# Computing periods of rational integrals

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# COMPUTING PERIODS OF RATIONAL INTEGRALS

## OBJECTIVE

*Input* $y(t) = \oint \mathrm{R}(t, z_0, \dots, z_n)\mathrm{d}z_0 \cdots \mathrm{d}z_n$

- Multiple integral
- of a rational function
- over a closed path.
- **This is period of rational integral.**

*Output* $c_m(t)y^{(m)}(t) + \cdots + c_0(t)y(t) = 0$

- Linear differential equation for $y$
- with polynomial coefficients.
- **This is the Picard-Fuchs equation.**

*[**Classical theorem** — Such a differential equation always exists.]*

## Computing a differential equation satisfied by a function ***is*** computing this function.

A differential equation and initial condition:

- defines uniquely the function,
- allows numerical evaluation, fast and with arbitrary precision,
- allows to compute asymptotics and power series expansions,
- can prove identities.

## MOTIVATIONS

• **Multiple binomial sums** can be computed as periods of rational functions through integral representations of generating functions. For example, with $u_n = \sum_{i=0}^{n}\sum_{j=0}^{n}\binom{i+j}{j}^2\binom{4n-2i-2j}{2n-2i}$

$$\sum_{n\geqslant 0} u_n t^n = \frac{1}{(2i\pi)^2}\oint \frac{(y-1)y^3}{(t-y^2(y-1)^2)((x-1)t - xy^2(y^2+x-1))}$$

The computation of the Picard-Fuchs equation gives a proof of the following identity:

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{i+j}{j}^2\binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}^2.$$

This is **end-to-end automated**, *cf.* Bruno Salvy's talk.

• In the context of **mirror symmetry**, topological invariants of a given variety, like Gromov-Witten invariants, can be computed with a Picard-Fuchs equation related to the mirror family.
⇝ Big integrals, hard to compute.

• And also combinatorics, number theory, physics.

## ELEMENTARY RELATIONS

**Integral of a derivative** $\displaystyle\oint \sum_{i=0}^{n}\frac{\partial \mathrm{C}_i}{\partial z_i}\mathrm{d}z_0\cdots\mathrm{d}z_m = 0$

**Integration by parts** $\displaystyle\oint \mathrm{F}\frac{\partial \mathrm{G}}{\partial z_i}\mathrm{d}z_0\cdots\mathrm{d}z_n = -\oint \frac{\partial \mathrm{F}}{\partial z_i}\mathrm{G}\,\mathrm{d}z_0\cdots\mathrm{d}z_m$

**Differentiation under $\oint$** $\displaystyle\frac{\partial}{\partial t}\oint \mathrm{F}\,\mathrm{d}z_0\cdots\mathrm{d}z_m = \oint \frac{\partial \mathrm{F}}{\partial t}\mathrm{d}z_0\cdots\mathrm{d}z_m$

## EXAMPLE

Perimeter of an ellipse as a function of its excentricity:

$$p(e) = \frac{1}{2i\pi}\oint \frac{\mathrm{d}x\mathrm{d}y}{1-\frac{1-e^2x^2}{(1-x^2)y^2}}$$

Then $(e-e^3)p'' + (1-e^2)p' + ep = 0$ (Euler, 1733)

*Proof.*

$$\overbrace{\left((e-e^3)\partial_e^2 + (1-e^2)\partial_e + e\right)}^{\text{operator in } e \text{ and } \partial_e \text{ only, commutes with } \oint} \cdot \overbrace{\left(\frac{1}{1-\frac{1-e^2x^2}{(1-x^2)y^2}}\right)}^{\text{the rational function being integrated}} =$$

$$\underbrace{\partial_x\left(-\frac{e\left(-1-x+x^2+x^3\right)y^2\left(-3+2x+y^2+x^2\left(-2+3e^2-y^2\right)\right)}{\left(-1+y^2+x^2(e^2-y^2)\right)^2}\right) + \partial_y\left(\frac{2e\left(-1+e^2\right)x\left(1+x^3\right)y^3}{\left(-1+y^2+x^2(e^2-y^2)\right)^2}\right)}_{\text{sum of derivatives, its integral over a closed path is zero}}$$

*QED.*

## NOTATIONS

- $\mathrm{R}(t, z_0, \dots, z_n) = a/f$ a rational function
- $\mathbb{K} = \mathbb{C}(t)$, the base field
- $\partial_i$ denotes the partial differentiation $\partial/\partial z_i$
- $\mathrm{R} \mapsto \oint \mathrm{R}\mathrm{d}\mathbf{z}$ denotes really the canonical $\mathbb{K}$-linear surjection

$$\mathbb{K}[z_0, \dots, z_n, \tfrac{1}{f}] \to \frac{\mathbb{K}[z_0, \dots, z_n, \frac{1}{f}]}{\sum_{i=0}^{n}\partial_i\mathbb{K}[z_0, \dots, z_n, \frac{1}{f}]}.$$

## HOMOGENEITY

Up to homogenization with a new variable, we can assume that R is homogeneous of degree $-n-1$, that is

$$\mathrm{R}(t, \lambda z_0, \dots, \lambda z_n) = \lambda^{-n-1}\mathrm{R}(t, z_0, \dots, z_n),$$

or equivalently,

$$\mathrm{R}(t, \lambda z_0, \dots, \lambda z_n)\mathrm{d}(\lambda z_0)\cdots\mathrm{d}(\lambda z_n) = \mathrm{R}(t, z_0, \dots, z_n)\mathrm{d}z_0\cdots\mathrm{d}z_n.$$

**Confinement** — For all polynomials $f$ and all integers $q$, the space of homogeneous fractions of degree $-n-1$ of the form $a/f^q$ is finite dimensional.

## CONTRIBUTIONS

- New and fast algorithm.
- Uses only commutative Gröbner basis and linear algebra, rather than $\mathcal{D}$-module theory.
- Based on a extension of Griffiths-Dwork reduction which handles singular cases.
- Implementation available in Magma.
  https://github.com/lairez/periods.git
- Computes quickly integrals that were previously out-of-reach.

## REDUCTION OF POLE ORDER

**Principle** — Rewrite $\oint \frac{*}{f^q}\mathrm{d}\mathbf{z}$ as $\oint \frac{*}{f^{q-1}}\mathrm{d}\mathbf{z}$ using integration by parts.

**How ?** — If $a = b_0\partial_0 f + \cdots b_n\partial_n f$, for some polynomials $b_i \in \mathbb{K}[\mathbf{z}]$, then

$$\oint \frac{a}{f^q}\mathrm{d}\mathbf{z} = \underbrace{\oint \sum_{i=0}^{n} b_i\frac{\partial_i f}{f^q}\mathrm{d}\mathbf{z} = \tfrac{1}{q-1}\oint \sum_{i=0}^{n}\frac{\partial_i b_i}{f^{q-1}}}_{\text{integration by parts}}.$$

**Rewriting rule** $\displaystyle\frac{\sum_i b_i\partial_i f}{f^q} \longrightarrow \frac{1}{q-1}\frac{\sum_i \partial_i b_i}{f^{q-1}}$

By contruction, if R rewrites into R′ by repeated application of the rewriting rule, then

$$\oint \mathrm{R}\mathrm{d}\mathbf{z} = \oint \mathrm{R}'\mathrm{d}\mathbf{z}.$$

**Definition** $a/f^q$ is *reducible* if there exists $b$ such that

$$\frac{a}{f^q} \longrightarrow \frac{b}{f^{q-1}}.$$

**Nota bene** $a/f^q$ is reducible if and only if $a$ is in the jacobian ideal of $f$, which is $\langle\partial_0 f, \dots, \partial_n f\rangle$.

## GRIFFITHS-DWORK THEOREM

Assume that $f$ defines a **smooth** hypersurface in the projective space $\mathbb{P}^n(\mathbb{K})$. Let $a/f^q$ a homogeneous fraction of degree $-n-1$.

- If $q > n$, then $a/f^q$ is reducible.
  In particular, there exists a polynomial $b$ such that
  $$\oint \frac{a}{f^q}\mathrm{d}\mathbf{z} = \oint \frac{b}{f^n}\mathrm{d}\mathbf{z}.$$
- If $\oint \frac{a}{f^q}\mathrm{d}\mathbf{z} = 0$ then $a/f^q$ is reducible.

## ALGORITHM, SMOOTH CASE

1. For each $k \geqslant 0$, compute $b_k$ such that
   $$\frac{\partial^k}{\partial t^k}\frac{a}{f} \longrightarrow \cdots \longrightarrow \frac{b_k}{f^n}.$$
   *[ Always possible, by Griffiths-Dwork theorem. ]*
2. Find a linear dependency relation over $\mathbb{K} = \mathbb{C}(t)$:
   $$c_0(t)\frac{b_0}{f^n} + \cdots + c_m(t)\frac{b_m}{f^n} = 0.$$
   *[ Exists for m large enough, by the confinement property. ]*
3. When found, return $c_m(t)y^{(m)}(t) + \cdots + c_0(t)y(t) = 0$.

# What if $f$ is *not* smooth?

*Generically* $f$ is smooth. In *practice*, most interesting integrals, if not all, have a **singular** $f$.
*generic case ≠ usual case*

## DIMCA'S THEOREM

Without assumptions on $f$, let $a/f^q$ be a homogeneous fraction of degree $-n-1$. Then there exists a polynomial $b$ such that

$$\oint \frac{a}{f^q}\mathrm{d}\mathbf{z} = \oint \frac{b}{f^n}\mathrm{d}\mathbf{z}.$$

But how to compute $b/f^n$ from $a/f^q$?

## REFERENCES


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- Alexandru Dimca. "On the de Rham cohomology of a hypersurface complement". In: *Amer. J. Math.* 113.4 (1991), pp. 763–771
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# Reduction rules for general case

## EXTENSION OF GRIFFITHS-DWORK REDUCTION

Consider again the rewriting rule

$$\frac{\sum_i b_i\partial_i f}{f^{q+1}} \longrightarrow \frac{1}{q}\frac{\sum_i \partial_i b_i}{f^q}.$$

The decomposition of a polynomial as $\sum_i b_i\partial_i f$ is not unique: zero decomposes non trivially. If $\sum_i b_i\partial_i f = 0$, then what about

$$\text{« } 0 \longrightarrow \frac{1}{q}\frac{\sum_i \partial_i b_i}{f^q} \text{ »?}$$

*Not a reduction* but indicates that

$$\oint \frac{\sum_i \partial_i b_i}{f^q}\mathrm{d}\mathbf{z} = 0$$

But it is possible that $(\sum_i \partial_i b_i)/f^q$ is not reducible!

**New rule** If $\displaystyle\sum_{i=0}^{n} b_i\partial_i f = 0$, then $\displaystyle\frac{\sum_i \partial_i b_i}{f^q} \longrightarrow 0$

Rank 1 rules $\overset{\text{def}}{=}$ Griffiths-Dwork rules
Rank 2 rules $\overset{\text{def}}{=}$ Rank 1 rules + New rules above

## FURTHER EXTENSION

Sometimes $\displaystyle\frac{a}{f^q} \overset{\text{rk 1}}{\longrightarrow} \frac{b}{f^{q-1}} \overset{\text{rk 2}}{\longrightarrow} 0$ but $b/f^{q-1}$ is not reducible at rank 2.

**New rule** If $\displaystyle\frac{a}{f^q} \overset{\text{rk 1}}{\longrightarrow} \frac{b}{f^{q-1}} \overset{\text{rk 2}}{\longrightarrow} 0$ then $\displaystyle\frac{b}{f^{q-1}} \longrightarrow 0$

Rank 3 rules $\overset{\text{def}}{=}$ Rank 2 rules + New rules above
And so on at rank 4, 5…

**Definition** $a/f^q$ is *reducible at rank r* if there exists $b$ such that

$$\frac{a}{f^q} \overset{\text{rk } r}{\longrightarrow} \frac{b}{f^{q-1}}.$$

## EXAMPLE

What do the new reduction rules bring?

$$f = 2xyz(w-x)(w-y)(w-z) - w^3(w^3 - w^2z + xyz) \quad (n=3)$$

$e(q,r) \overset{\text{def}}{=}$ # of independent rational homogeneous functions $a/f^q$ which are not reducible at rank $r$

| $q$ | 0 | 1 | 2 | 3 | 4 | $q>4$ |
|---|---|---|---|---|---|---|
| $e(q,0)$ | 0 | 10 | 165 | 680 | 1771 | $\sim 36q^3$ |
| $e(q,1)$ | 0 | 10 | 86 | 102 | 120 | $\sim 18q$ |
| $e(q,2)$ | 0 | 10 | 7 | 6 | 6 | 6 |
| $e(q,3)$ | 0 | 9 | 1 | 0 | 0 | 0 |

Interpretation of the reduction rules in terms of a *spectral sequence.*

## THEOREM

Without assumptions on $f$, there exists an integer $r$ depending only on $f$ such that for all homogeneous fraction $a/f^q$ of degree $-n-1$:

- If $q > n$, then $a/f^q$ is reducible at rank $r$.
- If $\oint \frac{a}{f^q}\mathrm{d}\mathbf{z} = 0$ then $a/f^q$ is reducible at rank $r$.

*[Compare with Griffiths-Dwork theorem.]*

## ALGORITHM, GENERAL CASE

1. Set $r = 1$.
2. For each $k \geqslant 0$, compute $b_k$, if possible, such that
   $$\frac{\partial^k}{\partial t^k}\frac{a}{f} \overset{\text{rk } r}{\longrightarrow} \cdots \overset{\text{rk } r}{\longrightarrow} \frac{b_k}{f^n}.$$
   If not possible, increase $r$ and start over.
   *[ Always possible for r large enough, by the theorem above. ]*
3. Find a linear dependency relation over $\mathbb{K} = \mathbb{C}(t)$:
   $$c_0(t)\frac{b_0}{f^n} + \cdots + c_m(t)\frac{b_m}{f^n} = 0.$$
   *[ Exists for m large enough, by the confinement property. ]*
4. When found, return $c_m(t)y^{(m)}(t) + \cdots + c_0(t)y(t) = 0$.

*[No* a priori *bound on r. Conjecture: r stays $\leq n+1$.]*

## FAST REDUCTION

**Rank 1** — Gröbner basis technology, applied to the ideal $\langle\partial_0 f, \dots, \partial_n f\rangle$
**Rank 2** — With Gröbner basis, computation of Koszul homology:

$$\frac{\text{syzygies}}{\text{trivial syzygies}} = \frac{\left\{(b_0, \dots, b_n) \in \mathbb{K}[z_0, \dots, z_n]^{n+1} \;\middle|\; \sum_{i=0}^{n} b_i\partial_i f = 0\right\}}{\mathbb{K}[z_0, \dots, z_n]\left\langle(\dots, \underbrace{\partial_j f}_{i^{\text{th}}\text{ pos.}}, \dots, \underbrace{-\partial_i f}_{j^{\text{th}}\text{ pos.}}, \dots)\right\rangle}.$$

Trivial syzygies do not bring new rules.
When $f$ is generic, all syzygies are trivial.
**Rank ⩾ 3** — Linear algebra