is bounded away from zero, our assumptions on the Bernoulli utilities imply that the ratio of marginal utilities equals the stochastic discount factor

$$\frac{u'_1(\bar{x}_1^i(y))}{u'_0(\bar{x}_0^i)} = \bar{\chi}(\sigma y),$$

for every i and y . As a consequence, the equilibrium equity prices can be written as

$$(11) \quad \bar{E} = \int_Y \bar{\chi}(\sigma y) y Q(dy, \bar{a}).$$

5. THE MARKET VALUE OF EACH FIRM

We now analyze how investment levels are chosen by firms in equilibrium. Each firm is assumed to be small relative to the aggregate economy and does not seek to manipulate prices. Under these competitive conditions, a natural objective function for a firm to maximize is its market value. When markets are complete with respect to primitive states (like in the standard Arrow–Debreu framework), the Arrow prices $\bar{p}(B)$ associated with any primitive event $B \subset \Omega$ are quoted in the market. Firm k 's manager can then take these prices as given and use them to compute the following conjectured equity value

$$\tilde{E}^k(a^k) \equiv \int_{\Omega} f^k(\omega, a^k) \bar{p}(d\omega)$$

associated with any out-of-equilibrium investment a^k . In that case, maximizing the standard competitive market value

$$(12) \quad \Pi^k(a^k) \equiv \tilde{E}^k(a^k) - a^k = \int_{\Omega} f^k(\omega, a^k) \bar{p}(d\omega) - a^k$$

leads to Pareto optimality.

In the absence of securities whose payoffs are contingent on exogenous events, a deep issue arises: how firms should assess their equity value for production plans different from the equilibrium ones? Or equivalently, how can we define the conjecture $\tilde{E}^k(a^k)$ when the prices $\bar{p}(d\omega)$ are not available?

5.1. Competitive conjectures conditional on aggregate production

Prices $\bar{\rho}(dz)$ contingent on endogenous events (aggregate production) do not have the same informational content as the state prices $\bar{p}(d\omega)$ do in the Arrow–Debreu environment. Our firms face the same prices for states of nature associated with the same equilibrium aggregate output. When firm k invests a^k , its average production across states ω associated with equilibrium aggregate output $z = \sigma f(\omega, \bar{a})$ is

$$(13) \quad \tilde{y}^k(a^k|z) \equiv \int_{Y^k} y^k Q(dy^k, a^k|z),$$

where

$$Q(B, a^k|z) \equiv P(\{f^k(a^k) \in B\} | \sigma f(\bar{a}) = z)$$

is the probability that firm k 's production belongs to $B \subset Y^k$ under the investment a^k and conditional on the equilibrium aggregate production being z .¹¹ We then propose the following definition to represent the competitive market value for the output-contingent economy:

$$(14) \quad V^k(a^k) \equiv \int_Z \tilde{y}^k(a^k|z) \bar{\rho}(dz) - a^k.$$

The key behavioral assumption behind Eqs. (13) and (14) is that firms and agents take prices as given and form competitive beliefs about the conditional expected production under different out-of-equilibrium investment levels. They understand that firm k 's output becomes the random variable $\omega \mapsto f^k(\omega, a^k)$ whenever it invests a^k . However, they also believe that firm k 's decisions do not affect the likelihood of aggregate production and, therefore, compute expected production $\tilde{y}^k(a^k|z)$ conditional on the event $\{\sigma f(\bar{a}) = z\}$. The conditioning event is evaluated at the equilibrium investment vector, *which includes the investment choice of firm k* . One could metaphorically think about this as if there was a continuum of firms so that the term $\tilde{y}^k(a^k|z)$ represented the conditional expected output when a firm invested a^k while all other infinite firms invested the equilibrium level.

REMARK 5.1 (Conditional risk-neutral valuation) We know from Proposition 4.1 that the equilibrium consumption plans and stochastic discount factor are constant across states of nature ω that are associated with the same aggregated output $z = \sigma f(\omega, \bar{a})$. At equilibrium, firms and agents become risk neutral conditional on the aggregate production z . This is behind the way we introduced the conditional expected production $\tilde{y}^k(a^k|z)$ in the competitive market value defined in Eq. (14).

¹¹For any event $B \in \mathcal{F}$ and any arbitrary random variable $g : \Omega \rightarrow Z$, we let $P(B|g)$ be the conditional expectation $\mathbb{E}[\mathbf{1}_B|g]$, where $\mathbf{1}_B$ is the indicator function of the set B . Since the random function $P(B|g)$ is g -measurable, there exists a measurable function $\Phi : Z \rightarrow \mathbb{R}$ such that $P(B|g) = \Phi(g)$. The number $\Phi(z)$ is denoted by $P(B|g = z)$.

REMARK 5.2 (Correctness at equilibrium) At equilibrium, our competitive definition of market value equals the equilibrium equity price minus investments. Indeed, we have

$$\begin{aligned}
V^k(\bar{a}^k) &= \int_Z \tilde{y}^k(\bar{a}^k|z) \bar{\chi}(z) \mu(dz, \bar{a}) - \bar{a}^k \\
&= \int_{z \in Z} \bar{\chi}(z) \left[\int_{Y^k} y^k Q(dy^k, \bar{a}^k|z) \right] \mu(dz, \bar{a}) - \bar{a}^k \\
&= \int_Y \bar{\chi}(\sigma y) y^k Q(dy, \bar{a}) - \bar{a}^k \\
&= \bar{E}^k - \bar{a}^k,
\end{aligned}$$

where these equalities follow from Eqs. (10), (11), (13), and (14).

5.2. Efficiency

We turn now to show the Pareto optimality of any allocation $((\bar{x}_0, \bar{x}_1), \bar{a})$ derived from a financial equilibrium in which investments maximize the competitive market value V^k . We have argued before that the impossibility to trade assets contingent on primitive states is an essential incompleteness of markets from the perspective of firms. Although consumers do not need to trade securities contingent on primitive states to perfectly share idiosyncratic risks, firms do need the information embedded in the state-contingent prices \bar{p} in order to compute the competitive present value $\Pi^k(a^k)$ associated with out-of-equilibrium investment plans $a^k \neq \bar{a}^k$, as defined in Eq. (12). Our next proposition shows that this incompleteness can be overcome if firms hold the competitive beliefs $\tilde{y}^k(a^k|z)$. To prove this, we show below the connection between the standard Arrow–Debreu concept of competitive equilibrium and our definition of financial equilibrium.

We recall that an **Arrow–Debreu equilibrium** is a list $(\bar{p}, (\bar{\xi}_0, \bar{\xi}_1), \bar{a})$ composed of: (i) a positive measure \bar{p} on (Ω, \mathcal{F}) representing state-contingent prices; (ii) a consumption plan $(\bar{\xi}_0, \bar{\xi}_1) \equiv (\bar{\xi}_0^i, \bar{\xi}_1^i)_{i \in I}$, where $\bar{\xi}_0^i \geq 0$ and $\bar{\xi}_1^i : \Omega \rightarrow \mathbb{R}_+$ is a random variable; and (iii) an investment vector $\bar{a} \equiv (a^k)_{k \in K}$ such that:

(a) the allocation $((\bar{\xi}_0, \bar{\xi}_1), \bar{a})$ is feasible, in the sense that

$$(15) \quad \sum_{i \in I} e_0^i - \bar{\xi}_0^i = \sum_{k \in K} \bar{a}^k$$

and

$$(16) \quad \sum_{i \in I} \bar{\xi}_1^i(\omega) = \sum_{k \in K} f^k(\omega, \bar{a}^k), \quad \text{for } P\text{-almost every } \omega \in \Omega;$$

(b) for each firm k , the investment \bar{a}^k maximizes the present-value function

$$\Pi^k(a^k) \equiv \int_{\Omega} f^k(\omega, a^k) \bar{p}(d\omega) - a^k;$$

(c) for each agent i , the consumption plan $(\bar{\xi}_0^i, \bar{\xi}_1^i)$ maximizes the expected utility

$$u_0(\bar{\xi}_0^i) + \int_{\Omega} u_1(\bar{\xi}_1^i(\omega)) P(d\omega)$$

subject to the present-value budget constraint

$$(17) \quad \xi_0^i + \int_{\Omega} \xi_1^i(\omega) \bar{p}(d\omega) \leq e_0^i + \Pi(\bar{a}) \cdot \delta^i.$$

where $\Pi(\bar{a}) \equiv (\Pi(\bar{a}^k))_{k \in K}$.

PROPOSITION 5.1 There exists an Arrow–Debreu equilibrium $(\bar{p}, (\bar{\xi}_0, \bar{\xi}_1), \bar{a})$ if, and only if, there exists a financial equilibrium $(\bar{E}, \bar{\rho}, (\bar{x}_0, \bar{x}_1, \bar{\eta}, \bar{\theta}))$ associated with the investment vector \bar{a} such that, for each k , \bar{a}^k maximizes the competitive market value V^k .

This proposition is proven in Appendix A.1. The proof consists in constructing the elements of a given equilibrium concept from the elements describing the alternative equilibrium. We notice that, in addition to investments \bar{a} , the equilibrium consumption plans coincide. This is to say that, for each i , we have $\bar{x}_0^i = \bar{\xi}_0^i$ and there is a Borel-measurable function $\bar{c}^i : Z \rightarrow \mathbb{R}_+$ such that

$$\bar{x}_1^i(y) = \bar{c}_1^i(\sigma y) \quad \text{and} \quad \bar{c}_1^i(\sigma f(\omega, \bar{a})) = \bar{\xi}_1^i(\omega).$$

Moreover, the state-contingent prices in the Arrow–Debreu equilibrium are

$$\bar{p}(d\omega) = \bar{\chi}(\sigma f(\omega, \bar{a})) P(d\omega),$$

while the prices in the financial equilibrium are

$$\bar{\rho}(dz) = \bar{\chi}(z) \mu(dz, \bar{a}) \quad \text{and} \quad \bar{E} = \int_{\Omega} f(\omega, \bar{a}) \bar{p}(d\omega).$$

The First Welfare Theorem holds true in the Arrow–Debreu framework, given the baseline assumptions of our model. It then follows from Proposition 5.1 that the allocation composed of an investment vector that maximizes each firm’s competitive value and the consumption profile from the corresponding financial equilibrium is Pareto optimal. This is proven in our next result.

THEOREM 5.1 *Let $(\bar{E}, \bar{\rho}, (\bar{x}_0, \bar{x}_1, \bar{\eta}, \bar{\theta}))$ be a financial equilibrium associated with an investment vector \bar{a} such that, for each k , \bar{a}^k maximizes V^k . Then, the allocation $((\bar{x}_0, \bar{x}_1), \bar{a})$ is Pareto optimal.*

PROOF: Take the financial equilibrium $(\bar{E}, \bar{\rho}, (\bar{x}_0, \bar{x}_1, \bar{\eta}, \bar{\theta}))$ associated with \bar{a} . Consider the corresponding Arrow–Debreu equilibrium $(\bar{p}, (\bar{\xi}_0, \bar{\xi}_1), \bar{a})$ defined by Proposition 5.1. Assume then, by way of contradiction, that there is an alternative output-contingent feasible allocation $((x_0, x_1), a)$ which Pareto dominates $((\bar{x}_0, \bar{x}_1), \bar{a})$. For each i , define the state-contingent consumption plan (ξ_0^i, ξ_1^i) by posing $\xi_0^i \equiv x_0^i$ and $\xi_1^i(\omega) \equiv x_1^i(\sigma f(\omega, a))$,

for every ω . The state-contingent allocation $((\xi_0, \xi_1), a)$ satisfies the feasibility constraints (15) and (16). Moreover, we have

$$\begin{aligned} u_0(\xi_0^i) + \int_{\Omega} u_1(\xi_1^i(\omega))P(d\omega) &= u_0(x_0^i) + \int_Y u_1(x_1^i(y))Q(dy, a) \\ &\geq u_0(\bar{x}_0^i) + \int_Z u_1(\bar{x}_1^i(y))Q(dy, \bar{a}) \\ &= u_0(\bar{\xi}_0^i) + \int_{\Omega} u_1(\bar{\xi}_1^i(\omega))P(d\omega), \end{aligned}$$

where the inequality is strict for at least one agent i . This means that the allocation $((\xi_0, \xi_1), a)$ Pareto dominates $((\bar{\xi}_0, \bar{\xi}_1), \bar{a})$, which contradicts the First Welfare Theorem applied to the Arrow–Debreu equilibrium $(\bar{p}, (\bar{\xi}_0, \bar{\xi}_1), \bar{a})$. *Q.E.D.*

5.3. Strategic conjectures

When analyzing the decision of non-marginal firms, Magill and Quinzii (2009) define

$$(18) \quad M^k(a^k) \equiv \int_Y \bar{\chi}(\sigma y) y^k Q(dy, (a^k, \bar{a}^{-k})) - a^k$$

as the objective function for firms.¹² Each firm k takes as given the equilibrium stochastic discount factor $\bar{\chi}$ and the investment vector \bar{a}^{-k} of all other firms. If we take a financial equilibrium $(\bar{E}, \bar{\rho}, (\bar{x}_0, \bar{x}_1, \bar{\eta}, \bar{\theta}))$ associated with \bar{a} , we will have $V^k(\bar{a}^k) = M^k(\bar{a}^k)$. However, $M^k(a^k)$ typically differs from our definition of competitive market value $V^k(a^k)$ for out-of-equilibrium investments $a^k \neq \bar{a}^k$.

Magill and Quinzii (2009) proved that the equilibrium in which each firm k sets an investment a^k to maximize $M^k(a^k)$ is not necessarily Pareto optimal. Given that there is no technological externality, one is led to think that this inefficiency result is related to a sort of pecuniary externality due to the specific output-contingent market structure under consideration. We proved that maximization of the competitive market value V^k restores efficiency and, therefore, no pecuniary externality results from the restriction imposed on the financial market.

By analyzing Eq. (18), one realizes that firm k implicitly anticipates the impact of the investment a^k over the probability distribution Q , which turns out to affect the value of all other firms through the probability $Q(dy, (a^k, \bar{a}^{-k}))$. To assess this issue from a different perspective, we use the exogenous shocks to write

$$M^k(a^k) = \int_{\Omega} f^k(\omega, a^k) \tilde{p}^k(d\omega, a^k) - a^k,$$

where

$$\tilde{p}^k(d\omega, a^k) \equiv \bar{\chi}(\sigma f(\omega, (a^k, \bar{a}^{-k})))P(d\omega).$$

¹²The term (a^k, \bar{a}^{-k}) represents a vector in which the k -th entry of \bar{a} is replaced by a^k .

Recall that if state-contingent claims were introduced in the market, all consumers would just be indifferent to buying or selling them if prices satisfied $\bar{p}(d\omega) = \bar{\chi}(\sigma f(\omega, \bar{a}))P(d\omega)$. Therefore, we can interpret $\bar{p}(d\omega)$ as the competitive market shadow price for event $d\omega \subset \Omega$. By replacing \bar{p} with $\tilde{p}^k(a^k)$, firm k 's manager implicitly anticipates the impact of investment decisions over the aggregate output distribution and manipulates the underlying state prices that affect the market values of all firms. The inefficiencies associated to the conjecture M^k are therefore due to the strategic behavior of the firm and are not related to market incompleteness. Actually, even if all state-contingent claims were available for trade, inefficiency would still arise if firms exerted the same market power through its investment decisions—using the price measure $\tilde{p}^k(d\omega, a^k)$ instead of the equilibrium price $\bar{p}(d\omega)$ to compute price conjectures.

A similar issue appears in economies in which the stochastic discount factor is constant but there are multiple goods whose equilibrium prices are affected by investments. [Magill, Quinzii, and Rochet \(2013\)](#) explore this issue in details and show that large firms exerting market power through investments should be stakeholder oriented and maximize a weighted sum of their shareholder value and their contributions to consumer and employee welfares.

6. ABOUT EXISTENCE

The objective of this paper is to investigate whether the maximization of a suitably defined competitive market value leads to efficient investment decisions by firms, even if investors can only write contracts on observable output. An important related issue is existence of a competitive equilibrium where firms maximize the corresponding competitive market value.

Given an arbitrary investment profile a , existence of a competitive financial equilibrium is assured by our assumptions on preferences and positiveness of date-0 endowments and of date-1 production outcomes. If, in addition to that, the sets A^k were convex, and the production functions $f^k(\omega, \cdot)$ were continuous and concave on A^k , for each ω , we could then assure existence of a competitive financial equilibrium associated with an investment profile \bar{a} that maximizes each firm's competitive market value V^k . This follows from Proposition 5.1 coupled with classic theorems on existence of Arrow–Debreu equilibrium—see [Bewley \(1972\)](#).

REMARK 6.1 (Finitely many states) When the support of the probability P is finite, the investment sets A^k are convex, and the production functions f^k are concave, then our assumption that securities' markets are complete with respect to aggregate output may (generically) yield complete markets with respect to the primitive states.¹³ This feature

¹³Indeed, concavity of the production function leads to a continuum of possible outcomes. Completeness with respect to aggregate output then requires (infinitely) many more securities than primitive states. We thank Martine Quinzii and Michael Magill for pointing this out.

is not generic when A^k is finite, as illustrated in Section 2. It also does not appear when we have a continuum of primitive states of nature. We refer the reader to Appendix A.2 for a simple technology for which, given any output profile y and investment vector a , there is a continuum of states ω satisfying $f(\omega, a) = y$.

Continuum of firms

Among the many technologies that do not satisfy the general assumptions for existence of an Arrow–Debreu equilibrium, there is one that deserves particular attention. The benchmark production model in contract theory is such that each firm either succeeds or fails to produce a certain amount of output, and the probability of success increases with the firm’s investment. This success-or-failure technology is represented by a non-convex production function and non-existence problems may arise. In Section 2, we constructed an equilibrium for an economy with this technology in which firms chose between two possible investment levels. However, as recently shown by Magill, Quinzii, and Rochet (2013), an Arrow–Debreu equilibrium may not exist when there is a continuum of possible investments but only finitely many possible outputs.

A traditional approach to overcome this issue considers a continuum of ex-ante identical firms with i.i.d. production draws. In this case, all variation in output that underlies this particular technology is eliminated and the objective of the firm ceases to be an issue. We use Appendix A.3 to discuss a model with a continuum of identical firms and perfectly correlated success-or-failure shocks.¹⁴ We explicitly compute an equilibrium under specific assumptions and also derive a general existence result. This illustrates an interesting way to keep variability in the (average) aggregate output while smoothing non-convexities through a continuum of firms. Notice also that modelling the productive sector with many small firms is consistent with the behavioral assumptions used along the paper.

7. CONCLUSION

There are two alternative traditions in economics to represent the outcome of risky enterprises. On one hand, the reference model in macroeconomics and general equilibrium uses the state-of-nature approach, which relies on random production functions that map investments and random primitive states of nature (with fixed objective probabilities) into realized outputs. On the other hand, the literature on contract theory relies on the probability approach in which production is modeled through transition functions mapping investments into probability measures over the set of possible outcomes.

As far as the description of production possibilities is concerned, the two approaches are equivalent. However, the two approaches differ on the financial contracts that are

¹⁴To simplify the presentation, we assume that all firms are identically affected by exogenous shocks. The more general case can be handled in a similar manner.

used to share risks and direct investments. By keeping states of nature hidden, the probability approach reminds us that writing contracts on the primitive states of nature is not realistic. It is sometimes difficult to describe these states in a contract or to verify them ex-post for execution. This is why most financial contracts available in practice are usually written on observed production outcomes or profits (e.g., stocks and options). It is then natural to ask whether this market incompleteness—generated by the lack of state-contingent claims—matters for efficiency.

It is well-known that the ability to contract upon primitive states of nature is not essential for an efficient allocation of resources in exchange economies with production risks only. If agents can sell their endowed stocks and trade claims written on the aggregate output, then the equilibrium consumption is efficient and only varies with aggregate risks. The interesting question concerns the ability of financial markets to efficiently direct firms' investments. We show that the difficulty raised by the lack of state prices can be overcome by considering an appropriate notion of *competitive* beliefs out of the equilibrium. This new concept describes the way each firm assesses the impact of alternative out-of-equilibrium investments without anticipating the effect of this decision over the distribution of aggregate variables. As a result, we obtain the classic recipe that shareholder value maximization leads to Pareto efficiency even for the more realistic financial structure in which all contracts are written on firms' outcomes.

Appendix A: APPENDIX

A.1. Proof of Proposition 5.1

From AD to FE

Let us take an Arrow–Debreu equilibrium $(\bar{p}, (\bar{\xi}_0, \bar{\xi}_1), \bar{a})$ and construct a financial equilibrium $(\bar{E}, \bar{\rho}, (\bar{x}_0, \bar{x}_1, \bar{\eta}, \bar{\theta}))$ associated with the investment vector \bar{a} . We define the equity and output-contingent prices as

$$\bar{E} \equiv \int_{\Omega} f(\omega, \bar{a}) \bar{p}(d\omega) \quad \text{and} \quad \bar{\rho}(B) \equiv \int_{\Omega} \mathbf{1}_B(\sigma f(\omega, \bar{a})) \bar{p}(d\omega),$$

for any Borelian set $B \subset Z$. It follows from strict concavity of u_1 that $\bar{\xi}_1(\omega)$ is constant across states ω associated with the same equilibrium aggregate output $\sigma f(\omega, \bar{a})$. This is to say that, for each agent i , there exists a measurable function $\bar{c}_1^i : Z \rightarrow \mathbb{R}_+$ satisfying $\bar{c}_1^i(z) = \bar{\xi}_1^i(\omega)$, for every ω such that $\sigma f(\omega, \bar{a}) = z$. We pose

$$(\bar{x}_0, \bar{x}_1(y)) \equiv (\bar{\xi}_0, \bar{c}_1(\sigma y)), \quad \text{for every } y.$$

The individual consumption plan $(\bar{\xi}_0^i, \bar{\xi}_1^i)$ must satisfy the Arrow–Debreu budget constraint

$$\bar{\xi}_0^i + \int_{\Omega} \bar{\xi}_1^i(\omega) \bar{p}(d\omega) \leq e_0^i + \Pi(\bar{a}) \cdot \delta^i.$$

Given the equity prices defined before, we have

$$(19) \quad \Pi^k(\bar{a}^k) = \bar{E}^k - \bar{a}^k, \quad \text{for every } k.$$

This allows us to rewrite the previous inequality as

$$\bar{x}_0^i + \int_Z \bar{c}_1^i(z) \bar{\rho}(dz) \leq e_0^i + (\bar{E} - \bar{a}) \cdot \delta^i.$$

Therefore, by taking $\bar{\eta}^i \equiv (1/I)\mathbf{1}$ and $\bar{\theta}^i(z) \equiv \bar{c}_1^i(z) - z/I$, we have that $(\bar{x}_0^i, \bar{x}_1^i, \bar{\eta}^i, \bar{\theta}^i)$ satisfies the budget constraints (4) and (5) at the equilibrium financial prices $(\bar{E}, \bar{\rho})$.

The consumption plans $(\bar{x}_0^i, \bar{x}_1^i)$ are optimal among all plans that can be financed by equity holdings and output-contingent contracts. To see this, let $(x_0^i, x_1^i, \eta^i, \theta^i)$ be an alternative plan satisfying the budget constraints (4) and (5) at the equilibrium prices $(\bar{E}, \bar{\rho})$. We construct the state-contingent plan $\xi_0^i \equiv x_0^i$ and $\xi_1^i(\omega) \equiv x_1^i(f(\omega, \bar{a}))$ and notice from Eq. (5) that

$$\begin{aligned} \int_{\Omega} x_1^i(f(\omega, \bar{a})) \bar{p}(d\omega) &= \int_{\Omega} \theta^i(\sigma f(\omega, \bar{a})) \bar{p}(d\omega) + \eta^i \cdot \int_{\Omega} f(\omega, \bar{a}) \bar{p}(d\omega) \\ &= \int_Z \theta^i(z) \bar{\rho}(dz) + \eta^i \cdot \bar{E}. \end{aligned}$$

By replacing this relation into Eq. (4) and using Eq. (19), we deduce that the consumption plan (ξ_0^i, ξ_1^i) satisfies the present-value budget constraint (17). Therefore, by optimality of $(\bar{\xi}_0^i, \bar{\xi}_1^i)$, we have

$$\begin{aligned} u_0(x_0^i) + \int_Y u_1(x_1^i(y)) Q(dy, \bar{a}) &= u_0(\xi_0^i) + \int_{\Omega} u_1(\xi_1^i(\omega)) P(d\omega) \\ &\leq u_0(\bar{\xi}_0^i) + \int_{\Omega} u_1(\bar{\xi}_1^i(\omega)) P(d\omega) \\ &= u_0(\bar{x}_0^i) + \int_Y u_1(\bar{x}_1^i(y)) Q(dy, \bar{a}). \end{aligned}$$

Feasibility of the allocation $((\bar{\xi}_0, \bar{\xi}_1), \bar{a})$ as defined by Eq. (15) and (16) implies feasibility of the allocation $((\bar{x}_0, \bar{x}_1), \bar{a})$ as defined by Eq. (6) and (7). Since the portfolio allocation $(\bar{\eta}, \bar{\theta})$ satisfies the market clearing condition (8), we can then deduce that $(\bar{E}, \bar{\rho}, (\bar{x}_0, \bar{x}_1, \bar{\eta}, \bar{\theta}))$ is a financial equilibrium associated with \bar{a} .

To conclude, we recall that \bar{a}^k maximizes Π^k and show that the objective function V^k is identical to Π^k given the equilibrium prices derived before. Indeed, standard first-order conditions applied to agent's problem in the Arrow–Debreu equilibrium imply that $\bar{p}(d\omega) = \bar{\chi}(\sigma f(\omega, \bar{a})) P(d\omega)$, where $\bar{\chi}(z) \equiv u_1'(\bar{c}_1^i(z))/u_0'(\bar{c}_0^i)$ for every z . We have

$\bar{\rho}(dz) = \bar{\chi}(z)\mu(dz, \bar{a})$, by construction, and then

$$\begin{aligned}
V^k(a^k) &= \int_Z \tilde{y}^k(a|z)\bar{\rho}(dz) - a^k \\
&= \int_Z \bar{\chi}(z)\tilde{y}^k(a|z)\mu(dz, \bar{a}) - a^k \\
&= \int_Z \bar{\chi}(z) \left[\int_{Y^k} y^k Q(dy^k, a^k|z) \right] \mu(dz, \bar{a}) - a^k \\
&= \int_Z \bar{\chi}(z) \left[\int_{\Omega} f^k(\omega, a^k) P(d\omega | \sigma f(\bar{a}) = z) \right] \mu(dz, \bar{a}) - a^k \\
&= \int_{\Omega} \bar{\chi}(\sigma f(\omega, \bar{a})) f^k(\omega, a^k) P(d\omega) - a^k \\
&= \Pi^k(a^k).
\end{aligned}$$

From FE to AD

We now take a financial equilibrium $(\bar{E}, \bar{\rho}, (\bar{x}_0, \bar{x}_1, \bar{\eta}, \bar{\theta}))$ associated with investments \bar{a} such that \bar{a}^k that maximizes V^k , for every k , and construct an Arrow–Debreu equilibrium $(\bar{p}, (\bar{\xi}_0, \bar{\xi}_1), \bar{a})$. Recall from the proof of Proposition 4.1 that there is a measurable function $\bar{\chi} : Z \rightarrow \mathbb{R}_+$ representing the stochastic discount factor in the sense that $\bar{\chi}(\sigma y) = u'_1(\bar{x}_1^i(y))/u'_0(\bar{x}_0^i)$ for every y and i . We then pose

$$\bar{p}(d\omega) \equiv \bar{\chi}(\sigma f(\omega, \bar{a}))P(d\omega), \quad \bar{\xi}_0^i \equiv \bar{x}_0^i, \quad \text{and} \quad \bar{\xi}_1^i(\omega) \equiv \bar{x}_1^i(f(\omega, \bar{a})).$$

The allocation $((\bar{\xi}_0, \bar{\xi}_1), \bar{a})$ satisfies the market clearing conditions (15) and (16). As for budget feasibility, we notice that, for each i , there is a measurable function $\bar{c}_1^i : Z \rightarrow \mathbb{R}_+$ satisfying $\bar{c}_1^i(z) = \bar{x}_1^i(\sigma y)$.¹⁵ The budget constraints (4) and (5) can be consolidated as

$$\bar{x}_0^i + \int_Z \bar{c}_1^i(z)\bar{\chi}(z)\mu(dz, \bar{a}) \leq e_0^i + (\bar{E} - \bar{a}) \cdot \delta^i.$$

The definitions of \bar{E} and $\bar{\rho}$ imply

$$\bar{\xi}_0^i + \int_{\Omega} \bar{\xi}_1^i(\omega)\bar{p}(d\omega) \leq e_0^i + \Pi(\bar{a}) \cdot \delta^i$$

and, then, the plan $(\bar{\xi}_0^i, \bar{\xi}_1^i)$ satisfies the present-value budget constraint (17).

To prove that $(\bar{\xi}_0^i, \bar{\xi}_1^i)$ is individually optimal, we fix an arbitrary consumption plan (ξ_0^i, ξ_1^i) satisfying the present-value budget constraint

$$(20) \quad \xi_0^i + \int_{\Omega} \xi_1^i(\omega)\bar{p}(d\omega) \leq e_0^i + \Pi(\bar{a}) \cdot \delta^i.$$

By posing

$$c_1^i(z) \equiv \int_{\Omega} \xi_1^i(\omega)P(d\omega | \sigma f(\bar{a}) = z),$$

¹⁵See the proof of Proposition 4.1.

we get from the definitions of \bar{E} and $\bar{\rho}$ that (20) writes as follows

$$\xi_0^i + \int_Z c_1^i(z) \bar{\rho}(dz) \leq e_0^i + (\bar{E} - \bar{a}) \cdot \delta^i.$$

By fixing $\theta^i(z) \equiv c^i(z)$ and $\eta^i \equiv 0$, we have that the plan $(x_0^i, x_1^i, \theta^i, \eta^i)$ must satisfy the budget restrictions (4) and (5). The optimality of $(\bar{x}_0^i, \bar{x}_1^i, \bar{\theta}^i, \bar{\eta}^i)$ then implies that

$$\begin{aligned} u_0(\xi_0^i) + \int_{\Omega} u_1(\xi_1^i(\omega)) P(d\omega) &= u_0(x_0^i) + \int_Y u_1(x_1^i(y)) Q(dy, \bar{a}) \\ &\leq u_0(\bar{x}_0^i) + \int_Y u_1(\bar{x}_1^i(y)) Q(dy, \bar{a}) \\ &= u_0(\bar{\xi}_0^i) + \int_{\Omega} u_1(\bar{\xi}_1^i(\omega)) P(d\omega). \end{aligned}$$

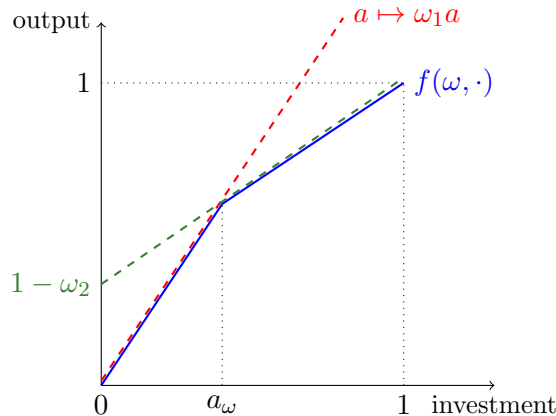
Finally, since \bar{a}^k maximizes V^k and $V^k = \Pi^k$, the Arrow–Debreu profit maximization condition is also satisfied.

A.2. Technology example

We consider the following technology to illustrate that completeness with respect to aggregate output does not yield (even generically) complete markets with respect to the states of nature. There is a single firm, and the primitive states $\omega = (\omega_1, \omega_2)$ lie in $\Omega \equiv [1, \infty) \times [0, 1)$. The production function maps investment $a \in [0, 1]$ to output $f(\omega, a) \in [0, 1]$ and displays piece-wise constant returns to scale as follows:

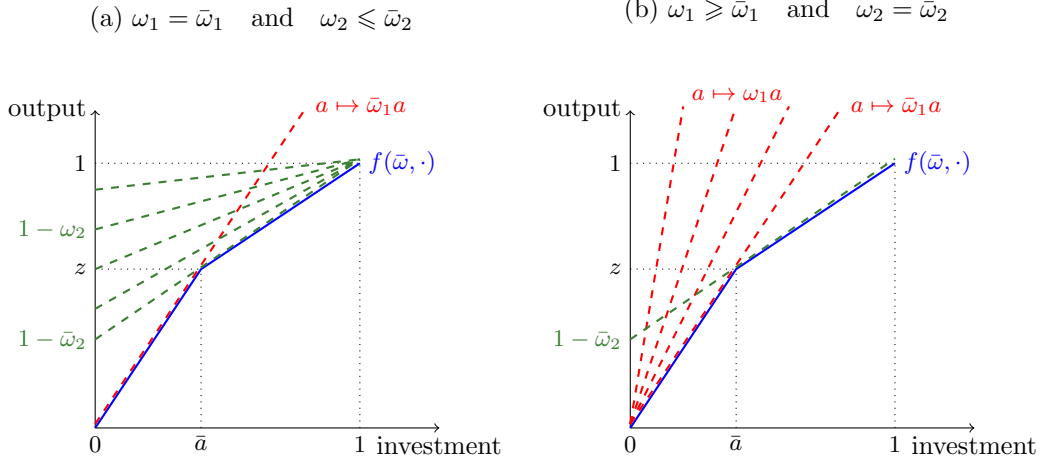
$$f(\omega, a) \equiv \begin{cases} \omega_1 a & \text{if } a \in [0, a_\omega] \\ \omega_2 a + (\omega_1 - \omega_2) a_\omega & \text{if } a \in [a_\omega, 1] \end{cases}$$

where the threshold is defined by $a_\omega \equiv (1 - \omega_2)/(\omega_1 - \omega_2)$. In words, our firm earns a high marginal productivity $\omega_1 \in [1, \infty)$ for low investment levels and a low marginal return $\omega_2 \in [0, 1]$ for high investment levels.



For each given equilibrium investment $\bar{a} \in [0, 1]$ and production $z > \bar{a}$, there are infinitely many possible shocks $\omega \in \Omega$ that generate $f(\omega, \bar{a}) = z$. To see this, take $z >$

\bar{a} , $\bar{\omega}_1 \equiv z/\bar{a}$, and $\bar{\omega}_2 \equiv (1-z)/(1-\bar{a})$. We obtain $f(\omega, \bar{a}) = z$ for any ω such that: (a) $\omega_1 = \bar{\omega}_1$ and $\omega_2 \leq \bar{\omega}_2$, or (b) $\omega_1 \geq \bar{\omega}_1$ and $\omega_2 = \bar{\omega}_2$. The next figures illustrate these cases. We conclude that observing the output z is not sufficient to pin down exactly the underlying productive shock.



A.3. Success-or-failure technology: continuum of firms

Consider an economy with a single firm which chooses an investment in the set $A \equiv [0, 1]$ and faces a success-or-failure production function. Two production levels are possible, $y_L > 0$ and $y_H > y_L$. The transition $Q(y, a)$ stands the probability of producing y when the firm invests a . Since there are only two possible output levels, we can simplify the notation by setting $Q_H(a) \equiv Q(y_H, a)$. We make the assumption that higher efforts increase the likelihood of success, i.e., $a \mapsto Q_H(a)$ is strictly increasing.

There is a canonical state-of-nature representation of this technology. We take the set Ω to be the interval $[0, 1]$ and the probability P to be the uniform measure. For every investment level a , we define $\omega(a) \equiv 1 - Q_H(a)$ and pose

$$(21) \quad f(\omega, a) = \begin{cases} y_L & \text{if } \omega \leq \omega(a) \\ y_H & \text{if } \omega > \omega(a). \end{cases}$$

Since $a \mapsto \omega(a)$ is strictly decreasing, we denote by $\omega \mapsto a(\omega)$ its inverse mapping from $[\omega(1), \omega(0)]$ to $[0, 1]$. For each state of nature $\omega \in [\omega(1), \omega(0)]$, the firm obtains the output

$$(22) \quad f(\omega, a) = \begin{cases} y_L & \text{if } a \leq a(\omega) \\ y_H & \text{if } a > a(\omega). \end{cases}$$

For states $\omega < \omega(1)$ and $\omega > \omega(0)$, the realized outputs are respectively y_L and y_H

regardless of the initial investment a .¹⁶

Notice that the production function $a \mapsto f(\omega, a)$ is not concave. We can then follow [Magill, Quinzii, and Rochet \(2013\)](#) and show that, for some specifications of the production technology, there does not exist a financial equilibrium in which the firm maximizes the competitive market value.¹⁷ To overcome the non-convexity of the success-or-failure technology, we propose to model perfect competition of the productive sector by considering the extreme case with a continuum $K \equiv [0, 1]$ of identical firms facing success-or-failure shocks that are perfectly correlated. This assumption is imposed to simplify the presentation.¹⁸

We pose a few remarks before proceeding. We first notice that the presence of a continuum of firms is consistent with our behavioral assumption that agents are convinced that a change in the investment of each firm does not affect the probability over the aggregate output. We also stress that shocks are not independent across firms. Independence would reduce the model to the case without aggregate uncertainty, where the choice of the firms' objective is not anymore an issue. The assumption that shocks are perfectly correlated allows us to keep output variability at equilibrium, which resembles the case with a single firm. Finally, shocks are not necessarily observable and contractible even when they affect all firms identically.¹⁹

We abuse notation and do not index firm-specific variables with the firm's name $k \in [0, 1]$. Since the production function of each firm is non-convex, we may have multiple solutions to the "representative" firm's maximization problem. In particular, ex-ante identical firms having different names may choose different investment levels. Therefore, we opt to represent firms' investment decisions using a probability measure α on the Borel σ -algebra of the set of investment levels $A \equiv [0, 1]$. The interpretation is that $\alpha(B)$ is the fraction of firms choosing investment in a Borelian set B of A . The corresponding (average) aggregate production contingent on the exogenous state of nature ω is denoted

¹⁶In that respect, the events $[0, \omega(1))$ and $(\omega(0), 1]$ correspond to the states ω_L and ω_H in the economy analyzed in Section 2.

¹⁷Formally, if the production technology is such that $a \mapsto Q_H(a)$ is decreasing, continuously differentiable and satisfies $Q'_H(0) = \infty$ and $Q'_H(1) = 0$, then there does not exist a financial equilibrium in which the firm maximizes the competitive market value.

¹⁸At the cost of notational complexity, we could have considered a slightly more general model allowing for different firms with imperfectly correlated production levels. For existence of an equilibrium, what matters to deal with non-convex production technologies is that we have a non-atomic measure space of firms.

¹⁹Examples of aggregate shocks include changes in a government's macroeconomic policy such as taxes and social security contributions on labor. Another possible aggregate shock is a general increase in labor productivity because of an easily accessible improvement in technological knowledge. Political instabilities in the Middle East that lead to changes in oil production or technological innovations in solar energy production are also examples of shocks affecting all firms. We hardly see contracts contingent on events like these.

by $\mathbb{E}_\alpha[f(\omega)]$. It follows from the production function represented in Eq. (22) that

$$(23) \quad \mathbb{E}_\alpha[f(\omega)] = \int_A f(\omega, a) \alpha(da) = y_L \alpha([0, a(\omega)]) + y_H (1 - \alpha([0, a(\omega)])).$$

Since we have infinitely many possible primitive states of nature ω , the set Z of aggregate production levels is now described by the interval $[y_L, y_H]$.²⁰

Given a distribution α of investment, we can define the distribution μ_α over the (average) aggregate production as follows

$$\mu_\alpha(B) \equiv P(\{\omega \in \Omega : \mathbb{E}_\alpha[f(\omega)] \in B\}),$$

for every Borelian set $B \subset [y_L, y_H]$. Having infinitely many firms, we need to consider infinitely many consumers. We assume that there is also a continuum $I \equiv [0, 1]$ of identical consumers, each one having the full ownership of a single firm. We also skip using names i to index consumer-specific variables.

Fix some equilibrium investment distribution $\bar{\alpha}$. Identical firms must have the same equilibrium initial value \bar{V} . We can assume without loss of generality that agents pool their asset holdings and make consumption plans $x_0 \in \mathbb{R}_+$ and $c_1 : Z \rightarrow \mathbb{R}_+$ in order to satisfy the following reduced-form budget constraint

$$(24) \quad x_0 + \int_Z c_1(z) \bar{\rho}(dz) \leq e_0 + \bar{V},$$

where $\bar{\rho}$ is the equilibrium measure representing output-contingent prices. Agents' problem have a unique optimal solution (\bar{x}_0, \bar{c}_1) in which $\bar{c}_1(z) = z$ and $\bar{x}_0 = e_0 - \bar{a}$.

In order to simplify the exposition, we assume hereafter that u_0 is a linear function with $u'_0 = 1$. The equilibrium stochastic discount factor becomes $\bar{\chi}(z) = u'_1(z)$. Firms maximize the same competitive market value function

$$V_{\bar{\alpha}}(a) \equiv \int_Z \tilde{y}_{\bar{\alpha}}(a|z) \bar{\rho}(dz) - a,$$

where

$$\tilde{y}_{\bar{\alpha}}(a|z) \equiv \int_\Omega f^k(\omega, a) P(d\omega | \mathbb{E}_{\bar{\alpha}}[f] = z)$$

is the conditional expected production under the investment distribution $\bar{\alpha}$ given an (average) aggregate output z . In equilibrium, firms may choose different investment levels, but they will all have the same market value \bar{V} . Formally, if we denote by $\text{supp}(\bar{\alpha})$ the support of the equilibrium investment distribution $\bar{\alpha}$, then we have $V(a) = \bar{V}$, for every

²⁰This illustrates that, when firms' outputs are not independent, considering a continuum of firms does not remove aggregate uncertainty. In fact, here, it potentially increases the set of possible aggregate outcomes.

investment $a \in \text{supp}(\bar{\alpha})$ and $V(a) \leq \bar{V}$, for $a \notin \text{supp}(\bar{\alpha})$. We can then write firms' competitive market value as follows:

$$\begin{aligned} V_{\bar{\alpha}}(a) &= \int_Z \bar{\chi}(z) \int_{\Omega} f^k(\omega, a) P(d\omega | \mathbb{E}_{\bar{\alpha}}[f] = z) \mu_{\bar{\alpha}}(dz) - a \\ &= \int_{\Omega} \bar{\chi}(\mathbb{E}_{\bar{\alpha}}[f(\omega)]) f(\omega, a) P(d\omega) - a. \end{aligned}$$

Given the production function represented in Eq. (21), we obtain

$$(25) \quad V_{\bar{\alpha}}(a) = y_L \int_0^{\omega(a)} \bar{\chi}(\mathbb{E}_{\bar{\alpha}}[f(\omega)]) P(d\omega) + y_H \int_{\omega(a)}^1 \bar{\chi}(\mathbb{E}_{\bar{\alpha}}[f(\omega)]) P(d\omega) - a.$$

Smooth probabilities

Let us now compute an equilibrium distribution $\bar{\alpha}$ for the production technology in which $a \mapsto Q_H(a)$ is decreasing, continuously differentiable and satisfies $Q'_H(0) = \infty$ and $Q'_H(1) = 0$.²¹ Recall that $\omega(a) \equiv 1 - Q_H(a)$ and notice that, for any $a \in (0, 1)$, we have

$$V'_{\bar{\alpha}}(a) = Q'_H(a) \bar{\chi}(\mathbb{E}_{\bar{\alpha}}[f(\omega(a))]) \Delta y - 1,$$

where $\Delta y \equiv y_H - y_L$.

Since Q'_H is continuous, there are limits \underline{b} and \bar{b} with $0 < \underline{b} < \bar{b} < 1$ such that

$$(26) \quad Q'_H(\underline{b}) \bar{\chi}(y_H) \Delta y = 1 \quad \text{and} \quad Q'_H(\bar{b}) \bar{\chi}(y_L) \Delta y = 1.$$

Moreover, since $\bar{\chi}(z) = u'_1(z)$ is continuously decreasing, there is a continuously decreasing function $a \mapsto \varphi(a)$ such that

$$\forall a \in [\underline{b}, \bar{b}], \quad Q'_H(a) \bar{\chi}(\varphi(a)) \Delta y = 1.$$

Naturally, we have $\varphi(\underline{b}) = y_H$ and $\varphi(\bar{b}) = y_L$.

Let us define the distribution $\bar{\alpha}$ to be such that

$$\varphi(a) = y_L \bar{\alpha}([0, a]) + y_H (1 - \bar{\alpha}([0, a])).$$

Since the function φ is continuous, the distribution $\bar{\alpha}$ is non-atomic. Indeed, for every $a \in [\underline{b}, \bar{b}]$, we have

$$\bar{\alpha}[0, a] = (y_H - \varphi(a)) / \Delta y,$$

where $\varphi(a) = \bar{\chi}^{-1}\left(\frac{-1}{Q'_H(a) \Delta y}\right)$. No firm chooses investment levels lower than \underline{b} or higher than \bar{b} . It is easy to see that the distribution $\bar{\alpha}$ has been constructed in order to set $V'_{\bar{\alpha}}(a) = 0$, for all $a \in [\underline{b}, \bar{b}]$. Notice also that $V'_{\bar{\alpha}}(a) > 0$, for $a < \underline{b}$ and $V'_{\bar{\alpha}}(a) < 0$, for $a > \bar{b}$. This concludes our argument and proves that $\bar{\alpha}$ is a competitive equilibrium investment profile for this economy.

²¹This corresponds to the production technology for which we do not have existence with a single firm.

REMARK A.1 Notice from Eq. (26) that the smaller the distance between $\bar{\chi}(y_H)$ and $\bar{\chi}(y_L)$, the narrower the interval $[\underline{b}, \bar{b}]$. In particular, as u_1 approaches a linear function, we find \underline{b} converging to \bar{b} and $\bar{\alpha}$ converging to a Dirac measure (symmetric equilibrium). Aggregate uncertainty in this situation closely approximates the case with a single firm—in which the (average) aggregate output is either y_L or y_H .

General existence theorem

We can relax the assumptions on the transition probability Q_H and still obtain the existence result for economies with a continuum of firms facing perfectly correlated shocks. The reasoning is somewhat more technical.

Let $\mathcal{M}(A)$ be the vector space of signed Borel measures on $A = [0, 1]$. An investment decision is a distribution α in $\mathcal{M}_+^1(A)$ the set of all positive measures with total mass 1. We make explicit the relation between α and each firm's competitive market value $V_\alpha(a)$ by defining

$$V_\alpha(a) \equiv \int_{\Omega} \bar{m}_\alpha(\omega) f(\omega, a) P(d\omega) - a,$$

where $\bar{m}_\alpha(\omega) \equiv \bar{\chi}(\mathbb{E}_\alpha[f(\omega)])$. We also denote by $G(\alpha)$ the set of optimal investment levels

$$G(\alpha) \equiv \operatorname{argmax}\{V_\alpha(a) : a \in A\}.$$

A distribution of investment $\bar{\alpha}$ corresponds to an equilibrium in which firms maximize the competitive market value when it only puts mass on optimal investment levels, i.e., when $\bar{\alpha}(G(\bar{\alpha})) = 1$.

THEOREM A.1 *There exists a competitive equilibrium distribution of investments.*

PROOF: Let $F : \mathcal{M}_+^1(A) \rightarrow \mathcal{M}_+^1(A)$ be the correspondence defined by

$$F(\alpha) \equiv \{\hat{\alpha} \in \mathcal{M}_+^1(A) : \hat{\alpha}(G(\alpha)) = 1\}.$$

A competitive equilibrium is a distribution $\bar{\alpha}$ of investment levels that is a fixed point of F , i.e., $\bar{\alpha} \in F(\bar{\alpha})$.

We propose to apply Kakutani's fixed-point theorem. The convex set $\mathcal{M}_+^1(A)$ is endowed with the weak-star topology of the duality $\langle \mathcal{M}_+^1(A), \mathcal{C}(A) \rangle$, where $\mathcal{C}(A)$ is the space of continuous real-valued functions defined on A . Since $\mathcal{C}(A)$ endowed with the sup-norm is separable and since $\mathcal{M}(A)$ is the topological dual of $\mathcal{C}(A)$, we get that $\mathcal{M}_+^1(A)$ is a compact metrizable space.

LEMMA A.1 *The correspondence $G : \mathcal{M}_+^1(A) \rightarrow A$ is upper semi-continuous for the weak-star topology.*

PROOF OF LEMMA A.1: Following Berge's Maximum Theorem, it is sufficient to show that $(a, \alpha) \mapsto V_\alpha(a)$ is continuous. Let $(a_n, \alpha_n)_{n \in \mathbb{N}}$ be a sequence in $A \times \mathcal{M}_+^1(A)$ converging to $(a, \alpha) \in A \times \mathcal{M}_+^1(A)$. We first show that $\lim_{n \rightarrow \infty} \mathbb{E}_{\alpha_n}[f(\omega)] = \mathbb{E}_\alpha[f(\omega)]$, for P -almost every state ω . Notice that

$$\mathbb{E}_{\alpha_n}[f(\omega)] = \alpha_n(\omega)y_L + (1 - \alpha_n(\omega))y_H = y_L + [y_H - y_L](1 - \alpha_n(\omega)),$$

where $\alpha_n(\omega) = \alpha_n([0, a(\omega)])$ is the measure of the interval $[0, a(\omega)]$. Since $(\alpha_n)_{n \in \mathbb{N}}$ converges for the weak-star topology to α , we have

$$\lim_{n \rightarrow \infty} \alpha_n([0, a]) = \alpha([0, a]),$$

for every $a \in A$ that is not an atom of α , i.e., for every a such that $\alpha(\{a\}) = 0$. Since there are at most countably many atoms of α and since $\omega \mapsto a(\omega)$ is strictly increasing, we obtain $\lim_{n \rightarrow \infty} \alpha_n(\omega) = \alpha(\omega)$, for P -almost every ω . This implies that $\lim_{n \rightarrow \infty} \mathbb{E}_{\alpha_n}[f(\omega)] = \mathbb{E}_\alpha[f(\omega)]$. By continuity of u'_1 , we find

$$\lim_{n \rightarrow \infty} m_{\alpha_n}(\omega) = m_\alpha(\omega),$$

for P -almost every state ω .

We now show that $\lim_{n \rightarrow \infty} V_{\alpha_n}(a_n) = V_\alpha(a)$. Recall that

$$V_{\alpha_n}(a_n) = -a_n + \int_{\Omega} m_{\alpha_n}(\omega) f(\omega, a_n) P(d\omega).$$

Since $\lim_{n \rightarrow \infty} f(\omega, a_n) = f(\omega, a)$, for P -almost every ω , we can apply the Lebesgue Dominated Convergence Theorem to obtain the desired result.²² *Q.E.D.*

LEMMA A.2 *The correspondence F is upper semi-continuous for the weak-star topology.*

PROOF OF LEMMA A.2: Since $\mathcal{M}_+^1(A)$ is compact, it is sufficient to show that F has a closed graph. Let $(\alpha'_n, \alpha_n)_{n \in \mathbb{N}}$ be a sequence converging to (α', α) and satisfying $\alpha'_n \in F(\alpha_n)$ for each n . Since $G(\alpha)$ is compact, there exists an open set K and compact set \bar{K} such that

$$G(\alpha) \subset K \subset \bar{K}.$$

Since G is upper semi-continuous, there exists N large enough such that for each $n \geq N$, we have $G(\alpha_n) \subset K$. In particular, $\alpha'_n(\bar{K}) = 1$. Since $(\alpha'_n)_{n \in \mathbb{N}}$ converges for the weak-star topology to α' we get that $\alpha'(\bar{K}) \geq \limsup_n \alpha'_n(\bar{K}) = 1$. We have thus proven that $\alpha'(\bar{K}) = 1$. Actually, we can construct a decreasing sequence $(K_n, \bar{K}_n)_{n \in \mathbb{N}}$ where K_n is open, \bar{K}_n is compact, $G(\alpha) \subset K_n \subset \bar{K}_n$ and $\bigcap_{n \in \mathbb{N}} \bar{K}_n = G(\alpha)$. It then follows that $\alpha'(G(\alpha)) = 1$. *Q.E.D.*

²²Notice that, for each n , we have $f(\omega, a_n) \leq y_H$ and $m_{\alpha_n}(\omega) \leq u'_1(y_L)$.

The correspondence F has non-empty values. Indeed, since the function $a \mapsto V_\alpha(a)$ is continuous and A is compact, the demand set $G(\alpha)$ is always non-empty. If \hat{a} is an element of $G(\alpha)$, then the Dirac measure on \hat{a} belongs to $F(\alpha)$. Since by construction the correspondence F has convex values, we can apply Kakutani's Fixed-point Theorem to the correspondence F . Q.E.D.

REFERENCES

- ARROW, K. J. (1953): "Le role des valeurs boursières pour la répartition la meilleure des risques," *Econométrie, Colloques Internationaux du Centre National de la Recherche Scientifique*, 40, 41–47.
- BEWLEY, T. F. (1972): "Existence of equilibria in economies with infinitely many commodities," *Journal of Economic Theory*, 4(3), 514–540.
- BISIN, A., P. GOTTARDI, AND G. RUTA (2014): "Equilibrium corporate finance and intermediation," Discussion Paper, European University Institute.
- DEBREU, G. (1959): *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*. John Wiley & Sons Inc., New York.
- DRÈZE, J. H. (1974): "Investment under private ownership: Optimality, equilibrium and stability," in *Allocation Under Uncertainty: Equilibrium and Optimality*, ed. by J. H. Drèze, pp. 129–165. Macmillan, New York.
- EKERN, S. (1975): "On the theory of the firm in an economy with incomplete markets: an addendum," *The Bell Journal of Economics and Management Science*, 6(1), 388–393.
- EKERN, S., AND R. WILSON (1974): "On the theory of the firm in an economy with incomplete markets," *The Bell Journal of Economics and Management Science*, 5(1), 171–180.
- GROSSMAN, S. J., AND O. D. HART (1979): "A theory of competitive equilibrium in stock market economies," *Econometrica*, 47(2), 293–329.
- HART, O. (1979): "On shareholder unanimity in large stock market economies," *Econometrica*, 47(5), 1057–1083.
- HAYEK, F. A. (1945): "The use of knowledge in society," *American Economic Review*, 35(4), 519–530.
- LELAND, H. E. (1974): "Production theory and the stock market," *The Bell Journal of Economics and Management Science*, 5(1), 125–144.
- MAGILL, M., AND M. QUINZII (2009): "The probability approach to general equilibrium with production," *Economic Theory*, 39(1), 1–41.
- (2010): "A comoment criterion for the choice of risky investment by firms," *International Economic Review*, 51(3), 723–744.
- MAGILL, M., M. QUINZII, AND J.-C. ROCHET (2013): "A critique of shareholder value maximization," Discussion Paper, electronic copy available at <http://papers.ssrn.com/abstract=2246797>.
- MAKOWSKI, L. (1983): "Competitive stock markets," *Review of Economic Studies*, 50(2),

305-330.