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# CONSTRAINED EFFICIENCY AND EQUILIBRIUM WITHOUT COMMITMENT<sup>1</sup>

V. Filipe Martins-da-Rocha<sup>a</sup> and Yiannis Vailakis<sup>b</sup>

We consider an infinite horizon economy where agents share income risks by trading a complete set of contingent claims but cannot commit to their promises. Allocations are restricted to be self-enforcing relative to autarchic reservation utilities. Under the assumption of uniform gains to trade, we characterize constrained Pareto efficiency, prove a constrained version of the Second Welfare Theorem and establish the existence of a constrained competitive equilibrium as defined by [Kehoe and Levine \(1993\)](#). Our results extend those in [Bloise and Reichlin \(2011\)](#) in several aspects.

## 1. INTRODUCTION

The paper studies infinite horizon exchange economies with complete contingent claims markets when there is no commitment and default induces permanent exclusion from future trading. As in [Eaton and Gersovitz \(1981\)](#) and [Kehoe and Levine \(1993\)](#), trade is subject to participation constraints that restrict allocations to be self-enforcing relative to autarchic reservation utilities.<sup>1</sup> The presence of such constraints and the associated imperfect risk-sharing imply that the economy cannot attain a social optimum. An important issue is then to explore under which conditions an allocation is constrained optimal, that is, to identify necessary and sufficient conditions that rule out benefits from redistributions given the participation constraints.

Following the classical approach in general equilibrium, [Bloise and Reichlin \(2011\)](#) provide an interesting treatment of this matter by characterizing constrained efficiency in terms of supporting linear functionals. They show that under uniform gains to trade, support by a linear functional is a necessary and sufficient condition for interior (uniformly bounded away from zero) allocations to be constrained efficient.<sup>2</sup> Furthermore, they show that any supporting linear functional admits a sequential representation, in

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<sup>1</sup>[Alvarez and Jermann \(2000\)](#) propose a sequential formulation of this model where agents trade a complete set of contingent bonds every period. The focus is on endogenously determined agent-specific debt limits that correspond to participation constraints at autarchic reservation utilities. Such limits on borrowing are referred in the literature as not-too-tight debt constraints.

<sup>2</sup>[Bloise and Reichlin \(2011\)](#) also provide an analysis of constrained efficiency in the absence of uniform gains to trade (but still restricting attention to allocations that are uniformly bounded away from zero). They identify a strong connection between constrained Pareto efficiency and a modified version of the Cass criterion originally proposed by [Cass \(1972\)](#) and others for stochastic overlapping generations economies. However, they only proved that the Modified Cass Criterion is a sufficient and necessary condition for constrained Pareto inefficiency, under the assumption of uniform gains to trade.

the sense that its purely additive part (bubble component) is null. This allows them to compare their characterization result with that provided by [Alvarez and Jermann \(2000\)](#) formulated in terms of high implied interest rates (i.e., finite present value of intertemporal aggregate endowment).<sup>3</sup>

This paper aims at looking on constrained Pareto efficiency from a fresh perspective. The main objective is to provide a complete characterization. More precisely, we show that under uniform gains to trade, high implied interest rates are necessary and sufficient for constrained Pareto efficiency, even if we dispense with the interiority restriction imposed in [Bloise and Reichlin \(2011\)](#). The novel aspects of our approach are as follows. First, to prove sufficiency, there is no need to assume uniform gains to trade. The proof relies on a new decentralization result that explores the concavity of Bernoulli functions and identifies sufficient conditions for state-contingent prices to belong to the constrained sub-differential of expected utilities. Second, to prove necessity, we show that if a linear functional supports a constrained Pareto efficient allocation, then it cannot have a bubble component and must coincide with the process of implied Arrow–Debreu prices (which turns out to display high interest rates).

Our contributions have direct implications for the validity of the Second Welfare Theorem. Indeed, standard arguments can be applied to show that any constrained Pareto optimal allocation is supported by some linear functional. Our analysis shows that this linear functional has no bubble component, and must coincide with the implied contingent-state prices. In particular, implied interest rates are high and our decentralization result can be applied to deduce that any constrained Pareto efficient allocation can be implemented as a constrained competitive equilibrium with high interest rates and endogenous transfers.

We also strengthen the Second Welfare Theorem by characterizing the set of consumption allocations implemented by constrained competitive equilibrium with high interest rates and zero initial transfers. We first prove that such equilibria always exist. A possible way to tackle existence is to employ the standard Negishi approach. Following this route one has to show that transfers are continuous functions of welfare weights. This is not a straightforward task since individual transfers involve an infinite sum. Combining the assumption of uniform gains to trade and our decentralization result, we show that the continuity of transfers is always ensured. We are also able to identify the subset of constrained Pareto efficient allocations that are implemented by constrained competitive equilibria with zero initial transfers. Inspired by the work of [Aliprantis et al. \(1987\)](#) we prove that this set coincides with the set of constrained Edgeworth equilibria.

The paper is organized as follows: Section 2 describes the environment and provides the definitions of the various concepts used throughout the paper. Section 3 contains our complete characterisation of constrained efficiency under uniform gains to trade.

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<sup>3</sup> [Bloise and Reichlin \(2011\)](#) show by means of an example that the assumption of uniform gains to trade is indispensable. Without it, constrained efficiency does not necessarily lead to high implied interest rates.

Section 4 proves a constrained version of the Second Welfare Theorem and establishes the existence of a constrained competitive equilibrium with zero initial transfers. One of the crucial assumption imposed in this paper is that there are uniform gains to trade. [Bloise and Reichlin \(2011\)](#) showed that this condition is always satisfied in stationary Markovian environments. Section 5 provides an alternative and simpler proof of this claim. Some technical results as well as a discussion on how our results differentiate from those in [Bloise and Reichlin \(2011\)](#) are collected in the Appendix.

## 2. THE MODEL

Here we present an infinite horizon general equilibrium model with lack of commitment and self-enforcing participation constraints, along the lines of [Kehoe and Levine \(1993\)](#) and [Bloise and Reichlin \(2011\)](#). Time and uncertainty are both discrete and there is a single non-storable consumption good. The economy consists of a finite set  $I$  of infinitely lived agents that share risks but cannot commit to future transfers.

### 2.1. Uncertainty

We use an event tree  $\Sigma$  to describe time, uncertainty and the revelation of information over an infinite horizon. There is a unique initial date-0 event  $s^0 \in \Sigma$  and for each date  $t \in \{0, 1, 2, \dots\}$  there is a finite set  $S^t \subset \Sigma$  of date- $t$  events  $s^t$ . Each  $s^t$  has a unique predecessor  $\sigma(s^t)$  in  $S^{t-1}$  and a finite number of successors  $s^{t+1}$  in  $S^{t+1}$  for which  $\sigma(s^{t+1}) = s^t$ . We use the notation  $s^{t+1} \succ s^t$  to specify that  $s^{t+1}$  is a successor of  $s^t$ . Event  $s^{t+\tau}$  is said to follow event  $s^t$ , also denoted  $s^{t+\tau} \succ s^t$ , if  $\sigma^{(\tau)}(s^{t+\tau}) = s^t$ . The set  $S^{t+\tau}(s^t) := \{s^{t+\tau} \in S^{t+\tau} : s^{t+\tau} \succ s^t\}$  denotes the collection of all date- $(t+\tau)$  events following  $s^t$ . Abusing notation, we let  $S^t(s^t) := \{s^t\}$ . The subtree of all events starting from  $s^t$  is then

$$\Sigma(s^t) := \bigcup_{\tau \geq 0} S^{t+\tau}(s^t).$$

We use the notation  $s^\tau \succeq s^t$  when  $s^\tau \succ s^t$  or  $s^\tau = s^t$ . In particular, we have  $\Sigma(s^t) = \{s^\tau \in \Sigma : s^\tau \succeq s^t\}$ .

### 2.2. Endowments and Preferences

Agents' endowments are subject to random shocks. We denote by  $e^i = (e^i(s^t))_{s^t \in \Sigma}$  agent  $i$ 's process of positive endowments  $e^i(s^t) > 0$  of the consumption good contingent to event  $s^t$ . Preferences over (non-negative) consumption processes  $c = (c(s^t))_{s^t \in \Sigma}$  are represented by the lifetime expected and discounted utility functional

$$U(c) := \sum_{t \geq 0} \beta^t \sum_{s^t \in S^t} \pi(s^t) u(c(s^t))$$

where  $\beta \in (0, 1)$  is the discount factor,  $\pi(s^t)$  is the unconditional probability of  $s^t$  and  $u : \mathbb{R}_+ \rightarrow [-\infty, \infty)$  is a Bernoulli function assumed to be strictly increasing, concave, continuous on  $\mathbb{R}_+$ , differentiable on  $(0, \infty)$ , bounded from above and satisfying Inada's condition at the origin.<sup>4</sup>

Given a date- $t$  event  $s^t$ , we denote by  $U(c|s^t)$  the lifetime continuation utility conditional to event  $s^t$ , defined by

$$U(c|s^t) := u(c(s^t)) + \sum_{\tau \geq 1} \beta^\tau \sum_{s^{t+\tau} \succ s^t} \pi(s^{t+\tau}|s^t) u(c(s^{t+\tau}))$$

where  $\pi(s^{t+\tau}|s^t) := \pi(s^{t+\tau})/\pi(s^t)$  is the conditional probability of  $s^{t+\tau}$  given  $s^t$ . We assume that  $U(e^i|s^0) > -\infty$  for every agent  $i$ .<sup>5</sup> Since the bernoulli function is bounded from above, we then get that  $U(e^i|s^t) > -\infty$  for all event  $s^t$ .

A collection  $(c^i)_{i \in I}$  of consumption processes is called an **allocation**. It is said to be **resource feasible** if  $\sum_{i \in I} c^i = \sum_{i \in I} e^i$ .

### 2.3. Self-Enforcing Consumption

A consumption process  $c^i$  may involve transfers contingent to an event  $s^t$  if  $c^i(s^t) < e^i(s^t)$ . We assume that agent  $i$  cannot commit to future transfers and has the option to walk away from a contract. We follow [Kehoe and Levine \(1993\)](#) (see also [Kocherlakota \(1996\)](#), [Alvarez and Jermann \(2000\)](#) and [Bloise and Reichlin \(2011\)](#)) and assume that autarky is the outside option for not fulfilling promises. A consumption process  $c^i$  is then said to be **self-enforcing**, if it satisfies the following participation constraints

$$U(c^i|s^t) \geq U(e^i|s^t), \quad \text{for all } s^t \succ s^0.$$

The set of consumption processes satisfying these constraints is denoted by  $\text{PC}^i$ . When the participation constraint is also satisfied at the initial event  $s^0$ , i.e.,  $U(c^i|s^0) \geq U(e^i|s^0)$ , then  $c^i$  is said to be **individually rational**.

### 2.4. Implied Interest Rates

A consumption process  $c^i$  is said to be **strictly positive** if  $c^i(s^t) > 0$  for every event  $s^t$ . In that case, we can define agent  $i$ 's marginal rate of substitution at event  $s^t$  by posing

$$\text{MRS}(c^i|s^t) := \beta \pi(s^t|\sigma(s^t)) \frac{u'(c^i(s^t))}{u'(c^i(\sigma(s^t)))}.$$

<sup>4</sup>The function  $u$  is said to satisfy the Inada's condition at the origin if  $\lim_{\varepsilon \rightarrow 0} [u(\varepsilon) - u(0)]/\varepsilon = \infty$ . This property is automatically satisfied if  $u(0) = -\infty$ . We assume that agents' preferences are homogenous. This is only for the sake of simplicity. All arguments can be adapted to handle the heterogenous case where the preference parameters  $(\beta, \pi, u)$  differ among agents.

<sup>5</sup>This assumption is automatically satisfied if either  $u(0) > -\infty$  or the allocation  $(e^i)_{i \in I}$  is uniformly bounded away from zero, in the sense that there exists  $\varepsilon > 0$  such that  $e^i(s^t) \geq \varepsilon$  for each agent  $i$  and event  $s^t$ .

Given a strictly positive allocation  $(c^i)_{i \in I}$ , we let  $p^* = (p^*(s^t))_{s^t \in \Sigma}$  be the process defined recursively by  $p^*(s^0) := 1$  and

$$\frac{p^*(s^t)}{p^*(\sigma(s^t))} := \max_{i \in I} \text{MRS}(c^i | s^t), \quad \text{for all } s^t \succ s^0.$$

Following [Alvarez and Jermann \(2000\)](#),  $p^*$  is called the process of **implied Arrow–Debreu prices**.

Given an arbitrary strictly positive process  $p = (p(s^t))_{s^t \succeq s^0}$  interpreted as Arrow–Debreu prices, we use  $\text{PV}(p; x | s^t)$  to denote the present value at date- $t$  event  $s^t$  of a process  $x$  restricted to the subtree  $\Sigma(s^t)$  and defined by

$$\text{PV}(p; x | s^t) := \frac{1}{p(s^t)} \sum_{s^{t+\tau} \in \Sigma(s^t)} p(s^{t+\tau}) x(s^{t+\tau}).$$

We say that  $p$  displays high interest rates when the present value of endowments under the price process  $p$  is finite, i.e.,  $\text{PV}(p; e^i | s^0) < \infty$ , for all  $i$ . A strictly positive allocation  $(c^i)_{i \in I}$  is said to have **high implied interest rates** when the implied Arrow–Debreu prices  $p^*$  display high interest rates.

## 2.5. Commodity and Price Space

Denote by  $\ell^\infty(e)$  the linear space of processes  $h \in \mathbb{R}^\Sigma$  satisfying

$$(2.1) \quad \exists \lambda \geq 0, \quad \forall s^t \in \Sigma, \quad |h(s^t)| \leq \lambda e(s^t)$$

where  $e := \sum_{i \in I} e^i$  is the process of aggregate endowments. The linear space  $\ell^\infty(e)$  is the natural commodity space since we necessarily have  $c^i \in \ell^\infty(e)$  for any resource feasible allocation  $(c^i)_{i \in I}$ .

**REMARK 2.1** Denote by  $\ell_+^\infty := \ell_+^\infty(\mathbf{1}_\Sigma)$  the space of non-negative processes that are uniformly bounded from above.<sup>6</sup> [Kehoe and Levine \(1993\)](#) and [Bloise and Reichlin \(2011\)](#) assume that endowments belong to  $\ell_+^\infty$  and restrict each agent to choose a consumption process in  $\ell_+^\infty$ . In addition, [Bloise and Reichlin \(2011\)](#) assume that the consumption and endowment processes are uniformly bounded away from zero.<sup>7</sup> In contrast to this literature, we do not impose any boundedness condition neither on endowments nor on consumption processes.

<sup>6</sup>For any subset  $A \subseteq \Sigma$ , we denote by  $\mathbf{1}_A$  the process  $x = (x(s^t))_{s^t \in \Sigma}$  defined by  $x(s^t) := 1$  if  $s^t \in A$  and  $x(s^t) := 0$  elsewhere.

<sup>7</sup>A process  $x = (x(s^t))_{s^t \in \Sigma}$  is said to be uniformly bounded away from zero whenever there exists  $\varepsilon > 0$  such that  $x(s^t) \geq \varepsilon$  for every  $s^t \in \Sigma$ . [Bloise and Reichlin \(2011\)](#) use the term “interior” for “uniformly bounded away from zero”. This is because a process  $x$  is uniformly bounded away from zero if, and only if, it belongs to the  $\|\cdot\|_{\mathbf{1}_\Sigma}$ -interior of  $\ell_+^\infty$ , where  $\|x\|_{\mathbf{1}_\Sigma} := \sup_{s^t \in \Sigma} |x(s^t)|$ .

We endow the space  $\ell^\infty(e)$  with the norm  $\|h\|_e$  defined as the lowest  $\lambda \geq 0$  satisfying (2.1). Equivalently, we have  $\|h\|_e := \sup_{s^t \in \Sigma} |h(s^t)/e(s^t)|$ . The cone of non-negative processes in  $\ell^\infty(e)$  is denoted by  $\ell_+^\infty(e)$ .

The  $\|\cdot\|_e$ -topological dual of  $\ell^\infty(e)$  is denoted by  $\text{ba}(e)$ , and the subset of non-negative linear functionals in  $\text{ba}(e)$  is denoted by  $\text{ba}_+(e)$ .<sup>8</sup> For any linear functional  $\varphi \in \text{ba}_+(e)$ , there exists a non-negative charge  $\nu^\varphi$  of bounded variation on the  $\sigma$ -algebra  $2^\Sigma$  (or, equivalently,  $\nu^\varphi$  is a finitely additive positive measure), such that

$$\varphi(h) = \int h_e d\nu^\varphi$$

where  $h_e$  is the process in  $\ell^\infty$  defined by  $h_e(s^t) := h(s^t)/e(s^t)$ . In particular, any  $\varphi \in \text{ba}_+(e)$  can be decomposed as follows

$$\varphi(h) = \text{PV}(p^\varphi; h|s^0) + \varphi^0(h), \quad \text{for every } h \in \ell^\infty(e)$$

for some non-negative process  $p^\varphi$  satisfying  $\text{PV}(p^\varphi; e|s^0) < \infty$  and some non-negative purely finitely additive linear functional  $\varphi^0$ .<sup>9</sup>

**REMARK 2.2** Any non-zero and non-negative linear functional defined on  $\ell^\infty(e)$  necessarily belongs to  $\text{ba}_+(e)$ . Indeed, continuity follows from the fact that  $e$  belongs to the  $\|\cdot\|_e$ -interior of  $\ell_+^\infty(e)$ .

A linear functional  $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$  is said to be strictly positive whenever  $\varphi(h) > 0$  for any non-zero  $h \in \ell_+^\infty(e)$ . Observe that if  $\varphi$  is strictly positive then  $\varphi$  is  $\|\cdot\|_e$ -continuous (i.e.,  $\varphi \in \text{ba}_+(e)$ ) and  $p^\varphi(s^t) > 0$  for any event  $s^t$ .

A **price system** is any arbitrary strictly positive linear functional  $\varphi$  normalized by the condition:  $p^\varphi(s^0) = 1$ . A **Bewley price process** is a strictly positive process  $p = (p(s^t))_{s^t \in \Sigma}$  such that  $p(s^0) = 1$  and  $\text{PV}(p; e|s^0)$  is finite.<sup>10</sup> Observe that any process  $p^\varphi$  associated to a price system  $\varphi$  is necessarily a Bewley price process. The purely finitely additive part  $\varphi^0 = \varphi - \text{PV}(p^\varphi; \cdot|s^0)$  is also called the **bubble component** of the price system  $\varphi$ .

## 2.6. Uniform Gains to Trade

Following [Bloise and Reichlin \(2011\)](#), we say that the economy exhibits **uniform gains to trade** if there is an individually rational and self-enforcing allocation  $(d^i)_{i \in I}$

<sup>8</sup>A linear functional  $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$  is said to be non-negative whenever  $\varphi(h) \geq 0$  for every  $h \in \ell_+^\infty(e)$ .

<sup>9</sup>The purely finitely additive linear functional  $\varphi^0$  can be characterized as follows: it is a linear and  $\|\cdot\|_e$ -continuous functional on  $\ell^\infty(e)$  such that  $\varphi^0(h) = \varphi^0(h^{[T]})$  where  $h^{[T]}$  is the tailed process defined by  $h^{[T]}(s^t) = h(s^t)$  if  $t \geq T$  and 0 elsewhere. Observe moreover that  $p^\varphi(s^t) = \varphi(\mathbf{1}_{\{s^t\}})$  for any event  $s^t$ .

<sup>10</sup>Observe that any Bewley price process  $p$  is such that  $p^e$  belongs to  $\ell_+^1$ —the set of convergent series defined on  $\Sigma$ —where  $p^e(s^t) := p(s^t)e(s^t)$ .

and  $\gamma > 0$  such that

$$(2.2) \quad \forall s^t \in \Sigma, \quad \sum_{i \in I} d^i(s^t) \leq (1 - \gamma)e(s^t)$$

where we recall that  $e(s^t) := \sum_{i \in I} e^i(s^t)$ .

This condition means that autarky can be Pareto improved, subject to participation constraints, even though a constant fraction of aggregate endowments is destroyed. [Bloise and Reichlin \(2011\)](#) prove that a stationary Markovian economy where autarky is not constrained Pareto efficient (see Definition 3.1 below) always exhibits uniform gains to trade. We propose in Section 5.1 an alternative proof of this result.

### 3. CONSTRAINED EFFICIENCY

We consider the following definition of Pareto dominance: an allocation  $(\tilde{c}^i)_{i \in I}$  **Pareto dominates** another allocation  $(c^i)_{i \in I}$  if  $U(\tilde{c}^i | s^0) \geq U(c^i | s^0)$  for every agent  $i$ , with a strict inequality for at least one agent. We first recall the concept of constrained Pareto efficiency introduced in [Kehoe and Levine \(1993\)](#).

**DEFINITION 3.1** An allocation  $(c^i)_{i \in I}$  is **constrained Pareto efficient** if it is resource feasible, self-enforcing, individually rational and if there is no other allocation  $(\tilde{c}^i)_{i \in I}$  that is also resource feasible, self-enforcing and individually rational which Pareto dominates  $(c^i)_{i \in I}$ .

We prove in Proposition A.1 (see the appendix) that if an allocation  $(c^i)_{i \in I}$  is constrained Pareto efficient then it must be strictly positive. In particular, the corresponding process  $p^*$  of implied Arrow–Debreu prices is well-defined. The objective of this section is to provide a complete characterization of constrained Pareto efficiency in terms of implied Arrow–Debreu prices under the assumption of uniform gains to trade.

#### 3.1. Constrained Efficiency and Supporting Price Systems

We first characterize constrained efficiency in terms of supporting price systems.

**DEFINITION 3.2** A linear functional  $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$  **supports** a resource feasible, self-enforcing and individually rational allocation  $(c^i)_{i \in I}$  if  $\varphi(\tilde{c}^i) \geq \varphi(c^i)$  for any self-enforcing and individually rational allocation  $(\tilde{c}^i)_{i \in I}$  that Pareto dominates  $(c^i)_{i \in I}$ .

Since preferences are strictly monotone, a supporting linear functional  $\varphi$  must be non-negative and  $\|\cdot\|_e$ -continuous, i.e.,  $\varphi$  must belong to  $\text{ba}_+(e)$ . Under uniform gains



to trade, the linear functional  $\varphi$  must be strictly positive.<sup>11</sup> Observe that if  $\varphi$  supports  $(c^i)_{i \in I}$  then for any  $\lambda > 0$ , the linear functional  $\lambda\varphi$  also supports  $(c^i)_{i \in I}$ . We can then, without any loss of generality, choose the normalization  $\varphi(\mathbf{1}_{\{s^0\}}) = p^\varphi(s^0) = 1$  and focus on supporting linear functionals that are price systems.

If a price system  $\varphi$  supports an individually rational, self-enforcing and resource feasible allocation  $(c^i)$ , then we can replace the inequality  $\varphi(\tilde{c}^i) \geq \varphi(c^i)$  by the strict inequality  $\varphi(\tilde{c}^i) > \varphi(c^i)$  whenever  $U(\tilde{c}^i|s^0) > U(c^i|s^0)$ .<sup>12</sup> In particular, we obtain the following sufficient condition for constrained Pareto efficiency. This result generalizes the sufficient condition of the first part of Lemma 5 in [Bloise and Reichlin \(2011\)](#) since we do not need to assume that there are uniform gains to trade.

**LEMMA 3.1** If a resource feasible, self-enforcing and individually rational allocation is supported by a price system, then it is constrained Pareto efficient.

**PROOF OF LEMMA 3.1:** Consider a price system  $\varphi$  and a self-enforcing and individually rational consumption process  $c^i$  such that  $\varphi(\tilde{c}^i) \geq \varphi(c^i)$  for any self-enforcing and individually rational consumption process  $\tilde{c}^i$  satisfying  $U(\tilde{c}^i|s^0) > U(c^i|s^0)$ . We propose to show that we must have  $\varphi(\tilde{c}^i) > \varphi(c^i)$ .

We first analyze the simple case where  $c^i(s^0) > 0$ . By continuity of the Bernoulli function, there exists  $\varepsilon \in (0, c^i(s^0))$  small enough such that  $U(\tilde{c}^i - \varepsilon \mathbf{1}_{\{s^0\}}|s^0) > U(c^i|s^0)$ . Observe that the consumption process  $\tilde{c}^i - \varepsilon \mathbf{1}_{\{s^0\}}$  is individually rational and self-enforcing. This implies that  $\varphi(\tilde{c}^i) - \varepsilon \geq \varphi(c^i)$  (recall that a price system is such that  $p(s^0) = 1$ ).

Assume now that  $c^i(s^0) = 0$ . We can apply Lemma A.2 in Appendix A to deduce that there exists an event  $s^\tau \succ s^0$  such that

- (a)  $\tilde{c}^i(s^\tau) > 0$  and  $U(\tilde{c}^i|s^\tau) > U(c^i|s^\tau)$ ;
- (b) for every intermediate event  $s^r$  satisfying  $s^0 \prec s^r \prec s^\tau$ , we have  $\tilde{c}^i(s^r) = 0$  and  $U(\tilde{c}^i|s^r) > U(c^i|s^r)$ .

Define the consumption process  $\hat{c}^i$  as follows:  $\hat{c}^i(s^t) := \tilde{c}^i(s^t)$  if  $s^t \neq s^\tau$  and  $\hat{c}^i(s^\tau) := \tilde{c}^i(s^\tau) - \varepsilon$ , where  $\varepsilon \in (0, \tilde{c}^i(s^\tau))$ . By continuity of the Bernoulli function, we can choose  $\varepsilon$  small enough such that for every event  $s^r$ ,

$$s^0 \prec s^r \prec s^\tau \implies U(\hat{c}^i|s^r) > U(c^i|s^r)$$

and such that  $U(\hat{c}^i|s^0) > U(c^i|s^0)$ . Observe that for any event  $s^t$  not satisfying  $s^0 \preceq s^t \prec s^\tau$ , we have  $U(\hat{c}^i|s^t) = U(\tilde{c}^i|s^t)$ . It follows that  $\hat{c}^i$  is individually rational, self-enforcing and satisfies  $U(\hat{c}^i|s^0) > U(c^i|s^0)$ . This implies that  $\varphi(\tilde{c}^i) - \varepsilon p(s^\tau) = \varphi(\hat{c}^i) \geq \varphi(c^i)$ . The

<sup>11</sup>See the proof of Lemma 5 in [Bloise and Reichlin \(2011\)](#). An alternative argument is as follows. From a standard convex separation argument, we can show that for every  $i$ , there exists  $\lambda^i, \mu^i \geq 0$  with  $(\lambda^i, \mu^i) \neq (0, 0)$  such that  $\lambda^i[U(\tilde{c}^i|s^0) - U(c^i|s^0)] \leq \mu^i\varphi(\tilde{c}^i - c^i)$  for any self-enforcing and individually rational consumption  $\tilde{c}^i$ . Since  $\varphi(e) > 0$ , it follows from the assumption of uniform gains to trade that there exists an agent  $k \in I$  such that  $\varphi(d^k) < \varphi(c^k)$ . This, in turn, implies that  $\lambda^k > 0$ . Then, by strict monotonicity of preferences, we get that  $\mu^k\varphi(v) > 0$  for any non-zero  $v \in \ell_+^\infty(e)$ .

<sup>12</sup>See the proof of Lemma 3.1 below.

desired conclusion follows from the fact that  $p(s^\tau) > 0$  (recall that as a price system,  $\varphi$  is strictly positive). *Q.E.D.*

We know from [Bloise and Reichlin \(2011\)](#) that a self-enforcing, individually rational and resource feasible allocation is constrained Pareto efficient only if it is supported by some price system. We show below that if a price system supports a constrained Pareto efficient allocation, then it must coincide with the process of implied Arrow–Debreu prices (which turns out to display high interest rates). This result generalizes Lemma 5 in [Bloise and Reichlin \(2011\)](#) since we prove that property (ii) is valid for all allocations, not only those that are uniformly bounded away from zero. Moreover, we do not need to assume that aggregate endowments are uniformly bounded away from zero.<sup>13</sup>

LEMMA 3.2 Assume there are uniform gains to trade.

- (i) A self-enforcing, individually rational and resource feasible allocation  $(c^i)_{i \in I}$  is constrained Pareto efficient if, and only if, it is supported by some price system  $\varphi$ .
- (ii) Any price system  $\varphi$  supporting a constrained Pareto efficient allocation  $(c^i)_{i \in I}$  involves no bubble component, i.e.,  $\varphi = \text{PV}(p^\varphi; \cdot | s^0)$ . Moreover  $p^\varphi$  coincides with  $p^*$  the process of implied Arrow–Debreu prices.

PROOF OF LEMMA 3.2: We first prove property (i).<sup>14</sup> The sufficiency part follows from Lemma 3.1. We only prove the necessity part. Fix a constrained Pareto optimal allocation  $(c^i)_{i \in I}$ . Let  $H$  be the set of all vectors  $h \in \ell^\infty(e)$  that can be written as

$$h = \sum_{i \in I} (\tilde{c}^i - c^i)$$

where  $(\tilde{c}^i)_{i \in I}$  is a self-enforcing and individually rational allocation which Pareto dominates  $(c^i)_{i \in I}$ . The set  $H$  is convex with a non-empty interior for the  $\|\cdot\|_e$ -topology.<sup>15</sup> The constrained Pareto efficiency of  $(c^i)_{i \in I}$  implies that 0 does not belong to  $H$ . Applying the Convex Separation Theorem, we get the existence of a non-zero and  $\|\cdot\|_e$ -continuous linear function  $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$  separating  $H$  and  $\{0\}$  in the sense that  $\varphi(h) \geq 0$  for every  $h \in H$ . This means that  $\varphi$  supports the allocation  $(c^i)_{i \in I}$ .

We now prove property (ii). Let  $\varphi$  be a price system supporting the allocation  $(c^i)_{i \in I}$ . Recall that  $\varphi$  belongs to  $\text{ba}_+(e)$ , is strictly positive and is normalized such that  $\varphi(\mathbf{1}_{\{s^t\}}) = p^\varphi(s^t) = 1$ .

<sup>13</sup>[Bloise and Reichlin \(2011\)](#) need these additional assumptions because their approach to prove property (ii) requires that marginal utilities of consumption at constrained Pareto efficient allocations are uniformly bounded from above and bounded away from zero.

<sup>14</sup>This corresponds to Proposition 5 in [Kehoe and Levine \(1993\)](#) and the first part of Lemma 5 in [Bloise and Reichlin \(2011\)](#). Since our environment is slightly more general (we do not arbitrarily restrict the commodity space to be  $\ell^\infty(\mathbf{1}_\Sigma)$ ), we provide the details of the proof.

<sup>15</sup>To see why the set  $H$  has non-empty interior for the  $\|\cdot\|_e$ -topology, we let  $\tilde{c}^i := c^i + e$ . Fix  $\varepsilon \in (0, 1)$  and choose any process  $g$  in  $\ell^\infty(e)$  with  $\|g\|_e < 1$ . We have that  $u(\tilde{c}^i(s^t) + g(s^t)) > u(c^i(s^t))$  which implies that  $\sum_{i \in I} (\tilde{c}^i + g - c^i) = \#I(e + g)$  belongs to  $H$ . Therefore,  $(\#I)e$  belongs to the interior of  $H$  for the  $\|\cdot\|_e$ -topology.

CLAIM 3.1 The process  $p^\varphi$  dominates  $p^*$ .

PROOF: To prove the desired result, it is sufficient to show that for each agent  $i$ , we have

$$\text{MRS}(c^i|s^t) \leq \frac{p^\varphi(s^t)}{p^\varphi(\sigma(s^t))}, \quad \text{for all } s^t \succ s^0.$$

This property follows from a standard variational argument. Indeed, fix an arbitrary event  $s^t$  and an arbitrary agent  $i$ . For some  $\chi > 1/\text{MRS}(c^i|s^t)$  and  $0 < \varepsilon < c^i(\sigma(s^t))$ , we can define the process  $\tilde{c}^i$  as follows

$$\forall s^\tau \in \Sigma, \quad \tilde{c}^i(s^\tau) := \begin{cases} c^i(\sigma(s^t)) - \varepsilon & \text{if } s^\tau = \sigma(s^t) \\ c^i(s^t) + \chi\varepsilon & \text{if } s^\tau = s^t \\ c^i(s^\tau) & \text{otherwise.} \end{cases}$$

Observe that the process  $\tilde{c}^i - c^i$  is different from zero only at the events  $\sigma(s^t)$  and  $s^t$ . This implies that it belongs to  $\ell^\infty(e)$ . Given the choice of  $\chi$ , we can choose  $\varepsilon > 0$  small enough such that  $\tilde{c}^i$  is self-enforcing and  $U(\tilde{c}^i|s^0) > U(c^i|s^0)$ .<sup>16</sup> This implies that

$$0 \leq \varphi(\tilde{c}^i - c^i) = -p^\varphi(\sigma(s^t))\varepsilon + p^\varphi(s^t)\varepsilon\chi.$$

Since  $\chi$  can be chosen arbitrarily close to  $1/\text{MRS}^i(c^i|s^t)$  we obtain the desired result. *Q.E.D.*

If agent  $i$ 's participation constraint is not binding at event  $s^t$ , i.e.,  $U(c^i|s^t) > U(e^i|s^t)$ , then we can replace  $\varepsilon > 0$  by  $-\varepsilon$  in the arguments of Claim 3.1 to show that

$$\frac{p^\varphi(s^t)}{p^\varphi(\sigma(s^t))} = \text{MRS}(c^i|s^t) = \frac{p^*(s^t)}{p^*(\sigma(s^t))}.$$

If we show that for any event  $s^t$ , there exists at least one agent for which the participation constraint is not binding, then we get the desired result:  $p^\varphi = p^*$ . This property is guaranteed by the assumption of uniform gains to trade.

CLAIM 3.2 At every event  $s^t \in \Sigma$ , there exists at least one agent with a non-binding participation constraint.

PROOF: Fix an event  $s^t$  and assume by way of contradiction that  $U(c^i|s^t) = U(e^i|s^t)$  for every  $i$ . Let  $(d^i)_{i \in I}$  be the individually rational and self-enforcing allocation in the

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<sup>16</sup>Since  $u$  is concave, we have

$$\pi(\sigma(s^t))u(\tilde{c}^i(\sigma(s^t))) + \pi(s^t)u(\tilde{c}^i(s^t)) > \pi(\sigma(s^t))u(c^i(\sigma(s^t))) + \pi(s^t)u(c^i(s^t)).$$

definition (2.2) of uniform gains to trade. We pose

$$x^i(s^\tau) := \begin{cases} c^i(s^\tau) & \text{if } s^\tau \notin \Sigma(s^t) \\ d^i(s^\tau) + (\gamma/\#I)e(s^\tau) & \text{if } s^\tau \succeq s^t. \end{cases}$$

Since  $d^i$  and  $c^i$  are self-enforcing, the consumption  $x^i$  is also self-enforcing. Moreover,

$$(3.1) \quad U(x^i|s^0) = \sum_{r=0}^{t-1} \sum_{s^r \in S^r} \beta^r \pi(s^r) u(c^i(s^r)) + \beta^t \pi(s^t) U(x^i|s^t) \\ + \sum_{\sigma^t \in S^t \setminus \{s^t\}} \beta^t \pi(\sigma^t) U(c^i|\sigma^t).$$

Since,  $U(x^i|s^t) > U(e^i|s^t) = U(c^i|s^t)$ , we get that that  $U(x^i|s^0) > U(c^i|s^0)$ . This contradicts the constrained Pareto efficiency of  $(c^i)_{i \in I}$ . *Q.E.D.*

At this point, we proved that  $p^\varphi = p^*$ . We now show that  $\varphi$  cannot have a bubble component, i.e.,  $\varphi^0(\cdot) := \varphi(\cdot) - \text{PV}(p^\varphi; \cdot|s^0) = 0$ . Assume, by way of contradiction, that there exists  $v \in \ell_+^\infty(e)$  such that  $\varphi^0(v) > 0$ . Since  $v$  belongs to  $\ell^\infty(e)$ , it follows from the assumption of uniform gains to trade that there exists an individually rational and self-enforcing allocation  $(f^i)_{i \in I}$  and  $\mu > 0$  such that<sup>17</sup>

$$(3.2) \quad \sum_{i \in I} f^i = -\mu v + \sum_{i \in I} c^i.$$

Fix  $\varepsilon > 0$  small enough such that  $\varphi^0(v) > \varepsilon/\mu$ . Since  $p^* = p^\varphi$  displays high interest rates, there exists a date  $\tau$  large enough such that

$$\sum_{s^\tau \in S^\tau} p^*(s^\tau) \text{PV}(p^*, e|s^\tau) \leq \varepsilon/2(\#I).$$

For every event  $s^t$  such that  $t < \tau$ , we pose  $x^i(s^t) := c^i(s^t)$ . Choose  $r > \tau$  and pose

$$x^i(s^t) := \begin{cases} e(s^t) & \text{if } \tau \leq t < r \\ f^i(s^t) & \text{if } t \geq r. \end{cases}$$

We have  $U(x^i|s^t) \geq U(f^i|s^t)$  for any event  $s^t$  with  $t \geq \tau$ . Observe that  $e(s^t) > c^i(s^t)$  for any event  $s^t$ . We can then choose  $r$  sufficiently large to get that  $U(x^i|s^\tau) > U(c^i|s^\tau)$  for any  $s^\tau \in S^\tau$ . This, in turn, implies  $U(x^i|s^t) > U(c^i|s^t)$  for any  $t < \tau$ . We have thus proved that the consumption process  $x^i$  is self-enforcing, individually rational and

<sup>17</sup>Indeed, we have  $v \leq \|v\|_e e$ . Choose  $\mu := \gamma/\|v\|_e$  and observe that

$$\sum_{i \in I} d^i + \gamma(e - (1/\|v\|_e)v) = -\mu v + \sum_{i \in I} c^i.$$

We can then pose  $f^i := d^i + \gamma^i(e - (1/\|v\|_e)v)$  where  $\gamma^i := \gamma/(\#I)$ .

satisfies  $U(x^i|s^0) > U(c^i|s^0)$ . It follows that  $h = \sum_{i \in I} (x^i - c^i)$  belongs to  $H$  in which case we have

$$(3.3) \quad 0 \leq \varphi(h) = \text{PV}(p^*; h) + \varphi^0(h).$$

Observe that

$$\begin{aligned} \text{PV}(p^*; h) &\leq \sum_{t \geq \tau} \sum_{s^t \in S^t} p^*(s^t) |h(s^t)| \\ &\leq (\#I) \sum_{t \geq \tau} \sum_{s^t \in S^t} p^*(s^t) e(s^t) \\ (3.4) \quad &\leq (\#I) \sum_{s^\tau \in S^\tau} p^*(s^\tau) \text{PV}(p^*; e|s^\tau) \leq \frac{\varepsilon}{2}. \end{aligned}$$

Since  $\varphi^0$  is purely finitely additive, we have<sup>18</sup>

$$(3.5) \quad \varphi^0(h) = \varphi^0(h^{[r]}) = -\mu\varphi^0(v^{[r]}) = -\mu\varphi^0(v) < -\varepsilon.$$

Combining (3.4) and (3.5) we get that  $\varphi(h) \leq -\varepsilon/2$  which contradicts (3.3). *Q.E.D.*

### 3.2. Constrained Efficiency and High Implied Interest Rates

The following lemma shows the necessity of high implied interest rates for constrained Pareto efficiency. Its proof follows as a direct corollary of Lemma 3.2. This is because the price process  $p^\varphi$  associated to a linear functional in  $\text{ba}_+(e)$  automatically satisfies  $\text{PV}(p^\varphi; e|s^0) < \infty$ .

**LEMMA 3.3** If there are uniform gains to trade, then every constrained Pareto efficient allocation exhibits high implied interest rates.<sup>19</sup>

Constrained Pareto efficiency obtains when there are no mutual gains from trading, including the trade opportunities involving transfers in the long run. Malinvaud efficiency, instead, is a weaker notion that requires the absence of any feasible welfare improvement subject to resource feasibility and participation constraints over any finite horizon.

**DEFINITION 3.3** An allocation  $(c^i)_{i \in I}$  is **constrained Malinvaud efficient** if it is resource feasible, self-enforcing, individually rational and if there is no other allocation  $(\tilde{c}^i)_{i \in I}$  that is also resource feasible, self-enforcing and individually rational which Pareto dominates  $(c^i)_{i \in I}$  and differs from  $(c^i)_{i \in I}$  only on finitely many events.

<sup>18</sup>Recall that for any process  $h = (h(s^t))_{s^t \in \Sigma}$ , we denote by  $h^{[r]}$  the tailed process at date  $r > 0$ , defined by  $h^{[r]}(s^t) = h(s^t)$  if  $t \geq r$  and  $h^{[r]}(s^t) = 0$  if  $t < r$ . In particular, for  $h = \sum_{i \in I} (x^i - c^i)$ , we have that  $h^{[r]} = \sum_{i \in I} (f^{i,[r]} - c^{i,[r]})$ .

<sup>19</sup>This result is related to Lemma 2 in [Bloise and Reichlin \(2011\)](#). We refer to Appendix B for a detailed discussion.

Every constrained Pareto efficient allocation is constrained Malinvaud efficient. We prove in Proposition A.1 (see the appendix) that if an allocation  $(c^i)_{i \in I}$  is Malinvaud efficient then it must be strictly positive. In particular, the corresponding process  $p^*$  of implied Arrow–Debreu prices is well-defined. In addition, constrained Malinvaud efficiency has a tractable characterization in terms of first order conditions: a resource feasible, self-enforcing and individually rational allocation is constrained Malinvaud efficient if, and only if,

$$(3.6) \quad \forall s^t \succ s^0, \quad U(c^i|s^t) > U(e^i|s^t) \implies \text{MRS}(c^i|s^t) = \frac{p^*(s^t)}{p^*(\sigma(s^t))}.$$

When the above property is satisfied at any strict successor event  $s^\tau \succ s^t$ , we say that **Euler equations** are satisfied at event  $s^t$ .

We next show that when implied interest rates are high, Euler equations are sufficient for constrained Pareto efficiency.

**LEMMA 3.4** Any constrained Malinvaud efficient allocation that exhibits high implied interest rates is constrained Pareto efficient.<sup>20</sup>

**PROOF:** Let  $(c^i)_{i \in I}$  be a constrained Malinvaud efficient allocation and assume that the associated process  $p^*$  of implied Arrow–Debreu prices exhibit high interest rates. Since  $(c^i)_{i \in I}$  is constrained Malinvaud efficient, we know that  $\text{MRS}(c^i|s^t) = p^*(s^t)/p^*(\sigma(s^t))$  if  $U(c^i|s^t) > U(e^i|s^t)$ . We can then apply Proposition A.2 to get that

$$\frac{1}{u'(c^i(s^0))} [U(\tilde{c}^i|s^0) - U(c^i|s^0)] \leq \text{PV}(p^*; \tilde{c}^i - c^i|s^0)$$

for any resource feasible, self-enforcing and individually rational allocation  $(\tilde{c}^i)_{i \in I}$ .<sup>21</sup> Summing over  $i$  the above inequalities, we get that

$$\sum_{i \in I} \frac{1}{u'(c^i(s^0))} [U(\tilde{c}^i|s^0) - U(c^i|s^0)] \leq 0.$$

It follows that  $(\tilde{c}^i)_{i \in I}$  cannot Pareto dominate  $(c^i)_{i \in I}$ . Therefore, we proved that  $(c^i)_{i \in I}$  is constrained Pareto efficient. *Q.E.D.*

Combining Lemma 3.3 and Lemma 3.4, we get a complete characterization of constrained efficiency under uniform gains to trade.

**PROPOSITION 3.1** Assume there are uniform gains to trade. A constrained Malinvaud efficient allocation is constrained Pareto efficient if, and only if, implied prices exhibit high interest rates.

<sup>20</sup>This result is related to Lemma 3 in [Bloise and Reichlin \(2011\)](#). We refer to Appendix B for a detailed discussion.

<sup>21</sup>If  $(\tilde{c}^i)_{i \in I}$  is a resource feasible allocation then each consumption process satisfies  $\tilde{c}^i \leq e$ , which implies that  $\text{PV}(p^*; \tilde{c}^i|s^0)$  is finite.

Proposition 3.1 generalizes Proposition 3 in Bloise and Reichlin (2011). Indeed, they only characterize the sub-class of allocations that are uniformly bounded away from zero.<sup>22</sup> Moreover, they assume that aggregate endowments are uniformly bounded from above and uniformly bounded away from zero. We do not need to impose these restrictions on allocations and endowments. We refer to Appendix B for a detailed comparison.

#### 4. CONSTRAINED COMPETITIVE EQUILIBRIUM

We next introduce the concept of a constrained competitive equilibrium (as defined in Kehoe and Levine (1993) and Alvarez and Jermann (2000)).

Recall that a price system is a linear functional  $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$  that is strictly positive (and therefore  $\|\cdot\|_e$ -continuous) and normalized by  $p^\varphi(s^0) = 1$ . Given a price system  $\varphi$  and a real number  $b$  representing an initial financial transfer, we denote by  $B_{\text{AD}}^i(\varphi; b)$  the (unconstrained) Arrow–Debreu budget set of all consumption processes  $c^i$  satisfying the following budget restriction

$$\varphi(c^i - e^i) \leq b.$$

The Kehoe–Levine budget set is then defined by

$$B_{\text{KL}}^i(\varphi; b) := B_{\text{AD}}^i(\varphi; b) \cap \text{PC}^i$$

where we recall that  $\text{PC}^i$  is the set of all consumption processes  $c^i$  satisfying the participation constraints  $U(c^i|s^t) \geq U(e^i|s^t)$  at every strict successor event  $s^t \succ s^0$ . Denote by  $d_{\text{KL}}^i(\varphi; b)$  the demand set associated to the budget set  $B_{\text{KL}}^i(\varphi; b)$ , i.e.,

$$d_{\text{KL}}^i(\varphi; b) := \text{argmax}\{U(c^i|s^0) : c^i \in B_{\text{KL}}^i(\varphi; b)\}.$$

**DEFINITION 4.1** A **Kehoe–Levine competitive equilibrium**  $(\varphi, (c^i)_{i \in I})$  is a family composed of a price system  $\varphi$  and a resource feasible and self-enforcing allocation  $(c^i)_{i \in I}$  such that  $c^i \in d_{\text{KL}}^i(\varphi; a^i(s^0))$  for every  $i$ , where  $(a^i(s^0))_{i \in I}$  is an allocation of initial transfers satisfying  $\sum_{i \in I} a^i(s^0) = 0$ .

**REMARK 4.1** A Kehoe–Levine competitive equilibrium  $(\varphi, (c^i)_{i \in I})$  with initial transfers  $(a^i(s^0))_{i \in I}$  need not be individually rational. If the allocation  $(c^i)_{i \in I}$  is individually rational, then we say that  $(a^i(s^0))_{i \in I}$  is an allocation of **individually rational transfers**. Observe that zero initial transfers are individually rational. This is because  $e^i$  belongs to  $B_{\text{KL}}^i(\varphi; 0)$ .

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<sup>22</sup>Recall that an allocation  $(c^i)_{i \in I}$  is said to be uniformly bounded away from zero when there exists  $\varepsilon > 0$  satisfying  $c^i(s^t) \geq \varepsilon$  for every  $i$  and  $s^t$ . To guarantee that feasible allocations are uniformly bounded away from zero, Bloise et al. (2013) imposed an additional boundary condition on  $u$  and  $\beta$ . We refer to Appendix B.4 where we discuss how restrictive this condition is.

Consider a Kehoe–Levine competitive equilibrium  $(\varphi, (c^i)_{i \in I})$ . If  $\varphi$  has no bubble component, in the sense that  $\varphi^0 = 0$ , then we say that the family  $(p^\varphi, (c^i)_{i \in I})$  is Kehoe–Levine competitive equilibrium with a Bewley price process.

**REMARK 4.2** An Arrow–Debreu competitive equilibrium  $(\varphi, (c^i)_{i \in I})$  is a family composed of a price system  $\varphi$  and a resource feasible (not necessarily self-enforcing) allocation  $(c^i)_{i \in I}$  such that  $c^i \in d_{\text{AD}}^i(\varphi; a^i(s^0))$  for every  $i$  where  $(a^i(s^0))_{i \in I}$  are market clearing initial transfers and  $d_{\text{AD}}^i(\varphi; b) := \operatorname{argmax}\{U(c^i|s^0) : c^i \in B_{\text{AD}}^i(\varphi; b)\}$ . Following the arguments in [Bewley \(1972\)](#) we can show that (1) there exists an Arrow–Debreu competitive equilibrium with zero transfers; and (2) for any Arrow–Debreu competitive equilibrium  $(\varphi, (c^i)_{i \in I})$ , the price system involves no bubble component. In particular, the family  $(p^\varphi, (c^i)_{i \in I})$  is an Arrow–Debreu competitive equilibrium with a Bewley price process. We show below that both results (1) and (2) extend to Kehoe–Levine competitive equilibria, provided there are uniform gains to trade.

#### 4.1. Welfare Theorems

It follows from standard arguments that a constrained version of the First Welfare Theorem is valid, i.e., every individually rational consumption allocation of a Kehoe–Levine competitive equilibrium is constrained Pareto efficient. As a direct corollary of [Lemma 3.4](#) and [Lemma 3.2](#), we obtain a version of the Second Welfare Theorem that is in the spirit of [Proposition 5](#) in [Kehoe and Levine \(1993\)](#). On one hand, they do not assume high implied interest rates, but, on the other hand, they only get existence of a quasi-equilibrium with a price system that may involve a bubble component.

**PROPOSITION 4.1** Let  $(c^i)_{i \in I}$  be a constrained Malinvaud efficient allocation such that the implied Arrow–Debreu prices  $p^*$  display high interest rates. Then the family  $(p^*, (c^i)_{i \in I})$  is a Kehoe–Levine competitive equilibrium with a Bewley price process and individually rational initial transfers defined by  $a^i(s^0) := \text{PV}(p^*; c^i - e^i|s^0)$ .

**PROOF OF PROPOSITION 4.1:** Let  $(c^i)_{i \in I}$  be a constrained Malinvaud efficient allocation such that the implied prices  $p^*$  display high interest rates.

Since  $(c^i)_{i \in I}$  is constrained Malinvaud efficient, we know that  $\text{MRS}(c^i|s^t) = p^*(s^t)/p^*(\sigma(s^t))$  if  $U(c^i|s^t) > U(e^i|s^t)$ . Since implied prices  $p^*$  display high interest rates, we have

$$\text{PV}(p^*; c^i|s^0) \leq \text{PV}(p^*; e^i|s^0) < \infty.$$

We can then apply [Proposition A.2](#) to get that

$$\frac{1}{u'(c^i(s^0))} [U(\tilde{c}^i|s^0) - U(c^i|s^0)] \leq \text{PV}(p^*; \tilde{c}^i - c^i|s^0)$$



for any self-enforcing consumption process  $\tilde{c}^i$  with  $\text{PV}(p^*; \tilde{c}^i | s^0)$  finite. This implies that if  $\tilde{c}^i$  is a self-enforcing consumption process such that

$$\text{PV}(p^*; \tilde{c}^i | s^0) \leq \text{PV}(p^*; c^i | s^0)$$

then we necessarily have  $U(\tilde{c}^i | s^0) \leq U(c^i | s^0)$ . We have thus proved that  $c^i$  belongs to  $d_{\text{KL}}^i(p^*, a^i(s^0))$ .

*Q.E.D.*

Combining Lemma 3.2 and Proposition 4.1, we obtain the following characterization of constrained Pareto optimality.

**PROPOSITION 4.2** Assume there are uniform gains to trade and consider an allocation  $(c^i)_{i \in I}$  that is resource feasible, self-enforcing and individually rational. The allocation  $(c^i)_{i \in I}$  is constrained Pareto optimal if, and only if, there exists a Kehoe–Levine competitive equilibrium  $(p, (c^i)_{i \in I})$  with a Bewley price process  $p$  and individually rational initial transfers  $a^i(s^0) := \text{PV}(p; c^i - e^i | s^0)$ . Moreover, the Bewley price process  $p$  must coincide with  $p^*$ , the process of implied Arrow–Debreu prices.

## 4.2. Existence

We have seen that the set of constrained Pareto efficient allocations coincides with the set of consumption allocations implemented by Kehoe–Levine competitive equilibria with individually rational transfers. Since existence of a constrained Pareto efficient allocation follows from standard continuity and compactness arguments, we derive the existence of Kehoe–Levine competitive equilibria with some initial transfers. We next show the existence of a Kehoe–Levine competitive equilibrium with zero initial transfers.

**THEOREM 4.1** *If there are uniform gains to trade, then there exists a Kehoe–Levine competitive equilibrium with zero initial transfers and high implied interest rates.*

**PROOF:** We follow the standard Negishi approach. Let

$$\Lambda := \{\lambda = (\lambda^i)_{i \in I} \in \mathbb{R}_+^I : \sum_{i \in I} \lambda^i = 1\}$$

be the set of welfare weights. For each  $\lambda \in \Lambda$  we can maximize the social welfare function

$$\sum_{i \in I} \lambda^i U(c^i | s^0)$$

among all feasible, self-enforcing and individually rational allocations. By strict concavity of the Bernoulli function, there exists a unique solution  $c(\lambda) = (c^i(\lambda))_{i \in I}$  which

is constrained Pareto efficient.<sup>23</sup> Denote by  $p^*(\lambda)$  the corresponding process of implied Arrow–Debreu prices. Applying Proposition 4.1, we get that  $(p^*(\lambda), c(\lambda))$  is a Kehoe–Levine competitive equilibrium with initial transfers

$$a^i(\lambda, s^0) := \text{PV}(p^*(\lambda); c^i(\lambda) - e^i | s^0).$$

The difficult step of the proof consists on proving that  $\lambda \mapsto a^i(\lambda, s^0)$  is continuous. Applying Berge’s Maximum Theorem, we get that each function  $\lambda \mapsto c^i(\lambda)$  is continuous for the product topology.<sup>24</sup> Since the Bernoulli function is continuously differentiable on  $(0, \infty)$  and the consumption processes are strictly positive, we also get that the function  $\lambda \mapsto p^*(\lambda)$  is continuous for the product topology. The continuity (for the product topology) of the mappings  $\lambda \mapsto p^*(\lambda)$  and  $\lambda \mapsto c^i(\lambda, s^0)$  is not a priori sufficient to get continuity of  $\lambda \mapsto a^i(\lambda, s^0)$  since its definition involves an infinite sum. However, if we prove that

$$(4.1) \quad \lim_{\tau \rightarrow \infty} \sup_{\lambda \in \Lambda} \text{PV}(p^*(\lambda); e | s^\tau) = 0$$

then we get continuity of the mapping  $\lambda \mapsto a^i(\lambda, s^0)$ . We show below that (4.1) is true.<sup>25</sup>

CLAIM 4.1 For any welfare weights  $\lambda \in \Lambda$ , we have

$$\text{PV}(p^*(\lambda); e | s^t) \leq \frac{1}{\gamma u'(e(s^t))} \sum_{i \in I} [U(e | s^t) - U(e^i | s^t)], \quad \text{for all } s^t.$$

PROOF: Fix an arbitrary vector  $\lambda \in \Lambda$ . Since  $p^*(\lambda)$  displays high interest rates and the Euler equations are satisfied at the allocation  $c(\lambda)$ , we can apply Proposition A.2 to deduce that

$$\frac{1}{u'(c^i(\lambda, s^t))} [U(\tilde{c}^i | s^t) - U(c^i(\lambda) | s^t)] \leq \text{PV}(p^*(\lambda); \tilde{c}^i - c^i(\lambda) | s^t)$$

for any self-enforcing consumption process  $\tilde{c}^i$  satisfying  $\text{PV}(p^*(\lambda); \tilde{c}^i | s^0) < \infty$ . Let  $(d^i)_{i \in I}$  be defined by the uniform gains to trade condition (2.2). We can choose  $\tilde{c}^i := d^i$  to get

$$\begin{aligned} \text{PV}(p^*(\lambda); c^i(\lambda) - d^i | s^t) &\leq \frac{1}{u'(c^i(\lambda, s^t))} [U(c^i(\lambda) | s^t) - U(d^i | s^t)] \\ &\leq \frac{1}{u'(e(s^t))} [U(e | s^t) - U(e^i | s^t)]. \end{aligned}$$

Summing over  $i$  and using (2.2), we get the desired result. Q.E.D.

<sup>23</sup>Uniqueness is straightforward if  $\lambda^i > 0$  for each  $i$ . For the general case, we can follow the arguments in the proof of Lemma 4 in Bloise et al. (2013).

<sup>24</sup>This is because the mapping  $c \mapsto U(c | s^0)$  is continuous for the product topology on the space of self-enforcing and individually rational consumption processes. See Lemma A.1 for details.

<sup>25</sup>Claim 4.1 is related to Lemma 3 in Kehoe and Levine (1993).

A direct consequence of Claim 4.1 is that each mapping  $\lambda \mapsto a^i(\lambda, s^0)$  is continuous. We can now follow the standard Negishi approach. Let  $G : \Lambda \rightarrow \Lambda$  be the correspondence defined by

$$G(\lambda) := \operatorname{argmin}\{\lambda' \cdot a(\lambda, s^0) : \lambda' \in \Lambda\}.$$

Applying Berge's Maximum Theorem, we get that the correspondence  $G$  is upper semi-continuous with non-empty convex values. By Kakutani's Fixed-Point Theorem, it admits a fixed point  $\lambda^*$ . We must have  $a^i(\lambda^*, s^0) = 0$  for each  $i$ . Indeed, assume by way of contradiction, that  $a^i(\lambda^*, s^0) > 0$  for some agent  $i$ .<sup>26</sup> Since  $\lambda^*$  minimizes the mapping  $\lambda \mapsto \lambda \cdot a(\lambda^*, s^0)$ , we must have  $\lambda^{i,*} = 0$ . Since  $c(\lambda^*)$  maximizes the social welfare function, we must have  $U(c^i(\lambda^*)|s^0) = U(e^i|s^0)$ . However, we proved in Proposition 4.1 that  $(p^*(\lambda^*), (c^i(\lambda^*))_{i \in I})$  is a Kehoe–Levine competitive equilibrium with initial transfers  $a(\lambda^*, s^0)$ . This means that

$$c^i(\lambda^*, s^0) \in d_{\text{KL}}^i(p^*(\lambda^*); a^i(\lambda^*, s^0)).$$

Since  $a^i(\lambda^*, s^0) > 0$ , we must have

$$\begin{aligned} U(c^i(\lambda^*)|s^0) &= \sup\{U(c^i|s^0) : c^i \in B_{\text{KL}}^i(p^*(\lambda^*); a^i(\lambda^*, s^0))\} \\ &> \sup\{U(c^i|s^0) : c^i \in B_{\text{KL}}^i(p^*(\lambda^*); 0)\} \\ &> U(e^i|s^0). \end{aligned}$$

The above inequality leads to a contradiction. We have thus proved that there exists a vector of welfare weights  $\lambda^*$  such that  $(p^*(\lambda^*), (c^i(\lambda^*))_{i \in I})$  is a Kehoe–Levine competitive equilibrium with zero initial transfers. *Q.E.D.*

### 4.3. Constrained Edgeworth Equilibria

We have proved the equivalence between constrained Pareto efficient allocations and the consumption allocations implemented by Kehoe–Levine competitive equilibria with individually rational transfers (see Proposition 4.2). The objective of this section is to identify the subset of constrained Pareto efficient allocations implemented by Kehoe–Levine competitive equilibria with zero initial transfers. We first adapt to our environment the concepts of core allocations and Edgeworth equilibria.

**DEFINITION 4.2** Consider a resource feasible, individually rational and self-enforcing allocation  $(c^i)_{i \in I}$ . A coalition  $S \subseteq I$  is said to improve upon the allocation  $(c^i)_{i \in I}$  whenever there exists an individually rational and self-enforcing allocation  $(\tilde{c}^i)_{i \in I}$  satisfying

$$(4.2) \quad \sum_{i \in S} \tilde{c}^i = \sum_{i \in S} e^i \quad \text{and} \quad U(\tilde{c}^i|s^0) > U(c^i|s^0), \quad \text{for all } i \in S.$$

<sup>26</sup>Recall that  $\sum_{i \in I} a^i(\lambda^*, s^0) = 0$ . If not all initial transfers are zero, then one of them must be positive.

The allocation  $(c^i)_{i \in I}$  that cannot be improved by any coalition is called a **constrained core allocation**.

It is straightforward to see that any consumption allocation implemented by a Kehoe–Levine competitive equilibrium with zero transfers is a constrained core allocation, and that any constrained core allocation is constrained Pareto efficient.

The conditions in (4.2) can be written as follows

$$(4.3) \quad \sum_{i \in I} \lambda^i \tilde{c}^i = \sum_{i \in I} \lambda^i e^i \quad \text{and} \quad \lambda^i U(\tilde{c}^i | s^0) > \lambda^i U(c^i | s^0), \quad \text{for all } i \in I$$

where  $\lambda^i := 1$  if  $i \in S$  and  $\lambda^i := 0$  if  $i \notin S$ . We may then extend the notion of coalition by saying that  $(\lambda^i)_{i \in I}$  is a  $r$ -coalition whenever  $\lambda^i \in \{0, 1/r, 2/r, \dots, (r-1)/r, 1\}$ . The interpretation is as follows. Consider a  $r$ -replica of our economy, in the sense that for each  $i$ , we have  $r$  identical agents with the same endowment process  $e^i$ . The ratio  $\lambda^i$  represents the rate of participation of type  $i$ 's agents in the  $r$ -coalition. We say that a  $r$ -coalition  $(\lambda^i)_{i \in I}$  improves upon the allocation  $(c^i)_{i \in I}$  when there exists an individually rational and self-enforcing allocation  $(\tilde{c}^i)_{i \in I}$  satisfying (4.3). We are now ready to adapt the concept of an Edgeworth equilibrium (see for instance Aliprantis et al. (1987)) to our environment.

**DEFINITION 4.3** A resource feasible, individually rational and self-enforcing allocation  $(c^i)_{i \in I}$  is a **constrained Edgeworth equilibrium** whenever for any replica  $r \in \mathbb{N}$ , it cannot be improved upon by any  $r$ -coalition.

Obviously, a constrained Edgeworth equilibrium is constrained Pareto efficient. It should also be noted that any consumption allocation implemented by a Kehoe–Levine competitive equilibrium with zero transfers is a constrained Edgeworth equilibrium. The contribution of this section is to prove that the converse is also true.

**THEOREM 4.2** *Assume there are uniform gains to trade. A consumption allocation is implemented by a Kehoe–Levine competitive equilibrium with zero transfers if, and only if, it is a constrained Edgeworth equilibrium.*

**PROOF:** We only need to prove the sufficiency part. Consider a constrained Edgeworth equilibrium  $(c^i)_{i \in I}$ . It is a resource feasible, individually rational and self-enforcing allocation such that there does not exist a family  $(\lambda^i)_{i \in I}$  of rational numbers in  $[0, 1] \cap \mathbb{Q}$  and an individually rational and self-enforcing allocation  $(\tilde{c}^i)_{i \in I}$  satisfying (4.3).<sup>27</sup> Using the continuity of preferences and the density of the set of rational numbers in the set of real number, we can show that

$$0 \notin G(c) := \left\{ \sum_{i \in I} \lambda^i (x^i - e^i) : x^i \in P^i(c^i), \lambda^i \geq 0 \text{ for each } i \text{ and } \sum_{i \in I} \lambda^i = 1 \right\},$$

<sup>27</sup>Observe that  $\bigcup_{r \in \mathbb{N}} \{0, 1/r, 2/r, \dots, (r-1)/r, 1\} = [0, 1] \cap \mathbb{Q}$ .

where  $P^i(c^i)$  is the set of all consumption processes  $x^i \in \ell^\infty(e)$  that are self-enforcing and satisfy  $U(x^i|s^0) > U(c^i|s^0)$ .<sup>28</sup> Since each set  $P^i(c^i)$  has non-empty  $\|\cdot\|_e$ -interior, the Convex Separation Theorem implies that there exists a non-zero and  $\|\cdot\|_e$ -continuous linear function  $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$  separating  $G(c)$  and  $\{0\}$ .<sup>29</sup> Therefore, for every  $i$ , we have  $\varphi(x^i) \geq \varphi(e^i)$  for any  $x^i \in P^i(c^i)$ . Since Bernoulli functions are strictly increasing, standard arguments show that  $\varphi(c^i) = \varphi(e^i)$ .<sup>30</sup> We have then proved that  $\varphi(x^i) \geq \varphi(c^i) = \varphi(e^i)$  for every  $x^i \in P^i(c^i)$ . In particular, the linear functional  $\varphi$  supports the allocation  $(c^i)_{i \in I}$ . Applying Lemma 3.2, we get that  $\varphi$  is a price system without a bubble component that coincides with  $\text{PV}(p^*; \cdot|s^0)$ , where  $p^*$  is the process of implied Arrow–Debreu prices. In particular,  $p^*$  displays high interest rates. Applying Proposition 4.1, we deduce that  $(p^*, (c^i)_{i \in I})$  is a Kehoe–Levine competitive equilibrium with initial transfers

$$a^i(s^0) = \text{PV}(p^*; c^i - e^i|s^0) = \varphi(c^i - e^i) = 0$$

and we get the desired result. Q.E.D.

We can apply Proposition 5.2.2 in Florenzano (2003) to prove that a constrained Edgeworth equilibrium always exists.<sup>31</sup> Our existence result in Theorem 4.1 then follows as a simple corollary of our equivalence result in Theorem 4.2.

## 5. ABOUT UNIFORM GAINS TO TRADE

One of the crucial assumption imposed in this paper is that there are uniform gains to trade. We propose to investigate two simple environments where this condition is always satisfied.

### 5.1. Stationary Markovian Economies

Here, we restrict attention to stationary Markovian endowments. Uncertainty is assumed to be represented by a simple Markov process on a finite state space  $S$ . An event  $s^t$  is then a  $t + 1$ -vector  $(s_0, s_1, \dots, s_t)$  where each shock  $s_\tau \in S$  and  $s_0 \in S$  is fixed. In

<sup>28</sup>The allocation  $(c^i)_{i \in I}$  satisfying the above property is called a fuzzy (or Aubin) core allocation of an economy where consumption processes are restricted to be self-enforcing. Under our assumptions, any Edgeworth equilibrium is a fuzzy core allocation. See the proof of Proposition 4.2.6 in Florenzano (2003) for details.

<sup>29</sup>To see why the set  $P^i(c^i)$  has non-empty  $\|\cdot\|_e$ -interior, we let  $x^i := c^i + e$ . Fix  $\varepsilon \in (0, 1)$  and choose any process  $z$  in  $\ell^\infty(e)$  with  $\|z\|_e < 1$ . We have that  $u(x^i(s^t) + z(s^t)) > u(c^i(s^t))$  which implies that  $x^i + z$  belongs to  $P^i(c^i)$ . Therefore,  $x^i$  belongs to the  $\|\cdot\|_e$ -interior of  $P^i(c^i)$ .

<sup>30</sup>Fix  $\varepsilon > 0$ . Observe that  $c^i + \varepsilon e \in P^i(c^i)$ . It follows that  $\varphi(c^i) + \varepsilon\varphi(e) \geq \varphi(e^i)$ . Passing to the limit when  $\varepsilon \rightarrow 0$ , we get that  $\varphi(c^i) \geq \varphi(e^i)$ . The desired result follows from market clearing.

<sup>31</sup>Apply Proposition 5.2.3 in Florenzano (2003) by choosing: the consumption set  $X^i = \text{PC}^i$ , the set  $P^i(c) := \{x^i \in X^i : U(x^i|s^0) > U(c^i|s^0)\}$ , the topology  $\sigma$  to be the product topology and the null production sector  $Y_j = Z = \{0\}$ . The assumptions in Proposition 5.2.2 follow from the following properties:  $U(\cdot|s^0)$  is concave and upper semi-continuous, the set  $\text{PC}^i$  is closed convex, and the set of resource feasible allocations is compact.

addition, the conditional probability  $\pi(s^{t+1}|s^t)$  is assumed to depend only on  $s_t$  and  $s_{t+1}$ . We abuse notation and denote this conditional probability by  $\pi(s_{t+1}|s_t)$ , implying that

$$\pi(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2}) \dots \pi(s_1|s_0).$$

A process  $z = (z(s^t))_{s^t \in \Sigma}$  is said to be **stationary Markovian** if  $z(s^t)$  is a time invariant function of the current shock  $s_t$ . We make a slight abuse of terminology and use the notation  $z(s_t)$  for  $z(s^{t-1}, s_t)$ .

We assume that agent  $i$ 's endowment process is stationary Markovian. For any event  $s^t = (s^{t-1}, s)$ , the endowment  $e^i(s^{t-1}, s)$  is denoted by  $e^i(s)$ . It follows that the reservation utility process is also stationary Markovian. Indeed, for every event  $s^t = (s^{t-1}, s)$  we have  $U(e^i|s^t) = V^i(s)$  where  $V^i = (V^i(s))_{s \in S} \in \mathbb{R}^S$  is the unique solution of the following recursive equations

$$\forall s \in S, \quad Y(s) = u(e^i(s)) + \beta \sum_{s' \in S} \pi(s'|s)Y(s').$$

When the process of endowments is stationary Markovian, the condition of uniform gains to trade is satisfied if, and only if, the allocation  $(e^i)_{i \in I}$  is not constrained Pareto efficient.<sup>32</sup> This result corresponds to Proposition 4 in [Bloise and Reichlin \(2011\)](#). The contribution of this section is to provide an alternative and simple proof that builds on the following intuitive result.

**LEMMA 5.1** Assume that  $(e^i)_{i \in I}$  is stationary Markovian and not constrained Pareto efficient. Then there exists an individually rational, self-enforcing and stationary Markovian allocation  $(d^i)_{i \in I}$  such that

$$\sum_{i \in I} d^i(s) < \sum_{i \in I} e^i(s), \quad \text{for all } s \in S.$$

**PROOF:** Since  $(e^i)_{i \in I}$  is not constrained Pareto efficient, there exists a constrained Pareto efficient allocation  $(c^i)_{i \in I}$  which Pareto dominates  $(e^i)_{i \in I}$ . This means that  $(c^i)_{i \in I}$  is individually rational, self-enforcing, resource feasible and satisfies  $U(c^i|s^0) \geq U(e^i|s^0)$  for every agent  $i$ , with a strict inequality for at least one agent. The stationarity of the endowment process implies that this Pareto dominance property is valid not only at the initial event  $s^0$ , but at any event  $s^1$  at date 1 (and actually at any event  $s^t$ ).

**CLAIM 5.1** For every  $s^1 \in S^1$ , we have  $U(c^i|s^1) \geq U(e^i|s^1)$  for every agent  $i$ , with a strict inequality for at least one agent.

**PROOF:** Fix an arbitrary event  $s^1$  at date 1. Since  $(c^i)_{i \in I}$  is self-enforcing, we have  $U(c^i|s^1) \geq U(e^i|s^1)$  for every agent  $i$ . Assume, by way of contradiction, that  $U(c^i|s^1) =$

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<sup>32</sup>The Pareto dominating consumption allocation  $(c^i)_{i \in I}$  need not be stationary Markovian.

$U(e^i|s^1)$  for every agent  $i$ . Since the Bernoulli function is strictly concave, this implies that  $c^i(s^t) = e^i(s^t)$  for every event  $s^t$  in the subtree  $\Sigma(s^1)$ . Otherwise, we would be able to Pareto dominate the constrained Pareto efficient allocation  $(c^i)_{i \in I}$ .<sup>33</sup> Now choose the date 2 event  $s^2 := (s^1, s_0)$  which is the successor of  $s^1$  corresponding to the shock  $s_0$ . Observe that the subtree  $\Sigma(s^2)$  is a copy of the whole tree  $\Sigma$ . In particular, any event  $s^t \in \Sigma(s^2)$  can be written  $s^t = (s^1, \sigma^{t-1})$  where  $\sigma^{t-1} = (s_0, \sigma_1, \dots, \sigma_{t-1}) \in \Sigma$ . Consider now the allocation  $(\tilde{c}^i)_{i \in I}$  defined by  $\tilde{c}^i(s^t) := c^i(s^t)$  if  $s^t \notin \Sigma(s^2)$ , and  $\tilde{c}^i(s^t) := c^i(\sigma^{t-1})$  for any event  $s^t = (s^1, \sigma^{t-1})$  where  $\sigma^{t-1} \in \Sigma$ . Since  $U(c^i|s^1) = U(e^i|s^1)$  for every agent  $i$ , we have that the allocation  $(\tilde{c}^i)_{i \in I}$  is individually rational, self-enforcing, resource feasible and Pareto dominates  $(c^i)_{i \in I}$ . *Q.E.D.*

Fix an arbitrary state  $s \in S$ . It follows from Claim 5.1 that

$$(5.1) \quad u(c^i(s_0, s)) + \beta \sum_{s' \in S} \pi(s'|s) U(c^i|(s_0, s, s')) \geq U(e^i|(s_0, s))$$

for every agent  $i$ , with a strict inequality for at least one agent. Recall that  $U(e^i|(s_0, s)) = V^i(s)$  and  $U(c^i|(s_0, s, s')) \geq V^i(s')$ . Therefore, for each  $i$  we can choose  $\delta^i(s) \in [0, c^i(s_0, s)]$  such that

$$(5.2) \quad \forall i \in I, \quad u(\delta^i(s)) + \beta \sum_{s' \in S} \pi(s'|s) V^i(s') \geq V^i(s)$$

together with<sup>34</sup>

$$(5.3) \quad \sum_{i \in I} \delta^i(s) < \sum_{i \in I} c^i(s_0, s) = \sum_{i \in I} e^i(s).$$

Denote by  $\mathcal{U}$  the set of vectors  $U = (U^i(s))_{(i,s) \in I \times S}$  in  $\mathbb{R}^{I \times S}$  satisfying

$$V^i(s) \leq U^i(s) \leq \frac{\sup u(\cdot)}{1 - \beta}, \quad \text{for every } (i, s) \in I \times S$$

and such that there exists  $x^i(s) \in [0, \delta^i(s)]$  satisfying

$$u(x^i(s)) + \beta \sum_{s' \in S} \pi(s'|s) U^i(s') \geq V^i(s), \quad \text{for every } (i, s) \in I \times S.$$

The set  $\mathcal{U}$  is non-empty. Indeed, it follows from (5.2) that the family  $(V^i(s))_{(i,s) \in I \times S}$  belongs to  $\mathcal{U}$ . It is straightforward to verify that  $\mathcal{U}$  is a compact and convex subset of

<sup>33</sup>Indeed, let  $(\tilde{c}^i)_{i \in I}$  be the allocation defined by  $\tilde{c}^i(s^t) := c^i(s^t)$  if  $s^t \notin \Sigma(s^1)$ , and  $\tilde{c}^i(s^t) := (1/2)c^i(s^t) + (1/2)e^i(s^t)$  for  $s^t \in \Sigma(s^1)$ . The allocation  $(\tilde{c}^i)_{i \in I}$  is individually rational, self-enforcing and resource feasible. Moreover, if there exists one agent  $i$  such that  $c^i$  and  $e^i$  do not coincide on the subtree  $\Sigma(s^1)$ , then  $(\tilde{c}^i)_{i \in I}$  Pareto dominates  $(c^i)_{i \in I}$ .

<sup>34</sup>If the inequality in (5.1) is strict, we can choose  $\delta^i(s) < c^i(s_0, s)$ , otherwise we take  $\delta^i(s) = c^i(s_0, s)$ .

$\mathbb{R}^{I \times S}$ . Let  $\Phi : \mathcal{U} \rightarrow \mathcal{U}$  be the correspondence where  $\Phi(U)$  is the set of all vectors  $\tilde{U}$  in  $\mathcal{U}$  for which there exists  $y^i(s) \in [0, \delta^i(s)]$  satisfying

$$V^i(s) \leq \tilde{U}^i(s) \leq u(y^i(s)) + \beta \sum_{s' \in S} \pi(s'|s) U^i(s'), \quad \text{for every } (i, s) \in I \times S.$$

The set  $\Phi(U)$  is a non-empty, convex and closed subset of  $\mathcal{U}$  and the correspondence  $\Phi$  is upper semi-continuous.<sup>35</sup> Applying Kakutani's Fixed-Point Theorem, there exist a vector  $(U^i(s))_{(i,s) \in I \times S}$  in  $\mathcal{U}$  and a vector  $(d^i(s))_{(i,s) \in I \times S}$  in  $\mathbb{R}^{I \times S}$  such that

$$(5.4) \quad V^i(s) \leq U^i(s) \leq u(d^i(s)) + \beta \sum_{s' \in S} \pi(s'|s) U^i(s'), \quad \text{for every } (i, s) \in I \times S$$

where  $d^i(s) \in [0, \delta^i(s)]$ . Abusing notations, we let  $(d^i)_{i \in I}$  be the consumption allocation associated to the vector  $(d^i(s))_{(i,s) \in I \times S}$  and defined by  $d^i(s^t) := d^i(s_t)$ . Applying recursively (5.4), we get

$$U(e^i|s^t) = V^i(s_t) \leq U^i(s_t) \leq U(d^i|s^t).$$

Therefore, the allocation  $(d^i)$  is self-enforcing, individually rational and stationary Markovian. Moreover, it follows from (5.3) that

$$\sum_{i \in I} d^i(s) \leq \sum_{i \in I} \delta^i(s) < \sum_{i \in I} c^i(s_0, s) = \sum_{i \in I} e^i(s).$$

*Q.E.D.*

It is straightforward to see that Lemma 5.1 implies that the economy exhibits uniform gains to trade since the process  $(e^i)_{i \in I}$  of endowments is uniformly bounded from above and uniformly bounded away from zero.

**PROPOSITION 5.1** If the process of endowments is stationary Markovian and not constrained Pareto efficient then there are uniform gains to trade.

**PROOF:** Define  $z := \sum_{i \in I} (e^i - d^i)$ , where  $(d^i)_{i \in I}$  is the individually rational, self-enforcing and stationary Markovian allocation  $(d^i)_{i \in I}$  of Lemma 5.1. The process  $z$  is strictly positive and stationary Markovian. Moreover, if we let

$$\gamma := \sup_{s \in S} \frac{z(s)}{e(s)}.$$

then  $\gamma \in (0, 1)$  and condition (2.2) is satisfied.

*Q.E.D.*

<sup>35</sup>Since  $U$  belongs to  $\mathcal{U}$ , the vector  $(V^i(s))_{(i,s) \in I \times S}$  belongs to  $\Phi(U)$ .



## 5.2. Seizable Assets

We discuss a simple extension of our environment. Following [Kehoe and Levine \(1993\)](#) and [Kocherlakota \(2008\)](#), we assume now that there is a physical asset (also called tree) in positive net supply delivering the dividend  $\xi(s^t) \geq 0$  at every event  $s^t$ . At the initial event  $s^0$ , each agent  $i$  holds a fraction  $\theta^i(s^0) \geq 0$  of the tree, with  $\sum_{i \in I} \theta^i(s^0) = 1$ . The aggregate resources to be allocated between agents at event  $s^t$  is now

$$\omega(s^t) := \xi(s^t) + \sum_{i \in I} e^i(s^t)$$

and an allocation  $(c^i)_{i \in I}$  is said to be resource feasible whenever

$$(5.5) \quad \sum_{i \in I} c^i = \omega = \xi + \sum_{i \in I} e^i.$$

Following the interpretation proposed by [Kehoe and Levine \(1993\)](#), the private endowment  $e^i(s^t)$  represents goods and services, such as labor, that cannot be physically dissociated from the agent. The shares of the physical asset (such as land) can change hands, and therefore can be seized in case of breach of contract. Therefore, the default option at event  $s^t$  is still represented by  $U(e^i|s^t)$  and the definition of an individually rational and self-enforcing allocation remains unchanged. In particular, the definition of a constrained Pareto efficient allocation remains the same except for the resource feasibility constraint that is now defined by (5.5). The commodity space  $\ell^\infty(e)$  is replaced by  $\ell^\infty(\omega)$  and the budget constraint  $\varphi(c^i) \leq \varphi(e^i)$  is replaced by  $\varphi(c^i) \leq \varphi(e^i + \theta^i \xi)$  in the definition of a Kehoe–Levine competitive equilibrium.

All the results presented above remain valid under the uniform gains to trade assumption which takes now the following form: the economy exhibits **uniform gains to trade** if there is an individually rational and self-enforcing allocation  $(d^i)_{i \in I}$  and  $\gamma > 0$  such that

$$(5.6) \quad \forall s^t \in \Sigma, \quad \sum_{i \in I} d^i(s^t) \leq (1 - \gamma)\omega(s^t).$$

The contribution of this section consists in showing that if dividends  $\xi$  of the tree is large enough with respect to the private aggregate endowments  $e$ , then the assumption of uniform gains to trade is automatically satisfied.

**PROPOSITION 5.2** If there exists a fraction  $\alpha > 0$  such that  $\xi \geq \alpha e$ , i.e.,

$$(5.7) \quad \xi(s^t) \geq \alpha \sum_{i \in I} e^i(s^t), \quad \text{for all event } s^t$$

then the economy exhibits uniform gains to trade.

PROOF: Observe that

$$\sum_{i \in I} e^i = e \leq (1 - \gamma)(\xi + e) = (1 - \gamma)\omega$$

where  $\gamma := \alpha/(1 + \alpha)$ . We can then choose  $(d^i)_{i \in I}$  such that  $\sum_{i \in I} d^i = (1 - \gamma)\omega$  and  $d^i \geq e^i$  for each  $i$ . Q.E.D.

Condition (5.7) is satisfied if dividends are uniformly bounded away from zero and private endowments are uniformly bounded from above. This is in particular the case if the economy is stationary Markovian and dividends are strictly positive. Therefore, the existence result in Proposition 6 of Kehoe and Levine (1993) is a particular case of our Theorem 4.1.

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## Appendix A: TECHNICAL RESULTS AND PROOFS

### A.1. Continuity Properties of the Utility Function

Since the Bernoulli function is bounded from above, it follows from standard arguments that the function  $U(\cdot|s^0) : c \mapsto U(c|s^0)$  is upper semi-continuous on  $\mathbb{R}_+^\Sigma$  for the product topology. It is also lower semi-continuous on the space of consumption processes that are self-enforcing and individually rational.

**LEMMA A.1** The utility function  $U(\cdot|s^0)$  is continuous for the product topology on the space of self-enforcing and individually rational consumption processes.

**PROOF:** Fix  $x$  and  $y$  two consumption processes in  $\mathbb{R}_+^\Sigma$ . For every date  $\tau \geq 1$ , we have

$$(A.1) \quad |U(x|s^0) - U(y|s^0)| \leq \sum_{t=0}^{\tau-1} \beta^t \sum_{s^t \in S^t} \pi(s^t) |u(x(s^t)) - u(y(s^t))| \\ + \beta^\tau \sum_{s^\tau \in S^\tau} \pi(s^\tau) |\Delta(x, y|s^\tau)|$$

where  $\Delta(x, y|s^\tau) := U(x|s^\tau) - U(y|s^\tau)$ . Observe that for any self-enforcing and individually rational consumption process  $z$ , we have

$$U(e^i|s^\tau) \leq U(z|s^\tau) \leq \sum_{s^\xi \succeq s^\tau} \beta^{\xi-\tau} \pi(s^\xi|s^\tau) \bar{u} \leq \frac{\bar{u}}{1-\beta}$$

where  $\bar{u} := \sup_{c \geq 0} u(c)$ . This implies that

$$(A.2) \quad \limsup_{\tau \rightarrow \infty} \beta^\tau \sum_{s^\tau \in S^\tau} \pi(s^\tau) U(z|s^\tau) \leq 0.$$

Since we assumed that  $U(e^i|s^0) > -\infty$  we must have

$$\lim_{\tau \rightarrow \infty} \beta^\tau \sum_{s^\tau \in S^\tau} \pi(s^\tau) U(e^i|s^\tau) = 0.$$

This implies that

$$(A.3) \quad \liminf_{\tau \rightarrow \infty} \beta^\tau \sum_{s^\tau \in S^\tau} \pi(s^\tau) U(z|s^\tau) \geq 0.$$

Combining (A.2) and (A.3), we can then conclude that

$$\lim_{\tau \rightarrow \infty} \beta^\tau \sum_{s^\tau \in S^\tau} \pi(s^\tau) |\Delta(x, y|s^\tau)| = 0$$

and the continuity of  $U(\cdot|s^0)$  follows from the continuity of the Bernoulli function. *Q.E.D.*

## A.2. Strictly Positive Consumption

LEMMA A.2 Consider a consumption process  $c^j$  that is self-enforcing and individually rational. Assume there is an event  $s^t \succeq s^0$  such that  $c^j(s^t) = 0$ . Then there must exist a successor event  $s^\tau \succ s^t$  such that

- (a)  $c^j(s^\tau) > 0$  and  $U(c^j|s^\tau) > U(e^j|s^\tau)$ ;
- (b) for every intermediate event  $s^r$  satisfying  $s^t \prec s^r \prec s^\tau$ , we have  $c^j(s^r) = 0$  and  $U(c^j|s^r) > U(e^j|s^r)$ .

PROOF OF LEMMA A.2: Assume by way of contradiction that there does not exist  $s^\tau \succ s^t$  satisfying (a) and (b). We let  $S_{\text{NB}}^{t+1}(s^t)$  be the set of date  $t+1$  events successors of  $s^t$  such that the participation constraint is non-binding. We must have  $c^j(s^{t+1}) = 0$  for every  $s^{t+1} \in S_{\text{NB}}^{t+1}(s^t)$  (otherwise (a) and (b) are satisfied). We denote by  $S_{\text{NB}}^{t+2}(s^t)$  the set defined by

$$S_{\text{NB}}^{t+2}(s^t) := \bigcup_{s^{t+1} \in S_{\text{NB}}^{t+1}(s^t)} \{s^{t+2} \in S^{t+2}(s^{t+1}) : U(c^j|s^{t+2}) > U(e^j|s^{t+2})\}.$$

For every event  $s^r$  we let  $\Delta U(c^j|s^r) := U(c^j|s^r) - U(e^j|s^r)$ . Observe that  $\pi(s^t)\Delta U(c^j|s^t)$  coincides with

$$\begin{aligned} \pi(s^t)[u(0) - u(e^j(s^t))] + \beta \sum_{s^{t+1} \in S_{\text{NB}}^{t+1}(s^t)} \pi(s^{t+1})[u(0) - u(e^j(s^{t+1}))] \\ + \beta^2 \sum_{s^{t+2} \in S_{\text{NB}}^{t+2}(s^t)} \pi(s^{t+2})\Delta U(c^j|s^{t+2}). \end{aligned}$$

Since  $\Delta U(c^j|s^t) \geq 0$ , the set  $S_{\text{NB}}^{t+2}(s^t)$  is non-empty. Again, we must have  $c^j(s^{t+2}) = 0$  for every  $s^{t+2} \in S_{\text{NB}}^{t+2}(s^t)$ . We can then define recursively a sequence  $(S_{\text{NB}}^{t+n}(s^t))_{n \geq 1}$  such that for every  $k \geq 2$

$$\begin{aligned} \pi(s^t)\Delta^j U(s^t) = \sum_{n=1}^{k-1} \beta^n \sum_{s^{t+n} \in S_{\text{NB}}^{t+n}(s^t)} \pi(s^{t+n})[u(0) - u(e^j(s^{t+n}))] + \\ \sum_{s^{t+k} \in S_{\text{NB}}^{t+k}(s^t)} \beta^k \pi(s^{t+k})\Delta^j U(s^{t+k}). \end{aligned}$$

Passing to the limit when  $k$  goes to infinite, we get  $\Delta U(c^j|s^t) < 0$  which contradicts the fact that  $c^j$  is individually rational and self-enforcing. *Q.E.D.*

PROPOSITION A.1 If  $(c^i)_{i \in I}$  is a constrained Malinvaud efficient allocation, then it is strictly positive.

PROOF: Let  $(c^i)_{i \in I}$  be a constrained Malinvaud efficient allocation. Fix a date- $t$  event  $s^t$  and assume by way of contradiction that there exists an agent  $j$  with  $c^j(s^t) = 0$ . From

the resource feasibility there exists an agent  $k \neq j$  with  $c^k(s^t) > 0$ . We propose to show that there exists a bilateral trade between agents  $j$  and  $k$  that is Pareto improving.

There exists a successor event  $s^\tau \succ s^t$  satisfying conditions (a) and (b) of Lemma A.2. Fix  $\varepsilon > 0$  small enough such that  $c^k(s^t) > \varepsilon$  and  $c^j(s^\tau) > \gamma\varepsilon$  where  $\gamma$  is arbitrarily chosen to satisfy

$$\gamma > \frac{\beta^\tau \pi(s^\tau) u'(c^k(s^\tau))}{\beta^t \pi(s^t) u'(c^k(s^t))}.$$

Define  $(\hat{c}^j, \hat{c}^k)$  as follows

$$\hat{c}^j(s^r) := \begin{cases} c^j(s^t) + \varepsilon & \text{if } s^r = s^t \\ c^j(s^\tau) - \gamma\varepsilon & \text{if } s^r = s^\tau \end{cases}, \quad \hat{c}^k(s^r) := \begin{cases} c^k(s^t) - \varepsilon & \text{if } s^r = s^t \\ c^k(s^\tau) + \gamma\varepsilon & \text{if } s^r = s^\tau \end{cases}$$

and  $(\hat{c}^j(s^r), \hat{c}^k(s^r)) := (c^j(s^r), c^k(s^r))$  for any  $s^r \notin \{s^t, s^\tau\}$ . By construction the consumption processes  $\hat{c}^j$  and  $\hat{c}^k$  satisfy participation constraints for every event  $s^r \succeq s^t$  that does not satisfy  $s^r \preceq s^\tau$  since we have  $U(\hat{c}^i|s^r) = U(c^i|s^r)$  for every  $i \in \{j, k\}$ .

The choice of  $\gamma$  is such that for  $\varepsilon > 0$  small enough we have

$$\pi(s^t) \beta^t u(\hat{c}^k(s^t)) + \pi(s^\tau) \beta^\tau u(\hat{c}^k(s^\tau)) > \pi(s^t) \beta^t u(c^k(s^t)) + \pi(s^\tau) \beta^\tau u(c^k(s^\tau)).$$

It follows that  $U(\hat{c}^k|s^r) > U(c^k|s^r)$  for any event  $s^r$  satisfying  $s^t \preceq s^r \preceq s^\tau$ . We have thus proved that  $\hat{c}^k$  is still self-enforcing and individually rational. Moreover, we have  $U(\hat{c}^k|s^t) > U(c^k|s^t)$ .

For every event  $s^r$  satisfying  $s^t \prec s^r \preceq s^\tau$ , agent  $j$ 's participation constraint is non-binding, i.e.,  $U(c^j|s^r) > U(e^j|s^r)$ . This implies that we can choose  $\varepsilon > 0$  small enough such that  $U(\hat{c}^j|s^r) \geq U(e^j|s^r)$  for any  $s^r$  satisfying  $s^t \prec s^r \preceq s^\tau$ . To get a contradiction, we only have to show that  $U(\hat{c}^j|s^t) > U(c^j|s^t)$ . Observe that

$$U(\hat{c}^j|s^t) - U(c^j|s^t) = [u(\varepsilon) - u(0)] + \frac{\beta^\tau \pi(s^\tau)}{\beta^t \pi(s^t)} [u(c^j(s^\tau) - \gamma\varepsilon) - u(c^j(s^\tau))].$$

Inada's condition implies that for  $\varepsilon > 0$  small enough we get the desired result. *Q.E.D.*

### A.3. Finite Present Value under Personalized Prices

LEMMA A.3 Consider an event  $s^t$ , a strictly positive consumption process  $c^i$  and a strictly positive process  $p = (p(s^t))_{s^t \in \Sigma}$  such that  $\text{PV}(p; e^i|s^0)$  is finite. Assume that, for every successor event  $s^\tau \succ s^t$ , the participation constraint is satisfied (i.e.,  $U(c^i|s^\tau) \geq U(e^i|s^\tau)$ ), and  $\text{MRS}^i(c^i|s^\tau) \leq p(s^\tau)/p(s^t)$ . Then

$$\text{PV}(p^i; c^i|s^\tau) < \infty$$

where  $p^i$  is the process of individual Arrow–Debreu prices defined recursively by  $p^i(s^0) := 1$  and  $p^i(s^t)/p^i(s^t) := \text{MRS}^i(c^i|s^t)$ .<sup>36</sup>

<sup>36</sup>In particular, we have  $p^i(s^t) = \beta^t \pi(s^t) u'(c^i(s^t)) / u'(c^i(s^0))$ .

PROOF: We provide a proof for  $s^t = s^0$ . The general case obtains replacing the tree  $\Sigma$  by the subtree  $\Sigma(s^t)$ . By concavity of the Bernoulli function we have

$$\begin{aligned} \frac{U(e^i|s^0) - U(c^i|s^0)}{u'(c^i(s^0))} &\leq \sum_{t=0}^{r-1} \sum_{s^t \in S^t} p^i(s^t) [e^i(s^t) - c^i(s^t)] \\ &\quad + \beta^r \sum_{s^r \in S^r} \pi(s^r) \frac{U(e^i|s^r) - U(c^i|s^r)}{u'(c^i(s^0))}. \end{aligned}$$

Since  $U(c^i|s^r) \geq U(e^i|s^r)$  we get that

$$\begin{aligned} \sum_{t=0}^{r-1} \sum_{s^t \in S^t} p^i(s^t) c^i(s^t) &\leq \sum_{t=0}^{r-1} \sum_{s^t \in S^t} p^i(s^t) e^i(s^t) + \frac{U(c^i|s^0) - U(e^i|s^0)}{u'(c^i(s^0))} \\ &\leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p(s^t) e^i(s^t) + \frac{U(c^i|s^0) - U(e^i|s^0)}{u'(c^i(s^0))} \end{aligned}$$

and the desired result follows from the assumption that  $PV(p; e^i|s^0)$  is finite. *Q.E.D.*

#### A.4. Decentralization Result

PROPOSITION A.2 Consider an event  $s^t$ , a strictly positive consumption process  $c^i$  and strictly positive process  $p = (p(s^t))_{s^t \in \Sigma}$  such that  $PV(p; e^i|s^0)$  is finite. Assume that, for every successor event  $s^\tau \succ s^t$ , the participation constraint is satisfied (i.e.,  $U(c^i|s^\tau) \geq U(e^i|s^\tau)$ ), and  $MRS^i(c^i|s^\tau) \leq p(s^\tau)/p(\sigma(s^\tau))$  with equality if  $U(c^i|s^\tau) > U(e^i|s^\tau)$ . Then  $c^i$  has finite present value, in the sense that  $PV(p; c^i|s^0) < \infty$ , and for every self-enforcing consumption process  $\tilde{c}^i$  with finite present value, we have

$$\frac{1}{u'(c^i(s^t))} [U(\tilde{c}^i|s^t) - U(c^i|s^t)] \leq PV(p; \tilde{c}^i - c^i|s^t).$$

PROOF: We provide a proof for  $s^t = s^0$ . The general case obtains replacing the tree  $\Sigma$  by the subtree  $\Sigma(s^t)$ . For every event  $s^\tau \succ s^0$ , we denote by  $q^i(s^\tau)$  the marginal rate of substitution  $MRS^i(c^i|s^\tau)$  and we let  $p^i$  be the associated ‘‘personalized’’ Arrow–Debreu price process defined by  $p^i(s^0) := 1$  and  $p^i(s^\tau) = q^i(s^\tau)p^i(\sigma(s^\tau))$ .<sup>37</sup> We first prove an intermediate result.

CLAIM A.1 For every self-enforcing consumption process  $\tilde{c}^i$  with finite present value, i.e.,  $PV(p; \tilde{c}^i|s^0) < \infty$ , we have

$$(A.4) \quad \forall \xi > 0, \quad p(s^0) b^i(s^0) \leq \sum_{\tau=0}^{\xi-1} \sum_{s^\tau \in S^\tau} p(s^\tau) (\tilde{c}^i(s^\tau) - c^i(s^\tau)) + \sum_{s^\xi \in S^\xi} p(s^\xi) b^i(s^\xi)$$

<sup>37</sup>Or equivalently  $p^i(s^\tau) = \beta^\tau \pi(s^\tau) u'(c^i(s^\tau)) / u'(c^i(s^0))$ .

where  $b^i(s^\tau) := \text{PV}(p^i; \tilde{c}^i - c^i | s^\tau)$ .<sup>38</sup> Moreover, we have

$$(A.5) \quad \lim_{\xi \rightarrow \infty} \sum_{s^\xi \in S^\xi} p(s^\xi) b^i(s^\xi) = 0.$$

PROOF OF CLAIM A.1: Concavity of the Bernoulli function implies that for every  $s^\tau \in \Sigma$ ,

$$(A.6) \quad \frac{\beta^\tau \pi(s^\tau)}{u'(c^i(s^0))} [U(\tilde{c}^i | s^\tau) - U(c^i | s^\tau)] \leq \sum_{s^r \in \Sigma(s^\tau)} p^i(s^r) (\tilde{c}^i(s^r) - c^i(s^r)) = p^i(s^\tau) b^i(s^\tau).$$

By definition of the process  $b^i$ , we get for every event  $s^\tau$ ,

$$b^i(s^\tau) = (\tilde{c}^i(s^\tau) - c^i(s^\tau)) + \sum_{s^{\tau+1} \succ s^\tau} q^i(s^{\tau+1}) b^i(s^{\tau+1}).$$

If  $U(c^i | s^{\tau+1}) > U(\tilde{c}^i | s^{\tau+1})$  then  $q^i(s^{\tau+1}) = q(s^{\tau+1})$ . If  $U(c^i | s^{\tau+1}) = U(\tilde{c}^i | s^{\tau+1})$  then we have  $U(\tilde{c}^i | s^{\tau+1}) \geq U(c^i | s^{\tau+1})$  since  $\tilde{c}^i$  is self-enforcing. It then follows from (A.6) that  $b^i(s^{\tau+1}) \geq 0$ . In both cases we have  $q^i(s^{\tau+1}) b^i(s^{\tau+1}) \leq q(s^{\tau+1}) b^i(s^{\tau+1})$ , implying that

$$b^i(s^\tau) \leq (\tilde{c}^i(s^\tau) - c^i(s^\tau)) + \sum_{s^{\tau+1} \succ s^\tau} q(s^{\tau+1}) b^i(s^{\tau+1}).$$

Multiplying by  $p(s^\tau)$  and summing the inequalities over all events  $s^\tau$  up to date  $\xi$ , we get Equation (A.4).

We now show that Equation (A.5) is satisfied. It is sufficient to prove that

$$\lim_{\xi \rightarrow \infty} \sum_{s^\xi \in S^\xi} p(s^\xi) \text{PV}(p^i; x | s^\xi) = 0$$

for any process  $x$  with finite present value under  $p$ , i.e.,  $\text{PV}(p; x | s^0) < \infty$ . Observe that

$$p(s^\xi) \text{PV}(p^i; x | s^\xi) = p(s^\xi) \sum_{s^r \in \Sigma(s^\xi)} \frac{p^i(s^r)}{p^i(s^\xi)} x(s^r).$$

For every  $s^r \in \Sigma(s^\xi)$  there exists a finite family of events  $(s^{\xi+1}, \dots, s^{r-1})$  such that

$$s^\xi \prec s^{\xi+1} \prec s^{\xi+2} \prec \dots \prec s^{r-1} \prec s^r.$$

In particular

$$\frac{p^i(s^r)}{p^i(s^\xi)} = q^i(s^{\xi+1}) \dots q^i(s^r) \leq \frac{p(s^r)}{p(s^\xi)}.$$

<sup>38</sup>To define  $b^i(s^\tau)$  we need to show that  $\text{PV}(p^i; \tilde{c}^i | s^0)$  and  $\text{PV}(p^i; c^i | s^0)$  are both finite. Since  $p^i \leq p$  we have that  $\text{PV}(p^i; \tilde{c}^i | s^0)$  is finite. The fact that  $\text{PV}(p^i; c^i | s^0)$  is finite follows from standard arguments. We refer to Appendix A.3 for details.

It follows that  $p(s^\xi) \text{PV}(p^i; x|s^\xi) \leq p(s^\xi) \text{PV}(p; x|s^\xi)$  and consequently

$$\sum_{s^\xi \in S^\xi} p(s^\xi) \text{PV}(p^i; x|s^\xi) \leq \sum_{r \geq \xi} \sum_{s^r \in S^r} p(s^r) x(s^r).$$

The desired result follows from the fact that  $\text{PV}(p; x|s^0)$  is finite. *Q.E.D.*

To prove that  $c^i$  has finite present value, we apply Claim A.1 choosing  $\tilde{c}^i := e^i$ . Observe that passing to the limit in (A.4) we get that

$$\frac{1}{u'(c^i(s^0))} [U(\tilde{c}^i|s^0) - U(c^i|s^0)] \leq \text{PV}(p; \tilde{c}^i - c^i|s^0).$$

*Q.E.D.*

## Appendix B: COMPARING WITH THE LITERATURE

Recall that if  $(c^i)_{i \in I}$  is a strictly positive consumption process, then the implied Arrow–Debreu price process  $p^*$  is defined recursively by  $p^*(s^0) := 1$  and

$$\frac{p^*(s^t)}{p^*(\sigma(s^t))} := \max_{i \in I} \text{MRS}(c^i|s^t), \quad \text{for all } s^t \succ s^0.$$

Following [Bloise and Reichlin \(2011\)](#), we say that an allocation satisfies the **Modified Cass Criterion** when there exists a non-null and uniformly bounded from above process  $v$  satisfying, for some  $\rho \in (0, 1)$ ,

$$\rho \sum_{s^{t+1} \succ s^t} p^*(s^{t+1}) v(s^{t+1}) \geq p^*(s^t) v(s^t), \quad \text{for all } s^t \succeq s^0.$$

When this condition holds true for  $\rho = 1$ , we say that the allocation satisfies the **Weak Modified Cass Criterion**.

[Bloise and Reichlin \(2011\)](#) introduce a different concept of “high implied interest rates” than the one we borrow from [Alvarez and Jermann \(2000\)](#). We say that implied interest rates are **BR-high** when

$$\sum_{s^t \in \Sigma} p^*(s^t) < \infty.$$

This concept coincides with the one we adopt if, and only if, the endowment process is uniformly bounded from above and uniformly bounded away from zero.<sup>39</sup>

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<sup>39</sup>In the sense that there exists  $\varepsilon > 0$  such that  $1/\varepsilon \leq e(s^t) \leq \varepsilon$  for any event  $s^t$ .



## B.1. Lemma 3.3

Lemma 3.3 is in the spirit of Lemma 2 in [Bloise and Reichlin \(2011\)](#) but cannot be compared to it. To illustrate this we state below their result using our terminology.

LEMMA B.1 (Lemma 2 in [Bloise and Reichlin \(2011\)](#)) Assume aggregate endowments are uniformly bounded from above. Consider an allocation  $c = (c^i)_{i \in I}$  that is feasible, self-enforcing, individually rational and uniformly bounded away from zero. If  $c$  is constrained Pareto optimal then it does not satisfy the Modified Cass Criterion.

On one hand, we do not restrict attention to allocations that are uniformly bounded away from zero. On the other hand, we have a stronger assumption (uniform gains to trade) but their conclusion is weaker than ours.

## B.2. Lemma 3.4

Lemma 3.4 is in the spirit of Lemma 3 in [Bloise and Reichlin \(2011\)](#) but cannot be compared to it. To illustrate this we state below their result using our terminology.

LEMMA B.2 (Lemma 3 in [Bloise and Reichlin \(2011\)](#)) Assume aggregate endowments are uniformly bounded from above. Consider an allocation that is constrained Malinvaud optimal and uniformly bounded away from zero. If it does not satisfy the Weak Modified Cass Criterion then it is constrained Pareto optimal.

On one hand, the sufficient condition we proposed in Lemma 3.4 is valid for any constrained Malinvaud efficient allocations, not necessarily those that are uniformly bounded away from zero. On the other hand, the sufficient condition of Lemma 3 in [Bloise and Reichlin \(2011\)](#) is weaker than ours.

## B.3. Proposition 3.1

Proposition 3.1 is in the spirit of Proposition 3 in [Bloise and Reichlin \(2011\)](#) that we state below using our terminology.

PROPOSITION B.1 (Proposition 3 in [Bloise and Reichlin \(2011\)](#)) Assume aggregate endowments are uniformly bounded from above, uniformly bounded away from zero, and there are uniform gains to trade. Consider an allocation that is constrained Malinvaud efficient and uniformly bounded away from zero. It is constrained Pareto efficient if, and only if, implied interest rates are high.

First, [Bloise and Reichlin \(2011\)](#) only characterize the sub-class of allocations that are uniformly bounded away from zero. Second, they assume that aggregate endowments

are uniformly bounded from above and uniformly bounded away from zero. They need these additional assumptions because their approach to prove the necessity part (see their Lemma 5) requires that marginal utilities of consumption at constrained Pareto efficient allocations are uniformly bounded from above and bounded away from 0.

#### B.4. Boundary Condition of [Bloise et al. \(2013\)](#)

In [Bloise et al. \(2013\)](#) (see also [Bidian and Bejan \(2014\)](#)), the following extra-conditions are imposed on primitives.

**(B)** The Bernoulli function is bounded from below (i.e.,  $u(0) \in \mathbb{R}$ ) and there exists  $\varepsilon > 0$  such that

$$\text{B.1. } \varepsilon \leq e^i(s^t) \leq 1/(\varepsilon \# I) \text{ for every } i \text{ and } s^t;$$

$$\text{B.2. } \beta u(0) + (1 - \beta)u(1/\varepsilon) < u(\varepsilon).$$

In this paper, we only assume that the endowment process is strictly positive, i.e.,  $e^i(s^t) > 0$ . In particular, we allow for the possibility of successive negative (positive) shocks on endowments such that  $e^i$  converge to 0 (to infinite) along some path. Moreover, we do not impose any consistency requirement between the subjective discount factor  $\beta$ , the Bernoulli function  $u$  and endowments. Observe that for any given pair  $(\beta, u)$ , we can construct an endowment process  $(e^i)_{I \in I}$  satisfying (B.1) but such that (B.2) is not satisfied. Indeed, (B.2) implies that

$$u(1/\varepsilon) < \frac{\beta}{1 - \beta}[u(\varepsilon) - u(0)] + u(\varepsilon)$$

passing to the limit when  $\varepsilon$  tends to 0, we get the contradiction  $\sup_{x \geq 0} u(x) \leq u(0)$ .