Topological sensitivity of a shape functional defined from a solution of a high order PDE

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December 19, 2014

Abstract

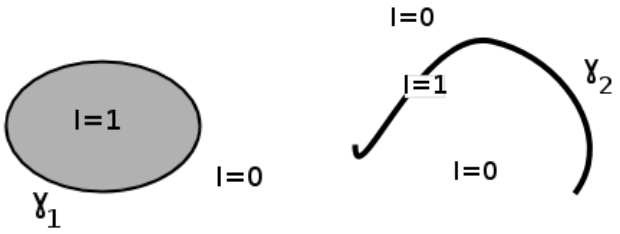
The topological gradient is defined as the leading term in the asymptotic expansion of a shape functional with respect to the size of a local perturbation. Introduced by Sokolowski [15] and Masmoudi [12], this notion has been intensively developed in recent years. There are many applications such as in mechanics of structures [3], in damage evolution modelling [2] and in image processing [6, 10, 5, 8]. This paper deals with the topological sensitivity of a cost function of the form $j(\Omega) = \int_\Omega |\nabla^m u_\Omega|^2$ where u_Ω solves a PDE of order $2m$ defined from the m-th Laplacian with Neumann boundary conditions. We place us in 2D and we consider a domain perturbed by a small crack.

1 Introduction

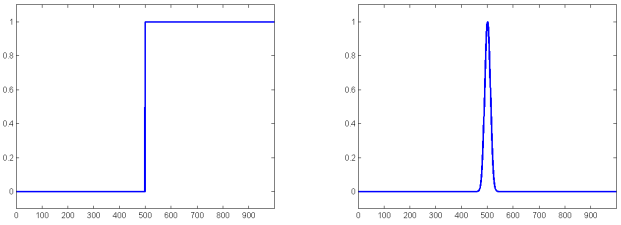
The goal of this paper is to generalise the topological gradient method studied and applied in segmentation of images [3, 6, 10, 5] to a more general and higher order problem adapted to object detection. Objects can of Lebesgue measure different to zero. In this case, we are interested in the detection of the boundary of the object commonly called edge. The discontinuity associated to an edge is a discontinuity with a jump of intensity across the structure. Objects can also be of zero Lebesgue measure (filaments, points in 2D) and in this case we want to detect the whole object. Such object is called fine structure and there is no jump of intensity across the structure (see Figure 1).

In [6, 10], the topological gradient has been applied to a problem of edge detection. It is well known that the detection of edges by using differential operators needs first order derivatives. It is no more true for fine structures which are discontinuities without jump of intensity across the structure. In [5, 8] authors justify theoretically and verify numerically that the detection of fine structures by using differential operators need second order derivatives. Hence they introduce a cost function involving second order derivatives of a regularization of the data solution of a fourth order PDE. In this paper we wonder what does happen for higher order problems with respect to these two kinds of structures ?

In 1D a contour can be modelled by the Heaviside function H. We can approximate H by a regular function $H_\eta \in C^p(\mathbb{R})$ which writes $H_\eta(x) = \varphi\left(\frac{x}{\eta}\right) \mathbb{1}_{|x|<\eta} + \mathbb{1}_{x\geq\eta}$ where $\varphi(x) = \sum_{k=0}^{p} a_k x^{2k+1}$ with $(a_k)_{1\leq k\leq p}$ such that $\varphi(1) = 1$ and $\varphi^{(k)}(1) = 0$ for $k \in [\![1, p]\!]$. Similarly, in 1D a fine structure can be modelled by $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$. It can be approximated by $f_\eta(x) = e^{-x^2/\eta^2}$. On Table 1, we study the m-th derivatives of the functions f_η and H_η with $m \in [\![0, p]\!]$ for $p = 4$ and $\eta = 1$. We see that derivatives of odd order penalize more edges than fine structures, while derivatives of even orders are extremal on fine structures and null on edges. We can generalize this reasoning in 2D by working on transverse cut. Keeping these considerations in mind, in this paper we propose to develop a topological gradient method based on m-th derivatives of a regularisation of the data. From a numerical point of view the high order of the PDE may seem a source of instabilities. However, the generalisation may be interesting theoretically to stand back from the two cases $[m = 1]$ [3] and $[m = 2]$ [5].



(a) An edge γ_1 (b) A filament γ_2



(c) I across the edge (d) I across the filament

Figure 1: Difference between an edge and a fine structure

Roughly speaking, the topological gradient is performed as follows : let Ω a regular domain of $\mathbb{R}^2$, $j(\Omega) = J(\Omega, u_\Omega)$ be a shape functional with u_Ω solution of a PDE defined on Ω and $J(\Omega, .)$ a cost function depending on Ω. For small $\epsilon > 0$, let $\Omega_\epsilon = \Omega \backslash \overline{\{x_0 + \epsilon\omega\}}$ where $x_0 \in \Omega$ and ω is a given subset of $\mathbb{R}^2$ (typically a crack or a ball). The topological sensitivity of $j(\Omega)$ is given by the leading term in the difference $j(\Omega_\epsilon) - j(\Omega)$ and generally it takes the form : $j(\Omega_\epsilon) - j(\Omega) = \varphi(\epsilon)\mathcal{I}(x_0, \omega) + o(\varphi(\epsilon))$ with $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such as $|\varphi| \searrow 0$ and $\mathcal{I}(x_0, \omega)$ is called the topological gradient associated to the cost function $j(\Omega)$. As said before, in [3, 5] authors have studied the topological sensitivity of a shape functional of the form $j(\Omega) = J(\Omega, u_\Omega) = \int_\Omega F(u_\Omega, \nabla u_\Omega, \nabla^2 u_\Omega)$ where u_Ω is solution of $Au_\Omega + u_\Omega = f$ where A denotes either the Laplacian or the bilaplacian with Neumann homogeneous boundary conditions and f stands for the data. In this paper we consider a general cost function full filling Hypotheses 1 and defined from the p-th derivatives ($0 \leq p \leq m$) of $2m$ order PDE solution. In particular, we apply our general study to quadratic cost functions of the form $J_p(\Omega, u) = \int_\Omega |\nabla^p u|^2$ with $1 \leq p \leq m$. Let notice that the PDE studied is $(-1)^m \Delta^m u_\Omega + u_\Omega = f$ with Neumann homogeneous boundary conditions and it is associated to the minimization problem $\min_{H^m(\Omega)} J_m(\Omega, u) + \|u - f\|^2_{0,\Omega}$. In image processing, f is generally the observed image eventually degraded by Gaussian noise and u_Ω can be seen as a regularization of f. A high order problem has already been studied in [4] but with homogeneous Dirichlet conditions of $\partial\Omega$ and the result given is known in function of polarization tensors which are not known in general and hard to evaluate. Here, we consider a cracked domain $\Omega_\epsilon = \Omega \backslash \overline{\{x_0 + \epsilon\sigma(\vec{n})\}}$, where $\sigma(\vec{n})$ is a straight crack centered, the origin and $x_0 \in \Omega$ and ϵ small enough to avoid a possible contact between the perturbation an the boundary of Ω. From these notations we have $\Omega_0 = \Omega$.
The paper reads as follows. First we determine the Euler equations associated to the minimization in $H^m(\Omega)$ of the energy function $J_\Omega(u) = \int_\Omega |\nabla^m u|^2$. Then we state the problem and we compute the topological gradient associated to a shape functional full filling Hypotheses 1 an defined from u_{Ω_ϵ} the solution of a $2m$ order PDE defined on Ω_ϵ. The final topological gradient expression is explicit in function of m, u_Ω and an adjoint state v_Ω defined on the unperturbed domain Ω. For numerical illustrations and applications in imaging we refer the reader to [6, 10] for $m = 1$ and to [8] for $m = 2$. In the sequel we assume that the center of the perturbation is $x_0 = 0$ and that the abscissa axe is given by the crack

direction (we place us in the local coordinate system to the crack).

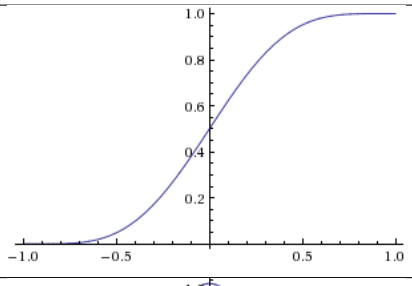
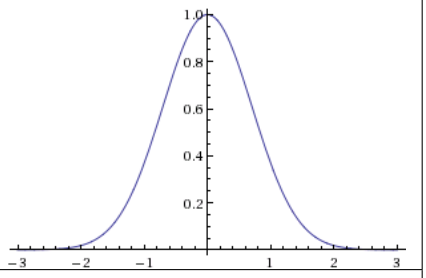
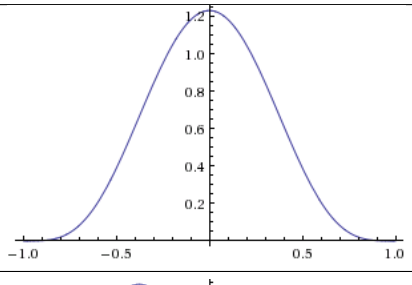
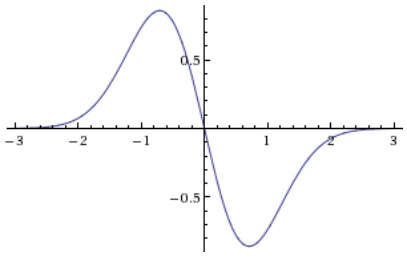
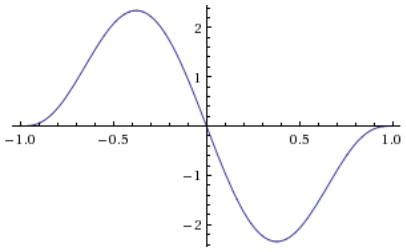
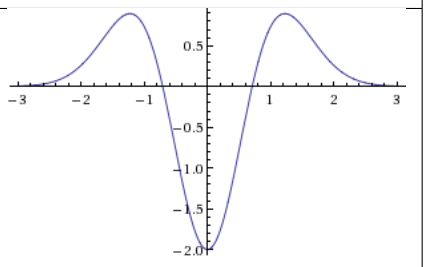
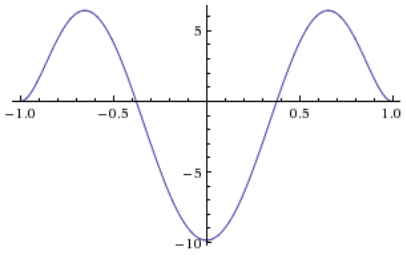
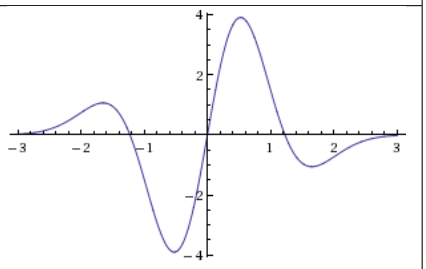
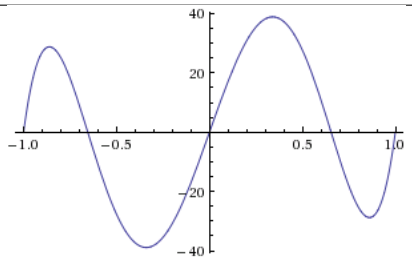
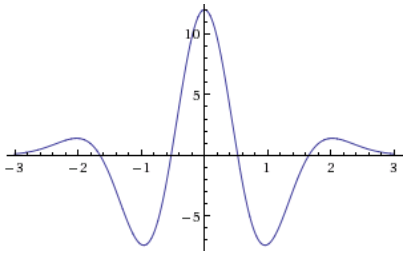
Values of m	Contour	Filament
$m = 0$		
$m = 1$		
$m = 2$		
$m = 3$		
$m = 4$		

Table 1: 1D study of the m-th derivatives of a contour and a filament regularizations for $m \in [\![0, 4]\!]$

We consider the function $J_\Omega : H^m(\Omega) \longrightarrow \mathbb{R}$:

$$\begin{aligned} J_\Omega(u) &= \int_\Omega \sum_{k=0}^{m} C_m^k \left(\frac{\partial^m u}{\partial x_1^k \partial x_2^{m-k}} \right)^2 \\ &= \int_{\Omega_\epsilon} |\nabla^m u|^2 \end{aligned} \tag{1.1}$$

Let $u_\Omega \in H(\Omega)$ defined by

$$u_\Omega = \underset{u\in H(\Omega)}{\operatorname{argmin}} \; J_\Omega(u) + \|u-f\|^2_{0,\Omega} \tag{1.2}$$

where $H(\Omega)$ is the Hilbert space $H(\Omega) = \left\{u \in L^2(\Omega), \nabla^m u \in L^2(\Omega)\right\}$. Gagliardo-Nirenberg inequalities (see [1] pp 75-79) lead to $H(\Omega) = H^m(\Omega)$.

We define the bilinear form $b_\Omega(u,v) = \frac{1}{2}DJ_\Omega(u).v : H^m(\Omega)\times H^m(\Omega) \longrightarrow \mathbb{R}$ defined by :

$$b_\Omega(u,v) = \int_\Omega \sum_{k=0}^{m} C_m^k \frac{\partial^m u}{\partial x_1^k \partial x_2^{m-k}} \frac{\partial^m v}{\partial x_1^k \partial x_2^{m-k}} \tag{1.3}$$

To shorten notations we denote $u_\epsilon = u_{\Omega_\epsilon}$. We define on $H^m(\Omega_\epsilon)$ the bilinear and linear forms $a_\epsilon(u,v)$ and $l_\epsilon(v)$:

$$\begin{aligned} a_\epsilon(u,v) &= b_{\Omega_\epsilon}(u,v) + \int_{\Omega_\epsilon} uv \\ l_\epsilon(v) &= \int_{\Omega_\epsilon} fv \end{aligned} \tag{1.4}$$

We introduce the problem $(\mathcal{P}_\epsilon)$ and its solution $u_\epsilon \in H^m(\Omega_\epsilon)$:

$$a_\epsilon(u_\epsilon, v) = l_\epsilon(v) \quad \forall v \in H^m(\Omega_\epsilon), \quad (\mathcal{P}_\epsilon) \tag{1.5}$$

In the sequel we denote by $u_{k,l} = \frac{\partial^{k+l}u}{\partial x_1^k \partial x_2^l}$ for $u \in H^m(\Omega_\epsilon)$ and $k,l \in \mathbb{N}$ such that $k+l \leq m$ and we show that the Euler equations associated to the minimization problem (1.2) in $H^m(\Omega)$ involve the m-th iterate of the Laplace operator Δ^m and m boundary operators that we explicit.

Théorème 1.1. *Let $b_\Omega(u,v)$ the bilinear form on $H^m(\Omega)$ defined by (1.3), we have*

$$b_\Omega(u,v) = (-1)^m \int_\Omega \Delta^m uv + \sum_{i=0}^{m-1} \int_{\partial\Omega} A_i(u) \frac{\partial^{m-1-i} v}{\partial n^{m-1-i}} d\sigma$$

where $A_i : H^m(\Omega) \longrightarrow H^{-i-1/2}(\partial\Omega)$ for $i \in [\![0, m-1]\!]$ is the differential operator of order $i+m$ defined by

$$\begin{aligned} A_i(u) = &\sum_{k=0}^{m} C_m^k \sum_{j=0}^{k-1} (-1)^j \sum_{\substack{0\leq p\leq k-j-1 \\ 0\leq q\leq m-k \\ p+q=i-j}} (-1)^q C_{k-j-1}^p C_{m-k}^q \frac{\partial^{p+q}}{\partial \tau^{p+q}} \left(n_1^{k-j-p+q} n_2^{m-k-q+p} u_{k+j,m-k} \right) \\ &+ \sum_{0\leq k+l\leq i} C_m^k (-1)^i C_{m-k-l-1}^{i-(k+l)} \frac{\partial^{i-(k+l)}}{\partial \tau^{i-(k+l)}} \left(n_1^{i-(k+l)} n_2^{m-i} u_{2k,m-k+l} \right) \end{aligned}$$

with the convention $\sum_{k=0}^{-1} = 0$. In the case of a straight crack $\sigma = \{(s,0), -1 < s < 1\}$, we have :

(i) For $i \in [\![0, \lfloor \frac{m-1}{2} \rfloor]\!]$

$$A_{2i} = \sum_{k=0}^{i} (-1)^{i-k} C_m^{2k} \left(\sum_{j=0}^{2k-1} C_{2k-j-1}^{k+i-j} C_{m-2k}^{i-k} \right) u_{2(k+i),m-2k} + \sum_{k=0}^{2i} C_m^k u_{2k,m-2k+2i}$$

(ii) For $i \in [\![0, \lfloor \frac{m-2}{2} \rfloor]\!]$:

$$A_{2i+1} = \sum_{k=0}^{i} (-1)^{i-k+1} C_m^{2k+1} \left(\sum_{j=0}^{2k-1} C_{2k-j}^{k+i-j+1} C_{m-2k-1}^{i-k} \right) u_{2(k+i+1),m-(2k+1)} - \sum_{k=0}^{2i+1} C_m^k u_{2k,m-2k+2i+1}$$

Remarque 1.2. *For $m = 1$, with these statements $A_0 = \frac{\partial}{\partial n} = \frac{\partial}{\partial x_2}$.*
For $m = 2$, we have (see the Kirchhoff thin plate equation [7, 5] with the Poisson ration $\nu = 0$), $A_0 = B_2$ and $A_1 = -B_1$.
In the case of the straight crack $\sigma = \{(s, 0), -1 < s < 1\}$, for $m \geq 1$, we have $A_0 = \frac{\partial^m}{\partial n^m} = \frac{\partial^m}{\partial x_2^m}$, and A_i are homogeneous differential operators of order $m + i$.

Proof. Keeping the same notations of the statements of the theorem, $b_\Omega(u, v)$ rewrites as :

$$b_\Omega(u, v) = \sum_{k=0}^{m} C_m^k \int_\Omega u_{k,m-k} v_{k,m-k} \tag{1.6}$$

Let $k \geq 1$, by integrating k times with respect to x_1 we get :

$$\int_\Omega u_{k,m-k} v_{k,m-k} = \sum_{j=0}^{k-1} \int_{\partial\Omega} (-1)^j \int_{\partial\Omega} n_1 u_{k+j,m-k} v_{k-j,m-k} + (-1)^k \int_\Omega u_{2k,m-k} v_{0,m-k}$$

Then by assuming that $k \leq m - 1$, we integrate $m - k$ times with respect to x_2 :

$$\int_\Omega u_{2k,m-k} v_{0,m-k} = \sum_{l=0}^{m-k-1} (-1)^l \int_{\partial\Omega} n_2 u_{2k,m-k} v_{0,m-k-l-1} + (-1)^{m-k} \int_\Omega u_{2k,2(m-k)} v$$

Hence for $k \in [\![0, m]\!]$ by using the convention $\sum_0^{-1} = 0$, we have :

$$\begin{aligned} \int_\Omega u_{k,m-k} v_{k,m-k} = &\sum_{j=0}^{k-1} \int_{\partial\Omega} (-1)^j \int_{\partial\Omega} n_1 u_{k+j,m-k} v_{k-j,m-k} + (-1)^k \sum_{l=0}^{m-k-1} (-1)^l \int_{\partial\Omega} n_2 u_{2k,m-k} v_{0,m-k-l-1} \\ &+ (-1)^m \int_\Omega u_{2k,2(m-k)} v \end{aligned} \tag{1.7}$$

We set $\vec{\tau} = \vec{n}^\perp$ such that $(\vec{n}, \vec{\tau})$ be an orthonormal basis. We express the $\frac{\partial^{k+l} u}{\partial x_1^k \partial x_2^l}$ in function of the $\frac{\partial^{i+j}}{\partial \tau^i \partial n^j}$ (where $i + j = k + l$). The relations between $\partial\tau$, ∂n, ∂x_1 and ∂x_2 are :

$$\begin{aligned} \frac{\partial}{\partial x_1} &= n_1 \frac{\partial}{\partial n} - n_2 \frac{\partial}{\partial \tau} \\ \frac{\partial}{\partial x_2} &= n_2 \frac{\partial}{\partial n} + n_1 \frac{\partial}{\partial \tau} \end{aligned}$$

Let $p_2, q_2 \in \mathbb{N}$, we have :

$$\begin{aligned} v_{p_2,q_2} &= \left(n_1 \frac{\partial}{\partial n} - n_2 \frac{\partial}{\partial \tau} \right)^{p_2} \left(n_2 \frac{\partial}{\partial n} + n_1 \frac{\partial}{\partial \tau} \right)^{q_2} v \\ &= \sum_{\substack{0 \leq p \leq p_2 \\ 0 \leq q \leq q_2}} (-1)^p C_{p_2}^p C_{q_2}^q n_1^{p_2 - p + q} n_2^{q_2 + p - q} \frac{\partial^{p_2+q_2} v}{\partial \tau^{p+q} \partial n^{p_2+q_2-p-q}} \end{aligned}$$

By using the integration by parts formula on a closed curve ($\int_{\partial\Omega} f \frac{\partial g}{\partial \tau} d\sigma = -\int_{\partial\Omega} \frac{\partial f}{\partial \tau} g d\sigma$), we get :

$$\int_{\partial\Omega} n_1 u_{p_1,q_1} v_{p_2,q_2} = \sum_{\substack{0 \leq p \leq p_2 \\ 0 \leq q \leq q_2}} (-1)^{\not{p}} C_{p_2}^p C_{q_2}^q (-1)^{\not{p}+q} \int_{\partial\Omega} \frac{\partial^{p+q}}{\partial \tau^{p+q}} \left(n_1^{p_2 - p + q + 1} n_2^{q_2 + p - q} u_{p_1,q_1} \right) \frac{\partial^{p_2+q_2-p-q} v}{\partial n^{p_2+q_2-p-q}} d\sigma \tag{1.8}$$

Similarly

$$\int_{\partial\Omega} n_2 u_{p_1,q_1} v_{p_2,q_2} = \sum_{\substack{0\le p\le p_2\\ 0\le q\le q_2}} (-1)^q C_{p_2}^p C_{q_2}^q \int_{\partial\Omega} \frac{\partial^{p+q}}{\partial\tau^{p+q}} \left(n_1^{p_2-p+q} n_2^{p+q_2-q+1} u_{p_1,q_1} \right) \frac{\partial^{p_2+q_2-p-q} v}{\partial n^{p_2+q_2-p-q}} d\sigma \tag{1.9}$$

From (1.8) with $p_1 = k+j$, $q_1 = m-k$, $p_2 = k-j-1$ and $q_2 = m-k$ we get :

$$\begin{aligned} &\int_{\partial\Omega} n_1 u_{k+j,m-k} v_{k-j-1,m-k} \\ &= \sum_{\substack{0\le p\le k-j-1\\ 0\le q\le m-k}} (-1)^q C_{k-j-1}^p C_{m-k}^q \int_{\partial\Omega} \frac{\partial^{p+q}}{\partial\tau^{p+q}} \left(n_1^{k-j-p+q} n_2^{m-k-q+p} u_{k+j,m-k} \right) \frac{\partial^{m-j-1-p-q} v}{\partial n^{m-j-1-p-q}} d\sigma \end{aligned} \tag{1.10}$$

From (1.9) with $p_1 = 2k$, $q_1 = m-k+l$, $p_2 = 0$ and $q_2 = m-k-l-1$ we have :

$$\int_{\partial\Omega} n_2 u_{2k,m-k+l} v_{0,m-k-l-1} = \sum_{0\le q\le m-k-l-1} (-1)^q C_{m-k-l-1}^q \int_{\partial\Omega} \frac{\partial^q}{\partial\tau^q} \left(n_1^q n_2^{m-k-l-q} u_{2k,m-k+l} \right) \frac{\partial^{m-k-l-1-q} v}{\partial n^{m-k-l-1-q}} d\sigma \tag{1.11}$$

By using (1.7), (1.10) and (1.11), we deduce :

$$\begin{aligned} &\int_\Omega u_{k,m-k} v_{k,m-k} \\ &= \sum_{j=0}^{k-1} (-1)^j \sum_{\substack{0\le p\le k-j-1\\ 0\le q\le m-k}} (-1)^q C_{k-j-1}^p C_{m-k}^q \int_{\partial\Omega} \frac{\partial^{p+q}}{\partial\tau^{p+q}} \left(n_1^{k-j-p+q} n_2^{m-k-q+p} u_{k+j,m-k} \right) \frac{\partial^{m-j-1-p-q} v}{\partial n^{m-j-1-p-q}} d\sigma \\ &+ (-1)^k \sum_{l=0}^{m-k-1} (-1)^l \sum_{0\le q\le m-k-l-1} (-1)^q C_{m-k-l-1}^q \int_{\partial\Omega} \frac{\partial^q}{\partial\tau^q} \left(n_1^q n_2^{m-k-l-q} u_{2k,m-k+l} \right) \frac{\partial^{m-k-l-1-q} v}{\partial n^{m-k-l-1-q}} d\sigma \\ &+ (-1)^m \int_\Omega u_{2k,2(m-k)} v \end{aligned} \tag{1.12}$$

From (1.6) and (1.12), we deduce the integration by parts formula of b_Ω :

$$\begin{aligned} b_\Omega(u,v) &= (-1)^m \sum_{k=0}^{m} C_m^k \int_\Omega u_{2k,2(m-k)} v \\ &+ \sum_{k=0}^{m} C_m^k \left\{ \sum_{j=0}^{k-1} (-1)^j \sum_{\substack{0\le p\le k-j-1\\ 0\le q\le m-k}} (-1)^q C_{k-j-1}^p C_{m-k}^q \int_{\partial\Omega} \frac{\partial^{p+q}}{\partial\tau^{p+q}} \left(n_1^{k-j-p+q} n_2^{m-k-q+p} u_{k+j,m-k} \right) \frac{\partial^{m-j-1-p-q} v}{\partial n^{m-j-1-p-q}} d\sigma \right. \\ &\left. + (-1)^k \sum_{l=0}^{m-k-1} (-1)^l \sum_{0\le q\le m-k-l-1} (-1)^q C_{m-k-l-1}^q \int_{\partial\Omega} \frac{\partial^q}{\partial\tau^q} \left(n_1^q n_2^{m-k-l-q} u_{2k,m-k+l} \right) \frac{\partial^{m-k-l-1-q} v}{\partial n^{m-k-l-1-q}} d\sigma \right\} \end{aligned} \tag{1.13}$$

We recognize the m-th Laplacian expression and we can rewrite (1.13) :

$$b_\Omega(u,v) = (-1)^m \int_\Omega \Delta^m u v + \sum_{i=1}^{2m-1} \int_{\partial\Omega} A_i(u) \frac{\partial^{2m-1-i} v}{\partial n^{2m-1-i}} d\sigma$$

with $A_i : H^m(\Omega) \longrightarrow H^{-i-1/2}(\partial\Omega)$ for $i \in [\![0, m-1]\!]$, the differential operator of order $i+m$ defined by :

$$A_i(u) = \sum_{k=0}^{m} C_m^k \sum_{j=0}^{k-1} (-1)^j \sum_{\substack{0\le p\le k-j-1\\ 0\le q\le m-k\\ p+q=i-j}} (-1)^q C_{k-j-1}^p C_{m-k}^q \frac{\partial^{p+q}}{\partial\tau^{p+q}} \left(n_1^{k-j-p+q} n_2^{m-k-q+p} u_{k+j,m-k} \right)$$

$$+ \sum_{0\le k+l\le i} C_m^k (-1)^k (-1)^l (-1)^{i-(k+l)} C_{m-k-l-1}^{i-(k+l)} \frac{\partial^{i-(k+l)}}{\partial\tau^{i-(k+l)}} \left(n_1^{i-(k+l)} n_2^{m-i} u_{2k,m-k+l} \right)$$

We can simplify the expression of A_i in the case of a straight crack. We assume that the crack writes as $\sigma = \{(s,0), -1 < s < 1\}$. For $x \in \sigma$, we have $n_1 = 0$, $n_2 = 1$ and $\partial_\tau = -\partial_{x_1}$. The term of the first sum over k is not null when $k-j-p+q = 0$ i.e. when $p-q = k-j$. By summing over subscripts $p, q \in \mathbb{N}$ such that $p+q = i-j$, we deduce that

$$p = \frac{1}{2}(k+i-2j), \ (a)$$
$$q = \frac{1}{2}(i-k), \ (b)$$

By using that $q \ge 0$ we deduce that $0 \le k \le i$ and (a) and (b) are full-filled if :

(a) i and k are even and $i \ge k$

(b) i et k are odd and $i \ge k$

Then the term of the second sum over k is not null when $i = k+l$ i.e. $l = i-k$. Hence we get the expressions of A_{2i} and A_{2i+1} :

(i) For $2i \le m-1$:

$$A_{2i} = \sum_{k=0}^{i} C_m^{2k} \sum_{j=0}^{2k-1} (-1)^{j+i-k} C_{2k-j-1}^{k+i-j} C_{m-2k}^{i-k} \frac{\partial^{2i-j}}{\partial\tau^{2i-j}} \left(u_{2k+j,m-2k} \right) + (-1)^{2i} \sum_{k=0}^{2i} C_m^k u_{2k,m-2k+2i}$$

$$= \sum_{k=0}^{i} (-1)^{i-k} C_m^{2k} \left(\sum_{j=0}^{2k-1} C_{2k-j-1}^{k+i-j} C_{m-2k}^{i-k} \right) u_{2(k+i),m-2k} + \sum_{k=0}^{2i} C_m^k u_{2k,m-2k+2i}$$

(ii) For $2i+1 \le m-1$:

$$A_{2i+1} = \sum_{k=0}^{i} (-1)^{i-k+1} C_m^{2k+1} \left(\sum_{j=0}^{2k-1} C_{2k-j}^{k+i-j+1} C_{m-2k-1}^{i-k} \right) u_{2(k+i+1),m-(2k+1)} - \sum_{k=0}^{2i+1} C_m^k u_{2k,m-2k+2i+1}$$

which ends the proof of this lemma.

□

2 Fundamental solution associated to the m-th Laplacian in 2D

Many solutions of linear differential problems can be expressed by using the fundamental solution associated to the differential operator. In this section we compute the fundamental solution associated to the m-th Laplacian Δ^m.

Théorème 2.1. *We denote by $E_m(x)$ the fundamental solution of the m-Laplacian which is defined by*

$$-\Delta^m E_m = \delta_0 \text{ in } D'(\mathbb{R}^2) \tag{2.1}$$

where δ_0 is the Dirac distribution. The expression of this solution is

$$E_m(x) = -\frac{1}{2^{2m-1}\pi((m-1)!)^2}|x|^{2(m-1)}log(|x|) \tag{2.2}$$

Proof. For $m = 1$, $E_1(x) = -\frac{1}{2\pi}log(|x|)$.
For $m = 2$, $E_2(x) = -\frac{1}{8\pi}|x|^2log(|x|)$.
We search $E_m(x)$ as follows

$$E_m(x) = a_m|x|^{p_m}log(|x|)$$

then ΔE_m needs to check

$$\Delta^{m-1}(\Delta E_m) = \delta_0 \tag{2.3}$$

We deduce that :

$$\Delta E_m = E_{m-1} + P_{2m-3}(x) = a_{m-1}|x|^{p_{m-1}}log(|x|) + P_{2m-3}(x)$$

where $P_{2m-3}(x)$ is a polynomial function of degree less or equal than $2m-3$. By standard computation :

$$\Delta\left(|x|^{p_m}log(|x|)\right) = |x|^{p_m-2}p_m^2log(|x|) + 2p_m|x|^{p_m-2}$$

Hence the following relations follow :

$$\begin{cases} p_{m-1} = p_m - 2 \text{ and } p_1 = 0 \\ a_{m-1} = a_m p_m^2 \text{ and } a_1 = -\dfrac{1}{2\pi} \\ p_m - 2 \leq 2m-3 \end{cases}$$

and we have

$$\begin{cases} p_m = 2(m-1) + p_1 \\ a_m = \dfrac{a_{m-1}}{p_m^2} \end{cases}$$

with $p_1 = 0$. First we get $p_m = 2(m-1)$ and then

$$a_m = -\frac{1}{2^{2m-1}\pi((m-1)!)^2}$$

We check that $|x|^{p_m-2} = |x|^{2(m-2)}$ is a polynomial function of degree $2m-4 \leq 2m-3$ and we deduce the expression (2.2). Reversely, if (2.3) is full-filled, then (2.1) is too. □

3 Statements of the problem and notations

Let $\Sigma \subset \mathbb{R}^2$ a regular open manifold of dimension 1 and $\widetilde{\Sigma}$ a closed and regular curve containing Σ (see Figure 2). We define the following functional spaces :

$$H_{00}^{1/2+i}(\Sigma) = \{u_{|\Sigma}, u \in H^{1/2+i}(\widetilde{\Sigma}), u_{|\widetilde{\Sigma}\setminus\overline{\Sigma}} = 0\}, \quad \forall i \in [\![0, m-1]\!]$$

We endow these spaces by the norms :

$$\|u_{|\Sigma}\|_{H_{00}^{1/2+i}(\Sigma)} = \|u\|_{H^{1/2+i}(\widetilde{\Sigma})}$$

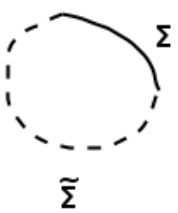
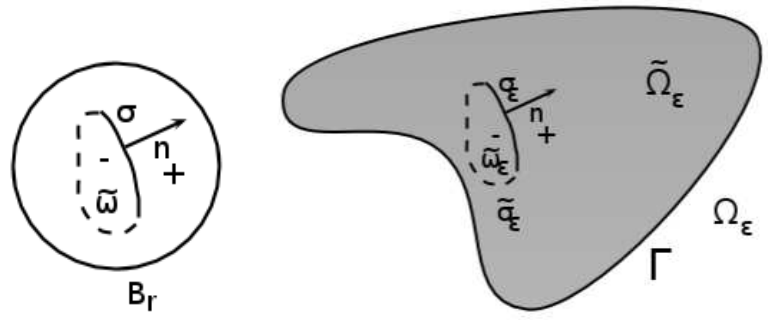


Figure 2: Σ and $\widetilde{\Sigma}$

Let $\sigma \subset \Omega$ a regular open manifold of dimension 1 containing the origin and of normal $\vec{n}$. We denote by $\vec{\tau}$ the vector such that $(\vec{n}, \vec{\tau})$ be orthonormal. $\partial\tau$ stands for the differentiation in the direction $\vec{\tau}$ and along σ. in this section we denote by $\Lambda = \mathbb{R}^2 \backslash \overline{\sigma}$ the exterior domain of the crack and we define the following weighted Sobolev space (see [13]) :

$$W^m(\Lambda) = \left\{ u, \frac{\nabla^k u}{(1+r^2)^{\frac{m-k}{2}} log(2+r^2)} \in L^2(\Lambda), \text{ for } k \in [\![0, m-1]\!], \nabla^m u \in L^2(\Lambda) \right\} \tag{3.1}$$

where $r = |x|$. $W^m(\Lambda)/\mathbb{P}_{m-1}$ stands for the quotient space of functions $W^m(\Lambda)$ defined up functions being $\mathbb{P}_{m-1}$. We assume that $\sigma_\epsilon = \{x, \frac{x}{\epsilon} \in \sigma\}$ does not touch $\partial\Omega$; thus we have $\partial\Omega_\epsilon = \sigma_\epsilon \cup \Gamma$. Let $\widetilde{\sigma}$ a closed and regular curve of same dimension of σ such that $\sigma \subset \widetilde{\sigma}$, and let $\widetilde{\omega}$ be the bounded domain of $\mathbb{R}^2$ such that $\partial\widetilde{\omega} = \widetilde{\sigma}$; we denote by $\widetilde{\omega_\epsilon} = \{x, \frac{x}{\epsilon} \in \widetilde{\omega}\}$, $\widetilde{\Omega_\epsilon} = \Omega \backslash \overline{\widetilde{\omega_\epsilon}}$ and we choose $r > 0$ and ϵ small enough such that $\widetilde{\omega} \subsetneq B_r \subset \frac{\Omega}{\epsilon}$ (see Figure 3).



(a) $\widetilde{\sigma} \supset \sigma$ and $B_r \supsetneq \widetilde{\omega}$ (b) Cracked domain

Figure 3: Cracked domain and extension of the crack by a closed curve

For $v \in H^m(\Omega_\epsilon)$ and $u \in H^{2m}(\Omega_\epsilon)$, by using the integration by parts formula given in Theorem 1.1 on $\Omega \backslash \overline{\widetilde{\omega_\epsilon}} \cup \widetilde{\omega_\epsilon}$, we get

$$\int_{\Omega_\epsilon} \left((-1)^m \Delta^m u + u\right) v = a_\epsilon(u, v) + \int_{\sigma_\epsilon} \sum_{i=0}^{m-1} A_i(u) \left[\frac{\partial^{m-1-i} v}{\partial n^{m-1-i}}\right] \tag{3.2}$$

where $a_\epsilon(u, v)$ is given in (1.4) and $\left[\frac{\partial^k v}{\partial n^k}\right] = \left(\frac{\partial^k v}{\partial n^k}\right)^+ - \left(\frac{\partial^k v}{\partial n^k}\right)^-$ denotes the jump of $\frac{\partial^k v}{\partial n^k}$ across σ_ϵ, by using notations described in Figure 3. From (1.5) and (3.2), and by assuming that $u_\epsilon \in H^{2m}(\Omega_\epsilon)$, u_ϵ is given by :

$$(\mathcal{P}_\epsilon) \begin{cases} (-1)^m \Delta^m u_\epsilon + u_\epsilon = f, \text{ in } \Omega_\epsilon \\ A_i(u_\epsilon) = 0, \text{ on } \sigma_\epsilon \cup \Gamma, \forall i \in [\![0, m-1]\!] \end{cases} \tag{3.3}$$

where $f \in L^2(\Omega_\epsilon)$.

We introduce the cost function $J(\Omega, u) : H^m(\Omega) \longrightarrow \mathbb{R}$ full-filling

Hypothèses 1. • *There exists a continuous linear map* $L_\epsilon : H^m(\Omega_\epsilon) \to \mathbb{R}$, $\delta J : \Omega \times S^1 \to \mathbb{R}$ *and* $\epsilon_0 > 0$ *such that* $\forall \epsilon_0 \geq \epsilon > 0$

$$J(\Omega_\epsilon, u_\epsilon) - J(\Omega, u_0) = L_\epsilon(u_\epsilon - u_0) + \epsilon^2 \delta J(x_0, \vec{n}) + o(\epsilon^2)$$

and we assume that $L_\epsilon(u)$ writes

$$L_\epsilon(u) = \int_{\Omega_\epsilon} l_0 u + \sum_{1\le i\le m-1} \int_{\sigma_\epsilon} B_i \left[\frac{\partial^{m-1-i} u}{\partial n^{m-1-i}}\right] + \sum_{0\le i\le m-1} \int_\Gamma D_i \frac{\partial^{m-1-i} u}{\partial n^{m-1-i}} \tag{3.4}$$

with $\|l_0\|_{0,\Omega_\epsilon} \le C$, $D_i \in H^{-i-1/2}(\Gamma)$, $B_i \in H_{00}^{i+1/2}(\sigma_\epsilon)'$ and $\|B_i(\epsilon x)\|_{H_{00}^{i+1/2}(\sigma)'} \le C$ where C is a constant not depending on ϵ.

To shorten notations, we denote $J_\epsilon(u) = J(\Omega_\epsilon, u)$ and δJ instead of $\delta J(x_0, \vec{n})$. In the sequel, to simplify we assume that the crack σ is straight and we assume that $\sigma = \{(s,0), -1 < s < 1\}$ (we place us in the local coordinate system associated to the crack). We compute the topological gradient by evaluating the leading term with respect to ϵ of the difference $J_\epsilon(u_\epsilon) - J_0(u_0)$ when $\epsilon \to 0$. By using the equations filled by u_ϵ and u_0 and Hypotheses 1, we have :

$$J_\epsilon(u_\epsilon) - J_0(u_0) = L_\epsilon(u_\epsilon - u_0) + \epsilon^2 \delta J + o(\epsilon^2) \tag{3.5}$$

where $L_\epsilon(u)$ is given by (3.4) and we set

$$\mathcal{J}_\epsilon = \epsilon^2 \delta J + o(\epsilon^2) \tag{3.6}$$

To compute (3.5), we introduce $v_\epsilon \in H^m(\Omega_\epsilon)$ solution of the adjoint problem :

$$a_\epsilon(u, v_\epsilon) = -L_\epsilon(u),\ \forall u \in H(\Omega_\epsilon) \tag{3.7}$$

From (3.7), (3.6) and (3.2), then (3.5) writes

$$\begin{aligned}
J_\epsilon(u_\epsilon) - J_0(u_0) &= -a_\epsilon(u_\epsilon - u_0, v_\epsilon) + \mathcal{J}_\epsilon \\
&= -l_\epsilon(v_\epsilon) + a_\epsilon(u_0, v_\epsilon) + \mathcal{J}_\epsilon \\
&= -\int_{\Omega_\epsilon} f v_\epsilon + \int_{\Omega_\epsilon} \left((-1)^m \Delta^m u_0 + u_0\right) v_\epsilon - \int_{\sigma_\epsilon} \sum_{i=0}^{m-1} A_i(u_0) \left[\frac{\partial^{m-1-i} v_\epsilon}{\partial n^{m-1-i}}\right] + \mathcal{J}_\epsilon \\
&= -\int_{\sigma_\epsilon} \sum_{i=0}^{m-1} A_i(u_0) \left[\frac{\partial^{m-1-i} v_\epsilon}{\partial n^{m-1-i}}\right] + \mathcal{J}_\epsilon
\end{aligned}$$

By setting $w_\epsilon = v_\epsilon - v_0$ with v_ϵ and v_0 given by (3.7) for $\epsilon > 0$ and $\epsilon = 0$; we rewrite $J_\epsilon(u_\epsilon) - J_0(u_0)$ in function of w_ϵ :

$$J_\epsilon(u_\epsilon) - J_0(u_0) = -\int_{\sigma_\epsilon} \sum_{i=m}^{2m-1} A_i(u_0) \left[\frac{\partial^{2m-1-i} w_\epsilon}{\partial n^{2m-1-i}}\right] + \mathcal{J}_\epsilon \tag{3.8}$$

Then, changing of variables and subscripts give

$$\begin{aligned}
J_\epsilon(u_\epsilon) - J_0(u_0) &= -\sum_{i=0}^{m-1} \epsilon \int_\sigma A_i(u_0)(\epsilon X) \left[\frac{\partial^{m-1-i} w_\epsilon}{\partial n^{m-1-i}}(\epsilon X)\right] d\sigma + \mathcal{J}_\epsilon \\
&= -\sum_{i=0}^{m-1} \epsilon \int_\sigma A_i(u_0)(\epsilon X) \frac{1}{\epsilon^{m-1-i}} \left[\frac{\partial^{m-1-i}\left(w_\epsilon(\epsilon X)\right)}{\partial n^{m-1-i}}\right] d\sigma + \mathcal{J}_\epsilon \\
&= -\sum_{k=0}^{m-1} \epsilon^{1-k} \int_\sigma A_{m-1-k}(u_0)(\epsilon X) \left[\frac{\partial^k}{\partial n^k}\left(w_\epsilon(\epsilon X)\right)\right] d\sigma + \mathcal{J}_\epsilon \\
&= -\sum_{k=0}^{m-1} \mathcal{I}_k + \mathcal{J}_\epsilon
\end{aligned} \tag{3.9}$$

where $\mathcal{I}_k$ for $k \in [\![0, m-1]\!]$ are defined by :

$$\mathcal{I}_k = \epsilon^{1-k} \int_\sigma A_{m-1-k}(u_0)(\epsilon X) \left[\frac{\partial^k}{\partial n^k} \left(w_\epsilon(\epsilon X) \right) \right] d\sigma \tag{3.10}$$

Now, we need to establish the asymptotic expansion of w_ϵ in $H^m(\Omega_\epsilon)$ norm. To do this, first we find the leading terms in w_ϵ which need to be compensated to have an asymptotic expansion in $o(\epsilon)$ in $H^m(\Omega_\epsilon)$ norm (see section 5).

4 Estimations of $A_i(u_0)(\epsilon X)$ for $X \in \sigma$

The following lemma gives the expansion with respect to ϵ at 0 of $A_i(u_0)(\epsilon X)$ for $X \in \sigma$.

Lemme 4.1. *Let v_0 solution of (3.7) with $\epsilon = 0$. In the case of a straight crack, by assuming that v_0 is regular (or equivalently that f is regular) we have the following estimations :*

$$A_0(v_0)(\epsilon X) = g_0(X) + O(\epsilon) \tag{4.1a}$$

$$A_i(v_0)(\epsilon X) = g_i(X) + O(1), \ \forall i \in [\![1, m-1]\!] \tag{4.1b}$$

with

$$\begin{cases} g_0(X) = \dfrac{\partial^m v_0}{\partial x_2^m}(0) \\ g_i(X) = 0, \ \textit{for } 1 \leq i \leq m-1 \end{cases} \tag{4.2}$$

Proof. With the expression of the A_i given in Theorem 1.1 for $i \in [\![1, m-1]\!]$ and for a straight crack, and by a Taylor expansion at 0 of the functions $\frac{\partial^{k+l} v_0}{\partial x_1^k \partial x_2^l}$ we get (4.1b). By using the expression of A_0 (see Remark 9.31) we deduce that :

$$A_0(v_0)(\epsilon X) = \frac{\partial^m v_0}{\partial x_2^m}(\epsilon X)$$

We conclude with a Taylor expansion of $\frac{\partial^m v_0}{\partial x_2^m}(\epsilon X)$ at 0. □

5 Asymptotic expansion of w_ϵ in $H^m(\Omega_\epsilon)$ norm

In this section we do the asymptotic expansion of w_ϵ with respect to ϵ in the sense of the $H^m(\Omega_\epsilon)$ norm. We recall that $w_\epsilon = v_\epsilon - v_0$ is solution of :

$$(\mathcal{Q}_\epsilon^c) \begin{cases} (-1)^m \Delta^m w_\epsilon + w_\epsilon = 0, \text{ in } \Omega_\epsilon \\ A_0(w_\epsilon) = -A_0(v_0), \text{ on } \sigma_\epsilon \\ A_i(w_\epsilon) = -A_i(v_0) - B_i, \text{ on } \sigma_\epsilon, \ \forall i \in [\![1, m-1]\!] \\ A_i(w_\epsilon) = 0, \text{ on } \Gamma, \ \forall i \in [\![0, m-1]\!] \end{cases} \tag{5.1}$$

To estimate w_ϵ we introduce the solution of the exterior problem $R \in W^m(\Lambda)/\mathbb{P}_{m-1}$:

$$(\mathcal{R}_{ext}^c) \begin{cases} \Delta^m R = 0, \ \text{in } \Lambda \\ A_i(R) = g_i, \ \text{on } \sigma, \ \forall i \in [\![0, m-1]\!] \end{cases} \tag{5.2}$$

where $\forall i \in [\![0, m-1]\!]$, $g_i \in H_{00}^{1/2+i}(\sigma)'$ is given by (4.2). Thanks to Theorem 9.6 given in Appendix, we deduce that the problem $(\mathcal{R}_{ext}^c)$ has a unique solution $R \in W^m(\Lambda)/\mathbb{P}_{m-1}$ which writes as follows :

$$R(x) = \sum_{i=0}^{m-1} \oint_\sigma \lambda_i(y) A_{i,y}(E(x-y)) d\sigma_y$$

where $\oint$ denotes the Cauchy principal value. Moreover we have :

$$(-1)^{m+1} \left[\frac{\partial^{m-1-i} R}{\partial_n^{m-1-i}} \right] = \lambda_i \quad \forall i \in [\![0, m-1]\!] \tag{5.3}$$

where

$$\begin{cases} \lambda_0(s) = \dfrac{(-1)^{m+1} 2^{2m-1}}{(2m-1) C_{2(m-1)}^{m-1}} \beta \sqrt{1-s^2} \quad \forall (s,0) \in \sigma \\ \lambda_i(s) = 0 \quad \forall i \in [\![1, m-1]\!] \quad \forall (s,0) \in \sigma \end{cases} \tag{5.4}$$

with $\beta = \frac{\partial^m v_0}{\partial x_2^m}(0)$. Thanks to Lemma 9.8 (see Appendix), we get

$$w_\epsilon = \epsilon^2 R\left(\frac{x}{\epsilon}\right) + e_\epsilon \text{ with } \|e_\epsilon\|_{H^m(\Omega_\epsilon)} = O(\phi_m(\epsilon)) \tag{5.5}$$

where

$$\phi_m(\epsilon) = \begin{cases} -\epsilon^2 log(\epsilon) & \text{for } m \geq 2 \\ \epsilon^2 \sqrt{-log(\epsilon)} & \text{for } m = 1 \end{cases}$$

In the sequel, we are showing that $\mathcal{I}_k \sim o(\epsilon^2)$ for $k \in [\![0, m-2]\!]$ and $\mathcal{I}_{m-1} \sim O(\epsilon^2)$.

6 Estimation of $\mathcal{I}_k$ for $k \in [\![0, m-1]\!]$

The following lemma give an estimation in $o(\epsilon^2)$ of the quantity $\mathcal{I}_k$ for $k \in [\![0, m-1]\!]$.

Lemme 6.1. *Let $\mathcal{I}_k$ defined by (3.10) for $k \in [\![0, m-1]\!]$. We have*

$$\mathcal{I}_k = o(\epsilon^2), \ \forall k \in [\![0, m-2]\!] \tag{6.1a}$$

$$\mathcal{I}_{m-1} = \epsilon^2 \frac{\partial^m u_0}{\partial x_2^m}(0)(-1)^{m+1} \int_\sigma \lambda_m(y) d\sigma + o(\epsilon^2) \tag{6.1b}$$

Proof. Let $k \in [\![0, m-2]\!]$, thanks to Lemma 9.3 and Lemma 4.1 applied to u_0, we get :

$$\begin{aligned} \mathcal{I}_k &= \epsilon^{1-k} \int_\sigma A_{m-1-k}(u_0)(\epsilon X) \left[\frac{\partial^k}{\partial n^k} \left(w_\epsilon(\epsilon X) \right) \right] d\sigma \\ &\leq C \epsilon^{1-k} |w_\epsilon(\epsilon X)|_{m, B_r \setminus \overline{\sigma}} \end{aligned}$$

By using a change of variable and Lemma 9.8 we deduce :

$$\mathcal{I}_k \leq C \epsilon^{m-k} |w_\epsilon|_{m, \Omega_\epsilon} \leq C \epsilon^{m-k+1} = o(\epsilon^2)$$

Now, let us consider $\mathcal{I}_{m-1}$; thanks to Lemma 9.8, Lemma 4.1 applied to u_0, and jump relations (5.3), we get :

$$\begin{aligned} \mathcal{I}_{m-1} &= \epsilon^{2-m} \int_\sigma A_0(u_0(\epsilon X)) \left[\frac{\partial^{m-1}(w_\epsilon(\epsilon X))}{\partial n^{m-1}} \right] d\sigma \\ &= \epsilon^{2-m} \frac{\partial^m u_0}{\partial x_2^m}(0) \int_\sigma \left[\frac{\partial^{m-1}(\epsilon^m R(X))}{\partial n^{m-1}} \right] d\sigma + \mathcal{E}_1 + \mathcal{E}_2 \\ &= \epsilon^2 \frac{\partial^m u_0}{\partial x_2^m}(0)(-1)^{m+1} \int_\sigma \lambda_m(y) d\sigma + \mathcal{E}_1 + \mathcal{E}_2 \end{aligned}$$

where

$$\mathcal{E}_1 = \epsilon^{2-m} \int_\sigma \left(A_0(u_0(\epsilon X)) - \frac{\partial^m u_0}{\partial x_2^m}(0) \right) \left[\frac{\partial^{m-1}(w_\epsilon(\epsilon X))}{\partial n^{m-1}} \right] d\sigma$$

$$\mathcal{E}_2 = \epsilon^{2-m} \frac{\partial^m u_0}{\partial x_2^m}(0) \int_\sigma \left[\frac{\partial^{m-1}(e_\epsilon(\epsilon X))}{\partial n^{m-1}} \right] d\sigma$$

We will show that $\mathcal{E}_1$ and $\mathcal{E}_2$ are $\sim o(\epsilon^2)$. By using Lemma 9.3, a change of variable, Lemma 4.1, a change of variable again and Lemma 9.8 we get

$$\mathcal{E}_1 \leq \epsilon^{2-m} \left\| A_0(u_0(\epsilon X)) - \frac{\partial^m u_0}{\partial x_2^m}(0) \right\|_{H_{00}^{1/2}(\sigma)'} |w_\epsilon(\epsilon X))|_{m,B_r\setminus\overline{\sigma}} \leq C\epsilon^2 |w_\epsilon|_{m,\Omega_\epsilon} \leq C\epsilon^3$$

Similarly we get :

$$\mathcal{E}_2 \leq \epsilon^{2-m} |e_\epsilon(\epsilon X)|_{m,B_r\setminus\overline{\sigma}} \leq C\epsilon |e_\epsilon|_{m,\Omega_\epsilon} \leq C\epsilon\phi_m(\epsilon)$$

with $\phi_m(\epsilon) = o(\epsilon)$. Hence, the following estimation holds :

$$\mathcal{I}_{m-1} = \epsilon^2 \frac{\partial^m u_0}{\partial x_2^m}(0)(-1)^{m+1} \int_\sigma \lambda_m(y) d\sigma + o(\epsilon^2)$$

□

7 Computation of the topological gradient

From (3.9) and by using estimations (6.1a) and (6.1b) we get :

$$J_\epsilon(u_\epsilon) - J_0(u_0) = \epsilon^2 \frac{\partial^m u_0}{\partial x_2^m}(0)(-1)^m \int_\sigma \lambda_m(y) d\sigma + \mathcal{J}_\epsilon + o(\epsilon^2)$$

By using the expression of λ_m (5.4) and the definition of $\mathcal{J}_\epsilon$ given in (3.6) we have :

$$\begin{aligned} J_\epsilon(u_\epsilon) - J_0(u_0) &= \epsilon^2 \frac{\partial^m u_0}{\partial x_2^m}(0)(-1)^m \int_{-1}^1 \frac{(-1)^{m+1} 2^{2m-1}}{(2m-1)C_{2(m-1)}^{m-1}} \frac{\partial^m v_0}{\partial x_2^m}(0) \sqrt{1-s^2} ds + \epsilon^2 \delta J + o(\epsilon^2) \\ &= -\epsilon^2 \frac{\partial^m u_0}{\partial x_2^m}(0) \frac{\partial^m v_0}{\partial x_2^m}(0) \frac{2^{2m-1}}{(2m-1)C_{2(m-1)}^{m-1}} \frac{\pi}{2} + \epsilon^2 \delta J + o(\epsilon^2) \\ &= -\epsilon^2 \pi \frac{2^{2(m-1)}}{(2m-1)C_{2(m-1)}^{m-1}} \frac{\partial^m u_0}{\partial x_2^m}(0) \frac{\partial^m v_0}{\partial x_2^m}(0) + \epsilon^2 \delta J + o(\epsilon^2) \end{aligned}$$

Therefore, the topological gradient written in the local coordinate system of the crack is

$$\mathcal{I}(0) = -\pi \frac{2^{2(m-1)}}{(2m-1)C_{2(m-1)}^{m-1}} \frac{\partial^m u_0}{\partial x_2^m}(0) \frac{\partial^m v_0}{\partial x_2^m}(0) + \delta J \tag{7.1}$$

8 Conclusion : general expression and some examples of cost functions

From (7.1), by a changing of coordinated system we deduce easily the topological gradient associated to a cost function $J(\Omega, u)$ full-filling Hypotheses 1 and to the PDE (3.3) for a domain perturbed by a straight crack of normal $\vec{n}$ and of center $x_0 \in \Omega$:

$$\mathcal{I}(x_0, \vec{n}) = -\pi \frac{2^{2(m-1)}}{(2m-1)C_{2(m-1)}^{m-1}} \nabla^m u_0(x_0)(\vec{n}, ..., \vec{n}) \nabla^m v_0(x_0)(\vec{n}, ..., \vec{n}) + \delta J(x_0, \vec{n}) \tag{8.1}$$

Now we give some cost functions examples and we compute the function $\delta J(x_0, \vec{n})$ in these cases. To detect edges or fine structures we can use

$$J_\epsilon(u) = \int_{\Omega_\epsilon} |\nabla^p u|^2 \tag{8.2}$$

with $p \in [\![1, m]\!]$.

(i) For $p = m$, by using the equation checked by u_ϵ and Lemma 9.8 applied to $u_\epsilon - u_0$, we get

$$J_\epsilon(u_\epsilon) - J_0(u_0) = L_\epsilon(u_\epsilon - u_0) + \mathcal{J}_\epsilon \tag{8.3}$$

with

$$L_\epsilon(u) = \int_{\Omega_\epsilon} (f - 2u_0)u$$
$$\mathcal{J}_\epsilon = -\|u_\epsilon - u_0\|^2_{0,\Omega_\epsilon} \leq C\phi_m(\epsilon)^2 = o(\epsilon^2)$$

We deduce that $\delta J = 0$.

(ii) For $p \in [\![1, m-1]\!]$, an integration by parts (Theorem 1.1) and Lemma 9.19 applied to $u_\epsilon - u_0$ lead to (8.3) with

$$L_\epsilon(u) = 2\int_{\Omega_\epsilon} \nabla^p u_0 . \nabla^p u$$
$$\mathcal{J}_\epsilon = |u_\epsilon - u_0|^2_{p,\Omega_\epsilon} = o(\epsilon^2)$$

Thanks to Theorem 1.1, we check that Hypotheses 1 are full-filled :

$$L_\epsilon(u) = \int_{\Omega_\epsilon} 2(-1)^p \Delta^p u_0 u + \sum_{i=0}^{p-2} \int_{\sigma_\epsilon} -2A^p_{p-1-i}(u_0) \left[\frac{\partial^i u}{\partial n^i}\right] + \int_\Gamma 2A^p_{p-1-i}(u_0) \frac{\partial^i u}{\partial n^i}$$

where the operators $A^p_i(u)$ for $i \in [\![0, p-1]\!]$ stand for the p Neumann conditions associated to the minimization in $H^p(\Omega)$ of $I^p_\Omega(u) = \int_\Omega |\nabla^p u|^2$. For contours an fine structures detection, we define the topological indicator at x_0 by

$$\mathcal{I}(x_0) = \max_{|\vec{n}|=1} |\mathcal{I}(x_0, \vec{n})|$$

$\mathcal{I}(x_0)$ is sensitive to discontinuities which make (8.2) very high (see Table 1).

Remarque 8.1. *(i) We check that for $p = m$, $m = 1$ and $m = 2$ we retrieve the topological gradient expressions given respectively in [3] and [5].*

(ii) In imaging detection the most interesting cost function is $J(\Omega, u) = \int_\Omega |\nabla^m u|^2$. Indeed, the more the order of the PDE is high the more numerical instabilities occurs, hence we should take the PDE of minimal order which lead to enough regular solutions to guarantee that $J(\Omega, u_\Omega)$ exists.

9 Annexe

In the following, we keep the notations and conventions described in Figure 3 and we choose $r > 0$ and ϵ small enough such that $\widetilde{\omega} \subsetneq B_r \subset \frac{\Omega}{\epsilon}$ (see Figure 3). We recall that we denote by B_r for $r > 0$ the ball of center 0 and of radius r, and $B = B_1$ is the unit ball. For a bounded domain $\omega \subset \mathbb{R}^2$, ω' stands for the unbounded domain $\mathbb{R}^2 \backslash \overline{\omega}$. Finally for a domain ω we denote by $\mathcal{D}(\omega)$ the set of functions $C^\infty(\omega)$ with compact support in ω. The following lemma is a consequence of Hardy (see [13]) and Deny-Lions (see [7] Lemma 5.2) inequalities.

Lemme 9.1 (Generalisation of Deny-Lions inequality)**.** *Let $\omega \subset B$, a regular subset of $\mathbb{R}^2$. We denote by $\mathcal{O} = \mathbb{R}^2 \backslash \overline{\omega}$ the exterior domain to ω. Let $u \in W^m(\mathcal{O})$, we have the following inequality*

$$\|u\|_{W^m(\mathcal{O})/\mathbb{P}_{m-1}} \leq C|u|_{m,\Lambda}$$

where C is a constant which depends on $\mathcal{O}$ and where $W^m(\mathcal{O})$ is a functional set defined in (3.1).

Proof. Let $\varphi \in C^m([0, +\infty[)$, the cut-off function :

$$\begin{cases} \varphi = 0, & \text{for } 0 \leq t \leq 1 \\ 0 \leq \varphi \leq 1 & \text{for } 1 \leq t \leq 2 \\ \varphi = 1 & \text{for } t \geq 2 \end{cases}$$

Let $\psi(x) = \varphi(|x|)$, then $u\psi \in W_0^m(B')$. On $W_0^m(B')$, thanks to Hardy inequality [13], we have

$$\|u\psi\|_{W_0^m(B')} \leq C|\psi u|_{m,B'}$$

By definition of ψ :

$$\|u\|_{W^m(B_2')} \leq \|\psi u\|_{W_0^m(B')}$$

We deduce that :

$$\|u\|_{W^m(B_2')} \leq C|\psi u|_{m,B'} \leq C|u|_{m,B_2'} + C\|u\|_{W^m(B_2\backslash\overline{B})} \leq C|u|_{m,B_2'} + C\|u\|_{W^m(B_2\backslash\overline{\omega})} \tag{9.1}$$

Then by using the definition of $W^m(B_2\backslash\overline{\omega})$ and by bounding weights, we get the equivalence between the $W^m(B_2\backslash\overline{\omega})$ and the $H^m(B_2\backslash\overline{\omega})$ norms. By considering the quotient space and thanks to Deny-Lions inequality, we get :

$$\|u\|_{W^m(B_2\backslash\overline{\omega})/\mathbb{P}_{m-1}} \leq C|u|_{m,B_2\backslash\overline{\omega}} \tag{9.2}$$

From (9.1) and (9.2), we have

$$\|u\|_{W^m(\mathcal{O})/\mathbb{P}_{m-1}} \leq C|u|_{m,\mathcal{O}}$$

which ends the proof of this lemma. □

Théorème (Gagliardo-Nirenberg inequality [1, 14])**.** *Let $m \geq 1$, the map $u \mapsto \left(\|u\|_{0,\Omega}^2 + |u|_{m,\Omega}^2\right)^{\frac{1}{2}}$ from $H^m(\Omega)$ to $\mathbb{R}$ is a norm on $H^m(\Omega)$ and more precisely we have*

$$\|u\|_{m,\Omega} \leq C(m,\Omega)\left(\|u\|_{0,\Omega}^2 + |u|_{m,\Omega}^2\right)^{1/2}$$

where $C(\Omega, m)$ is a constant depending on m and Ω.

Remarque 9.2. *The constant appearing in the Gagliardo-Nirenberg inequality depends on a constants which state the interior cone property (see [1] p 66 and pp 75-79). Let $\Omega_\epsilon = \Omega\backslash\overline{x_0 + \epsilon\omega}$, where ω is either a regular open manifold or a regular sub-domain of $\mathbb{R}^2$ and ϵ small enough to the perturbation $\{x_0 + \epsilon\omega\}$ does not contact the boundary of Ω. Then we can show that the Gagliardo-Nirenberg constant is bounded from ϵ when $\epsilon \to 0$.*

Lemme 9.3. *Let $k \in [\![0, m-1]\!]$, $g_k \in \left(H_{00}^{1/2+(m-1)-k}(\sigma)\right)'$ and $u \in H^m(B_r\backslash\overline{\sigma})$, we have*

$$\left|\int_\sigma g_k \left[\frac{\partial^k u}{\partial n^k}\right]\right| d\sigma \leq \|g_k\|_{H_{00}^{1/2+m-1-k}(\sigma)'} |u|_{m,B_r\backslash\overline{\sigma}}$$

where the spaces $H_{00}^{1/2+i}(\sigma)$ for $i \in \mathbb{N}$ are defined in section 3 and where we denote $H_{00}^{1/2+i}(\sigma)'$ the dual spaces.

Proof. By using the definition of the norm $\|.\|_{H_{00}^{1/2+m-1-k}(\sigma)}$ and by splitting the jump $\left[\frac{\partial^k u}{\partial n^k}\right]$; and finally by using the trace Theorem on $B_r\backslash\overline{\widetilde{\omega}}$ and on $\widetilde{\omega}$, we deduce that

$$\begin{aligned}
\int_\sigma g_k \left[\frac{\partial^k u}{\partial n^k}\right] &\leq \|g_k\|_{H_{00}^{1/2+(m-1)-k}(\sigma)'} \left\|\left[\frac{\partial^k u}{\partial n^k}\right]\right\|_{H_{00}^{1/2+(m-1)-k}(\sigma)} \\
&= \|g_k\|_{H_{00}^{1/2+(m-1)-k}(\sigma)'} \left\|\left[\frac{\partial^k u}{\partial n^k}\right]\right\|_{1/2+(m-1)-k,\widetilde{\sigma}} \\
&\leq C\|g_k\|_{H_{00}^{1/2+(m-1)-k}(\sigma)'} \left(\left\|\frac{\partial^k u}{\partial n^k}^- + \psi\right\|_{1/2+m-1-k,\widetilde{\sigma}} + \left\|\frac{\partial^k u}{\partial n^k}^+ + \psi\right\|_{1/2+m-1-k,\widetilde{\sigma}}\right) \\
&\leq C\|g_k\|_{H_{00}^{1/2+(m-1)-k}(\sigma)'} \left(\|u+\psi\|_{m,\widetilde{\omega}} + \|u+\psi\|_{m,B_r\backslash\overline{\widetilde{\omega}}}\right) \\
&= C\|g_k\|_{H_{00}^{1/2+(m-1)-k}(\sigma)'} \|u+\psi\|_{m,B_r\backslash\overline{\sigma}}
\end{aligned}$$

where ψ is a regular function defined on $\mathbb{R}^2$. Particularly, if we take $\psi \in \mathbb{P}_{m-1}$, by using the Deny-Lions inequality, we get

$$\begin{aligned}
\int_\sigma g_k \left[\frac{\partial^k u}{\partial n^k}\right] &\leq \|g_k\|_{H_{00}^{1/2+(m-1)-k}(\sigma)'} \|u\|_{H^m(B_r\backslash\overline{\sigma})/\mathbb{P}_{m-1}} \\
&\leq C\|g_k\|_{H_{00}^{1/2+(m-1)-k}(\sigma)'} |u|_{m,B_r\backslash\overline{\sigma}}
\end{aligned}$$

□

Lemme 9.4. *Let $u \in H^m(B_r\backslash\overline{\sigma})$ such that $\Delta^m u \in L^2(B_r\backslash\overline{\sigma})$ and $q_i \in H_{00}^{1/2+i}(\sigma)$ où $i \in [\![0, m-1]\!]$. Then we have the inequality :*

$$\sum_{i=0}^{m-1} \int_\sigma q_i A_i(u) \leq \sum_{i=0}^{m-1} \|q_i\|_{H_{00}^{1/2+i}(\sigma)} \left(|u|_{B_r\backslash\overline{\sigma}} + \|\Delta^m u\|_{B_r\backslash\overline{\sigma}}\right)$$

Proof. We extend the q_i by 0 on $\widetilde{\sigma}\backslash\overline{\sigma}$. We denote by $\widetilde{q_i} \in H^{1/2+i}(\widetilde{\sigma})$ these extensions. Let Q a continuous extension of $(\widetilde{q_0}, \widetilde{q_1}, ..., \widetilde{q_{m-1}})$ in $\widetilde{\omega}$. By integration by parts (see Theorem 1.1), we have :

$$\widetilde{b}(Q,u) = (-1)^m \int_{\widetilde{\omega}} \Delta^m u Q + \int_{\widetilde{\sigma}} \sum_{i=0}^{m-1} A_i(u) q_i, \ \forall u \in H^m(\widetilde{\omega})$$

where

$$\widetilde{b}(u,v) = \sum_{k=0}^{m} C_m^k \int_{\widetilde{\omega}} \frac{\partial^m u}{\partial x_1^k \partial x_2^{m-k}} \frac{\partial^m v}{\partial x_1^k \partial x_2^{m-k}}, \ \forall u, v \in H^m(\widetilde{\omega})$$

Moreover, we have :

$$\begin{aligned}
\left|\int_{\widetilde{\sigma}} \sum_{i=0}^{m-1} A_i(u) q_i\right| &\leq |b(Q,u)| + \left|\int_{\widetilde{\omega}} \Delta^m u Q\right| \\
&\leq C|u|_{2,\widetilde{\omega}} |Q|_{2,\widetilde{\omega}} + \|\Delta^m u\|_{0,\widetilde{\omega}} \|Q\|_{0,\widetilde{\omega}} \\
&\leq C\left(|u|_{2,\widetilde{\omega}} + \|\Delta^m u\|_{0,\widetilde{\omega}}\right) \|Q\|_{2,\widetilde{\omega}}
\end{aligned}$$

by using the continuity of the extension, the inclusion $B_r\backslash\overline{\sigma} \supset \widetilde{\omega}$ (see Figure 3), and by using the definitions of the $H_{00}^{1/2+i}(\sigma)$ norms for $i \in [\![0, m-1]\!]$, we get the result. □

Lemme 9.5. *Let $E(x)$ given in (2.2). Let $s \in \mathbb{N}^2$, we have*

$$\frac{\partial^{|s|}E}{\partial x^s} = \begin{cases} F_s(x) + G_s log(|x|), \ for\ 0 \leq |s| \leq 2m-2 \\ H_s, \ for\ |s| \geq 2m-1 \end{cases}$$

where F_s, G_s and H_s are rational functions homogeneous of degree $2(m-1)-|s|$ and we where denote for $s=(s_1,s_2)$, $\partial x^s = \partial x_1^{s_1}\partial x_2^{s_2}$ and $|s| = s_1+s_2$.

Proof. E writes as the product $E(x) = C_m g(x)h(x)$ where

$$g(x) = (x_1^2+x_2^2)^{m-1}$$
$$h(x) = log(x_1^2+x_2^2)$$

and C_m is a constant depending on m. $g(x)$ is a polynomial homogeneous function of degree $2(m-1)$, hence $\frac{\partial^{|\alpha|}g}{\partial x^\alpha}$ for $|\alpha| \leq 2(m-1)$ is a polynomial homogeneous function of degree $2(m-1)-|\alpha|$. Similarly, $\frac{\partial h}{\partial x_i}$ is homogeneous of degree -1, then $\frac{\partial^{|\alpha|}h}{\partial x^\alpha}$ is homogeneous of degree $-|\alpha|$. Thanks to Leibniz formula, we get

$$\frac{\partial^{|s|}(gh)}{\partial x^s} \sum_{0\leq\alpha\leq s} C_s^\alpha \frac{\partial^{|\alpha|}g}{\partial x^\alpha}\frac{\partial^{|s-\alpha|}h}{\partial x^{s-\alpha}}$$

where $C_s^\alpha = \prod C_{s_i}^{\alpha_i}$ where $\alpha \leq s \iff \alpha_i \leq s_i,\ \forall i \in \{1,2\}$. By splitting the sum in two parts we get :

$$\frac{\partial^{|s|}(gh)}{\partial x^s} \sum_{0\leq\alpha<s} C_s^\alpha \underbrace{\frac{\partial^{|\alpha|}g}{\partial x^\alpha}\frac{\partial^{|s-\alpha|}h}{\partial x^{s-\alpha}}}_{\text{homogeneous of degree } 2(m-1)-|s|} + \underbrace{\frac{\partial^{|s|}g}{\partial x^s}}_{\text{homogeneous of degree } 2(m-1)-|s| \text{ and null if } |s|\geq 2m-1} h$$

For $|s| \geq 2m-1$, $\frac{\partial^s E}{\partial x^s}$ is a rational homogeneous function of degree $2(m-1)-|s|$ □

Théorème 9.6. *Let $R \in W^m(\Lambda)/\mathbb{P}_{m-1}$ the solution of the following exterior problem :*

$$(\mathcal{R}_{ext}) \begin{cases} \Delta^m R = 0, \ in\ \Lambda \\ A_i(R) = g_i, \ on\ \sigma \quad \forall i \in [\![0, m-1]\!] \end{cases} \tag{9.3}$$

with $g_i \in \left(H_{00}^{1/2+i}(\sigma)\right)'$. We have :

1. *$R(x)$ is unique in $W^m(\Lambda)/\mathbb{P}_{m-1}$, and the map $(g_0,...,g_{m-1}) \mapsto R$ is continuous from $\left(H_{00}^{1/2}(\sigma)\right)' \times ... \times \left(H_{00}^{1/2+m-1}(\sigma)\right)'$ in $W^m(\Lambda)/\mathbb{P}_{m-1}$.*
2. *$R(x)$ writes in $W^m(\Lambda)/\mathbb{P}_{m-1}$ as a sum of multi-layer potentials :*

$$R(x) = \sum_{i=0}^{m-1} \int_\sigma \lambda_i(y) A_{i,y}(E(x-y))d\sigma_y, \ \forall x \in \Lambda$$

   with $\lambda_i \in H_{00}^{1/2+i}(\sigma)$ for $i \in [\![0, m-1]\!]$.

3. *We have the following jump relations across σ :*

$$(-1)^{m+1}\left[\frac{\partial^{m-1-i}R}{\partial n^{m-1-i}}\right] = \lambda_i, \ for\ i \in [\![0, m-1]\!]$$

4. *The densities λ_i are given by a system of m boundary integral equations :*

$$g_j(x) = \sum_{i=0}^{m-1} \oint_\sigma \lambda_i(y) A_{j,x} A_{i,y}(E(x-y)) d\sigma_y, \text{ for } j \in [\![0, m-1]\!]$$

where $\oint$ stands for the main Cauchy value.

5. *If $\sigma = \{(s,0), -1 < s < 1\}$, $g_0(x) = V$ where V is a constant and $g_i(x) = 0$ for $i \in [\![1, m-1]\!]$, the densities λ_i are given by*

$$\lambda_0(s) = \frac{(-1)^m 2^{2m-1}}{(2m-1)C^{m-1}_{2(m-1)}} V\sqrt{1-s^2}$$
$$\lambda_i(s) = 0, \forall i \in [\![1, m-1]\!]$$

Proof. **First point**
We keep the same notations and conventions as described in Figure 3. We introduce the two functional spaces

$$H^m(\Delta^m, \widetilde{\omega}) = \left\{u \in H^m(\widetilde{\omega}), \Delta^m u \in L^2(\widetilde{\omega})\right\}$$
$$W^m(\Delta^m, \widetilde{\omega}') = \left\{u \in W^m(\widetilde{\omega}'), (1+r^2)^{\frac{m}{2}} log(2+r^2)\Delta^m u \in L^2(\widetilde{\omega}')\right\}$$

where we recall that $\widetilde{\omega}' = \mathbb{R}^2 \backslash \overline{\widetilde{\omega}}$. We define the bilinear forms :
$\widetilde{b}(u,v)$ defined on $H^m(\widetilde{\omega}) \times H^m(\widetilde{\omega})$ by

$$\widetilde{b}(u,v) = \sum_{k=0}^{m} C^k_m \int_{\widetilde{\omega}} \frac{\partial^m u}{\partial x_1^k \partial x_2^{m-k}} \frac{\partial^m v}{\partial x_1^k \partial x_2^{m-k}}$$

and $\widetilde{b}'(u,v)$ defined on $W^m(\widetilde{\omega}') \times W^m(\widetilde{\omega}')$ by

$$\widetilde{b}'(u,v) = \sum_{k=0}^{m} C^k_m \int_{\widetilde{\omega}'} \frac{\partial^m u}{\partial x_1^k \partial x_2^{m-k}} \frac{\partial^m v}{\partial x_1^k \partial x_2^{m-k}}$$

An integration by parts $\widetilde{\omega}$ (see Theorem 1.1) gives :

$$\widetilde{b}(u,v) = (-1)^m \int_{\widetilde{\omega}} \Delta^m uv + \sum_{i=0}^{m-1} \int_{\partial\widetilde{\omega}} A_i(u) \frac{\partial^{m-1-i} v}{\partial n^{m-1-i}}, \ \forall v \in H^m(\widetilde{\omega})$$

Similarly

$$\widetilde{b}'(u,v) = (-1)^m \int_{\widetilde{\omega}'} \Delta^m uv - \sum_{i=0}^{m-1} \int_{\partial\widetilde{\omega}} A_i(u) \frac{\partial^{m-1-i} v}{\partial n^{m-1-i}} \ \forall v \in W^m(\widetilde{\omega}')$$

We introduce the functional space K defined by

$$K = \left\{u \in H^m(\Delta^m, \widetilde{\omega}) \times W^m(\Delta^m, \widetilde{\omega}')/\mathbb{P}_{m-1}, supp(\Delta^m u) = \sigma, [A_k(u)]_\sigma = 0, \left[\frac{\partial^k u}{\partial n^k}\right]_{\widetilde{\sigma}\backslash\overline{\sigma}} = 0, k \in [\![0, m-1]\!]\right\}$$

By bounding the weights used in the definition of $W^m(\Lambda)$ and by using the regularity property of functions $H^m_{loc}(\Lambda)$ we can rewrite K :

$$K = \{u \in W^m(\Delta^m, \Lambda)/\mathbb{P}_{m-1}, supp(\Delta^m u) = \sigma, [A_k(u)]_\sigma = 0, k \in [\![0, m-1]\!]\}$$

Therefore, the variational formulation of $(\mathcal{R}_{ext})$ writes :

$$\text{find } R \in K \text{ such that : } \quad b(R,v) = l(v),\ \forall v \in K, \quad (\mathcal{R}_{ext})$$

where $l(v)$ and $b(u,v)$ are respectively the linear and bilinear forms on K :

$$l(v) = \sum_{i=0}^{m-1} \int_\sigma g_i \left[\frac{\partial^{m-1-i} v}{\partial n^{m-1-i}}\right] \qquad \text{and} \qquad b(u,v) = \sum_{k=0}^{m} C_m^k \int_\Lambda \frac{\partial^m u}{\partial x_1^k \partial x_2^{m-k}} \frac{\partial^m v}{\partial x_1^k \partial x_2^{m-k}}$$

The problem $(\mathcal{R}_{ext})$ has a unique solution in K. Indeed, the problem is coercive on K :

$$b(u,u) \geq |u|^2_{W^m(\Lambda)} \tag{9.4}$$

and thanks to Lemma 9.1

$$\|u\|_K = \|u\|_{W^m(\Lambda)/\mathbb{P}_{m-1}} \leq C(\Lambda)|u|_{W^m(\Lambda)} \tag{9.5}$$

which shows that $b(u,v)$ is coercive on K. Thanks to Lemma 9.3, we have

$$|l(v)| \leq C \sum_{i=0}^{m-1} \|g_i\|_{H_{00}^{1/2+i}(\sigma)'} \|v\|_{W^m(\Lambda)/\mathbb{P}_{m-1}} \quad \forall v \in K \tag{9.6}$$

which proves that linear form $l(v)$ is continuous on K. As K is a closed subset of $W^m(\Lambda)/\mathbb{P}_{m-1}$ which is an Hilbert, we deduce that it is an Hilbert. Thanks to Lax-Milgram lemma, we get the existence and the uniqueness of the solution of problem $(\mathcal{R}_{ext})$. From the variational formulation of $(\mathcal{R}_{ext})$, (9.4), (9.5) and (9.6) we get the continuity of the map $(g_0, ..., g_{m-1}) \mapsto R$ for the topology associated with and we define the isomorphism :

$$J_0 : \begin{array}{c} (g_0, g_1, ... \qquad\qquad ..., g_{m-2}, g_{m-1}) \longmapsto R \\ \left(H_{00}^{1/2}(\sigma)\right)' \times ... \quad \times \left(H_{00}^{1/2+m-1}(\sigma)\right)' \longrightarrow K \end{array} \tag{9.7}$$

Second and third points

We consider the following problem : for $(q_0, q_1, ..., q_{m-1}) \in H_{00}^{1/2}(\sigma) \times ... \times H_{00}^{1/2+m-1}(\sigma)$

$$\text{trouver } Q \in K \text{ tel que } \left[\frac{\partial^{m-1-i} Q}{\partial n^{m-1-i}}\right] = q_i,\ \forall i \in [\![0, m-1]\!] \ (\mathcal{Q}_{ext})$$

Let $u, v \in K$, we have

$$b(u,v) = (-1)^m \int_\Lambda \Delta^m uv - \sum_{i=0}^{m-1} \int_\sigma A_i(v) \left[\frac{\partial^{m-1-i} u}{\partial n^{m-1-i}}\right]$$

The variational formulation of $(\mathcal{Q}_{ext})$ is :

$$\text{find } Q \in K \text{ such that : } \quad b(Q,v) = l'(v)\ , \forall v \in K, \quad (\mathcal{Q}_{ext})$$

where we define

$$l'(v) = -\sum_{i=0}^{m-1} \int_\sigma q_i A_i(v)$$

To show that $(\mathcal{Q}_{ext})$ is coercive we use the same reasoning as for $(\mathcal{R}_{ext})$. Thanks to Lemma 9.4, we get

$$l'(v) \leq C \sum_{i=0}^{m-1} \|q_i\|_{H_{00}^{1/2+i}(\sigma)} |v|_{m, B_r \setminus \overline{\sigma}}$$

By using the equivalence between the norm and the semi-norm on $W^2(\Lambda)/\mathbb{P}_1$, we deduce that the linear form $l'(v)$ is continuous on K. Thanks to Lax-Milgram lemma, we deduce that there exists an unique solution Q of $(\mathcal{Q}_{ext})$. From the variational formulation of $(\mathcal{Q}_{ext})$, we show that the map $(q_0, ...q_{m-1}) \mapsto Q$ is continuous for the topology associated with and we define the isomorphism :

$$J_1 : \begin{array}{cc} (q_0, q_1, ... & ..., q_{m-2}, q_{m-1}) \longmapsto Q \\ H_{00}^{1/2}(\sigma) \times ... & \times H_{00}^{1/2+m-1}(\sigma) \longrightarrow K \end{array} \tag{9.8}$$

We denote by $J = J_1^{-1} \circ J_0$ with J_0 defined in (9.7), the isomorphism :

$$J : \begin{array}{cc} (g_0, g_1, ... & ..., g_{m-2}, g_{m-1}) \longmapsto \\ \left(H_{00}^{1/2}(\sigma)\right)' \times ... & \times \left(H_{00}^{1/2+m-1}(\sigma)\right)' \longrightarrow \end{array} \begin{array}{cc} (q_0, q_1, ... & ..., q_{m-2}, q_{m-1}) \\ H_{00}^{1/2}(\sigma) \times ... & \times H_{00}^{1/2+m-1}(\sigma) \end{array}$$

J is the map corresponding to the Neumann to Dirichlet problem $(A_0(u), ..., A_{m-1}(u)) \to \left(\left[\frac{\partial^{m-1} u}{\partial n^{m-1}}\right], ..., [u]\right)$ where $u \in K$. Let $\bar{u}$ defined by :

$$\begin{aligned} \bar{u}(x) &= \sum_{i=0}^{m-1} \int_\sigma \lambda_i(y) A_{i,y}(E(x-y)) d\sigma(y) \\ &= \sum_{i=0}^{m-1} \int_{\partial\widetilde{\omega}} \widetilde{\lambda_i}(y) A_{i,y}(E(x-y)) d\sigma(y) \end{aligned}$$

where $\lambda_i \in H_{00}^{1/2+i}(\sigma)$ and $\widetilde{\lambda_i} \in H^{1/2+i}(\widetilde{\sigma})$ are the following extensions

$$\widetilde{\lambda_i} = \begin{cases} \lambda_i, \text{ on } \sigma \\ 0, \text{ on } \widetilde{\sigma} \backslash \overline{\sigma} \end{cases} \qquad \forall i \in [\![0, m-1]\!]$$

Let us show that $\Delta^m \bar{u} = 0$ on Λ. For $y \in \sigma$, the functions $A_{i,y}(E(. - y))$ are $C^\infty(\Lambda)$. Moreover

$$\Delta_x^m A_{i,y}(E(x-y)) = A_{i,y}(\Delta^m(E(x-y))) = 0$$

By using the regularity of the functions $A_{i,y}E(. - y)$ and the fact that their m-th Laplacian is null and so uniformly bounded with respect to x, we can switch the integral symbol and the Δ^m operator, which leads to :

$$\Delta^m \bar{u}(x) = 0, \, \forall x \in \Lambda$$

Thanks to a Taylor expansion of $E(. - y)$ at point $x \in \Lambda$ for $|x| \to \infty$, and by using the $A_i(u)$ expressions for a straight crack (Theorem 1.1) and Lemma 9.5 we have $\bar{u}(x) = O\left(|x|^{m-2} log(|x|)\right)$. We deduce that $\frac{\bar{u}}{(1+r^2)^{\frac{m}{2}} log(2+r^2)} \in L^2(\Lambda)$. Similarly, by $k \in [\![1, m]\!]$ derivations of $\bar{u}$, we get that $\frac{\nabla^k \bar{u}}{(1+r^2)^{\frac{m-k}{2}} log(2+r^2)} \in L^2(\Lambda)$. We conclude that $\bar{u} \in W^m(\Lambda)$. By considering $\bar{u}$ as an element of $W^m(\Lambda)/\mathbb{P}_1$ we get $\bar{u} \in K$. Thanks to Lemma 9.10, by considering the definition of $\widetilde{\omega}$ and the definitions of $\widetilde{\lambda_i}$, we have

$$\frac{\partial^{m-1-i} \bar{u}}{\partial n^{m-1-i}}^{\pm}(x) = \frac{\pm(-1)^{m+1}}{2} \lambda_i(x) + \sum_{j=0}^{m-1} \int_\sigma \lambda_j(y) \frac{\partial^{m-1-i}}{\partial n^{m-1-i}} \left(A_{j,y}(E(x-y))\, d\sigma_y, \; \forall x \in \sigma \quad \forall i \in [\![0, m-1]\!]\right.$$

We deduce the jump relations across σ :

$$\lambda_i = (-1)^{m+1} \left[\frac{\partial^{m-1-i} \bar{u}}{\partial n^{m-1-i}}\right], \text{ for } i \in [\![0, m-1]\!]$$

By setting $\lambda_i = (-1)^{m+1}\left[\frac{\partial^{m-1-i}R}{\partial n^{m-1-i}}\right]$, as J_1 is an isomorphism we get $\bar{u} = R$ which ends the proof of points 2 and 3.

Fourth point

By applying the $A_{i,x}$ operator to $\bar{u}$ for $i \in [\![0, m-1]\!]$, by doing $x \to \sigma$, and thanks to Lemma 9.5 we get

$$A_{i,x}A_{j,y}(E(x-y)) = O\left(\frac{1}{|x-y|^{2+i+j}}\right)$$

where $i, j \geq m$. By using the regularity of such potentials across σ, (see Lemma 9.10), we get the m boundary integral equations which define J^{-1} :

$$g_i(x) = \sum_{j=0}^{m-1} \oint_\sigma \lambda_j(y) A_{i,x}A_{j,y}(E(x-y))d\sigma_y, \ \forall i \in [\![0, m-1]\!] \tag{9.9}$$

where $\oint$ stands for the main Cauchy value.

Last point

In the straight crack case $\sigma = \{(s,0), -1 < s < 1\}$, by setting $x = (s,0)$ and $y = (t,0)$ and by using the expression of operators A_i for a straight crack (Theorem 1.1) and Lemma 9.5, we have :

$$A_{i,x}A_{j,y}(E(x-y)) = \frac{a_{i,j}}{(s-t)^{2+i+j}} \tag{9.10}$$

where a_{ij} are some constants. (9.9) rewrites as :

$$g_i(s) = \sum_{j=0}^{m-1} a_{ij} \oint_{-1}^{1} \frac{\lambda_j(t)}{(s-t)^{2+i+j}}, \text{ pour } i \in [\![0, m-1]\!] \tag{9.11}$$

We set

$$f_j(s) = \frac{1}{\pi}\int_{-1}^{1} \frac{\lambda_j(t)}{s-t}dt, \text{ for } -1 < s < 1 \text{ and } j \in [\![0, m-1]\!]$$

By derivation of f_j we show for $n \geq 0$ that

$$\frac{d^n f_j}{ds^n} = \frac{(-1)^n n!}{\pi}\int_{-1}^{1} \frac{\lambda_j(t)}{(s-t)^{n+1}}dt$$

By denoting $\frac{d^n f_j}{ds^n} = f_j^{(n)}$, (9.11) rewrites as :

$$\begin{aligned} g_i(s) &= \sum_{j=0}^{m-1} \frac{a_{ij}(-1)^{i+j+1}\pi}{(i+j+1)!} f_j^{(i+j+1)}(s) \\ &= \sum_{j=0}^{m-1} b_{ij} f_j^{(i+j+1)}(s) \quad \text{for} \quad i \in [\![0, m-1]\!] \end{aligned} \tag{9.12}$$

where we set

$$b_{ij} = \frac{a_{ij}(-1)^{i+j+1}\pi}{(i+j+1)!} \tag{9.13}$$

We rewrite (9.12) with the expressions of the g_i given at the fifth point of the theorem :

$$(\mathcal{S})\begin{cases} V = b_{0,0}f_0^{(1)} + b_{0,1}f_1^{(2)} + ... & +b_{0,m-1}f_{m-1}^{(m)} \\ 0 = b_{1,0}f_0^{(2)} + b_{1,1}f_1^{(3)} + ... & +b_{1,m-1}f_{m-1}^{(m+1)} \\ \vdots & \vdots \\ 0 = b_{m-1,0}f_0^{(m)} + b_{m-1,1}f_1^{(m+1)} + ... & +b_{m-1,m-1}f_{m-1}^{(2m-1)} \end{cases} \tag{9.14}$$

To solve $(\mathcal{S})$, we integrate i times the i-th row by taking as constants of integration 0 for the first $i-2$ integrations and $\frac{Vb_{i,0}}{b_{0,0}}$ for the $i-1$-th. The last constant is set to 0. In the sequel we will check that the constant $b_{0,0}$ is not null. The system $(\mathcal{S})$ becomes :

$$(\mathcal{S}')\begin{cases} Vs = b_{0,0}f_0 + b_{0,1}f_1^{(1)} + ... & +b_{0,m-1}f_{m-1}^{(m-1)} \\ \frac{Vb_{1,0}s}{b_{0,0}} = b_{1,0}f_0 + b_{1,m}f_1^{(1)} + ... & +b_{1,m-1}f_{m-1}^{(m-1)} \\ \vdots & \vdots \\ \frac{Vb_{2m-1,m}s}{b_{0,0}} = b_{m-1,0}f_0 + b_{m-1,1}f_1^{(1)} + ... & +b_{m-1,m-1}f_{m-1}^{(m-1)} \end{cases} \tag{9.15}$$

The unknowns of $(\mathcal{S}')$ are the $f_i^{(i)}$ for $i \in [\![0, m-1]\!]$. A trivial solution is

$$\begin{cases} f_0 = \dfrac{Vs}{b_{0,0}} \\ f_i = 0, \text{ for } i \in [\![1, m-1]\!] \end{cases}$$

We get m uncoupled boundary integral equations. To solve the first one, we use [11] and we get $\lambda_m(s) = \frac{V}{b_{0,0}}\sqrt{1-s^2}$ for $-1 < s < 1$. For more details we refer the reader to [9]. The other equations have the trivial solution $\lambda_i = 0$ for $i \in [\![1, m-1]\!]$. To sump up the λ_i are given by :

$$\begin{cases} \lambda_0 = \dfrac{V}{b_{0,0}}\sqrt{1-s^2}, \ -1 < s < 1 \\ \lambda_i = 0, \text{ for } -1 < s < 1 \text{ and } i \in [\![1, m-1]\!] \end{cases}$$

We check that $\lambda_i \in H_{00}^{1/2+i}(\sigma)$ and by using that J is injective, this solution is unique. From (9.13), we have $b_{0,0} = -\pi a_{0,0}$ and $a_{0,0}$ is given by (see (9.10)) :

$$A_{0,x}A_{0,y}(E(x-y)) = \frac{a_{0,0}}{(x_1-y_1)^2}, \text{ for } x, y \in \sigma$$

From Remark 1.2, we can explicit the operator A_0 and we get

$$(-1)^m \frac{\partial^{2m}}{\partial x_2^{2m}} E(x-y) = \frac{a_{m,m}}{(x_1-y_1)^2}, \text{ for } x, y \in \sigma \tag{9.16}$$

We show (see Lemma 9.9 with $n = m-1$) that

$$\frac{\partial^{2m}}{\partial x_2^{2m}} \left(|x|^{2(m-1)} log(|x|^2) \right)_{|x_2=0} = \frac{2(2m-1)!}{x_1^2}$$

Therefore, from (2.2) and (9.16) we deduce

$$a_{0,0} = \frac{(-1)^{m+1}(2m-1)!}{2^{2m-1}\pi((m-1)!)^2} \tag{9.17}$$

The expressions of $b_{0,0}$ and λ_0 are

$$b_{0,0} = \frac{(-1)^m(2m-1)}{2^{2m-1}} C^{m-1}_{2(m-1)}$$
$$\lambda_0(s) = \frac{(-1)^m 2^{2m-1}}{(2m-1)C^{m-1}_{2(m-1)}} V\sqrt{1-s^2}$$

which ends the proof of the last point and of this theorem.

□

Lemme 9.7. *Let $R(x)$ the solution of the problem (9.3).*
For $m \geq 1$, by using the convention $[\![0,-1]\!] = \emptyset$, we have the following estimations for $|x| \to \infty$ and $\epsilon \to 0$:

$$|\nabla^k R(x)| \leq C|x|^{m-2-k} log(|x|),\ k \in [\![0, m-2]\!]$$
$$|\nabla^{m-1} R(x)| \leq \frac{C}{|x|}$$
$$|\nabla^m R(x)| \leq \frac{C}{|x|^2}$$
$$\left\|\nabla^k R\left(\frac{x}{\epsilon}\right)\right\|_{0,\Omega_\epsilon} = O\left(-\epsilon^{-(m-2-k)} log(\epsilon)\right),\ \textit{for } k \in [\![0, m-2]\!]$$
$$\left\|\nabla^{m-1} R\left(\frac{x}{\epsilon}\right)\right\|_{0,\Omega_\epsilon} = O\left(\epsilon\sqrt{-log(\epsilon)}\right)$$
$$\left\|\nabla^m R\left(\frac{x}{\epsilon}\right)\right\|_{0,\Omega_\epsilon} = O\left(\epsilon\right)$$

Proof. From Theorem 9.6, $R(x)$ writes

$$R(x) = \sum_{i=0}^{m-1} \int_\sigma \lambda_i(y) A_{i,y}(E(x-y)) d\sigma_y \tag{9.18}$$

By using Theorem 1.1, the $A_{i,y}(E(x-y))$ writes as

$$A_{i,y}(E(x-y)) = \sum_{j=0}^{m-2} \varphi^i_j(y)|x-y|^{m-2-j} log(|x-y|) + \frac{\varphi^i_{m-1}(y)}{|x-y|}$$

Thanks to a Taylor expansion of order 1 at x for $|x| \longrightarrow \infty$ and to Lemma 9.5 we have

$$A_{i,y}(E(x-y)) = O\left(|x|^{m-2} log(|x|)\right)$$

By differentiating k times (9.18) and by using a Taylor expansion and Lemma 9.5 again we get the

asymptotic behaviour of $\nabla^k R$ for $k \in [\![0, m]\!]$. Let $r_2 > 0$ such that $\Omega \subset B_{r_2}$ for $0 \leq k \leq m-2$ we have :

$$\begin{aligned}
\left\|\nabla^k R\left(\frac{x}{\epsilon}\right)\right\|^2_{0,\Omega_\epsilon} &= \int_{\Omega_\epsilon} \left\|\nabla^k R\left(\frac{x}{\epsilon}\right)\right\|^2 dx = \epsilon^2 \int_{\frac{1}{\epsilon}\Omega\setminus\overline{\sigma}} \left\|\nabla^k R(y)\right\|^2 dy \\
&= \epsilon^2 \int_{B_r\setminus\overline{\sigma}} \left\|\nabla^k R(y)\right\|^2 dy + \epsilon^2 \int_{\frac{\Omega}{\epsilon}\setminus\overline{B_r}} \left\|\nabla^k R(y)\right\|^2 dy \\
&\leq \epsilon^2 \left(C + \int_{\frac{B_{r_2}}{\epsilon}\setminus\overline{B_r}} \left\|\nabla^k R(y)\right\|^2 dy \right) \\
&\leq \epsilon^2 \left(C + \int_2^{r_2/\epsilon} r^{2(m-2-k)} log(r)^2 r dr \right) \\
&\leq \epsilon^2 \left(C + \frac{1}{2(m-1-k)} \left[r^{2(m-1-k)} \left(log(r)^2 - \frac{1}{2(m-1-k)} log(r) + \frac{1}{(2(m-1))^2} \right) \right]_2^{r_2/\epsilon} \right) \\
&\leq C\epsilon^{-2(m-2-k)} log(\epsilon)^2
\end{aligned}$$

which shows the first estimation,

$$\begin{aligned}
\left\|\nabla^{m-1} R\left(\frac{x}{\epsilon}\right)\right\|^2_{0,\Omega_\epsilon} &= \int_{\Omega_\epsilon} \left\|\nabla^{m-1} R\left(\frac{x}{\epsilon}\right)\right\|^2 dx = \epsilon^2 \int_{\frac{1}{\epsilon}\Omega\setminus\overline{\sigma}} \left\|\nabla^{m-1} R(y)\right\|^2 dy \\
&\leq \epsilon^2 \left(\int_{B_r\setminus\overline{\sigma}} \left\|\nabla^{m-1} R(y)\right\|^2 dy + \int_{\frac{1}{\epsilon}B_{r_2}\setminus\overline{B_r}} \left\|\nabla^{m-1} R(y)\right\|^2 dy \right) \\
&\leq C\epsilon^2 + C\epsilon^2 \int_2^{r_2/\epsilon} \frac{1}{r^2} r dr \\
&= C\epsilon^2 + C\epsilon^2 \left[log(r) \right]_2^{\frac{r_2}{\epsilon}} \\
&\leq -C\epsilon^2 log(\epsilon)
\end{aligned}$$

and the second estimation is proved.

$$\begin{aligned}
\left\|\nabla^m R\left(\frac{x}{\epsilon}\right)\right\|^2_{0,\Omega_\epsilon} &= \int_{\Omega_\epsilon} \left\|\nabla^m R\left(\frac{x}{\epsilon}\right)\right\|^2 dx = \epsilon^2 \int_{\frac{1}{\epsilon}\Omega\setminus\overline{\sigma}} \left\|\nabla^m R(y)\right\|^2 dy \\
&\leq \epsilon^2 \left(\int_{B_r\setminus\overline{\sigma}} \left\|\nabla^m R(y)\right\|^2 dy + \int_{\frac{1}{\epsilon}B_{r_2}\setminus\overline{B_r}} \left\|\nabla^m R(y)\right\|^2 dy \right) \\
&\leq C\epsilon^2 + C\epsilon^2 \int_2^{r_2/\epsilon} \frac{1}{r^4} r dr \\
&= C\epsilon^2 + C\epsilon^2 \left[\frac{-2}{r^2} \right]_2^{\frac{r_2}{\epsilon}} \\
&\leq C\epsilon^2
\end{aligned}$$

which ends the proof. □

Lemme 9.8. *Let w_ϵ the solution of $(\mathcal{Q}^c_\epsilon)$ given in (5.1) for $m \geq 1$, and R the solution of $(\mathcal{R}_{ext})$ given in (5.2), we have the following asymptotic expansion*

$$w_\epsilon = \epsilon^m R\left(\frac{x}{\epsilon}\right) + e_\epsilon \tag{9.19}$$

with

$$\begin{aligned}\|e_\epsilon\|_{m,\Omega_\epsilon} &= O(\phi_m(\epsilon))\\ |w_\epsilon|_{k,\Omega_\epsilon} &= O(-\epsilon^2 log(\epsilon)) \quad \forall k \in [\![0, m-2]\!]\\ |w_\epsilon|_{m-1,\Omega_\epsilon} &= O(\epsilon^2\sqrt{-log(\epsilon)})\\ |w_\epsilon|_{m,\Omega_\epsilon} &= O(\epsilon)\end{aligned}$$

where

$$\phi_m(\epsilon) = \begin{cases} -\epsilon^2 log(\epsilon) & \text{for } m \geq 2\\ \epsilon^2\sqrt{-log(\epsilon)} & \text{for } m = 1\end{cases}$$

Proof. e_ϵ is defined by :

$$(\mathcal{E}_\epsilon)\begin{cases} (-1)^m\Delta^m e_\epsilon + e_\epsilon = -\epsilon^m R\left(\frac{x}{\epsilon}\right), \text{ in } \Omega_\epsilon\\ A_0(e_\epsilon) = \varphi_0(x) = O(|x|), \text{ on } \sigma_\epsilon\\ A_i(e_\epsilon) = -A_i(v_0) - B_i, \text{ on } \sigma_\epsilon \text{ for } i \in [\![1, m-1]\!]\\ A_i(e_\epsilon) = \psi_i^\epsilon(x) = O\left(\epsilon^2\right) \text{ on } \Gamma \text{ for } i \in [\![0, m-1]\!]\end{cases} \tag{9.20}$$

where $\varphi_0(x) = -A_0(v_0)(x) + A_0(v_0)(0) = O(|x|)$ (see (4.1a)), $\psi_i^\epsilon(x) = -\epsilon^m A_0\left(R\left(\frac{x}{\epsilon}\right)\right)$, $A_i(v_0)(x) = O(1)$ on σ_ϵ and B_i is defined in (3.4).

Proof of the estimation $\psi_i^\epsilon(x) = O\left(\epsilon^2\right)$

By using Theorem 1.1, the $A_j(u)$ for $j \in [\![0, m-1]\!]$ writes as

$$A_{j,y}(u) = \sum_{m \leq |p| \leq j+m} \varphi_p^j(y)\frac{\partial^{|p|}u}{\partial y^p} \tag{9.21}$$

From (9.21) and thanks to Lemma 9.5 for $x \in \Gamma$ and $y \in \sigma$ we get

$$\begin{aligned}A_{i,x}A_{j,y}\left(E\left(\frac{x}{\epsilon} - y\right)\right) &= \sum_{\substack{m \leq |p| \leq i+m\\ m \leq |q| \leq j+m}} \varphi_p^i(x)\varphi_q^j(y)\frac{\partial^{|p|+|q|}}{\partial x^p \partial x^q}\left(E\left(\frac{x}{\epsilon} - y\right)\right)\\ &= \sum_{\substack{m \leq |p| \leq i+m\\ m \leq |q| \leq j+m}} \varphi_p^i(x)\varphi_q^j(y)\frac{1}{\epsilon^{|p|}}\frac{\partial^{|p|+|q|}E}{\partial x^p \partial x^q}\left(\frac{x}{\epsilon} - y\right)\\ &= \sum_{\substack{m \leq |p| \leq i+m\\ m \leq |q| \leq j+m}} \varphi_p^i(x)\varphi_q^j(y)\frac{1}{\epsilon^{|p|}}G_{p+q}\left(\frac{x}{\epsilon} - y\right)\\ &= \sum_{\substack{m \leq |p| \leq i+m\\ m \leq |q| \leq j+m}} \varphi_p^i(x)\varphi_q^j(y)\frac{1}{\epsilon^{|p|}}\epsilon^{|p|+|q|-2m+2}G_{p+q}\left(x - \epsilon y\right)\\ &= \sum_{\substack{m \leq |p| \leq i+m\\ m \leq |q| \leq j+m}} \varphi_p^i(x)\varphi_q^j(y)\epsilon^{|q|-2m+2}G_{p+q}\left(x - \epsilon y\right)\\ &= O\left(\epsilon^{2-m}\right)\end{aligned}$$

where G_k is an homogeneous function of degree $2(m-1) - |k|$ and by using that $G_{p+q}\left(x - \epsilon y\right) = O\left(1\right)$ when $\epsilon \to 0$ for $x, y \in \Gamma$.

Splitting of $\|e_\epsilon\|_{m,\Omega_\epsilon}$

We split e_ϵ into the sum $e_\epsilon = e_\epsilon^1 + e_\epsilon^2$ with $e_\epsilon^1 \in H^m(\Omega_\epsilon)/\mathbb{P}_{m-1}$ solution of the following problem

$$(\mathcal{E}_\epsilon^1)\begin{cases} (-1)^m \Delta^m e_\epsilon^1 = 0, \text{ in } \Omega_\epsilon \\ A_0(e_\epsilon^1) = \varphi_0(x) = O(|x|), \text{ on } \sigma_\epsilon \\ A_i(e_\epsilon^1) = -A_i(v_0) - B_i, \text{ on } \sigma_\epsilon, \text{ for } i \in [\![1, m-1]\!] \\ A_i(e_\epsilon^1) = 0 \text{ on } \Gamma, \text{ for } i \in [\![1, m-1]\!] \end{cases}$$

and $e_\epsilon^2 \in H^m(\Omega_\epsilon)$ solution of the problem :

$$(\mathcal{E}_\epsilon^2)\begin{cases} (-1)^m \Delta^m e_\epsilon^2 + e_\epsilon^2 = -e_\epsilon^1 - \epsilon^m R\left(\frac{x}{\epsilon}\right), \text{ in } \Omega_\epsilon \\ A_i(e_\epsilon^2) = 0, \text{ on } \sigma_\epsilon, \text{ for } i \in [\![0, m-1]\!] \\ A_i(e_\epsilon^2) = \psi_i^\epsilon(x), \text{ on } \Gamma \end{cases}$$

Estimation of $\|e_\epsilon^1\|_{H^m(\Omega_\epsilon)/\mathbb{P}_1}$

The variational formulation of $(\mathcal{E}_\epsilon^1)$ is : find $e_\epsilon^1 \in H^m(\Omega_\epsilon)/\mathbb{P}_{m-1}$ such that

$$b_\epsilon(e_\epsilon^1, v) = -\int_{\sigma_\epsilon} \varphi_0 \left[\frac{\partial^{m-1} v}{\partial n^{m-1}}\right] + \sum_{k=1}^{m-1} \int_{\sigma_\epsilon} (A_k(v_0) + B_k) \left[\frac{\partial^{m-1-k} v}{\partial n^{m-1-k}}\right], \ \forall v \in H^m(\Omega_\epsilon)/\mathbb{P}_{m-1}$$

where

$$b_\epsilon(u, v) = \sum_{k=0}^{m} C_m^k \int_{\Omega_\epsilon} \frac{\partial^m u}{\partial x_1^k \partial x_2^{m-k}} \frac{\partial^m v}{\partial x_1^k \partial x_2^{m-k}}$$

We take as function test $v = e_\epsilon^1$ and we set for $k \in [\![1, m-1]\!]$

$$\mathcal{J}_k = \int_{\sigma_\epsilon} (A_k(v_0) + B_k) \left[\frac{\partial^{m-1-k} e_\epsilon^1}{\partial n^{m-1-k}}\right]$$

and

$$\mathcal{J}_0 = \int_{\sigma_\epsilon} \varphi_0 \left[\frac{\partial^{m-1} e_\epsilon^1}{\partial n^{m-1}}\right]$$

By using Lemma 9.3, the fact that $b_\epsilon(u, v)$ is coercive and the Deny-Lions inequality we get

$$\|e_\epsilon^1\|^2_{H^m(\Omega_\epsilon)/\mathbb{P}_{m-1}} \leq C|e_\epsilon^1|_{m,\Omega_\epsilon} \leq \sum_{k=0}^{m-1} \mathcal{J}_k \tag{9.22}$$

Let $k \in [\![1, m-1]\!]$, thanks to a change of variable, Lemma 9.3, Lemma 4.1 and a change of variable again we have

$$\begin{aligned} \mathcal{J}_k &= \epsilon \int_\sigma (A_k(v_0)(\epsilon X) + B_k(\epsilon X)) \left[\frac{\partial^{m-1-k} e_\epsilon^1(\epsilon X)}{\partial n^{m-1-k}}\right] d\sigma \\ &\leq C\epsilon^{1-(m-1-k)} |e_\epsilon^1(\epsilon X)|_{m, B_r \setminus \overline{\sigma}} \\ &\leq C\epsilon^{k+1} |e_\epsilon^1|_{m,\Omega_\epsilon} \end{aligned}$$

Since $k \geq 1$, we deduce that

$$\mathcal{J}_k \leq C\epsilon^2 |e_\epsilon^1|_{m,\Omega_\epsilon} \tag{9.23}$$

Similarly by using the estimation $\varphi_0(\epsilon X) = O(\epsilon)$, we have

$$\begin{aligned}\mathcal{J}_0 &= \epsilon \int_\sigma \varphi_0(\epsilon X) \left[\frac{\partial^{m-1} e_\epsilon^1}{\partial n^{m-1}}\right] \\ &\leq C\epsilon \|\varphi_0(\epsilon X)\|_{H_{00}^{1/2}(\sigma)'} \epsilon^{-(m-1)} |e_\epsilon^1(\epsilon X)|_{m,B_r\setminus\overline{\sigma}} \\ &\leq C\epsilon^{2-(m-1)} |e_\epsilon^1(\epsilon X)|_{m,B_r\setminus\overline{\sigma}} \\ &\leq C\epsilon^2 |e_\epsilon^1|_{m,\Omega_\epsilon}\end{aligned} \tag{9.24}$$

From (9.22) and by using (9.23) and (9.24), we get

$$\|e_\epsilon^1\|_{H^m(\Omega_\epsilon)/\mathbb{P}_{m-1}} \leq C\epsilon^2 \tag{9.25}$$

Estimation of $\|e_\epsilon^2\|_{m,\Omega_\epsilon}$

The variational formulation of $(\mathcal{E}_\epsilon^2)$ is : find $e_\epsilon^2 \in H^m(\Omega_\epsilon)$ such that

$$a_\epsilon(e_\epsilon^2, v) = \int_{\Omega_\epsilon} \left(-\epsilon^m R\left(\frac{x}{\epsilon}\right) - e_\epsilon^1\right) v + \sum_{i=0}^{m-1} \int_\Gamma \psi_i^\epsilon \frac{\partial^{m-1-i} v}{\partial n^{m-1-i}} \quad \forall v \in H^m(\Omega_\epsilon)$$

We take as test function $v = e_\epsilon^2$ and we set

$$\mathcal{K}_\epsilon = -\int_{\Omega_\epsilon} \left(\epsilon^m R\left(\frac{x}{\epsilon}\right) + e_\epsilon^1\right) e_\epsilon^2 \qquad \mathcal{L}_\epsilon = \sum_{i=0}^{m-1} \int_\Gamma \psi_i^\epsilon \frac{\partial^{m-1-i} e_\epsilon^2}{\partial n^{m-1-i}}$$

By using the definition of $a_\epsilon(u, v)$ (1.4), we deduce that

$$\|e_\epsilon^2\|_{m,\Omega_\epsilon}^2 = \mathcal{K}_\epsilon + \mathcal{L}_\epsilon \tag{9.26}$$

Lemma 9.7 and the estimation $\|e_\epsilon^1\|_{H^m(\Omega_\epsilon)/\mathbb{P}_{m-1}} = O(\epsilon^2)$ give

$$\mathcal{K}_\epsilon \leq C\phi_m(\epsilon) \|e_\epsilon^2\|_{0,\Omega_\epsilon} \tag{9.27}$$

Thanks to the estimation $\psi_i^\epsilon(x) = O(\epsilon^2)$ and to the trace Theorem on $\Omega\setminus\overline{B}$, we get

$$\mathcal{L}_\epsilon \leq C\epsilon^2 \|e_\epsilon^2\|_{m,\Omega\setminus\overline{B}} \leq C\epsilon^2 \|e_\epsilon^2\|_{m,\Omega_\epsilon} \leq C\epsilon^2$$

From this last inequality (9.26) and (9.27) we have

$$\|e_\epsilon^2\|_{m,\Omega_\epsilon} \leq C\phi_m(\epsilon) \tag{9.28}$$

By using (9.25) and (9.28) we get the first estimation :

$$\begin{aligned}\|e_\epsilon\|_{m,\Omega_\epsilon} &\leq \|e_\epsilon^1\|_{H^m(\Omega_\epsilon)/\mathbb{P}_{m-1}} + \|e_\epsilon^2\|_{m,\Omega_\epsilon} \\ &\leq C\phi_m(\epsilon)\end{aligned}$$

Finally, by differentiating m times (9.19) and by using Lemma 9.7 we get the second estimation :

$$\begin{aligned}|w_\epsilon|_{m,\Omega_\epsilon} &\leq \left\|\nabla^m R\left(\frac{x}{\epsilon}\right)\right\|_{0,\Omega_\epsilon} + |e_\epsilon|_{m,\Omega_\epsilon} \\ &\leq C\epsilon\end{aligned}$$

□

Lemme 9.9. *We have the following equality* [1] *:*

$$\frac{\partial^{2n+2}}{\partial x_2^{2n+2}}\left(|x|^{2n}log(|x|^2)\right)_{|x_2=0}=\frac{2(2n+1)!}{x_1^2}\quad for\quad x_1\neq 0$$

Proof. We set $f_n(x_1,x_2)=(x_1^2+x_2^2)^n log(x_1^2+x_2^2)$ and $\psi_n(r)=r^n log(r)$.
Step 1 : Let us show by recurrence :

$$\psi_n^{(k)}=\frac{n!}{(n-k)!}\psi_{n-k}+C(k,n)r^{n-k}\quad \forall 0\leq k\leq n$$

where $C(k,n)$ is a constant which depends on k and n.
For $k=0$ the property is right.
Let us assume that the property is right for $0<k<n$ then we have :

$$\begin{aligned}\psi_n^{(k+1)}&=\frac{n!}{(n-k)!}\left[(n-k)\psi_{n-k-1}+r^{n-k-1}\right]+(n-k)C(k,n)r^{n-k-1}\\&=\frac{n!}{(n-(k+1))!}\psi_{n-(k+1)}+\left[\frac{n!}{(n-k)!}+(n-k)C(k,n)\right]r^{n-(k+1)}\end{aligned}$$

The property is $0\leq k\leq n$ and the constants $C(k,n)$ are defined by the recurrence relation :

$$\begin{aligned}C(0,n)&=0\\C(k+1,n)&=\frac{n!}{(n-k)!}+(n-k)C(k,n)\quad \forall k<n\end{aligned}$$

For $k=n$, we have $\psi_n^{(n)}(r)=n!log(r)+C(n,n)$. By recurrence we have $\psi_n^{(n+1)}(r)=\frac{n!}{r}$, $\psi_n^{(n+2)}(r)=-\frac{n!}{r^2}$, $\psi_n^{(n+3)}(r)=2\frac{n!}{r^3}$, ... i.e. :

$$\psi_n^{(n+k)}=\frac{(-1)^{k+1}n!(k-1)!}{r^k}$$

To conclude, we have

$$\psi_n^{(k)}(r)=\begin{cases}\dfrac{n!}{(n-k)!}\psi_{n-k}+C(k,n)r^{n-k} & \text{si } k\leq n\\ \dfrac{(-1)^{k-n-1}n!(k-n-1)!}{r^{k-n}} & \text{si } k>n\end{cases}\tag{9.29}$$

with

$$\begin{cases}C(0,n)=0\\ C(k+1,n)=\dfrac{n!}{(n-k)!}+(n-k)C(k,n)\quad \forall k<n\end{cases}$$

Step 2 : We set $f(x)=\varphi\left(\frac{|x|^2}{2}\right)$, by recurrence we show :

$$\frac{\partial^k f}{\partial x_2^k}=\sum_{0\leq 2i\leq k}a_{i,k}\varphi^{(k-i)}\left(\frac{|x|^2}{2}\right)x_2^{k-2i}\quad \forall k\geq 0\tag{9.30}$$

[1] Thanks to Arthur Vavasseur for the proof of this Lemma

For $k=0:\ 0\leq 2i\leq k \Leftrightarrow i=0$ and $f(x)=\varphi^{(0)}\left(\frac{|x|^2}{2}\right)x_2^0$.
Let us suppose the equality right for $k\in\mathbb{N}$, then at rank $k+1$ we have

$$\begin{aligned}
\frac{\partial^{k+1}f}{\partial x_2^{k+1}} &= \sum_{0\leq 2i\leq k} a_{i,k}\frac{\partial}{\partial x_2}\left[\varphi^{(k-i)}\left(\frac{|x|^2}{2}\right)x_2^{k-2i}\right]\\
&= \sum_{0\leq 2i\leq k-1} a_{i,k}\left[\varphi^{(k-i+1)}\left(\frac{|x|^2}{2}\right)x_2^{k-2i+1}+\varphi^{(k-i)}\left(\frac{|x|^2}{2}\right)(k-2i)x_2^{k-2i-1}\right]\\
&+ a_{\frac{k}{2},k}\left[\frac{\partial}{\partial x_2}\left(\varphi^{(k-\frac{k}{2})}\left(\frac{|x|^2}{2}\right)x_2^{k-2\frac{k}{2}}\right)\right]\mathbb{1}_{\{k\in 2\mathbb{N}\}}\\
&= \sum_{0\leq 2i\leq k-1} a_{i,k}\varphi^{(k-i+1)}\left(\frac{|x|^2}{2}\right)x_2^{k+1-2i}+\sum_{0\leq 2i\leq k-1} a_{ik}\varphi^{(k-i)}\left(\frac{|x|^2}{2}\right)(k-2i)x_2^{k+1-2(i+1)}\\
&+ a_{\frac{k}{2},k}\left[\varphi^{(\frac{k}{2}+1)}\left(\frac{|x|^2}{2}\right)x_2\right]\mathbb{1}_{\{k\in 2\mathbb{N}\}}\\
&= \sum_{0\leq 2i\leq k-1} a_{i,k}\varphi^{(k+1-i)}\left(\frac{|x|^2}{2}\right)x_2^{k+1-2i}+\sum_{2\leq 2i\leq k+1} a_{i-1,k}\varphi^{(k+1-i)}\left(\frac{|x|^2}{2}\right)(k-2i+2)x_2^{k+1-2i}\\
&+ a_{\frac{k}{2},k}\left[\varphi^{(k+1-\frac{k}{2})}\left(\frac{|x|^2}{2}\right)x_2\right]\mathbb{1}_{\{k\in 2\mathbb{N}\}}\\
&= a_{0,k}\varphi^{(k+1)}\left(\frac{|x|^2}{2}\right)x_2^{k+1}+\sum_{2\leq 2i\leq k-1}\left[a_{i,k}+(k-2i+2)a_{i-1,k}\right]\varphi^{(k+1-i)}\left(\frac{|x|^2}{2}\right)x_2^{k+1-2i}\\
&+ a_{j(k)-1,k}\left[\varphi^{(k+1-j(k))}\left(\frac{|x|^2}{2}\right)x_2^{k+1-2j(k)}(k-2j(k)+2)\right]+a_{\frac{k}{2},k}\left[\varphi^{(k+1-\frac{k}{2})}\left(\frac{|x|^2}{2}\right)x_2\right]\mathbb{1}_{\{k\in 2\mathbb{N}\}}
\end{aligned}$$

with $j(k)=\frac{1}{2}\{k,k+1\}\cap 2\mathbb{N}=\lfloor\frac{k+1}{2}\rfloor$. We set $a_{i,k+1}=\begin{cases}a_{0,k}=1,\text{ if } i=0 \quad\text{by recurrence}\\ a_{i,k}+(k-2i+2)a_{i-1,k} \quad \text{if } 2\leq 2i\leq k-1\end{cases}$
We have :

$$\left.\begin{aligned}
\frac{\partial^{k+1}f}{\partial x_2^{k+1}} &= \sum_{0\leq 2i\leq k-1} a_{i,k+1}\varphi^{(k+1-i)}\left(\frac{|x|^2}{2}\right)x_2^{k+1-2i}\\
&+ a_{\lfloor\frac{k+1}{2}\rfloor,k}\varphi^{(k+1-\lfloor\frac{k+1}{2}\rfloor)}\left(\frac{|x|^2}{2}\right)x_2^{k+1-2\lfloor\frac{k+1}{2}\rfloor}\left(k-2\lfloor\frac{k+1}{2}\rfloor+2\right)\\
&+ a_{\frac{k}{2},k}\left[\varphi^{(k+1-\frac{k}{2})}\left(\frac{|x|^2}{2}\right)x_2^{k+1-2\frac{k}{2}}\right]\mathbb{1}_{\{k\in 2\mathbb{N}\}}
\end{aligned}\right\} = g(x)$$

We discuss the two last terms with respect to the parity of k.

(i) If $k\in 2\mathbb{N}$ then $\lfloor\frac{k+1}{2}\rfloor=\frac{k}{2}$ and

$$\begin{aligned}
g(x) &= a_{\frac{k}{2}-1,k}\varphi^{k+1-\frac{k}{2}}\left(\frac{|x|^2}{2}\right)x_2^{k+1-2\frac{k}{2}}\left(k-2\frac{k}{2}+2\right)+a_{\frac{k}{2},k}\varphi^{k+1-\frac{k}{2}}\left(\frac{|x|^2}{2}\right)x_2^{k+1-2\frac{k}{2}}\\
&= \left(2a_{\frac{k}{2}-1,k}+a_{\frac{k}{2},k}\right)\varphi^{k+1-\frac{k}{2}}\left(\frac{|x|^2}{2}\right)x_2^{k+1-2\frac{k}{2}}
\end{aligned}$$

and $\{k,k+1\}\cap 2\mathbb{N}=k$ so with $a_{\frac{k}{2},k+1}=2a_{\frac{k}{2}-1,k}+a_{\frac{k}{2},k}$, we retrieve the equality at rank $k+1$.

(ii) If $k \in 2\mathbb{N}+1$, $\lfloor \frac{k+1}{2} \rfloor = \frac{k+1}{2}$ and

$$\begin{aligned} g(x) &= a_{\frac{k+1}{2}-1,k}\varphi^{k+1-\frac{k+1}{2}}\left(\frac{|x|^2}{2}\right)x_2^{k+1-2\frac{k+1}{2}}\left(k-2\frac{k+1}{2}+2\right) \\ &= \left(a_{\frac{k-1}{2},k}+a_{\frac{k}{2},k}\right)\varphi^{k+1-\frac{k+1}{2}}\left(\frac{|x|^2}{2}\right)x_2^{k+1-2\frac{k+1}{2}} \end{aligned}$$

and $\{k, k+1\} \cap 2\mathbb{N} = k+1$ so with $a_{\frac{k+1}{2},k+1} = a_{\frac{k-1}{2},k}$, we retrieve the equality at rank $k+1$. Finally, (9.30) is right and the $a_{i,k}$ are given by

$$\left\{\begin{array}{l} a_{0,k} = 1, \quad \forall k \in \mathbb{N} \\ a_{i,k+1} = a_{i,k} + (k-2i+2)a_{i-1,k}, \text{ if } 2 \le 2i \le k-1 \\ \left.\begin{array}{r} a_{\frac{k}{2},k+1} = a_{\frac{k}{2},k} + 2a_{\frac{k}{2}-1,k}, \text{ if } k \in 2\mathbb{N} \\ a_{\frac{k+1}{2},k+1} = a_{\frac{k-1}{2},k} \text{ if } k \in 2\mathbb{N}+1 \end{array}\right\} \text{ if } k \le 2i \le k+1 \end{array}\right. \tag{9.31}$$

Step 3 We are going to use the two formula (9.29) and (9.31) by setting $f_n(x) = (x_1^2+x_2^2)^n log(x_1^2+x_2^2) = \varphi_n\left(\frac{|x|^2}{2}\right)$ where $\varphi_n(r) = \psi_n(2r) = (2r)^n log(2r)$. By using the two recurrence formula proved in the steps 1 and 2 we have :

(i) If $k \le n$

$$\begin{aligned} \frac{\partial^k f_n}{\partial x_2^k} &= \sum_{0\le 2i\le k} a_{i,k}\varphi_n^{(k-i)}\left(\frac{|x|^2}{2}\right)x_2^{k-2i} \\ &= \sum_{0\le 2i\le k} a_{i,k}2^{k-i}\left[\frac{(k-i)!}{(n-k+i)!}\varphi_{n-k+i}\left(\frac{|x|^2}{2}\right) + C(k-i,n)|x|^{2(n-k+i)}\right]x_2^{k-2i} \end{aligned}$$

with $0 \le k-i \le n$. Particularly for $x_2 = 0$ and $x_1 \ne 0$, the only non-zero term is obtained for $k = 2i$:

$$\frac{\partial^k f_n}{\partial x_2^k} = \quad = a_{\frac{k}{2},k}2^{\frac{k}{2}}\left[\frac{(\frac{k}{2})!}{(n-\frac{k}{2})!}\varphi_{n-\frac{k}{2}}\left(\frac{|x|^2}{2}\right) + C(\frac{k}{2},n)|x|^{2(n-\frac{k}{2})}\right]\mathbb{1}_{\{k\in 2\mathbb{N}\}}$$

(ii) If $n < k < 2n+2$

$$\begin{aligned} \frac{\partial^k f_n}{\partial x_2^k} &= \sum_{0\le 2i\le k} a_{i,k}\varphi_n^{(k-i)}\left(\frac{|x|^2}{2}\right)x_2^{k-2i} \\ &= \sum_{0\le 2i\le k} a_{i,k}2^{k-i}\left[\frac{(k-i)!}{(n-k+i)!}\varphi_{n-k+i}\left(\frac{|x|^2}{2}\right) + C(k-i,n)|x|^{2(n-k+i)}\right]x_2^{k-2i}\mathbb{1}_{\{i\ge k-n\}} \\ &+ \sum_{0\le 2i\le k} a_{i,k}2^{k-i}\frac{(-1)^{k-i-n-1}n!(k-i-n-1)!}{|x|^{2(k-i-n)}}\mathbb{1}_{\{i<k-n\}}x_2^{k-2i} \end{aligned}$$

with

$$(0 \le k-i < 2n+2) \wedge (k-i \le n) \Longleftrightarrow (i \ge k-n)$$

and where we have used (9.29). Particularly for $x_2 = 0$ and $x_1 \ne 0$ the only non-zero term is obtained for $k = 2i$ and in this case $i \le k-n$:

$$\frac{\partial^k f_n}{\partial x_2^k}(x_1,0) = a_{\frac{k}{2},k}2^{\frac{k}{2}}\left[\frac{(\frac{k}{2})!}{(n-\frac{k}{2})!}\varphi_{n-\frac{k}{2}}\left(\frac{|x|^2}{2}\right) + C(\frac{k}{2},n)|x|^{2(n-\frac{k}{2})}\right]\mathbb{1}_{\{k\in 2\mathbb{N}\}}$$

(iii) If $k = 2n+2$

$$\frac{\partial^{2n+2} f_n}{\partial x_1^{2n+2}} = \sum_{0\le 2i\le 2n+2} a_{i,2n+2}\varphi_n^{(2n+2-i)}\left(\frac{|x|^2}{2}\right) x_2^{2n+2-2i}$$
$$= \sum_{0\le i\le n+1} a_{i,2n+2}\varphi_n^{(2n+2-i)}\left(\frac{|x|^2}{2}\right) x_2^{2(n-i+1)}$$

$0 \le i \le n+1$ so $2n-i+2 \ge n+1 > n$ and from the recurrence formula (9.29) for $k > n$:

$$\varphi_n^{(2n+2-i)}\left(\frac{|x|^2}{2}\right) = \frac{2^{2n+2-i} n!(n+1-i)!(-1)^{n+1-i}}{|x|^{2(n+2-i)}}$$

therefore

$$\frac{\partial^{2n+2} f_n}{\partial x_1^{2n+2}} = \sum_{0\le i\le n+1} a_{i,2n+2}\frac{2^{2n+2-i} n!(n+1-i)!(-1)^{n+1-i}}{|x|^{2(n+2-i)}} x_2^{2(n-i+1)}$$

For $x_2 = 0$ and $x_1 \neq 0$ the only-non zero term is obtained for $i = n+1$:

$$\frac{\partial^{2n+2} f_n}{\partial x_1^{2n+2}}(x_1, 0) = \frac{2^{n+1} n! a_{n+1,2n+2}}{x_1^2} \tag{9.32}$$

Step 4 : We compute the coefficients $a_{i,k}$ defined by the recurrence formula (9.31). We conjecture the following formula :

$$a_{i,k} = \frac{k!}{2^i i!(k-2i)!} \tag{9.33}$$

Let us prove this equality for $(i,k) \in \left\{(i,k) \in \mathbb{N}^2, 2i \le k\right\}$. Let $\mathcal{P}(k) = \left\{a_{i,k} = \frac{k!}{2^i i!(k-2i)!}, \forall 2i \le k\right\}$.
$\mathcal{P}(0)$ is easily checked.
Let us suppose $\mathcal{P}(k)$ and show $\mathcal{P}(k+1)$. From (9.31) :

(i) for $i = 0$ we have $a_{0,k+1} = 1 = \frac{(k+1)!}{(k+1)!}$

(ii) for $2 \le 2i \le k-1$ we have :

$$a_{i,k+1} = a_{i,k} + (k-2i+2)a_{i-1,k} = \frac{k!}{2^i i!(k-2i)!} + (k-(2i-2))\frac{k!}{2^{i-1}(i-1)!(k-(2i-2))!}$$
$$= \frac{k!}{2^{i-1}(i-1)!(k-2i)!}\left(\frac{1}{2i} + \frac{1}{k-2i+1}\right)$$
$$= \frac{(k+1)!}{2^i i!((k+1)-2i)!}$$

(iii) for $k \le 2i \le k+1$ we have :

(a) if $k \in 2\mathbb{N}$:

$$a_{\frac{k}{2},k+1} = a_{\frac{k}{2},k} + 2a_{\frac{k}{2}-1,k} = \frac{k!}{2^{\frac{k}{2}}\frac{k}{2}!0!} + 2\frac{k!}{2^{\frac{k}{2}-1}(\frac{k}{2}-1)!2!} = \frac{k!}{2^{\frac{k}{2}}(\frac{k}{2}-1)!2!}\left(\frac{2}{k}+2\right)$$
$$= \frac{(k+1)!}{2^{\frac{k}{2}}\frac{k}{2}!}$$

(b) if $k \in 2\mathbb{N}+1$:

$$a_{\frac{k+1}{2},k+1} = a_{\frac{k-1}{2},k} = \frac{k!}{2^{\frac{k-1}{2}}\left(\frac{k-1}{2}\right)!\left(k-2\frac{k-1}{2}\right)!} = \frac{k!}{2^{\frac{k-1}{2}}\left(\frac{k-1}{2}\right)!} = \frac{k+1}{2^{\frac{k+1}{2}}}\frac{k!}{2^{\frac{k-1}{2}}\left(\frac{k-1}{2}\right)!}$$
$$= \frac{(k+1)!}{2^{\frac{k+1}{2}}\left(\frac{k+1}{2}\right)!}$$

Hence $\mathcal{P}(k+1)$ is right and $\mathcal{P}(n)$ is right for all $n \in \mathbb{N}$.

Now we come back to (9.32) and we apply (9.33) for $(i,k) = (n+1, 2n+2)$:

$$\frac{\partial^{2n+2} f_n}{\partial x_1^{2n+2}}(x_1, 0) = \frac{2^{n+1} n! a_{n+1,2n+2}}{x_1^2} = \frac{2(2n+1)!}{x_1^2}$$

which ends the proof of this lemma. □

Lemme 9.10. *Let Ω a bounded domain of $\mathbb{R}^2$, $f \in H^\alpha(\partial\Omega)$ with $\alpha > 0$, and let $V_k(f) : H^\alpha(\partial\Omega) \to H^{\alpha+2m-k}(\partial\Omega)$ for $k \in [\![1, 2m]\!]$ the multi-layer potentials :*

$$V_k(f) = \begin{cases} \displaystyle\int_{\partial\Omega} f(y) \frac{\partial^{k-1}(E(x-y))}{\partial n_y^{k-1}} d\sigma(y) \text{ for } 1 \le k \le m \\ \displaystyle\int_{\partial\Omega} f(y) A_{k-m-1,y}(E(x-y)) d\sigma(y) \text{ for } m+1 \le k \le 2m \end{cases}$$

where $E(x)$ is the fundamental solution of $-\Delta^m$ given in (2.2). By denoting

$$V_k^-(f)(x) = V_k(f)(x) \text{ for } x \in \Omega$$
$$V_k^+(f)(x) = V_k(f)(x) \text{ for } x \in \Omega^c$$

and for $x \in \partial\Omega$ we set

$$V_k^+(f)(x) = \lim_{\substack{x \to y \\ x \in \Omega^c}} V_k^+(f)(y), \qquad V_k^-(f)(x) = \lim_{\substack{x \to y \\ x \in \Omega}} V_k^-(f)(y)$$

We have the following jump relations across $\partial\Omega$ and for $x \in \partial\Omega$:

$$(\mathcal{R}_1) \begin{cases} \frac{\partial^k}{\partial n^k} V_j^+(f)(x) = \frac{\partial^k}{\partial n^k} V_j^-(f)(x) = \frac{\partial^k}{\partial n^k} V_j(f)(x), \text{ in } H^{\alpha+2m-k-j}(\partial\Omega) \text{ for } k \in [\![0, m-1]\!] \wedge j \ne 2m-k \\ \frac{\partial^k}{\partial n^k} V_{2m-k}^{\pm}(f)(x) = \pm\frac{(-1)^{m+1}}{2} f(x) + V_{2m-k}(f), \text{ in } H^\alpha(\partial\Omega) \text{ for } k \in [\![0, m-1]\!] \end{cases}$$

$$(\mathcal{R}_2) \begin{cases} A_k V_j^{\pm}(f)(x) = A_k V_j(f)(x), \text{ dans } H^{\alpha+m-k-j}(\partial\Omega) \text{ for } k \in [\![0, m-1]\!] \wedge j \ne m-k \\ A_k V_{m-k}^{\pm}(f)(x) = \pm\dfrac{(-1)^m}{2} f(x) + A_k V_{m-k}(f)(x), \text{ dans } H^\alpha(\partial\Omega) \text{ for } k \in [\![0, m-1]\!] \end{cases}$$

where have denoted respectively in $H^{\alpha+2m-k-j}(\partial\Omega)$ and $H^{\alpha+m-k-j}(\partial\Omega)$ the following potentials (when kernels are singular for $x \in \partial\Omega$, these boundary integrals are defined in the sense of the main Cauchy value) for $k \in [\![0, m-1]\!]$ and $j \in [\![1, 2m]\!]$:

$$\frac{\partial^k}{\partial n_x^k} V_j(f)(x) = \begin{cases} \displaystyle\int_{\partial\Omega} \frac{\partial^k}{\partial n^k} \frac{\partial^{j-1}}{\partial n_y^{j-1}} (E(x-y)) f(y) d\sigma(y) \text{ for } 1 \le j \le m \\ \displaystyle\int_{\partial\Omega} \frac{\partial^k}{\partial n_x^k} A_{j-m-1,y}(E(x-y)) f(y) d\sigma(y) \text{ for } m+1 \le j \le 2m \end{cases}$$

and

$$A_kV_j(f)(x) = \begin{cases} \displaystyle\int_{\partial\Omega} A_{k,x}\frac{\partial^{j-1}}{\partial n_y^{j-1}}(E(x-y))f(y)d\sigma(y) \text{ for } 1 \le j \le m \\ \displaystyle\int_{\partial\Omega} A_{k,x}A_{j-m-1,y}(E(x-y))f(y)d\sigma(y) \text{ for } m+1 \le j \le 2m \end{cases}$$

Proof. The proof of this lemma is in part heuristic and is inspired from [[7] pp 384-388]. We show the jump properties for particular densities which are in the following space (the image of the Calderòn projector associated to the operator Δ^m) :

$$\mathcal{P} = \left\{(u_{|\partial\Omega}, \partial_n u_{|\partial\Omega}, ..., \partial_n^{m-1}u_{|\partial\Omega}, A_0u_{|\partial\Omega}, ..., A_{m-1}u_{|\partial\Omega}), u \in H^m(\Omega) \text{ and } \Delta^m u = 0\right\}$$

Moreover, we use in this proof the kernel expressions in the case of a straight manifold to discuss about the singularity of the kernels. From Theorem 1.1 we have the following equality :

$$\begin{aligned} u(x) &= \int_\Omega \delta(x-y)u \\ &= -\int_\Omega \Delta^m u E(x-y)dy + \sum_{i=0}^{m-1}\int_{\partial\Omega}\left\{A_{i,y}(E(x-y))\frac{\partial^{m-1-i}u(y)}{\partial n^{m-1-i}} - A_{i,y}(u)\frac{\partial^{m-1-i}E(x-y)}{\partial n^{m-1-i}}\right\}d\sigma(y) \end{aligned} \tag{9.34}$$

Let $w \in H^{2m}(\Omega)$ such that $\Delta^m w = 0$ in Ω. Let $F_k \in H^{\alpha+2m-k}(\partial\Omega)$ for $1 \le k \le 2m$ such that

$$\begin{cases} \Delta^m w = 0, \text{ in } \Omega \\ \quad F_k = -A_{m-k}(w), \text{ on } \partial\Omega \text{ for } 1 \le k \le m \\ F_{2m-k} = \dfrac{\partial^k w}{\partial n^k}, \text{ on } \partial\Omega \text{ for } 0 \le k \le m-1 \end{cases} \tag{9.35}$$

For $x \in \Omega$, thanks to (9.34), we get

$$w(x) = \sum_{i=1}^{2m} V_i^-(F_i)(x) \tag{9.36}$$

First we extend the vectors fields $\vec{n}(x)$ and $\vec{\tau}(x)$ in C^∞ vectors fields on a neighbourhood of $\partial\Omega$ that we denote respectively $\widetilde{n}(x)$ and $\widetilde{\tau}(x)$. Thus we extend the operators $f \mapsto \frac{\partial^{2m-1-k}f}{\partial n^{2m-1-k}}$ and $f \mapsto A_k(f)$ for $k \in [\![m, 2m-1]\!]$ by using the vectors fields $\widetilde{n}(x)$ and $\widetilde{\tau}(x)$.

(i) **Proof of the relations** $(\mathcal{R}_1)$
Consider $0 \le k \le m-1$, from (9.36), we get

$$\frac{\partial^k w}{\partial\widetilde{n}^k} = (-1)^m\sum_{i=1}^{2m}\frac{\partial^k}{\partial\widetilde{n}^k}V_i^-(F_i)(x), \quad x \in \Omega$$

Letting x tend nontangentially to $\partial\Omega$ in (9.34), we have only half of the contribution of the Dirac function $\delta(x-y)$ at $y = x \in \partial\Omega$ so

$$\frac{1}{2}\frac{\partial^k w}{\partial n^k}(x) = (-1)^m\sum_{i=1}^{2m}\frac{\partial^k}{\partial n^k}V_i(F_i)(x), \quad x \in \partial\Omega \tag{9.37}$$

Thanks to Lemma 9.5 and Theorem 1.1 we get the following singularities estimations in the case of a straight manifold

$$\begin{aligned}
&\text{for } k+j<2m \quad \frac{\partial^k}{\partial n^k}V_j^-(x)=O\left(|x-y|^{2m-1-(k+j)}log(|x-y|\right)\\
&\text{for } k+j\geq 2m \quad \frac{\partial^k}{\partial n^k}V_j^-(x)=O\left(\frac{1}{|x-y|^{k+j+1-2m}}\right)
\end{aligned}$$

We deduce that

(a) For $k+j<2m$ kernels are integrable for $x\in\partial\Omega$ and we can switch the integral and the limit symbols

$$\frac{\partial^k}{\partial n^k}V_j^-(x)=\frac{\partial^k}{\partial n^k}V_j(x)$$

(b) For $k+j>2m$ thanks to [[7], Lemma 6.4], we get the same equality in the sense of the main Cauchy value.

By using (9.35), (9.37) is only possible if

$$\begin{cases}
\dfrac{\partial^k}{\partial n^k}V_{2m-k}^-(F_{2m-k})(x)=\dfrac{(-1)^m}{2}F_{2m-k}(x)+\dfrac{\partial^k}{\partial n^k}V_{2m-k}(F_{2m-k})(x)\\
\dfrac{\partial^k}{\partial n^k}V_j^-(x)=\dfrac{\partial^k}{\partial n^k}V_j(x),\ \text{for } 1\leq j\leq 2m \text{ and } j\neq 2m-k
\end{cases}$$

For $x\in B_R\backslash\overline{\Omega}$, (9.36) is changed in

$$w(x)=-\sum_{i=1}^{2m}V_i^+(F_i)(x)+G(x)$$

with $G(x)$ defined similarly by using multi-layer potentials using the boundary ∂B_R and regular in a neighbourhood of $\partial\Omega$. By the same reasoning we have

$$\begin{cases}
\dfrac{\partial^k}{\partial n^k}V_{2m-k}^+(F_{2m-k})(x)=-\dfrac{(-1)^m}{2}F_{2m-k}(x)+\dfrac{\partial^k}{\partial n^k}V_{2m-k}(F_{2m-k})(x)\\
\dfrac{\partial^k}{\partial n^k}V_j^+(x)=\dfrac{\partial^k}{\partial n^k}V_j(x),\ \text{for } 1\leq j\leq 2m \text{ and } j\neq 2m-k
\end{cases}$$

which proves the relations $(\mathcal{R}_1)$.

(ii) **Proof of relations** $(\mathcal{R}_2)$

Consider $0\leq k\leq m-1$, similarly we get for $x\in\partial\Omega$:

$$\frac{A_kw(x)}{2}=(-1)^m\sum_{i=1}^{2m}A_kV_i(F_i)(x),\qquad x\in\partial\Omega$$

And by the same reasoning used for (i), and by using (9.35), this equality is only possible if

$$\begin{cases} A_k V^-_{m-k}(F_{m-k})(x) = -\dfrac{(-1)^m}{2} F_{m-k}(x) + A_k V_{m-k}(F_{m-k})(x) \\ \qquad A_k V^-_j(F_j)(x) = A_k V_j(F_j)(x), \text{ for } 1 \leq j \leq 2m \text{ and } j \neq m-k \end{cases}$$

By the same way we get

$$\begin{cases} A_k V^+_{m-k}(F_{m-k})(x) = \dfrac{(-1)^m}{2} F_{m-k}(x) + A_k V_{m-k}(F_{m-k})(x) \\ \qquad A_k V^+_j(F_j)(x) = A_k V_j(F_j)(x), \text{ for } 1 \leq j \leq 2m \text{ and } j \neq m-k \end{cases}$$

which ends the proof of the relations $(\mathcal{R}_2)$ and the lemma's one.

□

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