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Abstract. The recently developed theory of Lagrangian flows for transport equations with low regularity coefficients enables to consider non BV vector fields. We apply this theory to prove existence and stability of global Lagrangian solutions to the repulsive Vlasov-Poisson system with only integrable initial distribution function with finite energy. These solutions have a well-defined Lagrangian flow. An a priori estimate on the smallness of the superlevels of the flow in three dimensions is established in order to control the characteristics.

1. Introduction

We consider the Cauchy problem for the classical Vlasov-Poisson system

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \tag{1.1}$$

$$f(0,x,v) = f^0(x,v)\,, \tag{1.2}$$

where $f(t,x,v) \geqslant 0$ is the distribution function, $t \geqslant 0$, $x, v \in \mathbb{R}^N$, and

$$E(t,x) = -\nabla_x U(t,x) \tag{1.3}$$

is the force field. The potential U satisfies the Poisson equation

$$-\Delta_x U = \omega(\rho(t,x) - \rho_b(x)), \tag{1.4}$$

with $\omega = +1$ for the electrostatic (repulsive) case, $\omega = -1$ for the gravitational (attractive) case, and where the density ρ of particles is defined through

$$\rho(t,x) = \int_{\mathbb{R}^N} f(t,x,v)dv, \tag{1.5}$$

and $\rho_b \geqslant 0$, $\rho_b \in L^1(\mathbb{R}^N)$ is an autonomous background density. Since we are in the whole space, the relation (1.3) together with the Poisson equation (1.4) yield the equivalent relation

$$E(t,x) = \frac{\omega}{|S^{N-1}|}\frac{x}{|x|^N} * (\rho(t,x) - \rho_b(x)), \tag{1.6}$$

where the convolution is in the space variable.

The Vlasov-Poisson system has been studied for long. Existence of local in time smooth solutions in dimension $N = 3$ has been obtained in [21] after the results of [8]. Global smooth solutions have been proved to exist in [9, 32, 24], with improvements on the growth in time in [33, 37, 19, 34, 28, 30]. Related results are [35, 27, 15]. These solutions need a sufficiently smooth

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initial datum f^0. In particular, the following theorem is a classical result due to Pfaffelmoser [32].

Theorem 1.1. *If $N = 3$, let f^0 be a non-negative C^1 function of compact support defined on $\mathbb{R}^6$. Then there are a non-negative $f \in C^1(\mathbb{R}^7)$ and $U \in C^2(\mathbb{R}^4)$, which tends to zero at infinity for every fixed t, satisfying equations (1.1)-(1.5) with $\rho_b = 0$. For each fixed t, the function $f(t, x, v)$ has compact support. The solution is determined uniquely by the initial datum f^0.*

In a different spirit, global weak solutions were proved to exist in [6, 16, 18], with only $f^0 \in L^1(\mathbb{R}^6)$, $f^0 \log^+ f^0 \in L^1$, $|v|^2 f^0 \in L^1$, $E^0 \in L^2$ (and $\rho_b = 0$, $\omega = +1$). Related results with *weak* initial data have been obtained in [31, 22, 38, 29].

In this paper we would like mainly to extend the existence result of [16] to initial data in L^1 with finite energy (in the repulsive case $\omega = +1$), avoiding the $L \log^+ L$ assumption. Our existence result is Theorem 8.4. It involves a well-defined flow. Even weaker solutions were considered in [39, 25, 26], where the distribution function is a measure. However, these solutions do not have well-defined characteristics.

Our approach uses the theory of Lagrangian flows for transport equations with vector fields having weak regularity, developed in [17, 1, 4, 14, 5], and recently in [13, 2, 10]. It enables to consider force fields that are not in $W^{1,1}_{loc}$, nor in BV_{loc}. In this context we prove stability results with strongly or weakly convergent initial distribution function. The flow is proved to converge strongly anyway. Our main results were announced in [12]. Related results can be found in [3].

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2. Conservation of mass and energy

We would like here to recall some basic identities related to the VP system. They hold for smooth solutions, and a priori not for weak solutions, for which only the associated a priori bounds remain valid.

Integrating (1.1) with respect to v and noting that the last term is in v-divergence form we obtain the *local conservation of mass*

$$\partial_t \rho(t, x) + \operatorname{div}_x(J(t, x)) = 0, \tag{2.1}$$

where the current J is defined by

$$J(t, x) = \int_{\mathbb{R}^N} v f(t, x, v)\, dv. \tag{2.2}$$

Integrating again with respect to x, we obtain the global conservation of mass

$$\frac{d}{dt} \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} f(t, x, v) dx dv = \frac{d}{dt} \int\limits_{\mathbb{R}^N} \rho(t, x) dx = 0. \tag{2.3}$$

Multiplying (1.1) by $\frac{|v|^2}{2}$, integrating in x and v, we get after integration by parts in v

$$\frac{d}{dt}\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|v|^2}{2}f(t,x,v)dxdv-\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}E\cdot vf\,dxdv=0. \tag{2.4}$$

Using (1.6) and (2.1), one has

$$\partial_t E+\sum_{k=1}^{N}\frac{\partial}{\partial x_k}\left(\frac{\omega}{|S^{N-1}|}\frac{x}{|x|^N}*J_k\right)=0, \tag{2.5}$$

or in other words

$$\partial_t E=\omega\nabla_x(-\Delta_x)^{-1}\mathrm{div}_x J, \tag{2.6}$$

which means that $\partial_t E$ is the the gradient component of $-\omega J$ (Helmholtz projection). We deduce that

$$\int_{\mathbb{R}^N}E\cdot\partial_t E\,dx=-\omega\int_{\mathbb{R}^N}E\cdot J\,dx. \tag{2.7}$$

Using (2.2) in (2.4), we obtain the *conservation of energy*

$$\frac{d}{dt}\left[\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|v|^2}{2}f(t,x,v)dxdv+\frac{\omega}{2}\int\limits_{\mathbb{R}^N}|E(t,x)|^2dx\right]=0\,. \tag{2.8}$$

The total conserved energy is the sum of the kinetic energy and of the potential energy multiplied by the factor $\omega=\pm1$. In particular, in the electrostatic case $\omega=+1$ we deduce from (2.8) a uniform bound in time on both the kinetic and the potential energy, assuming that they are finite initially. In the gravitational case $\omega=-1$ it is not possible to exclude that the individual terms of the kinetic and potential energy become unbounded in finite time, while the sum remains constant. Indeed it is known that it does not happen in three dimensions as soon as f^0 is sufficiently integrable, but we cannot exclude this a priori for only L^1 solutions.

Note that the assumption $E^0\in L^2$ is satisfied in 3 dimensions as soon as $\rho^0-\rho_b\in L^{6/5}$. However, in one or two dimensions, for E^0 to be in L^2 it is necessary that $\int(\rho^0-\rho_b)dx=0$, as is easily seen in Fourier variable. It is also necessary that $\rho^0-\rho_b$ has enough decay at infinity. Thus in one or two dimensions, in order to have finite energy, ρ_b cannot be zero identically.

3. Regularity of the force field for L^1 densities

3.1. **Singular integrals.** We recall here the basic properties of singular integral operators, that can be found in [36] for example.

Definition 3.1. A function K is a *singular kernel of fundamental type* in $\mathbb{R}^N$ if the following properties hold:

(1) $K|_{\mathbb{R}^N\setminus\{0\}}\in C^1(\mathbb{R}^N\setminus\{0\})$.
(2) There exists a constant $C_0\geqslant 0$ such that

$$|K(x)|\leqslant\frac{C_0}{|x|^N},\qquad x\in\mathbb{R}^N\setminus\{0\}. \tag{3.1}$$

(3) There exists a constant $C_1 \geqslant 0$ such that

$$|\nabla K(x)| \leqslant \tfrac{C_1}{|x|^{N+1}}, \qquad x \in \mathbb{R}^N \backslash \{0\}. \tag{3.2}$$

(4) There exists a constant $A_2 \geqslant 0$ such that

$$\left| \int_{R_1<|x|<R_2} K(x)dx \right| \leqslant A_2, \tag{3.3}$$

for every $0 < R_1 < R_2 < \infty$.

Theorem 3.2 (Calderón-Zygmund [36])**.** *A singular kernel of fundamental type K has an extension as a distribution on $\mathbb{R}^N$ (still denoted by K), unique up to a constant times Dirac delta at the origin, such that $\hat{K} \in L^\infty(\mathbb{R}^N)$. Define*

$$Su = K * u, \quad \text{for } u \in L^2(\mathbb{R}^N), \tag{3.4}$$

in the sense of multiplication in the Fourier variable. Then we have the estimates for $1 < p < \infty$

$$||Su||_{L^p(\mathbb{R}^N)} \leqslant C_{N,p}(C_0 + C_1 + ||\hat{K}||_{L^\infty})||u||_{L^p(\mathbb{R}^N)}, \quad u \in L^p \cap L^2(\mathbb{R}^N). \tag{3.5}$$

If K is a singular kernel of fundamental type, we call the associated operator S a singular integral operator on $\mathbb{R}^N$. We define then the Fréchet space $\mathscr{R}(\mathbb{R}^N) = \cap_{m\in\mathbb{N}, 1<p<\infty} W^{m,p}(\mathbb{R}^N)$ and its dual $\mathscr{R}'(\mathbb{R}^N) \subset \mathscr{S}'(\mathbb{R}^N)$, where $\mathscr{S}'(\mathbb{R}^N)$ is the space of tempered distributions on $\mathbb{R}^N$. Since all singular integral operators are bounded on $\mathscr{R}(\mathbb{R}^N)$, by duality we can define the operator S also $\mathscr{R}'(\mathbb{R}^N) \to \mathscr{R}'(\mathbb{R}^N)$. In particular it enables to define Su for $u \in L^1(\mathbb{R}^N)$ or for u a measure. The result Su is in $\mathscr{R}'(\mathbb{R}^N) \subset \mathscr{S}'(\mathbb{R}^N)$.

3.2. **The split vector field.** Let $\rho(t,x) \in L^\infty((0,T); L^1(\mathbb{R}^N))$. We denote by

$$b(t,x,v) = (b_1, b_2)(t,x,v) = (v, E(t,x)) = \Big(v, -\omega\nabla_x(-\Delta_x)^{-1}(\rho(t,x) - \rho_b(x))\Big) \tag{3.6}$$

the associated vector field on $(0,T) \times \mathbb{R}^N \times \mathbb{R}^N$. Then the Vlasov equation can be written in the form of the transport equation $\partial_t f + b \cdot \nabla_{x,v} f = 0$. In the following subsections we establish bounds on the vector field b.

3.3. **Local integrability.** For L^1 densities, using the weak Hardy-Littlewood-Sobolev inequality (see Lemma 4.5.7 in [20]) we have that

$$\begin{aligned} &|||\nabla(-\Delta)^{-1}(\rho(t,x) - \rho_b(x))|||_{M^{\frac{N}{N-1}}(\mathbb{R}^N)} \\ &\leqslant \left|\left|\left| \frac{1}{|S^{N-1}|} |x|^{1-N} * |\rho(t,x) - \rho_b(x)| \right|\right|\right|_{M^{\frac{N}{N-1}}(\mathbb{R}^N)} \\ &\leqslant c_N ||\rho(t,x) - \rho_b(x)||_{L^1(\mathbb{R}^N)}, \end{aligned} \tag{3.7}$$

where

$$|||u|||_{M^p(\mathbb{R}^N)} \equiv \sup_{\gamma>0} \gamma\, \mathscr{L}^N(\{x \in \mathbb{R}^N \, s.t. \, |u(x)| > \gamma\})^{1/p}, \tag{3.8}$$

and $\mathscr{L}^N$ denotes the Lebesgue measure in $\mathbb{R}^N$. It follows that

$$|||E|||_{L^\infty((0,T);M^{\frac{N}{N-1}}(\mathbb{R}^N))} \leqslant c_N ||\rho - \rho_b||_{L^\infty((0,T);L^1(\mathbb{R}^N))}, \tag{3.9}$$

and using the inclusion $M^{\frac{N}{N-1}}(\mathbb{R}^N) \subset L^p_{\rm loc}(\mathbb{R}^N)$ for $1 \leqslant p < \frac{N}{N-1}$ we conclude that $b \in L^\infty((0,T); L^p_{\rm loc}(\mathbb{R}^N_x \times \mathbb{R}^N_v))$ for any $1 \leqslant p < \frac{N}{N-1}$, since $v \in L^p_{\rm loc}(\mathbb{R}^N_x \times \mathbb{R}^N_v)$ for any p.

3.4. **Spatial regularity.** Since $b_1 = v$ is smooth, the only non-trivial gradient is the one of $b_2 = E$, indeed the differential matrix of the vector field is given by

$$Db = \begin{pmatrix} D_x b_1 & D_v b_1 \\ D_x b_2 & D_v b_2 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{Id} \\ D_x E & 0 \end{pmatrix}. \tag{3.10}$$

We have by (3.6)

$$(D_x E)_{ij} \equiv \partial_{x_j} E_i = -\omega \partial^2_{x_i x_j}((-\Delta_x)^{-1}(\rho - \rho_b)) \quad \text{for } 1 \leqslant i,j \leqslant N. \tag{3.11}$$

It is well-known that the operator $\partial^2_{x_i x_j}(-\Delta_x)^{-1}$ is a singular integral operator. Its kernel is

$$K_{ij}(x) = -\frac{1}{|S^{N-1}|} \frac{\partial}{\partial_{x_j}} \left(\frac{x_i}{|x|^N} \right), \tag{3.12}$$

it is given outside of the origin by

$$K_{ij}(x) = \frac{1}{|S^{N-1}|} \left(N \frac{x_i x_j}{|x|^{N+2}} - \frac{\delta_{ij}}{|x|^N} \right), \qquad \text{for } x \in \mathbb{R}^N \backslash \{0\}. \tag{3.13}$$

The kernel satisfies the conditions of Subsection 3.1, and $\hat{K}_{ij}(\xi) = -\xi_i \xi_j / |\xi|^2$. Thus (each component of) $D_x E$ is a singular integral of an $L^\infty((0,T); L^1(\mathbb{R}^N))$ function.

3.5. **Time regularity.** According to (2.6), $\partial_t E$ is a singular integral of the current J defined by (2.2). Using the bounds available for solutions with finite mass and energy

$$||f(t,\cdot)||_{L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v)}, \iint |v|^2 f(t,x,v) dx dv \leqslant C, \tag{3.14}$$

and since $|v| \leqslant 1 + |v|^2$, we get that $J \in L^\infty((0,T); L^1(\mathbb{R}^N_x))$. Hence $\partial_t E$ is a singular integral of an $L^\infty((0,T); L^1(\mathbb{R}^N_x))$ function. In particular,

$$\partial_t E \in L^\infty((0,T); \mathscr{S}'(\mathbb{R}^N)). \tag{3.15}$$

4. Lagrangian flows

Suppose that f is a smooth solution to (1.1)-(1.5) with $f(0,x,v) = f^0$. Then f is constant along the characteristics $(X(s,t,x,v), V(s,t,x,v))$, which solve the system of equations

$$\begin{cases} \dfrac{dX}{ds}(s,t,x,v) = V(s,t,x,v), \\ \\ \dfrac{dV}{ds}(s,t,x,v) = E(s, X(s,t,x,v), V(s,t,x,v)), \end{cases} \tag{4.1}$$

with initial data $X(t,t,x,v) = x$ and $V(t,t,x,v) = v$. Thus the solution can be expressed as $f(t,x,v) = f^0(X(0,t,x,v), V(0,t,x,v))$.

In order to extend this notion of characteristics to non-smooth solutions, we define regular Lagrangian flows, which are defined in an almost everywhere sense.

Definition 4.1. Let $b \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^{2N}; \mathbb{R}^{2N})$, and $t \in [0,T)$. A map $Z : [t,T] \times \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is a regular Lagrangian flow starting at time t for the vector field b if

(1) for a.e. $z \in \mathbb{R}^{2N}$ the map $s \mapsto Z(s,z)$ is an absolutely continuous integral solution of $\dot{\beta}(s) = b(s, \beta(s))$ for $s \in [t,T]$ with $\beta(t) = z$.
(2) There exists a constant L independent of s such that

$$\mathscr{L}^{2N}(Z(s,\cdot)^{-1}(A)) \leqslant L\mathscr{L}^{2N}(A)$$

for every Borel set $A \subset \mathbb{R}^{2N}$.

The constant L is called the compressibility constant of Z.

Definition 4.2. Define the sublevel of the flow as the set

$$G_\lambda = \{z \in \mathbb{R}^{2N} : |Z(s,z)| \leqslant \lambda \text{ for almost all } s \in [t,T]\}. \tag{4.2}$$

5. Stability estimate for Lagrangian flows

We summarize the main result from [13] in the following regularity setting of the vector field in arbitrary dimension. We say that a vector field b satisfies **(R1)** if b can be decomposed as

$$\frac{b(t,z)}{1+|z|} = \tilde{b}_1(t,z) + \tilde{b}_2(t,z) \tag{5.1}$$

where $\tilde{b}_1 \in L^1((0,T); L^1(\mathbb{R}^{2N}))$, $\tilde{b}_2 \in L^1((0,T); L^\infty(\mathbb{R}^{2N}))$.

We assume also that b satisfies **(R2)**: for every $j = 1, \dots, 2N$,

$$\partial_{z_j} b = \sum_{k=1}^{m} S_{jk} g_{jk} \tag{5.2}$$

where S_{jk} are singular integrals of fundamental type on $\mathbb{R}^{2N}$ and $g_{jk} \in L^1((0,T); L^1(\mathbb{R}^{2N}))$. Moreover, we assume condition **(R3)**, that is

$$b \in L^p_{\mathrm{loc}}([0,T] \times \mathbb{R}^{2N}), \qquad \text{for some } p > 1. \tag{5.3}$$

We recall the following stability theorem from [13], where we denote by B_r the ball with center 0 and radius r in $\mathbb{R}^{2N}$.

Theorem 5.1. *Let b, $\bar{b}$ be two vector fields satisfying* **(R1)**, *b satisfying also* **(R2)**, **(R3)**. *Fix $t \in [0,T)$ and let Z and $\bar{Z}$ be regular Lagrangian flows starting at time t associated to b and $\bar{b}$ respectively, with compression constants L and $\bar{L}$. Then the following holds.*
For every $\gamma > 0$, $r > 0$ and $\eta > 0$ there exist $\lambda > 0$ and $C_{\gamma,r,\eta} > 0$ such that

$$\mathscr{L}^{2N}\left(B_r \cap \{|Z(s,\cdot) - \bar{Z}(s,\cdot)| > \gamma\}\right) \leqslant C_{\gamma,r,\eta} ||b - \bar{b}||_{L^1((0,T)\times B_\lambda)} + \eta$$

for all $s \in [t,T]$. The constants λ and $C_{\gamma,r,\eta}$ also depend on

- *The equi-integrability in $L^1((0,T); L^1(\mathbb{R}^{2N}))$ of g_{jk} coming from* **(R2)**,
- *The norms of the singular integral operators S_{jk} from* **(R2)**,
- *The norm $||b||_{L^p((0,T)\times B_\lambda)}$ corresponding to* **(R3)**,
- *The $L^1(L^1)$ and $L^1(L^\infty)$ norms of the decompositions of b and $\bar{b}$ in* **(R1)**,
- *The compression constants L and $\bar{L}$.*

We would like now to state a variant of this theorem, where **(R1)** and **(R2)** are replaced by **(R1a)** and **(R2a)**.

We consider the following weakened assumption **(R1a)**: for all regular Lagrangian flows $Z : [t,T] \times \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ relative to b starting at time t with compression constant L, and for all $r, \lambda > 0$,

$$\mathscr{L}^{2N}(B_r \backslash G_\lambda) \leqslant g(r,\lambda), \quad \text{with } g(r,\lambda) \to 0 \text{ as } \lambda \to \infty \text{ at fixed } r, \tag{5.4}$$

where G_λ denotes the sublevel of the flow Z, defined in (4.2).

We next consider a splitting of the variables, $\mathbb{R}^{2N}_z = \mathbb{R}^N_x \times \mathbb{R}^N_v$, and we denote $Z(t,x,v) = (X,V)(t,x,v)$. Noticing the special form (3.6), we assume the condition that b satisfies **(R2a)**:

$$b(t,x,v) = (b_1,b_2)(t,x,v) = (b_1(v), b_2(t,x)), \tag{5.5}$$

with

$$b_1 \in \mathrm{Lip}(\mathbb{R}^N_v), \tag{5.6}$$

and where b_2 involves singular kernels only in the first set of variables x, that is for every $j = 1, ..., N$,

$$\partial_{x_j} b_2 = \sum_{k=1}^{m} S_{jk} g_{jk}, \tag{5.7}$$

where S_{jk} are singular integrals of fundamental type on $\mathbb{R}^N$ and $g_{jk} \in L^1((0,T); L^1(\mathbb{R}^N))$.

Theorem 5.2. *Let b, $\bar{b}$ be two vector fields satisfying* **(R1a)**, *b satisfying also* **(R2a)**, **(R3)**. *Fix $t \in [0,T)$ and let Z and $\bar{Z}$ be regular Lagrangian flows starting at time t associated to b and $\bar{b}$ respectively, with compression constants L and $\bar{L}$, and sublevels G_λ and $\bar{G}_\lambda$. Then the following holds. For every $\gamma > 0$, $r > 0$ and $\eta > 0$, there exist $\lambda > 0$ and $C_{\gamma,r,\eta} > 0$ such that*

$$\mathscr{L}^{2N}\left(B_r \cap \{|Z(s,\cdot) - \bar{Z}(s,\cdot)| > \gamma\}\right) \leqslant C_{\gamma,r,\eta} ||b - \bar{b}||_{L^1((0,T)\times B_\lambda)} + \eta$$

for all $s \in [t,T]$. The constants λ and $C_{\gamma,r,\eta}$ also depend on

- *The equi-integrability in $L^1((0,T); L^1(\mathbb{R}^N))$ of g_{jk} coming from* **(R2a)**,
- *The norms of the singular integral operators S_{jk} from* **(R2a)**,
- *The Lipschitz constant of b_1 from* **(R2a)**,
- *The norm $||b||_{L^p((0,T)\times B_\lambda)}$ corresponding to* **(R3)**,
- *The rate of decay of $\mathscr{L}^{2N}(B_r \backslash G_\lambda)$ and $\mathscr{L}^{2N}(B_r \backslash \bar{G}_\lambda)$ from* **(R1a)**,
- *The compression constants L and $\bar{L}$.*

Proof. We summarize the modifications of the proof of Theorem 5.1 from [13]. The main added difficulty is the singular integral operators on $\mathbb{R}^N_x$ instead of $\mathbb{R}^{2N}$. More general situations with singular integrals of measures instead of L^1 functions are considered in [10]. Let $z = (x,v) \in \mathbb{R}^N \times \mathbb{R}^N$. We estimate the quantity

$$\Phi_\delta(s) = \int\limits_{B_r \cap G_\lambda \cap \bar{G}_\lambda} \log\left(1 + \frac{|Z(s,z) - \bar{Z}(s,z)|}{\delta}\right) dz\,. \tag{5.8}$$

Differentiating with respect to time, we get

$$\begin{aligned}\Phi'_\delta(s) \leqslant & \int\limits_{B_r\cap G_\lambda\cap \bar G_\lambda} \frac{|b(s,Z(s,z))-\bar b(s,\bar Z(s,z))|}{\delta+|Z(s,z)-\bar Z(s,z)|}dz\\ \leqslant & \frac{\bar L}{\delta}||b(s,\cdot)-\bar b(s,\cdot)||_{L^1(B_\lambda)}\\ & + \int\limits_{B_r\cap G_\lambda\cap \bar G_\lambda} \min\Big\{\frac{|b(s,Z(s,z))|+|b(s,\bar Z(s,z))|}{\delta},\\ & \frac{|b_1(s,Z(s,z))-b_1(s,\bar Z(s,z))|}{|Z(s,z)-\bar Z(s,z)|}+\frac{|b_2(s,Z(s,z))-b_2(s,\bar Z(s,z))|}{|Z(s,z)-\bar Z(s,z)|}\Big\}dz.\end{aligned}$$

Using the special form of b from **(R2a)**, we obtain

$$\begin{aligned}\Phi'_\delta(s) \leqslant & \frac{\bar L}{\delta}||b(s,\cdot)-\bar b(s,\cdot)||_{L^1(B_\lambda)}\\ & + \int\limits_{B_r\cap G_\lambda\cap \bar G_\lambda} \min\Big\{\frac{|b(s,Z(s,z))|+|b(s,\bar Z(s,z))|}{\delta},\\ & \frac{|b_1(V(s,z))-b_1(\bar V(s,z))|}{|V(s,z)-\bar V(s,z)|}+\frac{|b_2(s,X(s,z))-b_2(s,\bar X(s,z))|}{|X(s,z)-\bar X(s,z)|}\Big\}dz\\ \leqslant & \frac{\bar L}{\delta}||b(s,\cdot)-\bar b(s,\cdot)||_{L^1(B_\lambda)}+\mathrm{Lip}(b_1)\mathscr{L}^{2N}(B_r)\\ & + \int\limits_{B_r\cap G_\lambda\cap \bar G_\lambda} \min\Big\{\frac{|b(s,Z(s,z))|+|b(s,\bar Z(s,z))|}{\delta},\\ & \frac{|b_2(s,X(s,z))-b_2(s,\bar X(s,z))|}{|X(s,z)-\bar X(s,z)|}\Big\}dz.\end{aligned} \tag{5.9}$$

Using assumption **(R2a)**, we can use the estimate of [13] on the difference quotients of b_2,

$$\frac{|b_2(s,X(s,z))-b_2(s,\bar X(s,z))|}{|X(s,z)-\bar X(s,z)|}\leqslant \mathscr{U}(s,X(s,z))+\mathscr{U}(s,\bar X(s,z)), \tag{5.10}$$

where $\mathscr{U}(s,.)\in M^1(\mathbb{R}^N)$ for fixed s is indeed given by

$$\mathscr{U}(s,\cdot)=\sum_{j=1}^N\sum_{k=1}^m M_j(S_{jk}g_{jk}(s,.)), \tag{5.11}$$

with M_j a smooth maximal operator on $\mathbb{R}^N$. Next, we can define the function

$$\mathscr{Z}(s,x,v)=\mathscr{U}(s,x)\mathbb{1}_{(x,v)\in\overline{B_\lambda}}, \tag{5.12}$$

and we notice that since the above integrals are over $G_\lambda\cap\bar G_\lambda$, we can replace the right-hand side of (5.10) by $\mathscr{Z}(s,Z(s,z))+\mathscr{Z}(s,\bar Z(s,z))$. Then, for given $\varepsilon>0$, we can decompose $g_{jk}=g^1_{jk}+g^2_{jk}$, with $\|g^1_{jk}\|_{L^1((0,T)\times\mathbb{R}^N)}\leqslant\varepsilon$ and $\|g^2_{jk}\|_{L^2((0,T)\times\mathbb{R}^N)}\leqslant C_\varepsilon$, g^2_{jk} having support in a set A_ε of finite measure. This gives rise to two type of terms $\mathscr{Z}^1$ and $\mathscr{Z}^2$. Since all the $\mathscr{Z}$ terms have

compact support in v, this allows to perform the same estimates as in [13]. We finally get

$$\begin{aligned}&\mathscr{L}^{2N}(B_r \cap \{|Z(\tau,\cdot) - \bar{Z}(\tau,\cdot)| > \gamma\})\\ \leqslant &\frac{1}{\log(1+\gamma/\delta)} \int_t^\tau \Phi'_\delta(s)\, ds + \mathscr{L}^{2N}(B_r \backslash G_\lambda) + \mathscr{L}^{2N}(B_r \backslash \bar{G}_\lambda).\end{aligned} \tag{5.13}$$

By choosing λ large first, then ε small, and finally δ small, we conclude the proof as in [13]. □

6. Control of superlevels

In order to apply Theorem 5.2, we need to satisfy **(R1a)**. Therefore, we seek an upper bound on the size of $B_r \backslash G_\lambda$.

6.1. **The case of low space dimension.** We first recall the following lemma from [13].

Lemma 6.1. *Let $b : (0,T) \times \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ be a vector field satisfying* **(R1)**. *Then b satisfies* **(R1a)**, *where the function $g(r,\lambda)$ depends only on $L, ||\tilde{b}_1||_{L^1((0,T);L^1(\mathbb{R}^{2N}))}, ||\tilde{b}_2||_{L^1((0,T);L^\infty(\mathbb{R}^{2N}))}$.*

This lemma allows us to control the superlevels of b in 1 or 2 dimensions.

Proposition 6.2. *Let b be the vector field in* (3.6), *with $E \in L^\infty((0,T);L^2(\mathbb{R}^N))$. For $N = 2$ or $N = 1$, b satisfies* **(R1)**, *hence also* **(R1a)**.

Proof. It is clear that

$$\frac{v}{1+|x|+|v|} \in L^\infty_t(L^\infty_{x,v}), \tag{6.1}$$

and

$$\frac{E(t,x)}{1+|x|+|v|} = \frac{E(t,x)}{1+|x|+|v|}\mathbb{1}_{|v|\leqslant|E(t,x)|} + \frac{E(t,x)}{1+|x|+|v|}\mathbb{1}_{|v|>|E(t,x)|} \equiv \tilde{E}_1 + \tilde{E}_2. \tag{6.2}$$

Clearly $\tilde{E}_2 \in L^\infty_t(L^\infty_{x,v})$, and if $N = 2$, $\tilde{E}_1 \in L^\infty((0,T);L^1_{x,v})$ since

$$\begin{aligned}&\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|E(t,x)|}{1+|x|+|v|}\mathbb{1}_{|v|\leqslant|E(t,x)|}dxdv\\ &\leqslant \int\limits_{\mathbb{R}^2} |E(t,x)|\Big(\int\limits_{|v|\leqslant|E(t,x)|} \frac{1}{|v|}dv\Big)dx = 2\pi\int\limits_{\mathbb{R}^2}|E(t,x)|^2dx.\end{aligned} \tag{6.3}$$

In the case $N = 1$, we have directly that $E(t,x)/(1+|v|) \in L^\infty((0,T);L^2_{x,v})$. □

6.2. **The case of three space dimensions.** The condition **(R1)** being not satisfied in 3 dimensions (the above computation would require $E \in L^1_t(L^3_x)$), we need an estimate on $|Z|$ in order to control the superlevels. For getting this we integrate in space a function growing slower at infinity than $\log(1+|Z|)$ (this corresponding to the case **(R1)**).

Proposition 6.3. *Let b be as in* (3.6) *with $N = 3$, $E \in L^\infty((0,T);L^2(\mathbb{R}^N))$, satisfying* (2.6) *with $J \in L^\infty((0,T);L^1(\mathbb{R}^N))$. Furthermore, assume that $\omega = +1$, $\rho \geqslant 0$, and $\rho_b \in L^1 \cap L^p(\mathbb{R}^3)$ for some $p > 3/2$. Then* **(R1a)** *holds, where the function g depends only on L, T, $||E||_{L^\infty_t(L^2_x)}$, $||J||_{L^\infty_t(L^1_x)}$, $\|(-\Delta)^{-1}\rho_b\|_{L^\infty}$, and one has $g(r,\lambda) \to 0$ as $\lambda \to \infty$ at fixed r.*

Proof. Step 1.) Let $Z : [t,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ be a regular Lagrangian flow relative to b starting at time t, with compression constant L and sublevel G_λ. Denoting $Z = (X,V)$, we have the ODEs

$$\begin{cases} \dot{X}(s,x,v) = V(s,x,v), \\ \dot{V}(s,x,v) = E(s,X(s,x,v)). \end{cases} \tag{6.4}$$

Recalling that $E = -\nabla_x U$, one has

$$\begin{aligned} \partial_s \frac{|V(s,x,v)|^2}{2} &= V(s,x,v) \cdot \partial_s V(s,x,v) = E(s,X(s,x,v)) \cdot \partial_s X(s,x,v) \\ &= -\partial_s[U(s,X(s,x,v))] + \partial_t U(s,X(s,x,v)). \end{aligned} \tag{6.5}$$

This computation is indeed related to the form of the Hamiltonian for (1.1), $\mathscr{H} = |v|^2/2 + U(t,x)$.

We are going to bound the superlevels of $V(s,x,v)$. We claim that

$$\iint\limits_{B_r} \sup_{s\in[t,T]} \Big(1 + \log\Big(1 + |V(s,x,v)|^2/2\Big)\Big)^\alpha \, dxdv \leqslant A, \tag{6.6}$$

where $0 < \alpha < 1/3$, and for some constant A depending on L, T, r, α, and on the norms $||E||_{L^\infty_t(L^2_x)}$, $||J||_{L^\infty_t(L^1_x)}$, $\|(-\Delta)^{-1}\rho_b\|_{L^\infty}$. Assume for the moment that this holds. From the lower bound

$$\begin{aligned} &\iint\limits_{B_r} \sup_{s\in[t,T]} \Big(1 + \log\Big(1 + |V(s,x,v)|^2/2\Big)\Big)^\alpha \, dxdv \\ &\geqslant \mathscr{L}^6(B_r \backslash \widetilde{G}_\lambda)(1 + \log(1 + \lambda^2/2))^\alpha, \end{aligned} \tag{6.7}$$

with $\widetilde{G}_\lambda$ the sublevel of V, we get that

$$\mathscr{L}^6(B_r \backslash \widetilde{G}_\lambda) \leqslant \frac{A}{(1 + \log(1 + \lambda^2/2))^\alpha}. \tag{6.8}$$

Next, we remark that by the first equation in (6.4), whenever $(x,v) \in \widetilde{G}_\lambda$ one has $|X(s,x,v)| \leqslant |x| + |s-t|\lambda$, and $|Z(s,x,v)| \leqslant |x| + (1+T)\lambda$. Thus for $\lambda > r$, one has $B_r \backslash G_\lambda \subset B_r \backslash \widetilde{G}_{(\lambda - r)/(1+T)}$, which enables to conclude the proposition (for $\lambda \leqslant r$ we can just bound $\mathscr{L}^6(B_r \backslash G_\lambda)$ by $\mathscr{L}^6(B_r)$).
Step 2.) By Step 1, it is enough to prove that we have a decomposition

$$\left(1 + \log\left(1 + \frac{|V(s,\cdot,\cdot)|^2}{2}\right)\right)^\alpha \leqslant f_1 + f_2 \in L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v) + L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v), \tag{6.9}$$

for $(x,v) \in B_r$, where f_1, f_2 are independent of $s \in [t,T]$. Let

$$\beta(y) = (1 + \log(1+y))^\alpha, \quad \text{for } y \geqslant 0. \tag{6.10}$$

Then

$$\begin{aligned}\beta'(y) &= \frac{\alpha(1+\log(1+y))^{\alpha-1}}{1+y},\\ 0 < -\beta''(y) &\leqslant \frac{(1+\log(1+y))^{\alpha-1}}{(1+y)^2}.\end{aligned} \tag{6.11}$$

Using (6.5), we compute

$$\begin{aligned}&\partial_s\left[\beta\left(\frac{|V(s,x,v)|^2}{2}\right)\right]\\ =&\Big(-\partial_s[U(s,X(s,x,v))] + \partial_t U(s,X(s,x,v))\Big)\beta'\left(\frac{|V(s,x,v)|^2}{2}\right)\\ =&-\partial_s\left[U(s,X(s,x,v))\beta'\left(\frac{|V(s,x,v)|^2}{2}\right)\right]\\ &+U(s,X(s,x,v))\beta''\left(\frac{|V(s,x,v)|^2}{2}\right)V(s,x,v)\cdot E(s,X(s,x,v))\\ &+\partial_t U(s,X(s,x,v))\beta'\left(\frac{|V(s,x,v)|^2}{2}\right).\end{aligned} \tag{6.12}$$

Thus, integrating between t and s,

$$\begin{aligned}&\left(1+\log\left(1+\frac{|V(s,x,v)|^2}{2}\right)\right)^{\alpha}\\ =&-\frac{\alpha U(s,X(s,x,v))}{\left(1+\frac{|V(s,x,v)|^2}{2}\right)\left(1+\log\left(1+\frac{|V(s,x,v)|^2}{2}\right)\right)^{1-\alpha}}\\ &+\frac{\alpha U(t,x)}{\left(1+\frac{|v|^2}{2}\right)\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)^{1-\alpha}}+\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)^{\alpha}\\ &+\int_t^s\left\{U(\tau,X(\tau,x,v))V(\tau,x,v)\cdot E(\tau,X(\tau,x,v))\beta''\left(\frac{|V(\tau,x,v)|^2}{2}\right)\right.\\ &\left.+\partial_t U(\tau,X(\tau,x,v))\beta'\left(\frac{|V(\tau,x,v)|^2}{2}\right)\right\}d\tau.\end{aligned} \tag{6.13}$$

Step 3.) Since $E(t,\cdot)\in L^2(\mathbb{R}^3)$, we have by the Sobolev embedding that $U(t,\cdot)\in L^6(\mathbb{R}^3)$. Thus clearly

$$\frac{U(t,x)}{1+\frac{|v|^2}{2}}\in L^6(\mathbb{R}^3_x\times\mathbb{R}^3_v)\subset L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)+L^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_v). \tag{6.14}$$

Next, since $\omega=+1$ and $\rho\geqslant 0$, one has $U=U_\rho-U_{\rho_b}$, with $U_\rho=\frac{1}{4\pi|x|}*\rho\geqslant 0$. Thus $U\geqslant -\|U_{\rho_b}\|_{L^\infty}$. Thus the first three terms in the expansion (6.13) are upper bounded in $L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)+L^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_v)$. It remains to estimate the integral. We can bound it by $\Phi_1+\Phi_2$, with

$$\Phi_1:=\int_t^T\left|U(\tau,X(\tau,x,v))V(\tau,x,v)\cdot E(\tau,X(\tau,x,v))\beta''\left(\frac{|V(\tau,x,v)|^2}{2}\right)\right|d\tau, \tag{6.15}$$

$$\Phi_2:=\int_t^T\left|\partial_t U(\tau,X(\tau,x,v))\beta'\left(\frac{|V(\tau,x,v)|^2}{2}\right)\right|d\tau. \tag{6.16}$$

Note that Φ_1, Φ_2 are independent of s. We estimate Φ_1 in $L^{3/2}(\mathbb{R}^3_x \times \mathbb{R}^3_v) \subset L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v) + L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v)$. Passing the $L^{3/2}$ norm under the integral and changing $(X(\tau,x,v), V(\tau,x,v))$ to (x,v), this gives (up to a factor L)

$$\begin{aligned}
&\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \left| \frac{U(\tau,x)v\cdot E(\tau,x)}{\left(1+\frac{|v|^2}{2}\right)^2 \left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)^{1-\alpha}} \right|^{3/2} dxdv \\
&\leqslant \int\limits_{\mathbb{R}^3} |U(\tau,x)E(\tau,x)|^{3/2} dx \int\limits_{\mathbb{R}^3} \frac{2^{3/4}\, dv}{\left(1+\frac{|v|^2}{2}\right)^{9/4} \left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)^{3(1-\alpha)/2}} \\
&\leqslant c||U(\tau,\cdot)||^{3/2}_{L^6(\mathbb{R}^3)} ||E(\tau,\cdot)||^{3/2}_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Thus $\Phi_1 \in L^{3/2}(\mathbb{R}^3_x \times \mathbb{R}^3_v)$.

Step 4.) For Φ_2, we notice that E satisfies (2.6) and $E = -\nabla_x U$, thus

$$\partial_t U = -\omega(-\Delta_x)^{-1}\mathrm{div}_x J. \tag{6.17}$$

Since $J(\tau,\cdot) \in L^1(\mathbb{R}^3_x)$, we deduce by the Hardy Littlewood Sobolev inequality that

$$|||\partial_t U(\tau,\cdot)|||_{M^{3/2}(\mathbb{R}^3)} \leqslant c||J(\tau,\cdot)||_{L^1(\mathbb{R}^3)}. \tag{6.18}$$

Therefore, we estimate

$$\begin{aligned}
&\left|\left|\left| \frac{\partial_t U(\tau, X(\tau,x,v))}{\left(1+\frac{|V(\tau,x,v)|^2}{2}\right) \left(1+\log\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)\right)^{1-\alpha}} \right|\right|\right|_{M^{3/2}(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \\
&\leqslant L^{2/3} \left|\left|\left| \frac{\partial_t U(\tau,x)}{\left(1+\frac{|v|^2}{2}\right) \left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)^{1-\alpha}} \right|\right|\right|_{M^{3/2}(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \\
&\leqslant L^{2/3} \left|\left|\left| \frac{\partial_t U(\tau,x)}{\left(1+\frac{|v|^2}{2}\right) \left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)^{1-\alpha}} \right|\right|\right|_{L^{3/2}(\mathbb{R}^3_v; M^{3/2}(\mathbb{R}^3_x))} \\
&\leqslant L^{2/3} |||\partial_t U(\tau,\cdot)|||_{M^{3/2}(\mathbb{R}^3_x)} \left(\int\limits_{\mathbb{R}^3} \frac{dv}{\left(1+\frac{|v|^2}{2}\right)^{3/2} \left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)^{3(1-\alpha)/2}} \right)^{2/3} \\
&\leqslant c||J(\tau,\cdot)||_{L^1(\mathbb{R}^3)},
\end{aligned} \tag{6.19}$$

where the last integral is convergent since $3(1-\alpha)/2 > 1$. From the inclusion $M^{3/2}(\mathbb{R}^3_x \times \mathbb{R}^3_v) \subset L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v) + L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v)$, and integrating (6.16) over B_r, we get (6.9) as desired. □

7. Renormalized solutions and Lagrangian solutions

We recall the different notions of weak solutions for the Vlasov-Poisson system. We shall always assume that $1 \leqslant N \leqslant 3$, and we consider an initial datum $f^0 \in L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v)$, $f^0 \geqslant 0$. We introduce first renormalized solutions, following [16, 18].

Definition 7.1. We say that $f \in L^\infty((0,T);L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v))$, $f \geqslant 0$, is a solution to the Vlasov equation (1.1) in the renormalized sense if for all test functions $\beta \in C^1([0,\infty))$ with β bounded, we have that

$$\partial_t \beta(f) + v \cdot \nabla_x \beta(f) + \mathrm{div}_v\Big(E(t,x)\beta(f)\Big) = 0, \tag{7.1}$$

in $\mathscr{D}'((0,T)\times\mathbb{R}^N_x\times\mathbb{R}^N_v)$.

We next introduce the notion of Lagrangian solutions.

Definition 7.2. Let be given a vector field $b(t,x,v) = (v, E(t,x))$ as in (3.6) for some $\rho \in L^\infty((0,T);L^1(\mathbb{R}^N))$, $\rho \geqslant 0$, and $\rho_b \in L^1(\mathbb{R}^N)$. We assume that $E \in L^\infty((0,T);L^2(\mathbb{R}^N))$, and that (2.6) holds with $J \in L^\infty((0,T);L^1(\mathbb{R}^N))$. We assume furthermore that either $N = 1$ or 2, or $N = 3$ and $\omega = +1$, $\rho_b \in L^p(\mathbb{R}^3)$ for some $p > 3/2$. We consider regular Lagrangian flows Z as in Definition 4.1, except that now $s \in [0,T]$ instead of $s \in [t,T]$ (forward-backward flow), and with compression constant L independent of $t \in [0,T]$. Then according to Subsections 3.3, 3.4, Proposition 6.2, Proposition 6.3, the vector field b satisfies assumptions **(R1a), (R2a), (R3)**. Therefore, Theorem 5.2 yields the uniqueness of the forward-backward regular Lagrangian flow $Z = (X,V)$. The whole theory of [13] then applies indeed, with very little modifications in the proofs. In particular there is existence and uniqueness of the forward-backward regular Lagrangian flow, with compression constant 1, and stability. We can thus define in accordance with [13] a Lagrangian solution f to the Vlasov equation (1.1) by

$$f(t,x,v) = f^0\Big(X(s=0,t,x,v), V(s=0,t,x,v)\Big), \qquad \text{for all } t \in [0,T], \tag{7.2}$$

for arbitrary $f^0 \in L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v)$. It verifies in particular $f \in C([0,T];L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v))$, and it is indeed also a renormalized solution.

Definition 7.3. We define a Lagrangian solution to the Vlasov-Poisson system as a couple (f,E) such that

(1) $f \in C([0,T];L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v))$, $f \geqslant 0$, $|v|^2 f \in L^\infty((0,T);L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v))$,
(2) $E(t,x)$ is given by the convolution (1.6) with $\rho(t,x) = \int f(t,x,v)dv$, $\rho_b \in L^1(\mathbb{R}^N)$, $\rho_b \geqslant 0$ (and if $N = 3$, $\omega = +1$, $\rho_b \in L^p(\mathbb{R}^3)$ for some $p > 3/2$),
(3) $E \in L^\infty((0,T);L^2(\mathbb{R}^N_x))$,
(4) The relation (2.6) holds with $J(t,x) = \int v f(t,x,v)dv$,
(5) f is a Lagrangian solution to the Vlasov equation, in the sense of (7.2).

8. Existence of Lagrangian solutions

8.1. **Compactness.** In this subsection we prove two compactness results, Theorems 8.2 and 8.3, for families of Lagrangian solutions to the Vlasov-Poisson system, with strongly or weakly convergent initial data.

Lemma 8.1. *Let $g(x) = \frac{x}{|x|^N}$ for $x \in \mathbb{R}^N$, and denote by $\tau_h g(x) = g(x+h)$. Then for any $1 < p < \frac{N}{N-1}$,*

$$\|\tau_h g(x) - g(x)\|_{L^p(\mathbb{R}^N)} \leqslant c|h|^\alpha, \tag{8.1}$$

with $\alpha = 1 - N + N/p > 0$, and where c depends on N, p.

Proof. Fix $h \in \mathbb{R}^N$, $h \neq 0$. For $|x| > 2|h|$, we have for all $0 \leqslant \theta \leqslant 1$, $|x + \theta h| \geqslant |x| - \theta|h| > |x|/2$, thus we have

$$\begin{aligned} |\tau_h g(x) - g(x)| &\leqslant |h| \sup_{0\leqslant\theta\leqslant 1} |\nabla g(x + \theta h)| \\ &\leqslant |h| \sup_{0\leqslant\theta\leqslant 1} \frac{c_N}{|x + \theta h|^N} \leqslant c_N \frac{|h|}{|x|^N}. \end{aligned} \tag{8.2}$$

Then we estimate

$$\int\limits_{|x|>2|h|} \frac{|h|^p}{|x|^{Np}} dx = c_N |h|^p \int\limits_{2|h|}^{\infty} r^{N-1-Np} dr = c_N |h|^p \frac{(2|h|)^{N-Np}}{Np - N} = c_{N,p} |h|^{p-Np+N}.$$

Next, for $|x| \leqslant 2|h|$, we write

$$|\tau_h g(x) - g(x)| \leqslant \frac{1}{|x + h|^{N-1}} + \frac{1}{|x|^{N-1}}, \tag{8.3}$$

and clearly

$$\begin{aligned} &\int\limits_{|x|\leqslant 2|h|} \left(\frac{1}{|x + h|^{(N-1)p}} + \frac{1}{|x|^{(N-1)p}} \right) dx \\ &\leqslant 2 \int\limits_{|y|\leqslant 3|h|} \frac{dy}{|y|^{(N-1)p}} = c_N \int\limits_0^{3|h|} r^{N-Np+p-1} dr = c_{N,p} |h|^{N-Np+p}, \end{aligned} \tag{8.4}$$

since the last integral is convergent for $p < \frac{N}{N-1}$. □

Theorem 8.2. *Let (f_n, E_n) be a sequence of Lagrangian solutions to the Vlasov-Poisson system satisfying*

$$f_n^0 \to f^0 \text{ in } L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v), \tag{8.5}$$

and

$$\iint |v|^2 f_n(t, x, v) dx dv + \int |E_n(t, x)|^2 dx \leqslant C, \qquad \text{for all } t \in [0, T]. \tag{8.6}$$

Then, up to a subsequence f_n converges strongly in $C([0, T]; L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v))$ to f, E_n converges in $C([0, T]; L^1_{loc}(\mathbb{R}^N))$ to E, and (f, E) is a Lagrangian solution to the Vlasov-Poisson system with initial datum f^0. Moreover, the regular forward-backward Lagrangian flow $Z_n(s, t, x, v)$ converges to $Z(s, t, x, v)$ locally in measure in $\mathbb{R}^N \times \mathbb{R}^N$, uniformly in $s, t \in [0, T]$.

Proof. Step 1.) (Equi-integrability)
Because of (7.2) and (8.5) we have

$$\|f_n(t, \cdot, \cdot)\|_{L^1(\mathbb{R}^N \times \mathbb{R}^N)} = \|f_n^0\|_{L^1(\mathbb{R}^N \times \mathbb{R}^N)} \leqslant M. \tag{8.7}$$

Then because of the bounds (8.7), (8.6), and applying Propositions 6.2, 6.3, one has for any $r > 0$

$$\mathscr{L}^{2N}\{(x, v) \in B_r : \sup_{0\leqslant s\leqslant T} |Z_n(s, t, x, v)| > \gamma\} \to 0, \text{ as } \gamma \to \infty, \text{uniformly in } t, n. \tag{8.8}$$

Since the sequence f_n^0 is uniformly equi-integrable, and since Z_n is measure-preserving, we have by (7.2) and (8.8) that $f_n(t,\cdot)$ is equi-integrable, uniformly in t, n. Consequently, $\rho_n(t,\cdot)$ is also equi-integrable, uniformly in t, n. Using the bound (8.6), $vf_n(t,\cdot)$ is also equi-integrable, and therefore $J_n(t,\cdot)$ is also equi-integrable, uniformly in t, n.

Step 2.) (Compactness of the field)
In order to prove that $E_n(t,x) \to E(t,x)$ in $L^1_{\rm loc}((0,T)\times\mathbb{R}^N_x)$, we first look at the compactness in x. Denote by $\tau_h E_n(t,x) = E_n(t,x+h)$. Then using (1.6),

$$\begin{aligned} &||\tau_h E_n(t,\cdot) - E_n(t,\cdot)||_{L^p(\mathbb{R}^N)} \\ &\leqslant c \left\| \tau_h \frac{x}{|x|^N} - \frac{x}{|x|^N} \right\|_{L^p(\mathbb{R}^N)} ||\rho_n(t,\cdot) - \rho_b(\cdot)||_{L^1(\mathbb{R}^N)}. \end{aligned} \tag{8.9}$$

Thus according to Lemma 8.1, we get for any $1 < p < \frac{N}{N-1}$ that

$$||E_n(t,x+h) - E_n(t,x)||_{L^p(\mathbb{R}^N)} \to 0, \qquad \text{as } h \to 0, \text{ uniformly in } t, n. \tag{8.10}$$

Then, because of (2.6), we have that $\partial_t E_n$ is bounded in $L^\infty((0,T);\mathscr{S}'(\mathbb{R}^N))$. Applying the Aubin-Lions-Kruzkov lemma A.1, we conclude that E_n is compact in $L^1_{\rm loc}([0,T]\times\mathbb{R}^N_x)$. Thus after extraction of a subsequence, $E_n(t,x) \to E(t,x)$ strongly in $L^1((0,T);L^1_{\rm loc}(\mathbb{R}^N_x))$.

Step 3.) (Convergence of the flow)
Because of the bound (8.6), one has $E \in L^\infty((0,T),L^2(\mathbb{R}^N))$. Also, using the uniform bounds on ρ_n, J_n in $L^\infty((0,T);L^1(\mathbb{R}^N))$ and the uniform equi-integrability obtained in Step 1, one has up to a subsequence $\rho_n \to \rho$, $J_n \to J$ in the sense of distributions, with $\rho, J \in L^\infty((0,T);L^1(\mathbb{R}^N))$. We can pass to the limit in (1.6) and (2.6). Therefore, $b = (v,E)$ satisfies the assumptions **(R1a), (R2a), (R3)** and Definition 7.2 applies. According to [13, Lemma 6.3], since (8.8) holds, and ρ_n are equi-integrable, we deduce the convergence of Z_n to Z locally in measure in $\mathbb{R}^N_x \times \mathbb{R}^N_v$, uniformly with respect to $s,t \in [0,T]$, where Z is the regular forward-backward Lagrangian flow associated to b.

Step 4.) (Convergence of f)
Using the convergence (8.5), we can apply [13, Proposition 7.3], and we conclude that $f_n \to f$ in $C([0,T];L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v))$, where f is the Lagrangian solution to the Vlasov equation with coefficient b and initial datum f^0. It follows that $\rho_n \to \rho = \int f dv$ in $C([0,T];L^1(\mathbb{R}^N_x))$. By lower semi-continuity, we get from (8.6) that

$$\iint |v|^2 f(t,x,v)dxdv + \int |E(t,x)|^2 dx \leqslant C, \qquad \text{for all } t \in [0,T]. \tag{8.11}$$

The bound (8.6) gives also that $J_n \to J = \int vf dv$ in $C([0,T];L^1(\mathbb{R}^N_x))$. Therefore, (f,E) is a Lagrangian solution to the Vlasov-Poisson system. Using (1.6), we get that $E_n \to E$ in $C([0,T];L^1_{loc}(\mathbb{R}^N))$, which concludes the proof. □

Theorem 8.3. *Let (f_n, E_n) be a sequence of Lagrangian solutions to the Vlasov-Poisson system satisfying*

$$f_n^0 \rightharpoonup f^0 \text{ weakly in } L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v), \tag{8.12}$$

and the bound (8.6). *Then, up to a subsequence* f_n *converges in* $C([0,T];\text{weak}-L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v))$ *to* f, E_n *converges in* $C([0,T];L^1_{loc}(\mathbb{R}^N))$ *to* E, *and* (f,E) *is a Lagrangian solution to the Vlasov-Poisson system with initial datum* f^0. *Moreover, the regular forward-backward Lagrangian flow* $Z_n(s,t,x,v)$ *converges to* $Z(s,t,x,v)$ *locally in measure in* $\mathbb{R}^N\times\mathbb{R}^N$, *uniformly in* $s,t\in[0,T]$.

Proof. It is the same as that of Theorem 8.2, except the last step 4. Instead we apply [13, Proposition 7.7] and conclude that $f_n\rightharpoonup f$ in $C([0,T];\text{weak}-L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v))$, where f is the Lagrangian solution to the Vlasov equation with coefficient b and initial datum f^0. It follows that $\rho_n\rightharpoonup\rho=\int f dv$ in $C([0,T];\text{weak}-L^1(\mathbb{R}^N_x))$. By lower semi-continuity, we get again from (8.6) the energy bound (8.11). The bound (8.6) also enables to conclude that $J_n\rightharpoonup J=\int vf dv$ in $C([0,T];\text{weak}-L^1(\mathbb{R}^N_x))$. Therefore, (f,E) is a Lagrangian solution to the Vlasov-Poisson system. Using (1.6) and the compactness estimate (8.10), we get that $E_n\to E$ in $C([0,T];L^1_{loc}(\mathbb{R}^N))$, which concludes the proof. □

8.2. **Existence.** We conclude this section by the existence of Lagrangian solutions to the Vlasov-Poisson system for initial datum in L^1 with finite energy, in the repulsive case.

Theorem 8.4. *Let* $N=1,2$ *or* 3, *and let* $f^0\in L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v)$, $f^0\geqslant 0$. *Define* ρ^0 *and* E^0 *by*

$$\rho^0(x)=\int f^0(x,v)dv,\qquad E^0(x)=\frac{\omega}{|S^{N-1}|}\frac{x}{|x|^N}*(\rho^0(x)-\rho_b(x)),\quad(8.13)$$

with $\omega=+1$ *(repulsive case),* $\rho_b\in L^1(\mathbb{R}^N)$, $\rho_b\geqslant 0$, *and in the case* $N=3$ $\rho_b\in L^p(\mathbb{R}^3)$ *for some* $p>3/2$. *Assume that the initial energy is finite,*

$$\iint|v|^2f^0(x,v)dxdv+\int|E^0(x)|^2dx<\infty.\quad(8.14)$$

Then there exists a Lagrangian solution (f,E) *to the Vlasov-Poisson system defined for all time, having* f^0 *as initial datum, and satisfying for all* $t\geqslant 0$

$$\iint|v|^2f(t,x,v)dxdv+\int|E(t,x)|^2dx\leqslant\iint|v|^2f^0(x,v)dxdv+\int|E^0(x)|^2dx.\quad(8.15)$$

Proof. We use the classical way of getting global weak solutions to the Vlasov-Poisson system, i.e. we approximate the initial datum f^0 by a sequence of smooth data $f^0_n\geqslant 0$ with compact support. We approximate also ρ_b by smooth $\rho^n_b\geqslant 0$ with compact support (with $\int(\rho^0_n-\rho^n_b)dx=0$ if $N=1,2$). It is possible to do that with the upper bounds

$$\begin{aligned}&\limsup_{n\to\infty}\iint|v|^2f^0_n(x,v)dxdv\leqslant\iint|v|^2f^0(x,v)dxdv,\\&\limsup_{n\to\infty}\int|E^0_n(x)|^2dx\leqslant\int|E^0(x)|^2dx.\end{aligned}\quad(8.16)$$

Then, for each n, there exists a smooth classical solution (f_n,E_n) with initial datum f^0_n, to the Vlasov-Poisson system, defined for all time $t\geqslant 0$. Note that we can alternatively consider a regularized Vlasov-Poisson system with

energy identity, as in [11]. Since $\omega = +1$, the conservation of energy (2.8) gives for all $t \geqslant 0$,

$$\iint |v|^2 f_n(t,x,v)dxdv + \int |E_n(t,x)|^2 dx = \iint |v|^2 f_n^0(x,v)dxdv + \int |E_n^0(x)|^2 dx. \tag{8.17}$$

The couple (f_n, E_n) is in particular a Lagrangian solution to the Vlasov-Poisson system, for all intervals $[0,T]$. We can therefore apply Theorem 8.2. Extracting a diagonal subsequence, we get the convergence of (f_n, E_n) to (f, E) as stated in Theorem 8.2, where (f, E) is a Lagrangian solution to the Vlasov-Poisson system defined for all time, with f^0 as initial datum. The bound (8.11), together with (8.16), gives (8.15). □

Let us end with a remark on measure densities. When considering a sequence of solutions to the Vlasov-Poisson system, the vector fields $b_n = (v, E_n)$ have a gradient in (x,v) of the form

$$Db_n = \begin{pmatrix} D_1 b_n^1 & D_2 b_n^1 \\ D_1 b_n^2 & D_2 b_n^2 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{Id} \\ S(\rho_n - \rho_b) & 0 \end{pmatrix}, \tag{8.18}$$

where the index 1 stands for x, 2 for v, and where S is a singular integral operator. If we require only that $D_1 b_n^2$ converges in the sense of distributions to $D_1 b^2 = S(\rho - \rho_b)$, for some measure $\rho \in \mathscr{M}(\mathbb{R}^N)$, then we are in the setting of [10]. If ρ_n is uniformly bounded in $L^1((0,T); \mathscr{M}(\mathbb{R}^N))$, and $b_n \to b$ strongly in $L^1((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_v))$ with b satisfying (8.18), we conclude that $Z_n \to Z$ strongly, where Z is the regular Lagrangian flow associated to b. However, we are not able to define the push forward (7.2) of a measure f^0. This prevents from applying the context of [10] to the Vlasov-Poisson system with measure data.

Appendix: The weak Aubin-Lions-Kruzkov lemma

In this appendix we recall a classical weak form of the Aubin-Lions compactness lemma [7], in the spirit of Kruzkov [23, Lemma 5].

Lemma A.1. *Let Ω be an open subset of $\mathbb{R}^N$ and $T > 0$, $1 \leqslant p < \infty$. Assume that S is a bounded subset of $L^p((0,T) \times \Omega)$, such that*

(i) S is locally L^p-precompact in space, i.e. for any compact subset $K \subset \Omega$,

$$\int_0^T \int_K |u(t,x+h) - u(t,x)|^p \, dxdt \to 0 \text{ as } h \to 0, \text{ uniformly for } u \in S. \tag{A.1}$$

(ii) For $u \in S$, $\partial_t u$ is bounded in $L^\infty((0,T); \mathscr{D}'(\Omega))$. This means that for any $\varphi \in C_c^\infty(\Omega)$ and any $u \in S$, the map $t \mapsto \langle \partial_t u, \varphi \rangle$ belongs to $L^\infty((0,T))$ and

$$|\langle \partial_t u, \varphi \rangle| \leqslant C_\varphi, \text{ for a.e. } t \in (0,T) \text{ and all } u \in S, \tag{A.2}$$

where $C_\varphi \geqslant 0$ depends on the support of φ and a finite number of L^∞ norms of derivatives of φ (but not on u).

Then S is precompact in $L^p_{\mathrm{loc}}((0,T) \times \Omega)$.

Proof. According to the Riesz-Fréchet-Kolmogorov criterion of precompactness in L^p and taking into account (i), we have to prove that for any compact sets $L \subset (0,T)$ and $K \subset \Omega$,

$$\int_L \int_K |u(t+\tau,x) - u(t,x)|^p \, dxdt \to 0 \text{ as } \tau \to 0, \text{ uniformly for } u \in S. \quad \text{(A.3)}$$

For $\varepsilon > 0$, define $u_\varepsilon = \rho_\varepsilon *_x u$, where ρ_ε is a mollifier sequence in space. Then because of (i), $u_\varepsilon - u$ can be made arbitrary small in $L^p((0,T) \times K)$ for ε small enough, uniformly for $u \in S$. But for fixed ε, because of (ii), $\partial_t u_\varepsilon$ is bounded in $L^\infty((0,T) \times K)$, uniformly for $u \in S$. It follows that $\|u_\varepsilon(.+\tau,.) - u_\varepsilon\|_{L^p(L\times K)} \leqslant C_\varepsilon |\tau|$. Decomposing

$$u(.+\tau,.) - u = (u(.+\tau,.) - u_\varepsilon(.+\tau,.)) + (u_\varepsilon(.+\tau,.) - u_\varepsilon) + (u_\varepsilon - u), \quad \text{(A.4)}$$

we conclude that $u(.+\tau,.) - u \to 0$ in $L^p(L \times K)$ as $\tau \to 0$, uniformly for $u \in S$, i.e. (A.3) holds, and this concludes the proof of the lemma. □

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