Constrained Efficiency Without Commitment

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We consider an infinite horizon economy where agents share income risks by trading a complete set of contingent claims but cannot commit to their promises. Allocations are restricted to be self-enforcing relative to autarchic reservation utilities. We provide a general characterization of constrained Pareto efficiency without assuming that there are uniform gains to trade. Our results extend those in Bloise and Reichlin (2011) in several aspects.

1. Introduction

The paper studies infinite horizon exchange economies with complete contingent claims markets when there is no commitment and default induces permanent exclusion from future trading. As in Eaton and Gersovitz (1981) and Kehoe and Levine (1993), trade is subject to participation constraints that restrict allocations to be self-enforcing relative to autarchic reservation utilities.[1] The presence of such constraints and the associated imperfect risk-sharing imply that the economy cannot attain a social optimum. An important issue is then to explore under which conditions an allocation is constrained Pareto efficient, that is, to identify necessary and sufficient conditions that rule out benefits from redistributions given the participation constraints.

Following the classical approach in general equilibrium, Bloise and Reichlin (2011) provide an interesting treatment of this matter by characterizing constrained Pareto efficiency in terms of supporting linear functionals. They show that under uniform gains to trade, the support by a linear functional is a necessary and sufficient condition for interior (uniformly bounded away from zero) allocations to be constrained Pareto efficient. Furthermore, they show that any supporting linear functional admits a sequential representation, in the sense that its purely additive part (bubble component) is null. This allows them to compare their characterization result with that provided by Alvarez and Jermann (2000) formulated in terms of high implied interest rates (i.e., state-contingent prices that imply a finite present value of the intertemporal aggregate endowment).[2]

[1]We would like to thank Gaetano Bloise for comments and suggestions. The financial support of CNPq is gratefully acknowledged by V. Filipe Martins-da-Rocha. Yiannis Vailakis acknowledges the financial support of an ERC starting grant (FP7, DCFM 240983) and of an ANR research grant (Project Novo Tempus).

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[1]Alvarez and Jermann (2000) propose a sequential formulation of this model where agents trade a complete set of contingent bonds every period. The focus is on endogenously determined agent-specific debt limits that correspond to participation constraints at autarchic reservation utilities. Such limits on borrowing are referred in the literature as not-too-tight debt constraints.

[2]Bloise and Reichlin (2011) show by means of an example that the assumption of uniform gains to trade is indispensable. Without it, constrained Pareto efficiency does not necessarily lead to high implied interest rates.

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Bloise and Reichlin (2011) also study constrained Pareto efficiency in the absence of uniform gains to trade (still restricting attention to allocations that are uniformly bounded away from zero). They identify a strong connection between constrained Pareto inefficiency and two variants of the Cass criterion originally proposed for stochastic overlapping generations economies. However, they only identify a necessary and sufficient condition under the assumption of uniform gains to trade.

This paper aims at looking at constrained Pareto efficiency from a fresh perspective. The main objective is to provide a complete characterization, free of the assumption of uniform gains to trade. More precisely, we show that an allocation is constrained Pareto efficient if, and only if, it can be approximated by a sequence of constrained Pareto efficient allocations associated to perturbed economies where a physical asset in positive net supply is introduced. The proof of this result unfolds in two steps.

First, we prove that under uniform gains to trade, high implied interest rates are necessary and sufficient for constrained Pareto efficiency, even if we dispense with the interiority restriction imposed in Bloise and Reichlin (2011). The novel aspects of this step are as follows. On one hand, to prove sufficiency, there is no need to assume uniform gains to trade. We propose a new decentralization result that explores the concavity of Bernoulli functions and identifies sufficient conditions for state-contingent prices to belong to the constrained sub-differential of expected utilities. On the other hand, to prove necessity, we show that if a linear functional supports a constrained Pareto efficient allocation, then it cannot have a bubble component and must coincide with the process of implied Arrow–Debreu prices (which turns out to display high interest rates).

It is easy to verify that the assumption of uniform gains to trade is automatically satisfied in any perturbed economy in which the dividend process is sufficiently large with respect to the aggregate endowment process. The second step then involves to exploit the characterization result of the first step to prove that an allocation that satisfies the Euler equations is constrained Pareto efficient if, and only if, it is the limit of allocations exhibiting high implied interest rates. We say that such allocations exhibit "almost high implied interest rates".

We illustrate the usefulness of our general characterisation result for the standard stationary Markovian economy with two agents, two shocks and no aggregate uncertainty. Constrained Pareto efficiency of the autarchic allocation is then characterized by means of the income dispersion parameter. We also compare our results with those in Bloise and Reichlin (2011) and show that, in the absence of uniform gains to trade, our necessary and sufficient condition is strictly weaker than their sufficient condition and strictly stronger than their necessary condition.

We show in a supplementary material (Martins-da-Rocha and Vailakis (2015)) that our characterization result has direct implications for the validity of the Second Welfare Theorem. Indeed, standard arguments can be applied to show that any constrained Pareto efficient allocation is supported by some linear functional. This linear functional has no bubble component, and must coincide with the implied contingent-state prices. In particular, implied interest rates are high and we can deduce (using our decentralization result) that any constrained Pareto efficient allocation can be implemented as a constrained competitive equilibrium with high interest rates and endogenous transfers. We also strengthen the Second Welfare Theorem by characterizing the set of consumption allocations implemented as constrained competitive equilibria with high interest rates and zero initial transfers.

The paper is organized as follows: Section 2 describes the environment and provides the definitions of the various concepts used throughout the paper. As an intermediate step, Section 3 contains our characterization of constrained Pareto efficiency under uniform gains to trade. The general characterization follows in Section 4. A discussion on how our results differentiate from those in Bloise and Reichlin (2011) are collected in the Appendix. The details of some technical results can be found in the supplementary material.

2. The Model

Here we present an infinite horizon exchange economy with lack of commitment and self-enforcing participation constraints, along the lines of Kehoe and Levine (1993) and Bloise and Reichlin (2011). Time and uncertainty are both discrete and there is a single non-storable consumption good. The economy consists of a finite set I of infinitely lived agents that share risks but cannot commit to future transfers.

2.1. Uncertainty

We use an event tree Σ to describe time, uncertainty and the revelation of information over an infinite horizon. There is a unique initial date-0 event $s^0 \in \Sigma$ and for each date $t \in \{0,1,2,\ldots\}$ there is a finite set $S^t \subset \Sigma$ of date-t events s^t. Each s^t has a unique predecessor $\sigma(s^t)$ in S^{t-1} and a finite number of successors s^{t+1} in S^{t+1} for which $\sigma(s^{t+1}) = s^t$. We use the notation $s^{t+1} \succ s^t$ to specify that s^{t+1} is a successor of s^t. Event $s^{t+\tau}$ is said to follow event s^t, also denoted $s^{t+\tau} \succ s^t$, if $\sigma^{(\tau)}(s^{t+\tau}) = s^t$. The set $S^{t+\tau}(s^t) := \{s^{t+\tau} \in S^{t+\tau} : s^{t+\tau} \succ s^t\}$ denotes the collection of all date-$(t+\tau)$ events following s^t. Abusing notation, we let $S^t(s^t) := \{s^t\}$. The subtree of all events starting from s^t is then

$$\Sigma(s^t) := \bigcup_{\tau \geq 0} S^{t+\tau}(s^t).$$

We use the notation $s^\tau \succeq s^t$ when $s^\tau \succ s^t$ or $s^\tau = s^t$. In particular, we have $\Sigma(s^t) = \{s^\tau \in \Sigma : s^\tau \succeq s^t\}$.

2.2. Endowments and Preferences

Agents' endowments are subject to random shocks. We denote by $e^i = (e^i(s^t))_{s^t \in \Sigma}$ agent i's process of positive endowments $e^i(s^t) > 0$ of the consumption good contingent to event s^t. Preferences over (non-negative) consumption processes $c = (c(s^t))_{s^t \in \Sigma}$ are represented by the lifetime expected and discounted utility functional

$$U(c) := \sum_{t \geq 0} \beta^t \sum_{s^t \in S^t} \pi(s^t) u(c(s^t))$$

where $\beta \in (0,1)$ is the discount factor, $\pi(s^t)$ is the unconditional probability of s^t and $u : \mathbb{R}_+ \to [-\infty, \infty)$ is a Bernoulli function assumed to be strictly increasing, concave, continuous on $\mathbb{R}_+$, differentiable on $(0,\infty)$, bounded from above and satisfying Inada's condition at the origin.[3]

Given a date-t event s^t, we denote by $U(c|s^t)$ the lifetime continuation utility conditional to event s^t, defined by

$$U(c|s^t) := u(c(s^t)) + \sum_{\tau \geq 1} \beta^\tau \sum_{s^{t+\tau} \succ s^t} \pi(s^{t+\tau}|s^t) u(c(s^{t+\tau}))$$

where $\pi(s^{t+\tau}|s^t) := \pi(s^{t+\tau})/\pi(s^t)$ is the conditional probability of $s^{t+\tau}$ given s^t. We assume that $U(e^i|s^0) > -\infty$ for every agent i.[4] Since the bernoulli function is bounded from above, we then get that $U(e^i|s^t) > -\infty$ for all event s^t.

A collection $(c^i)_{i \in I}$ of consumption processes is called an **allocation**. It is said to be **resource feasible** if $\sum_{i \in I} c^i = \sum_{i \in I} e^i$.

2.3. Self-Enforcing Consumption

A consumption process c^i may involve transfers contingent to an event s^t if $c^i(s^t) < e^i(s^t)$. We assume that agent i cannot commit to future transfers and has the option to walk away from a contract. We follow Kehoe and Levine (1993) (see also Bloise and Reichlin (2011)) and assume that autarky is the outside option for not fulfilling promises. A consumption process c^i is then said to be **self-enforcing** if it satisfies the following participation constraints

$$U(c^i|s^t) \geq U(e^i|s^t), \quad \text{for all } s^t \succ s^0.$$

When the participation constraint is also satisfied at the initial event s^0, i.e., $U(c^i|s^0) \geq U(e^i|s^0)$, then c^i is said to be **individually rational**.

[3]The function u is said to satisfy the Inada's condition at the origin if $\lim_{\varepsilon \to 0}[u(\varepsilon) - u(0)]/\varepsilon = \infty$. This property is automatically satisfied if $u(0) = -\infty$. We assume that agents' preferences are homogenous. This is only for the sake of simplicity. All arguments can be adapted to handle the heterogenous case where the preference parameters (β, π, u) differ among agents.

[4]This assumption is automatically satisfied if either $u(0) > -\infty$ or the allocation $(e^i)_{i \in I}$ is uniformly bounded away from zero, in the sense that there exists $\varepsilon > 0$ such that $e^i(s^t) \geq \varepsilon$ for each agent i and event s^t.

2.4. Implied Interest Rates

A consumption process c^i is said to be **strictly positive** if $c^i(s^t) > 0$ for every event s^t. In that case, we can define agent i's marginal rate of substitution at event s^t by posing

$$\mathrm{MRS}(c^i|s^t) := \beta\pi(s^t|\sigma(s^t))\frac{u'(c^i(s^t))}{u'(c^i(\sigma(s^t)))}.$$

Given a strictly positive allocation $(c^i)_{i\in I}$, we let $p^\star = (p^\star(s^t))_{s^t\in\Sigma}$ be the process defined recursively by $p^\star(s^0) := 1$ and

$$\frac{p^\star(s^t)}{p^\star(\sigma(s^t))} := \max_{i\in I}\mathrm{MRS}(c^i|s^t), \quad \text{for all } s^t \succ s^0.$$

Following Alvarez and Jermann (2000), $p^\star$ is called the process of **implied Arrow–Debreu prices**.

Given an arbitrary strictly positive process $p = (p(s^t))_{s^t\succeq s^0}$ interpreted as Arrow–Debreu prices, we use $\mathrm{PV}(p;x|s^t)$ to denote the present value at date-t event s^t of a process x restricted to the subtree $\Sigma(s^t)$ and defined by

$$\mathrm{PV}(p;x|s^t) := \frac{1}{p(s^t)}\sum_{s^{t+\tau}\in\Sigma(s^t)} p(s^{t+\tau})x(s^{t+\tau}).$$

We say that p displays high interest rates when the present value of endowments under the price process p is finite, i.e., $\mathrm{PV}(p;e^i|s^0) < \infty$, for all i.[5] A strictly positive allocation $(c^i)_{i\in I}$ is said to have **high implied interest rates** when the implied Arrow–Debreu prices $p^\star$ display high interest rates.

2.5. Commodity and Price Space

Denote by $\ell^\infty(e)$ the linear space of processes $h \in \mathbb{R}^\Sigma$ satisfying

$$\exists\lambda \geq 0, \quad \forall s^t \in \Sigma, \quad |h(s^t)| \leq \lambda e(s^t) \tag{2.1}$$

where $e := \sum_{i\in I} e^i$ is the process of the aggregate endowment. The linear space $\ell^\infty(e)$ is the natural commodity space since we necessarily have $c^i \in \ell^\infty(e)$ for any resource feasible allocation $(c^i)_{i\in I}$.

Remark 2.1 Denote by $\ell^\infty_+ := \ell^\infty_+(\mathbf{1}_\Sigma)$ the space of non-negative processes that are uniformly bounded from above.[6] Kehoe and Levine (1993) and Bloise and Reichlin (2011) assume that endowments belong to ℓ^∞_+ and restrict each agent to choose a consumption

[5] Observe that p displays high interest rates if, and only if, the process p^e belongs to ℓ^1_+–the set of convergent series defined on Σ–where $p^e(s^t) := p(s^t)e(s^t)$.

[6] For any subset $A \subseteq \Sigma$, we denote by $\mathbf{1}_A$ the process $x = (x(s^t))_{s^t\in\Sigma}$ defined by $x(s^t) := 1$ if $s^t \in A$ and $x(s^t) := 0$ elsewhere.

process in ℓ^∞_+. In addition, Bloise and Reichlin (2011) assume that the consumption and endowment processes are uniformly bounded away from zero.[7] In contrast to those papers, we do not impose any boundedness condition neither on endowments nor on consumption processes.

We endow the space $\ell^\infty(e)$ with the norm $\|h\|_e$ defined as the lowest $\lambda \geq 0$ satisfying (2.1). Equivalently, we have $\|h\|_e := \sup_{s^t\in\Sigma} |h(s^t)/e(s^t)|$. The cone of non-negative processes in $\ell^\infty(e)$ is denoted by $\ell^\infty_+(e)$.

The $\|\cdot\|_e$-topological dual of $\ell^\infty(e)$ is denoted by $\mathrm{ba}(e)$, and the subset of non-negative linear functionals in $\mathrm{ba}(e)$ is denoted by $\mathrm{ba}_+(e)$.[8] For any linear functional $\varphi \in \mathrm{ba}_+(e)$, there exists a non-negative charge ν^φ of bounded variation on the σ-algebra 2^Σ (or, equivalently, ν^φ is a finitely additive positive measure), such that

$$\varphi(h) = \int h_e d\nu^\varphi$$

where h_e is the process in ℓ^∞ defined by $h_e(s^t) := h(s^t)/e(s^t)$. In particular, any $\varphi \in \mathrm{ba}_+(e)$ can be decomposed as follows

$$\varphi(h) = \mathrm{PV}(p^\varphi; h|s^0) + \varphi^0(h), \quad \text{for every } h \in \ell^\infty(e)$$

for some non-negative process p^φ satisfying $\mathrm{PV}(p^\varphi; e|s^0) < \infty$ and some non-negative purely finitely additive linear functional φ^0.[9]

Remark 2.2 Any non-zero and non-negative linear functional defined on $\ell^\infty(e)$ necessarily belongs to $\mathrm{ba}_+(e)$. Indeed, continuity follows from the fact that e belongs to the $\|\cdot\|_e$-interior of $\ell^\infty_+(e)$.

A linear functional $\varphi : \ell^\infty(e) \to \mathbb{R}$ is said to be strictly positive whenever $\varphi(h) > 0$ for any non-zero $h \in \ell^\infty_+(e)$. Observe that if φ is strictly positive then φ is $\|\cdot\|_e$-continuous (i.e., $\varphi \in \mathrm{ba}_+(e)$) and $p^\varphi(s^t) > 0$ for any event s^t.

A **price system** is any arbitrary strictly positive linear functional φ normalized by the condition: $p^\varphi(s^0) = 1$. A **Bewley price process** is a strictly positive process $p = (p(s^t))_{s^t\in\Sigma}$ such that $p(s^0) = 1$ and $\mathrm{PV}(p; e|s^0)$ is finite (i.e., p displays high interest rates). Observe that any process p^φ associated to a price system φ is necessarily a Bewley price process. The purely finitely additive part $\varphi^0 = \varphi - \mathrm{PV}(p^\varphi; \cdot|s^0)$ is also called the **bubble component** of the price system φ.

[7] A process $x = (x(s^t))_{s^t\in\Sigma}$ is said to be uniformly bounded away from zero whenever there exists $\varepsilon > 0$ such that $x(s^t) \geq \varepsilon$ for every $s^t \in \Sigma$. Bloise and Reichlin (2011) use the term "interior" for "uniformly bounded away from zero". This is because a process x is uniformly bounded away from zero if, and only if, it belongs to the $\|\cdot\|_{1_\Sigma}$-interior of ℓ^∞_+, where $\|x\|_{1_\Sigma} := \sup_{s^t\in\Sigma} |x(s^t)|$.

[8] A linear functional $\varphi : \ell^\infty(e) \to \mathbb{R}$ is said to be non-negative whenever $\varphi(h) \geq 0$ for every $h \in \ell^\infty_+(e)$.

[9] The purely finitely additive linear functional φ^0 can be characterized as follows: it is a linear and $\|\cdot\|_e$-continuous functional on $\ell^\infty(e)$ such that $\varphi^0(h) = \varphi^0(h^{[T]})$ where $h^{[T]}$ is the tailed process defined by $h^{[T]}(s^t) = h(s^t)$ if $t \geq T$ and 0 elsewhere. Observe moreover that $p^\varphi(s^t) = \varphi(\mathbf{1}_{\{s^t\}})$ for any event s^t.

2.6. Uniform Gains to Trade

Following Bloise and Reichlin (2011), we say that the economy exhibits **uniform gains to trade** if there is an individually rational and self-enforcing allocation $(d^i)_{i\in I}$ and $\gamma > 0$ such that

$$\forall s^t \in \Sigma, \quad \sum_{i\in I} d^i(s^t) \leq (1-\gamma)e(s^t) \tag{2.2}$$

where we recall that $e(s^t) := \sum_{i\in I} e^i(s^t)$.

This condition means that autarky can be Pareto improved, subject to participation constraints, even though a constant fraction of aggregate endowments is destroyed. In Section 4.1 we consider a simple extension of our environment with a seizable physical asset and provide sufficient conditions ensuring that the economy exhibits uniform gains to trade.

3. Constrained Pareto Efficiency under Uniform Gains to Trade

We consider the following definition of Pareto dominance: an allocation $(\tilde{c}^i)_{i\in I}$ **Pareto dominates** another allocation $(c^i)_{i\in I}$ if $U(\tilde{c}^i|s^0) \geq U(c^i|s^0)$ for every agent i, with a strict inequality for at least one agent. We first recall the concept of constrained Pareto efficiency introduced in Kehoe and Levine (1993).

Definition 3.1 An allocation $(c^i)_{i\in I}$ is **constrained Pareto efficient** if it is resource feasible, self-enforcing, individually rational and if there is no other allocation $(\tilde{c}^i)_{i\in I}$ that is also resource feasible, self-enforcing and individually rational which Pareto dominates $(c^i)_{i\in I}$.

Remark 3.1 If an allocation $(c^i)_{i\in I}$ is constrained Pareto efficient then it must be strictly positive.[10] In particular, the corresponding process $p^\star$ of implied Arrow–Debreu prices is well-defined.

The objective of this section is to provide a complete characterization of constrained Pareto efficiency in terms of implied Arrow–Debreu prices under the assumption of uniform gains to trade.

3.1. Constrained Pareto Efficiency and Supporting Price Systems

We first characterize constrained Pareto efficiency in terms of supporting price systems.

Definition 3.2 A linear functional $\varphi : \ell^\infty(e) \to \mathbb{R}$ **supports** a resource feasible, self-enforcing and individually rational allocation $(c^i)_{i\in I}$ if $\varphi(\tilde{c}^i) \geq \varphi(c^i)$ for any self-enforcing and individually rational allocation $(\tilde{c}^i)_{i\in I}$ that Pareto dominates $(c^i)_{i\in I}$.

[10] See Proposition 2.1 in Martins-da-Rocha and Vailakis (2015) for a detailed proof of this claim.

REMARK 3.2 Since preferences are strictly monotone, a supporting linear functional φ must be non-negative and $\|\cdot\|_e$-continuous, i.e., φ must belong to $\mathrm{ba}_+(e)$. Under uniform gains to trade, the linear functional φ must be strictly positive.[11] Observe that if φ supports $(c^i)_{i\in I}$ then for any $\lambda > 0$, the linear functional $\lambda\varphi$ also supports $(c^i)_{i\in I}$. We can then, without any loss of generality, choose the normalization $\varphi(\mathbf{1}_{\{s^0\}}) = p^\varphi(s^0) = 1$ and focus on supporting linear functionals that are price systems.

It is shown below that if a price system φ supports an individually rational, self-enforcing and resource feasible allocation $(c^i)_{i\in I}$, then we can replace the inequality $\varphi(\tilde{c}^i) \geq \varphi(c^i)$ by the strict inequality $\varphi(\tilde{c}^i) > \varphi(c^i)$ whenever $U(\tilde{c}^i|s^0) > U(c^i|s^0)$. This allows us to obtain a sufficient condition for constrained Pareto efficiency in terms of supporting price systems.

LEMMA 3.1 If a resource feasible, self-enforcing and individually rational allocation is supported by a price system, then it is constrained Pareto efficient.

PROOF OF LEMMA 3.1: Consider a resource feasible, self-enforcing and individually rational allocation $c = (c^i)_{i\in I}$ that is supported by a price system φ. Assume, by way of contradiction, that there exists resource feasible, self-enforcing and individually rational allocation $\tilde{c} = (\tilde{c}^i)_{i\in I}$ that Pareto dominates c. To get a contradiction, it suffices to show that $\varphi(\tilde{c}^i) > \varphi(c^i)$ for each i. We can assume, without any loss of generality, that $\tilde{c}$ is constrained Pareto efficient. In particular, it is strictly positive and for each i, we have $\tilde{c}^i(s^0) > 0$.[12] By continuity of the Bernoulli function, there exists $\varepsilon \in (0, \tilde{c}^i(s^0))$ small enough such that $U(\tilde{c}^i - \varepsilon\mathbf{1}_{\{s^0\}}|s^0) > U(c^i|s^0)$. Observe that the consumption process $\tilde{c}^i - \varepsilon\mathbf{1}_{\{s^0\}}$ is individually rational and self-enforcing. Since the allocation c is supported by the price system φ, this implies that $\varphi(\tilde{c}^i) - \varepsilon \geq \varphi(c^i)$ (recall that a price system is such that $p^\varphi(s^0) = 1$). *Q.E.D.*

We next show that, under uniform gains to trade, the support by a price system is a necessary condition for a self-enforcing, individually rational and resource feasible allocation to be constrained Pareto efficient. In addition, any of such supporting price system must coincide with the process of implied Arrow–Debreu prices (which turns out to display high interest rates).

LEMMA 3.2 Assume there are uniform gains to trade.

[11] See the proof of Lemma 5 in Bloise and Reichlin (2011). An alternative argument is as follows. From a standard convex separation argument, we can show that for every i, there exists $\lambda^i, \mu^i \geq 0$ with $(\lambda^i, \mu^i) \neq (0,0)$ such that $\lambda^i[U(\tilde{c}^i|s^0) - U(c^i|s^0)] \leq \mu^i\varphi(\tilde{c}^i - c^i)$ for any self-enforcing and individually rational consumption $\tilde{c}^i$. Since $\varphi(e) > 0$, it follows from the assumption of uniform gains to trade that there exists an agent $k \in I$ such that $\varphi(d^k) < \varphi(c^k)$. This, in turn, implies that $\lambda^k > 0$. Then, by strict monotonicity of preferences, we get that $\mu^k\varphi(v) > 0$ for any non-zero $v \in \ell^\infty_+(e)$.

[12] See Proposition 2.1 in Martins-da-Rocha and Vailakis (2015).

(i) A self-enforcing, individually rational and resource feasible allocation is constrained Pareto efficient only if it is supported by some price system.

(ii) Any price system φ supporting a constrained Pareto efficient allocation involves no bubble component, i.e., $\varphi(\cdot) = \mathrm{PV}(p^\varphi; \cdot|s^0)$. Moreover p^φ coincides with $p^\star$ the process of implied Arrow–Debreu prices.[13]

Proof of Lemma 3.2: We first prove property (i). Fix a constrained Pareto efficient allocation $(c^i)_{i\in I}$. Let H be the set of all vectors $h \in \ell^\infty(e)$ that can be written as

$$h = \sum_{i\in I} \left(\tilde{c}^i - c^i\right)$$

where $(\tilde{c}^i)_{i\in I}$ is a self-enforcing and individually rational allocation which Pareto dominates $(c^i)_{i\in I}$. The set H is convex with a non-empty interior for the $\|\cdot\|_e$-topology.[14] The constrained Pareto efficiency of $(c^i)_{i\in I}$ implies that 0 does not belong to H. Applying the Convex Separation Theorem, we get the existence of a non-zero and $\|\cdot\|_e$-continuous linear function $\varphi : \ell^\infty(e) \to \mathbb{R}$ separating H and $\{0\}$ in the sense that $\varphi(h) \geq 0$ for every $h \in H$. This means that φ supports the allocation $(c^i)_{i\in I}$. Uniform gains to trade ensure that φ is a price system (see Remark 3.2).

We now prove property (ii). Let φ be a price system supporting the allocation $(c^i)_{i\in I}$. Recall that φ belongs to $\mathrm{ba}_+(e)$, is strictly positive and is normalized such that $\varphi(\mathbf{1}_{\{s^0\}}) = p^\varphi(s^0) = 1$.

Claim 3.1 The process p^φ dominates $p^\star$.

Proof: To prove the desired result, it is sufficient to show that for each agent i, we have

$$\mathrm{MRS}(c^i|s^t) \leq \frac{p^\varphi(s^t)}{p^\varphi(\sigma(s^t))}, \quad \text{for all } s^t \succ s^0.$$

This property follows from a standard variational argument. Indeed, fix an arbitrary event s^t and an arbitrary agent i. For some $\chi > 1/\mathrm{MRS}(c^i|s^t)$ and $0 < \varepsilon < c^i(\sigma(s^t))$, we

[13] Property (i) corresponds to Proposition 5 in Kehoe and Levine (1993). The result generalizes the necessity part of Lemma 5 in Bloise and Reichlin (2011) since (a) we prove that property (ii) is valid for all allocations, not only for those that are uniformly bounded away from zero, and (b) we do not need to assume that aggregate endowments are uniformly bounded from above and uniformly bounded away from zero. Bloise and Reichlin (2011) need these additional assumptions because their approach to prove property (ii) requires that marginal utilities of consumption at constrained Pareto efficient allocations are uniformly bounded from above and uniformly bounded away from zero.

[14] To see why the set H has non-empty interior for the $\|\cdot\|_e$-topology, we let $\tilde{c}^i := c^i + e$. Choose any process g in $\ell^\infty(e)$ with $\|g\|_e < 1$. We have that $u(\tilde{c}^i(s^t) + g(s^t)) > u(c^i(s^t))$ which implies that $\sum_{i\in I}(\tilde{c}^i + g - c^i) = \#I(e+g)$ belongs to H. Therefore, $(\#I)e$ belongs to the interior of H for the $\|\cdot\|_e$-topology.

can define the process $\tilde{c}^i$ as follows

$$\forall s^\tau \in \Sigma, \quad \tilde{c}^i(s^\tau) := \begin{cases} c^i(\sigma(s^t)) - \varepsilon & \text{if } s^\tau = \sigma(s^t) \\ c^i(s^t) + \chi\varepsilon & \text{if } s^\tau = s^t \\ c^i(s^\tau) & \text{otherwise.} \end{cases}$$

Observe that the process $\tilde{c}^i - c^i$ is different from zero only at the events $\sigma(s^t)$ and s^t. This implies that it belongs to $\ell^\infty(e)$. Given the choice of χ, we can choose $\varepsilon > 0$ small enough such that $\tilde{c}^i$ is self-enforcing and $U(\tilde{c}^i|s^0) > U(c^i|s^0)$.[15] This implies that

$$0 \leq \varphi(\tilde{c}^i - c^i) = -p^\varphi(\sigma(s^t))\varepsilon + p^\varphi(s^t)\varepsilon\chi.$$

Since χ can be chosen arbitrarily close to $1/\operatorname{MRS}^i(c^i|s^t)$ we obtain the desired result. *Q.E.D.*

If agent i's participation constraint is not binding at event s^t, i.e., $U(c^i|s^t) > U(e^i|s^t)$, then we can replace $\varepsilon > 0$ by $-\varepsilon$ in the arguments of Claim 3.1 to show that

$$\frac{p^\varphi(s^t)}{p^\varphi(\sigma(s^t))} = \operatorname{MRS}(c^i|s^t) = \frac{p^\star(s^t)}{p^\star(\sigma(s^t))}.$$

If we show that for any event s^t, there exists at least one agent for which the participation constraint is not binding, then we get the desired result: $p^\varphi = p^\star$. This property is guaranteed by the assumption of uniform gains to trade.

Claim 3.2 At every event of Σ, there exists at least one agent with a non-binding participation constraint.

Proof: Fix an event s^t and assume by way of contradiction that $U(c^i|s^t) = U(e^i|s^t)$ for every i. Let $(d^i)_{i\in I}$ be the individually rational and self-enforcing allocation satisfying the inequality (2.2) of uniform gains to trade. We pose

$$x^i(s^\tau) := \begin{cases} c^i(s^\tau) & \text{if } s^\tau \notin \Sigma(s^t) \\ d^i(s^\tau) + (\gamma/\#I)e(s^\tau) & \text{if } s^\tau \succeq s^t. \end{cases}$$

Since d^i and c^i are self-enforcing, the consumption x^i is also self-enforcing. Moreover,

$$U(x^i|s^0) = \sum_{r=0}^{t-1} \sum_{s^r \in S^r} \beta^r \pi(s^r) u(c^i(s^r)) + \beta^t \pi(s^t) U(x^i|s^t) + \sum_{\sigma^t \in S^t \setminus \{s^t\}} \beta^t \pi(\sigma^t) U(c^i|\sigma^t). \tag{3.1}$$

[15]Since u is concave, we have

$$\pi(\sigma(s^t))u(\tilde{c}^i(\sigma(s^t))) + \pi(s^t)u(\tilde{c}^i(s^t)) > \pi(\sigma(s^t))u(c^i(\sigma(s^t))) + \pi(s^t)u(c^i(s^t)).$$

Since, $U(x^i|s^t) > U(e^i|s^t) = U(c^i|s^t)$, we get that that $U(x^i|s^0) > U(c^i|s^0)$. This contradicts the constrained Pareto efficiency of $(c^i)_{i\in I}$. *Q.E.D.*

At this point, we proved that $p^\varphi = p^\star$. We now show that φ cannot have a bubble component, i.e., $\varphi^0(\cdot) := \varphi(\cdot) - \mathrm{PV}(p^\varphi; \cdot|s^0) = 0$. Assume, by way of contradiction, that there exists $v \in \ell^\infty_+(e)$ such that $\varphi^0(v) > 0$. Since v belongs to $\ell^\infty(e)$, it follows from the assumption of uniform gains to trade that there exists an individually rational and self-enforcing allocation $(f^i)_{i\in I}$ and $\mu > 0$ such that[16]

$$(3.2) \qquad \sum_{i\in I} f^i = -\mu v + \sum_{i\in I} c^i.$$

Fix $\varepsilon > 0$ small enough such that $\varphi^0(v) > \varepsilon/\mu$. Since $p^\star = p^\varphi$ displays high interest rates, there exists a date τ large enough such that

$$\sum_{s^\tau\in S^\tau} p^\star(s^\tau)\,\mathrm{PV}(p^\star, e|s^\tau) \le \varepsilon/2(\#I).$$

For every event s^t such that $t < \tau$, we pose $x^i(s^t) := c^i(s^t)$. Choose $r > \tau$ and pose

$$x^i(s^t) := \begin{cases} e(s^t) & \text{if } \tau \le t < r \\ f^i(s^t) & \text{if } t \ge r. \end{cases}$$

We have $U(x^i|s^t) \ge U(f^i|s^t)$ for any event s^t with $t \ge \tau$. Observe that $e(s^t) > c^i(s^t)$ for any event s^t. We can then choose r sufficiently large to get that $U(x^i|s^\tau) > U(c^i|s^\tau)$ for any $s^\tau \in S^\tau$. This, in turn, implies $U(x^i|s^t) > U(c^i|s^t)$ for any $t < \tau$. We have thus proved that the consumption process x^i is self-enforcing, individually rational and satisfies $U(x^i|s^0) > U(c^i|s^0)$. It follows that $h = \sum_{i\in I}\left(x^i - c^i\right)$ belongs to H in which case we have

$$(3.3) \qquad 0 \le \varphi(h) = \mathrm{PV}(p^\star; h) + \varphi^0(h).$$

Observe that

$$\begin{aligned} \mathrm{PV}(p^\star; h) &\le \sum_{t\ge\tau}\sum_{s^t\in S^t} p^\star(s^t)|h(s^t)| \\ &\le (\#I)\sum_{t\ge\tau}\sum_{s^t\in S^t} p^\star(s^t)e(s^t) \\ (3.4) \qquad &\le (\#I)\sum_{s^\tau\in S^\tau} p^\star(s^\tau)\,\mathrm{PV}(p^\star; e|s^\tau) \le \frac{\varepsilon}{2}. \end{aligned}$$

[16] Indeed, we have $v \le \|v\|_e\, e$. Choose $\mu := \gamma/\|v\|_e$ and observe that

$$\sum_{i\in i} d^i + \gamma(e - (1/\|v\|_e)v) = -\mu v + \sum_{i\in I} c^i.$$

We can then pose $f^i := d^i + \gamma^i(e - (1/\|v\|_e)v)$ where $\gamma^i := \gamma/(\#I)$.

Since φ^0 is purely finitely additive, we have[17]

$$\varphi^0(h) = \varphi^0(h^{[r]}) = -\mu\varphi^0(v^{[r]}) = -\mu\varphi^0(v) < -\varepsilon. \tag{3.5}$$

Combining (3.4) and (3.5) we get that $\varphi(h) \leq -\varepsilon/2$ which contradicts (3.3). *Q.E.D.*

3.2. Constrained Pareto Efficiency and High Implied Interest Rates

The following lemma shows the necessity of high implied interest rates for constrained Pareto efficiency. Its proof follows as a direct corollary of Lemma 3.2. This is because the price process p^φ associated to a linear functional in $\text{ba}_+(e)$ automatically satisfies $\text{PV}(p^\varphi; e|s^0) < \infty$.

Lemma 3.3 If there are uniform gains to trade, then every constrained Pareto efficient allocation exhibits high implied interest rates.[18]

Constrained Pareto efficiency obtains when there are no mutual gains from trading, including the trade opportunities involving transfers in the long run. Malinvaud efficiency, instead, is a weaker notion that requires the absence of any feasible welfare improvement subject to resource feasibility and participation constraints over any finite horizon.

Definition 3.3 An allocation $(c^i)_{i\in I}$ is **constrained Malinvaud efficient** if it is resource feasible, self-enforcing, individually rational and if there is no other allocation $(\tilde{c}^i)_{i\in I}$ that is also resource feasible, self-enforcing and individually rational which Pareto dominates $(c^i)_{i\in I}$ and differs from $(c^i)_{i\in I}$ only on finitely many events.

Remark 3.3 Every constrained Pareto efficient allocation is constrained Malinvaud efficient. If an allocation $(c^i)_{i\in I}$ is constrained Malinvaud efficient then it must be strictly positive.[19] In particular, the corresponding process $p^\star$ of implied Arrow–Debreu prices is well-defined. In addition, constrained Malinvaud efficiency has a tractable characterization in terms of first order conditions: a resource feasible, self-enforcing and individually rational allocation is constrained Malinvaud efficient if, and only if,

$$\forall s^t \succ s^0, \quad U(c^i|s^t) > U(e^i|s^t) \Longrightarrow \text{MRS}(c^i|s^t) = \frac{p^\star(s^t)}{p^\star(\sigma(s^t))}. \tag{3.6}$$

When the above property is satisfied at any strict successor event $s^\tau \succ s^t$, we say that **Euler equations** are satisfied at event s^t.

[17] Recall that for any process $h = (h(s^t))_{s^t\in\Sigma}$, we denote by $h^{[r]}$ the tailed process at date $r > 0$, defined by $h^{[r]}(s^t) = h(s^t)$ if $t \geq r$ and $h^{[r]}(s^t) = 0$ if $t < r$. In particular, for $h = \sum_{i\in I}(x^i - c^i)$, we have that $h^{[r]} = \sum_{i\in I}(f^{i,[r]} - c^{i,[r]})$.

[18] This result is related to Lemma 2 in Bloise and Reichlin (2011). We refer to Appendix A for a detailed discussion.

[19] See Proposition 2.1 in Martins-da-Rocha and Vailakis (2015).

We next show that when implied interest rates are high, Euler equations are sufficient for constrained Pareto efficiency.

LEMMA 3.4 Any constrained Malinvaud efficient allocation that exhibits high implied interest rates is constrained Pareto efficient.[20]

PROOF: Let $(c^i)_{i\in I}$ be a constrained Malinvaud efficient allocation and assume that the associated process $p^\star$ of implied Arrow–Debreu prices exhibits high interest rates. Since $(c^i)_{i\in I}$ is constrained Malinvaud efficient, we know that $\mathrm{MRS}(c^i|s^t) = p^\star(s^t)/p^\star(\sigma(s^t))$ if $U(c^i|s^t) > U(e^i|s^t)$. We can then apply Proposition 2.2 in Martins-da-Rocha and Vailakis (2015) to get that

$$\frac{1}{u'(c^i(s^0))}\left[U(\tilde{c}^i|s^0) - U(c^i|s^0)\right] \le \mathrm{PV}(p^\star; \tilde{c}^i - c^i|s^0)$$

for any resource feasible, self-enforcing and individually rational allocation $(\tilde{c}^i)_{i\in I}$.[21] Summing over i the above inequalities, we get that

$$\sum_{i\in I}\frac{1}{u'(c^i(s^0))}\left[U(\tilde{c}^i|s^0) - U(c^i|s^0)\right] \le 0.$$

It follows that $(\tilde{c}^i)_{i\in I}$ cannot Pareto dominate $(c^i)_{i\in I}$. Therefore, we proved that $(c^i)_{i\in I}$ is constrained Pareto efficient. *Q.E.D.*

Combining Lemma 3.3 and Lemma 3.4, we get a complete characterization of constrained Pareto efficiency under uniform gains to trade.

PROPOSITION 3.1 Assume there are uniform gains to trade. A constrained Malinvaud efficient allocation is constrained Pareto efficient if, and only if, implied prices exhibit high interest rates.[22]

4. A GENERAL CHARACTERIZATION OF CONSTRAINED PARETO EFFICIENCY

The crucial assumption imposed so far is that there are uniform gains to trade. The objective of this section is to provide a necessary and sufficient condition for constrained Pareto efficiency in full generality by dropping this condition. In particular, we show that an allocation is constrained Pareto efficient if, and only if, it can be approximated by a sequence of constrained Pareto efficient allocations associated to perturbed economies

[20] This result is related to Lemma 3 in Bloise and Reichlin (2011). We refer to Appendix A for a detailed discussion.

[21] If $(\tilde{c}^i)_{i\in I}$ is a resource feasible allocation then each consumption process satisfies $\tilde{c}^i \le e$, which implies that $\mathrm{PV}(p^\star; \tilde{c}^i|s^0)$ is finite.

[22] Proposition 3.1 generalizes Proposition 3 in Bloise and Reichlin (2011). We refer to Appendix A for a detailed comparison.

where the assumption of uniform gains to trade is always satisfied. The perturbed economies are a particular case of the following simple extension of our environment where a physical and seizable asset is introduced.[23]

4.1. Seizable Assets

Following Kehoe and Levine (1993) and Kehoe and Levine (2001), we assume that there is a physical asset (a tree) in positive net supply delivering the dividend $\xi(s^t) \geq 0$ at every event s^t. At the initial event s^0, each agent i holds a fraction $\theta^i(s^0) \geq 0$ of the tree, with $\sum_{i\in I} \theta^i(s^0) = 1$. The aggregate resources to be allocated among agents at event s^t are now

$$\omega(s^t) := \xi(s^t) + \sum_{i\in I} e^i(s^t)$$

and an allocation $(c^i)_{i\in I}$ is said to be resource feasible whenever

$$\sum_{i\in I} c^i = \omega = \xi + \sum_{i\in I} e^i. \tag{4.1}$$

Following the interpretation proposed by Kehoe and Levine (1993), the private endowment $e^i(s^t)$ represents goods and services, such as labor, that cannot be physically dissociated from the agent. The shares of the physical asset (such as land) can change hands, and therefore can be seized in case of breach of contract. Therefore, the default option at event s^t is still represented by $U(e^i|s^t)$ and the definition of an individually rational and self-enforcing allocation remains unchanged. In particular, the definition of a constrained Pareto efficient allocation remains the same except for the resource feasibility constraint that is now defined by (4.1). The commodity space $\ell^\infty(e)$ is replaced by $\ell^\infty(\omega)$.

All the results presented above remain valid under the uniform gains to trade assumption which takes now the following form: the economy exhibits **uniform gains to trade** if there is an individually rational and self-enforcing allocation $(d^i)_{i\in I}$ and $\gamma > 0$ such that

$$\forall s^t \in \Sigma, \quad \sum_{i\in I} d^i(s^t) \leq (1-\gamma)\omega(s^t). \tag{4.2}$$

We show below that if the dividend process ξ is large enough with respect to the private aggregate endowment process e, then the assumption of uniform gains to trade is automatically satisfied.

Proposition 4.1 If there exists a fraction $\alpha > 0$ such that $\xi \geq \alpha e$, i.e.,

$$\xi(s^t) \geq \alpha \sum_{i\in I} e^i(s^t), \quad \text{for any event } s^t \tag{4.3}$$

[23]By physical asset we mean an asset in positive net supply.

then the economy exhibits uniform gains to trade.

Proof: Observe that

$$\sum_{i\in I} e^i = e \le (1-\gamma)(\xi + e) = (1-\gamma)\omega$$

where $\gamma := \alpha/(1+\alpha)$. We can then choose $(d^i)_{i\in I}$ such that $\sum_{i\in I} d^i = (1-\gamma)\omega$ and $d^i \ge e^i$ for each i. *Q.E.D.*

Remark 4.1 Condition (4.3) is satisfied if dividends are uniformly bounded away from zero and private endowments are uniformly bounded from above. This is the case in stationary Markovian economies with strictly positive dividends.

4.2. Almost High Implied Interest Rates

We have seen that any constrained Malinvaud allocation that exhibits high implied interest rates is constrained Pareto efficient. However, unless the economy exhibits uniform gains to trade, the converse is not true in general (see Section 4.3 for an example). Our aim is to propose a weaker requirement than high implied interest rates. For this purpose, we introduce the concept of ε-perturbed economies.

Definition 4.1 We denote by $\mathcal{E}(\varepsilon)$ the economy in which we introduce a seizable physical asset in positive net supply that delivers the fraction $\varepsilon e(s^t)$ of the aggregate endowment at every event s^t. The economy $\mathcal{E}(\varepsilon)$ is called the **ε-perturbed economy**.

A consumption allocation $c = (c^i)_{i\in I}$ is said to be ε-constrained Pareto (Malinvaud) efficient if it is constrained Pareto (Malinvaud) efficient in the economy $\mathcal{E}(\varepsilon)$.

Remark 4.2 In any ε-perturbed economy $\mathcal{E}(\varepsilon)$, the outside option has the same value as in the original economy $\mathcal{E} := \mathcal{E}(0)$ without seizable physical assets. This implies that a consumption plan is individually rational and self-enforcing in $\mathcal{E}(\varepsilon)$ if, and only if, it is individually rational and self-enforcing in $\mathcal{E}$. In particular, an allocation $c = (c^i)_{i\in I}$ is ε-constrained Malinvaud efficient if, and only if, (a) it is individually rational and self-enforcing; (b) it satisfies Euler equations; (c) it is ε-feasible.

We focus on allocations that can be approximated by a sequence of constrained Pareto efficient allocations of perturbed economies.

Definition 4.2 A constrained Malinvaud efficient allocation $c = (c^i)_{i\in I}$ is said to exhibit **almost high implied interest rates** if there exist a non-negative sequence (ε_n) decreasing to 0 and a sequence (c_n) of ε_n-constrained Malinvaud efficient allocations, each exhibiting high implied interest rates, that converges (for the product topology) to the allocation c.

Observe that if an allocation c exhibits high implied interest rates, then it also exhibits almost high implied interest rates (it suffices to set $\varepsilon_n := 0$ and $c_n := c$). The converse is not true as shown in the example presented in Section 4.3.

We can now formally state the main result of this paper: "almost high implied interest rates" is a necessary and sufficient condition for constrained Pareto efficiency.

THEOREM 4.1 *Any constrained Malinvaud efficient allocation is constrained Pareto efficient if, and only if, it exhibits almost high implied interest rates.*

PROOF: We first show that "almost high implied interest rates" is a sufficient condition. Let $c = (c^i)_{i\in I}$ be a constrained Malinvaud efficient allocation and assume that there exist a non-negative sequence (ε_n) decreasing to 0 and a sequence (c_n) of ε_n-constrained Malinvaud efficient allocations exhibiting high interest rates such that each (c^i_n) converges (for the product topology) to c^i. It follows from Lemma 3.4 that the allocation $(c^i_n)_{i\in I}$ is constrained Pareto efficient for the economy $\mathcal{E}(\varepsilon_n)$. This property is sufficient to get the desired result. Indeed, assume by way of contradiction, that the allocation $(c^i)_{i\in I}$ is not constrained Pareto efficient. Then, there exists an alternative feasible, individually rational and self-enforcing allocation $(\tilde{c}^i)_{i\in I}$ that Pareto dominates $(c^i)_{i\in I}$. Without any loss of generality, we can assume that $(\tilde{c}^i)_{i\in I}$ strongly Pareto dominates $(c^i)_{i\in I}$ in the sense that $U(\tilde{c}^i|s^0) > U(c^i|s^0)$ for each agent i.[24] Since (c^i_n) converges to c^i and the mapping $U(\cdot|s^0)$ is continuous on the set of individually rational and self-enforcing consumption processes (see Lemma 2.1 in Martins-da-Rocha and Vailakis (2015)), we can deduce that there exists n large enough such that $(\tilde{c}^i)_{i\in I}$ strongly Pareto dominates the allocation $(c^i_n)_{i\in I}$. This contradicts the constrained Pareto efficiency of $(c^i_n)_{i\in I}$ in the ε_n-perturbed economy.

We now prove that "almost high implied interest rates" is a necessary condition. Let $c = (c^i)_{i\in I}$ be a constrained Pareto efficient allocation. For each integer $n \geq 1$, there exists a $(1/n)$-feasible allocation $\hat{c}_n = (\hat{c}^i_n)_{i\in I}$ satisfying $\hat{c}^i_n \geq c^i$.[25] We can assume without any loss of generality that $\hat{c}_n$ is a $(1/n)$-constrained Pareto efficient allocation. Applying Proposition 4.1 and Lemma 3.3 we deduce that each allocation $\hat{c}_n$ exhibits high implied interest rates.

By feasibility, the sequence $(\hat{c}_n)$ belongs to a compact set and there exists a strictly increasing function $\kappa : \mathbb{N} \to \mathbb{N}$ such that the subsequence $(c_n) := (\hat{c}_{\kappa(n)})$ converges to some allocation $f = (f^i)_{i\in I}$ which is feasible, individually rational and self-enforcing. We claim that $f = c$. Indeed, since $U(\hat{c}^i_n|s^0) \geq U(c^i|s^0)$ for each i, passing to the limit we get that the allocation f weakly Pareto dominates c. Since c is constrained Pareto efficient, this implies that $U(f^i|s^0) = U(c^i|s^0)$ for each i. Strict-concavity of Bernoulli functions

[24] Indeed, we can choose $(\tilde{c}^i)_{i\in I}$ to be constrained Pareto efficient. It then follows from Proposition 2.1 in Martins-da-Rocha and Vailakis (2015) that $\tilde{c}^i(s^0) > 0$ for each i. We can then make a transfer from unconstrained agents to constrained agents and obtain a strong Pareto improvement.

[25] It suffices to allocate the physical asset's delivery $(1/n)e(s^t)$ among the consumers.

implies that we must have $f = c$. Letting $\varepsilon_n := 1/\kappa(n)$, we get the desired result. *Q.E.D.*

Bloise and Reichlin (2011) introduced two variants of the Cass Criterion and show that one of them is necessary and the other one is sufficient for constrained Pareto efficiency. We show in Appendix A, by means of an example, that none of these conditions is simultaneously sufficient and necessary. This implies that our criterion is strictly stronger than their necessary condition and strictly weaker than their sufficient condition.[26] In that perspective, our Theorem 4.1 fills a gap in the literature. Moreover, we provide below an example to illustrate the applicability of our characterization result.

4.3. An Application to Stationary Markovian Economies

Here, we restrict attention to stationary Markovian economies. Uncertainty is assumed to be represented by a simple Markov process on a finite state space Z. An event s^t is then a $t+1$-vector $(s_0, s_1, \ldots, s_t)$ where each shock $s_\tau \in Z$ and $s_0 \in Z$ is fixed. In addition, the conditional probability $\pi(s^{t+1}|s^t)$ is assumed to depend only on s_t and s_{t+1}. We abuse notation and denote this conditional probability by $\pi(s_{t+1}|s_t)$, implying that

$$\pi(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2}) \ldots \pi(s_1|s_0).$$

A process $x = (x(s^t))_{s^t \in \Sigma}$ is said to be **stationary Markovian** if $x(s^t)$ is a time invariant function of the current shock s_t. We make a slight abuse of terminology and use the notation $x(s_t)$ for $x(s^{t-1}, s_t)$.

We assume that agent i's endowment process is stationary Markovian. For any event $s^t = (s^{t-1}, s_t)$, the endowment $e^i(s^{t-1}, s_t)$ is denoted by $e^i(s_t)$. It follows that the reservation utility process is also stationary Markovian. Indeed, for every event $s^t = (s^{t-1}, s_t)$ we have $U(e^i|s^t) = V^i(s_t)$ where $V^i = (V^i(z))_{z \in Z} \in \mathbb{R}^Z$ is the unique solution of the following recursive equations

$$\forall z \in Z, \quad Y(z) = u(e^i(z)) + \beta \sum_{z' \in Z} \pi(z'|z) Y(z').$$

When the process of endowments is stationary Markovian, the condition of uniform gains to trade is satisfied if, and only if, the autarchic allocation $(e^i)_{i \in I}$ is not constrained Pareto efficient. This result corresponds to Proposition 4 in Bloise and Reichlin (2011). However, this result (combined with the other results in Bloise and Reichlin (2011)) does not allow to deduce a full characterization of constrained Pareto efficiency for stationary Markovian economies. Indeed, if we know that the autarchic allocation $(e^i)_{i \in I}$ is constrained Pareto inefficient, then any constrained Malinvaud efficient allocation is constrained Pareto efficient, if and only if, it exhibits high interest rates. However, Bloise and Reichlin (2011) do not provide any necessary and sufficient condition in terms of

[26]When there are uniform gains to trade, the three conditions coincide with the property that "implied interest rates are high".

prices to determine whether the autarchic allocation is constrained Pareto inefficient. To illustrate this problem, we consider the standard stationary Markovian symmetric economy with two agents and two shocks.

There are two agents $I = \{i_1, i_2\}$. In each period, one agent receives the high endowment $1+\sigma$ and the other receives the low endowment $1-\sigma$ where $\sigma \in [0,1]$. Agents switch endowments with probability $1-\delta$ where $\delta \in (0,1)$. Formally, uncertainty is captured by the Markov process s_t, with state space $Z = \{z_{i_1}, z_{i_2}\}$ and symmetric transition probabilities

$$\text{Prob}(s_{t+1} = z_i | s_t = z_i) = \delta.$$

The endowment $e^i(s^t)$ only depends on the current shock s_t, with $e^i(s^t) = 1+\sigma$ if $s_t = z_i$ and $e^i(s^t) = 1-\sigma$ if $s_t \neq z_i$. The question at issue is to determine whether the autarchic allocation is constrained Pareto efficient. The answer naturally depends on the values of the dispersion parameter σ. We show below that there exists some threshold σ_{M} such that

(i) if $\sigma > \sigma_{\text{M}}$ then the autarchic allocation is constrained Pareto inefficient since we can exhibit a simple stationary Markovian allocation that Pareto dominates the autarchic allocation;

(ii) if $\sigma < \sigma_{\text{M}}$, then the autarchic allocation is constrained Pareto efficient since it displays high interest rates (the implied risk-less interest rate is state invariant and strictly positive);[27]

(iii) if $\sigma = \sigma_{\text{M}}$, then the implied risk-less interest rate is zero. No conclusion can be drawn from the results stated in the literature, but we can apply our characterization result to prove that the autarchic allocation is constrained Pareto efficient.[28]

To provide a formal proof of properties (i)–(iii), we introduce the following notations. Define $V_h(\sigma) := U(e^i|(s^{t-1}, z_i))$ and $V_\ell(\sigma) := U(e^i|(s^{t-1}, z_j))$ the autarchic continuation utility in the high-endowment and low-endowment state respectively, where $z_j \neq z_i$. We easily compute

$$(1-\beta)V_h(\sigma) = \overline{\alpha}u(1+\sigma) + \underline{\alpha}u(1-\sigma) \quad \text{and} \quad (1-\beta)V_\ell(\sigma) = \underline{\alpha}u(1+\sigma) + \overline{\alpha}u(1-\sigma)$$

where

$$\overline{\alpha} = \frac{1-\beta\delta}{(1-\beta\delta)+(\beta-\beta\delta)} \quad \text{and} \quad \underline{\alpha} = \frac{\beta-\beta\delta}{(1-\beta\delta)+(\beta-\beta\delta)}.$$

The function V_ℓ is strictly decreasing. Moreover, there exists σ_{M} such that V_h is strictly increasing on $[0, \sigma_{\text{M}}]$ and strictly decreasing on $[\sigma_{\text{M}}, 1]$. To simplify the presentation, we assume that the primitives (β, δ, u) are such that $\overline{\alpha}u(2) + \underline{\alpha}u(0) < u(1)$. It then follows that there exists $\sigma_{\text{FB}} \in (\sigma_{\text{M}}, 1)$ such that $(1-\beta)V_h(\sigma_{\text{FB}}) = u(1)$.

[27] Recall that any autarchic allocation is a Malinvaud optimum and Lemma 3.4 applies.

[28] The sufficient condition proposed in Bloise and Reichlin (2011) is not satisfied. Indeed, since risk-less interest rate is zero, the process $v = (v(s^t))_{s^t \in \Sigma})$ defined by $v(s^t) := 1$ satisfies the Weak Modified Cass Criterion introduced by Bloise and Reichlin (2011). See Appendix A for details.

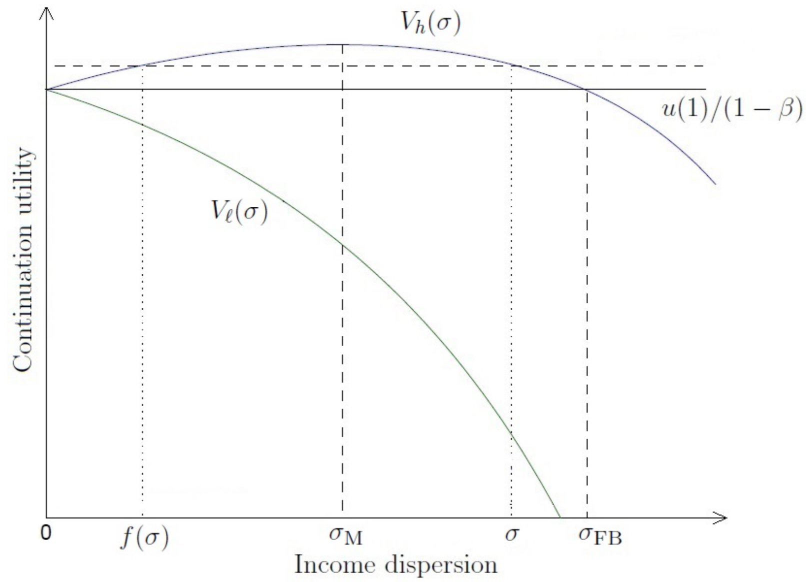

FIGURE 1.— Constrained efficient allocations

Observe that implied Arrow–Debreu prices satisfy

$$\frac{p^\star(s^{t+1})}{p^\star(s^t)} = \beta\delta =: q^{nc}, \quad \text{if } s_{t+1} = s_t$$

and

$$\frac{p^\star(s^{t+1})}{p^\star(s^t)} = \beta(1-\delta)\frac{u'(1-\sigma)}{u'(1+\sigma)} =: q^c(\sigma), \quad \text{if } s_{t+1} \neq s_t.$$

The risk-less interest rate $q^{nc} + q^c(\sigma)$ is state invariant and satisfies[29]

$$\forall \sigma < \sigma_{\mathrm{M}}, \quad q^{nc} + q^c(\sigma) < q^{nc} + q^c(\sigma_{\mathrm{M}}) = 1.$$

This implies that if $\sigma < \sigma_{\mathrm{M}}$, then the autarchic allocation displays high implied interest rates and is therefore constrained Pareto efficient. We have thus proved property (ii).

Observe that if $\sigma \geq \sigma_{\mathrm{FB}}$, then full insurance is a feasible, individually rational and self-enforcing allocation, which implies that the autarchic allocation is constrained Pareto inefficient. Take now σ in the interval $(\sigma_{\mathrm{M}}, \sigma_{\mathrm{FB}})$. There exists $f(\sigma) \in (0, \sigma_{\mathrm{M}})$ such that $V_h(f(\sigma)) = V_h(\sigma)$. Consider the stationary Markovian allocation $(c^i)_{i\in I}$ defined by $c^i(s^t) := 1 + f(\sigma)$ if the endowment is high (i.e., $s_t = z_i$) and $c^i(s^t) := 1 - f(\sigma)$ if the endowment is low (i.e., $s_t \neq z_i$). This allocation is individually rational, self-enforcing,

[29]Observe that the first order conditions imply that $\frac{u'(1-\sigma_{\mathrm{M}})}{u'(1+\sigma_{\mathrm{M}})} = \frac{\overline{\alpha}}{\underline{\alpha}} = \frac{1-\beta\delta}{\beta-\beta\delta}$.

feasible and Pareto dominates the autarchic allocation. We have thus proved that the autarchic allocation is constrained Pareto inefficient if $\sigma > \sigma_{\mathrm{M}}$. This corresponds to property (i).

The interesting case corresponds to $\sigma = \sigma_{\mathrm{M}}$. We will show that we can apply our Theorem 4.1 to prove that the autarchic allocation is constrained Pareto efficient. We would like to stress that neither our Lemma 3.4 nor Lemma 3 in Bloise and Reichlin (2011) can be applied to prove this result. This is because the risk-less interest rates is zero.[30]

Consider the economy where there is a seizable physical asset in positive net supply that delivers $\varepsilon \in (0, \sigma_{\mathrm{M}})$ at every event s^t. For any $\xi \in [0, \sigma_{\mathrm{M}} - \varepsilon/2]$, we let $c_{\varepsilon,\xi}$ be the ε-feasible stationary Markovian allocation defined by

$$c^i_{\varepsilon,\xi}(s^t) := \begin{cases} 1 + \sigma_{\mathrm{M}} - \xi, & \text{if endowment is high, i.e., } s_t = z_i \\ 1 - \sigma_{\mathrm{M}} + \varepsilon + \xi, & \text{otherwise.} \end{cases}$$

Define $V^\varepsilon_h(\xi) := U(c^i_{\varepsilon,\xi}|(s^{t-1}, z_i))$ and $V^\varepsilon_\ell(\xi) := U(c^i_{\varepsilon,\xi}|(s^{t-1}, z_j))$ the continuation utility in the high-endowment and low-endowment state respectively, where $z_j \neq z_i$. We easily compute

$$(1-\beta)V^\varepsilon_h(\xi) = \overline{\alpha}u(1 + \sigma_{\mathrm{M}} - \xi) + \underline{\alpha}u(1 - \sigma_{\mathrm{M}} + \varepsilon + \xi)$$

and

$$(1-\beta)V^\varepsilon_\ell(\xi) = \underline{\alpha}u(1 + \sigma_{\mathrm{M}} - \xi) + \overline{\alpha}u(1 - \sigma_{\mathrm{M}} + \varepsilon + \xi).$$

Observe that the participation constraint at the low endowment shock is satisfied for any value of $\xi \in [0, \sigma_{\mathrm{M}} - \varepsilon/2]$.[31] Moreover, the function $V^\varepsilon_h(\cdot)$ is strictly decreasing on the interval $[0, \sigma_{\mathrm{M}} - \varepsilon/2]$.[32] We let $\xi^\star(\varepsilon)$ be the largest value in $[0, \sigma_{\mathrm{M}} - \varepsilon/2]$ for which the participation constraint at the high endowment shock is satisfied. In other words, we decrease the consumption at the high endowment shock from $1 + \sigma_{\mathrm{M}}$ until we either achieve the symmetric first best (for $\xi = \sigma_{\mathrm{M}} - \varepsilon/2$) or until the participation constraint binds. We claim that the associated allocation $(c_{\varepsilon,\xi^\star(\varepsilon)})$ is ε-constrained Pareto efficient. It has been constructed to be ε-feasible, individually rational and self-enforcing. Denote by $p^\star_\varepsilon$ the implied Arrow–Debreu prices associated to the allocation $(c_{\varepsilon,\xi^\star(\varepsilon)})$. We have

$$\frac{p^\star_\varepsilon(s^{t+1})}{p^\star_\varepsilon(s^t)} = \beta\delta =: q^{nc}, \quad \text{if } s_{t+1} = s_t$$

[30] The autarchic allocation satisfies the Weak Modified Cass Criterion introduced in Bloise and Reichlin (2011). See Section A in the appendix for details.

[31] This is because $V^\varepsilon_\ell(\xi) > V_\ell(\sigma_{\mathrm{M}} - \xi) \geq V_\ell(\sigma_{\mathrm{M}})$.

[32] This is because

$$(1-\beta)\left[V^\varepsilon_h\right]'(\xi) = -\overline{\alpha}u'(1 + \sigma_{\mathrm{M}} - \xi) + \underline{\alpha}u'(1 - \sigma_{\mathrm{M}} + \varepsilon + \xi) < -\overline{\alpha}u'(1 + \sigma_{\mathrm{M}}) + \underline{\alpha}u'(1 - \sigma_{\mathrm{M}}) = 0.$$

and

$$\frac{p^\star_\varepsilon(s^{t+1})}{p^\star_\varepsilon(s^t)} = \beta(1-\delta)\frac{u'(1-\sigma_{\mathrm{M}}+\varepsilon+\xi^\star(\varepsilon))}{u'(1+\sigma_{\mathrm{M}}-\xi^\star(\varepsilon))} =: q^c_\varepsilon, \quad \text{if } s_{t+1} \neq s_t.$$

Since

$$q^c_\varepsilon = \beta(1-\delta)\frac{u'(c^i(s^{t+1})}{u'(c^i(s^t))}$$

where i is the agent with the low endowment at event s^{t+1}, we get that Euler equations are satisfied (or, equivalently, that $(c_{\varepsilon,\xi^\star(\varepsilon)})$ is ε-constrained Malinvaud efficient). Moreover, the risk-less interest rate satisfies

$$q^{nc}+q^c_\varepsilon \leq q^{nc}+\beta(1-\delta)\frac{u'(1-\sigma_{\mathrm{M}}+\varepsilon)}{u'(1+\sigma_{\mathrm{M}})} < q^{nc}+\beta(1-\delta)\frac{u'(1-\sigma_{\mathrm{M}})}{u'(1+\sigma_{\mathrm{M}})} = q^{nc}+q^c(\sigma_{\mathrm{M}}) = 1.$$

We then get that the interest rates implied by the allocation $(c_{\varepsilon,\xi^\star(\varepsilon)})$ are high. Finally, when ε tends to 0, the optimal value $\xi^\star(\varepsilon)$ also converges to 0. This implies that

$$\forall i \in I, \quad \lim_{\varepsilon\to 0} c^i_{\varepsilon,\xi^\star(\varepsilon)} = e^i.$$

Applying Theorem 4.1, we get that the autarchic allocation is constrained Pareto efficient.

5. Conclusion

We propose a complete characterization of constrained Pareto efficiency under limited commitment that does not require the assumption of "uniform gains to trade". We show that an allocation is constrained Pareto efficient if, and only if, it can be approximated by a sequence of constrained Pareto efficient allocations associated to perturbed economies where a physical and seizable asset in positive net supply is introduced. We illustrate the usefulness of our general characterization result for the standard stationary Markovian economy with two agents, two shocks and no aggregate uncertainty. We also compare our results with those in Bloise and Reichlin (2011) and show that, in the absence of uniform gains to trade, our necessary and sufficient condition is strictly weaker than their sufficient condition and strictly stronger than their necessary condition.

Appendix A: Comparing with the literature

Recall that if $(c^i)_{i\in I}$ is a strictly positive consumption process, then the implied Arrow–Debreu price process $p^\star$ is defined recursively by $p^\star(s^0) := 1$ and

$$\frac{p^\star(s^t)}{p^\star(\sigma(s^t))} := \max_{i\in I} \mathrm{MRS}(c^i|s^t), \quad \text{for all } s^t \succ s^0.$$

Following Bloise and Reichlin (2011), we say that an allocation satisfies the **Modified Cass Criterion** when there exists a non-null, non-negative and uniformly bounded from above process v satisfying, for some $\rho \in (0,1)$,

$$\rho \sum_{s^{t+1} \succ s^t} p^\star(s^{t+1}) v(s^{t+1}) \geq p^\star(s^t) v(s^t), \quad \text{for all } s^t \succeq s^0.$$

When this condition holds true for $\rho = 1$, we say that the allocation satisfies the **Weak Modified Cass Criterion**.

Bloise and Reichlin (2011) introduce a different concept of "high implied interest rates" than the one we borrow from Alvarez and Jermann (2000). We say that implied interest rates are **BR-high** when

$$\sum_{s^t \in \Sigma} p^\star(s^t) < \infty.$$

This concept coincides with the one we adopt if, and only if, the endowment process is uniformly bounded from above and uniformly bounded away from zero.[33]

A.1. Lemma 3.3

Lemma 3.3 is in the spirit of Lemma 2 in Bloise and Reichlin (2011) but cannot be compared to it. To illustrate this we state below their result using our terminology.

Lemma A.1 (Lemma 2 in Bloise and Reichlin (2011)) Assume aggregate endowments are uniformly bounded from above. Consider an allocation $c = (c^i)_{i \in I}$ that is feasible, self-enforcing, individually rational and uniformly bounded away from zero. If c is constrained Pareto efficient then it does not satisfy the Modified Cass Criterion.

On one hand, we do not restrict attention to allocations that are uniformly bounded away from zero. On the other hand, we have a stronger assumption (uniform gains to trade) but their conclusion is weaker than ours.

A.2. Lemma 3.4

Lemma 3.4 is in the spirit of Lemma 3 in Bloise and Reichlin (2011) but cannot be compared to it. To illustrate this we state below their result using our terminology.

Lemma A.2 (Lemma 3 in Bloise and Reichlin (2011)) Assume aggregate endowments are uniformly bounded from above. Consider an allocation that is constrained Malinvaud efficient and uniformly bounded away from zero. If it does not satisfy the Weak Modified Cass Criterion then it is constrained Pareto efficient.

[33] In the sense that there exists $\varepsilon > 0$ such that $1/\varepsilon \leq e(s^t) \leq \varepsilon$ for any event s^t.

On one hand, the sufficient condition we proposed in Lemma 3.4 is valid for any constrained Malinvaud efficient allocations, not necessarily those that are uniformly bounded away from zero. On the other hand, the sufficient condition of Lemma 3 in Bloise and Reichlin (2011) is weaker than ours.

A.3. Proposition 3.1

Proposition 3.1 is in the spirit of Proposition 3 in Bloise and Reichlin (2011) that we state below using our terminology.

PROPOSITION A.1 (Proposition 3 in Bloise and Reichlin (2011)) Assume aggregate endowments are uniformly bounded from above, uniformly bounded away from zero, and there are uniform gains to trade. Consider an allocation that is constrained Malinvaud efficient and uniformly bounded away from zero. It is constrained Pareto efficient if, and only if, implied interest rates are high.

First, Bloise and Reichlin (2011) only characterize the sub-class of allocations that are uniformly bounded away from zero. To guarantee that feasible allocations are uniformly bounded away from zero, Bloise et al. (2013) imposed an additional boundary condition on u and β. In the following section we discuss how restrictive this condition is. Second, they assume that aggregate endowments are uniformly bounded from above and uniformly bounded away from zero. They need these additional assumptions because their approach to prove the necessity part (see their Lemma 5) requires that marginal utilities of consumption at constrained Pareto efficient allocations are uniformly bounded from above and uniformly bounded away from zero.

A.4. Boundary Condition of Bloise et al. (2013)

In Bloise et al. (2013), the following extra-conditions are imposed on primitives.

(B) The Bernoulli function is bounded from below (i.e., $u(0) \in \mathbb{R}$) and there exists $\varepsilon > 0$ such that

B.1. $\varepsilon \leq e^i(s^t) \leq 1/(\varepsilon \# I)$ for every i and s^t;

B.2. $\beta u(0) + (1-\beta)u(1/\varepsilon) < u(\varepsilon)$.

In this paper, we only assume that the endowment process is strictly positive, i.e., $e^i(s^t) > 0$. In particular, we allow for the possibility of successive negative (positive) shocks on endowments such that e^i converge to 0 (to infinite) along some path. Moreover, we do not impose any consistency requirement between the subjective discount factor β, the Bernoulli function u and endowments. Observe that for any given pair (β, u), we can construct an endowment process $(e^i)_{I\in I}$ satisfying (B.1) but such that (B.2) is not satisfied. Indeed, (B.2) implies that

$$u(1/\varepsilon) < \frac{\beta}{1-\beta}[u(\varepsilon) - u(0)] + u(\varepsilon)$$

passing to the limit when ε tends to 0, we get the contradiction $\sup_{x \geq 0} u(x) \leq u(0)$.

A.5. Theorem 4.1

Our characterization result Theorem 4.1 extends Proposition 1 and Proposition 3 in Bloise and Reichlin (2011) since we identify a necessary and sufficient condition without imposing the additional requirement that there exist uniform gains to trade. To provide a direct comparison with the conditions used by Bloise and Reichlin (2011), we introduce the following concept.

Definition A.1 A constrained Malinvaud allocation c is said to **almost satisfy the Modified Cass Criterion** if either it does satisfy the Modified Cass Criterion, or there exists $\bar{\varepsilon} > 0$ and a neighborhood W (for the product topology) of the allocation c such that for every $\varepsilon \in (0, \bar{\varepsilon})$, any ε-constrained Malinvaud efficient allocation in the neighborhood W of c satisfies the Modified Cass Criterion.[34]

We rephrase our Theorem 4.1 to characterize constrained Pareto inefficiency.

Theorem A.1 *Assume that each endowment process e^i belongs to ℓ^∞. Consider a constrained Malinvaud efficient allocation that is uniformly bounded away from zero. It is constrained Pareto inefficient if, and only if, it almost satisfies the Modified Cass Criterion.*

Combining the above theorem and Lemma 2 in Bloise and Reichlin (2011), we deduce that for any allocation that is constrained Malinvaud efficient and uniformly bounded away from zero, if it satisfies the Modified Cass Criterion then it also almost satisfies the Modified Cass Criterion. The converse is not always valid as illustrated by the following example.

Example A.1 Consider the following deterministic economy borrowed from the appendix of Bloise and Reichlin (2011). There are two agents $I = \{e, o\}$ (e for even and o for odd). Let $x_e > 0$ and $x_o > 0$ satisfy $x_e + x_o = 1$ and $u'(x_e) = \beta u'(x_o)$. Fix a strictly positive and strictly decreasing sequence $(\xi_t)_{t \geq 0}$ satisfying

$$u(x_e) + \beta u(x_o) = u(x_e + \xi_t) - \beta u(x_o - \xi_{t+1}).$$

[34] Without any loss of generality, the pair $(\bar{\varepsilon}, W)$ can be chosen such that, for some large enough time period τ, the set W is composed of all allocations $\tilde{c}$ satisfying $|\tilde{c}^i(s^t) - c^i(s^t)| < \bar{\varepsilon}$ for every i and any event s^t with $t \leq \tau$.

The autarchic allocation $e = (e_e, e_o)$ is defined by

$$e_e := (x_e+\xi_0, x_o-\xi_1, x_e+\xi_2, x_o-\xi_3, \ldots) \quad \text{and} \quad e_o := (x_o-\xi_0, x_e+\xi_1, x_o-\xi_2, x_e+\xi_3, \ldots).$$

Bloise and Reichlin (2011) proved that the autarchic allocation e is constrained Pareto inefficient. In particular, it almost satisfied the Modified Cass Criterion. We claim that it does not satisfy the Modified Cass Criterion. Indeed, assume by way of contradiction that there exists a non-null, non-negative and uniformly bounded from above sequence $v = (v_t)_{t \geq 0}$ satisfying, for some $\rho \in (0,1)$,

$$\rho \frac{p^\star_{t+1}}{p^\star_t} v_{t+1} \geq v_t, \quad \text{for all } t \geq 0.$$

Observe that implied Arrow–Debreu prices satisfy

$$\frac{p^\star_{t+1}}{p^\star_t} = \beta \frac{u'(x_o - \xi_{t+1})}{u'(x_e + \xi_t)} := q_{t+1}.$$

Fix any $\beta' \in (\beta, 1)$. Since $\lim_{t \to \infty} q_{t+1} = 1$, there exists a time period τ large enough such that $\beta q_{t+1} \leq \beta'$ for every $t \geq \tau$. We then deduce that $v = 0$ and get a contradiction.

Combining Theorem A.1 and Lemma 3 in Bloise and Reichlin (2011), we deduce that for any allocation that is constrained Malinvaud efficient and uniformly bounded away from zero, if it almost satisfies the Modified Cass Criterion then it also satisfies the Weak Modified Cass Criterion. The converse is not always valid. Indeed, consider the stationary Markovian economy of Section 4.3 with $\sigma = \sigma_{\mathrm{M}}$. The autarchic allocation is constrained Pareto efficient. In particular, it does not almost satisfy the Modify the Modified Cass Criterion. However, it does satisfy the Weak Modified Cass Criterion since the risk-less interest rate is zero.

References

Alvarez, F. and Jermann, U. J.: 2000, Efficiency, equilibrium, and asset pricing with risk of default, *Econometrica* **68**(4), 775–797.

Bloise, G. and Reichlin, P.: 2011, Asset prices, debt constraints and inefficiency, *Journal of Economic Theory* **146**(4), 1520–1546.

Bloise, G., Reichlin, P. and Tirelli, M.: 2013, Fragility of competitive equilibrium with risk of default, *Review of Economic Dynamics* **16**(2), 271–295.

Eaton, J. and Gersovitz, M.: 1981, Debt with potential repudiation: Theoretical and empirical analysis, *Review of Economic Studies* **48**(2), 289–309.

Kehoe, T. J. and Levine, D. K.: 1993, Debt-constrained asset markets, *Review of Economic Studies* **60**(4), 865–888.

Kehoe, T. J. and Levine, D. K.: 2001, Liquidity constrained markets versus debt constrained markets, *Econometrica* **69**(3), 575–598.

Martins-da-Rocha, V. F. and Vailakis, Y.: 2015, Supplement to 'constrained efficiency without commitment'. Supplemental Material, electronic copy available at https://sites.google.com/site/filipeecon/research/DebtConstrainedSupp.pdf.