Self-enforcing Debt, Reputation, and the Role of Interest Rates

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How domestic costs of default do interact with the threat of exclusion from credit markets to determine interest rates and sovereign debt sustainability? In this paper, we address this question in the context of a stochastic general equilibrium model with lack of commitment and self-enforcing debt in which default has two consequences: loss of access to international borrowing and output costs. In contrast to Bulow and Rogoff (1989), we show that part of the ability to borrow is merely attributed to the threat of credit exclusion, or equivalently, to the loss of the sovereign's reputation. Apart from the limit case–analyzed by Hellwig and Lorenzoni (2009)–where output costs are absent, equilibrium interest rates are always higher than growth rates, implying that the way "reputation for repayment" supports debt does not depend on whether debt limits allow agents to exactly roll over existing debt period by period.

1. Introduction

Unlike domestic loans, sovereign loans are not usually secured by any form of collateral or enforced by a supranational legal system. This leads to the question of why sovereign governments ever repay their debts, or, equivalently, why investors accept to lend to sovereigns. One of the most advocated reason is the well-known reputational argument (first formalized by Eaton and Gersovitz (1981)): sovereign states who repudiate their debts may tarnish their reputation, be denied access to credit, and, therefore, loose part of their ability to smooth consumption. However, the exclusion from credit markets is not the only consequence of default. Recent evidence from emerging markets suggests that default episodes are usually accompanied by direct costs on the sovereign economy. For instance, when the country cannot discriminate between debts owned by foreigner creditors and debts owned by its own creditors, default may be damaging for the domestic financial system. Similarly, default may trigger trade sanctions that alter the sovereign's international trade pattern. To the extent that trade or banking is essential for production, such costs may cause a decline in the domestic output.[1] A question then arises: How domestic costs of default do interact with the threat of exclusion from credit markets to determine interest rates and borrowing levels?

This paper addresses this question in the context of a stochastic general equilibrium model with lack of commitment and self-enforcing debt. It is assumed that default consequences consist of two components: loss of access to international borrowing and direct

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[1]See among others Cohen and Sachs (1986), Cohen (1992) and Tomz and Wright (2007, 2013).

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output costs modeled as the loss of an exogenous fraction of income.[2] Following Alvarez and Jermann (2000) and Hellwig and Lorenzoni (2009), agents have access to a complete set of one period contingent bonds and face debt limits that are compatible with maximal risk sharing subject to debt repayment being individual rational. We fully characterise self-enforcing debt limits and obtain two striking implications: (1) unless output costs are zero, interest rates are always higher than agents' growth rates; (2) part of the ability to borrow–captured by equilibrium self-enforcing debt limits–must necessarily reflect the threat of credit exclusion, or equivalently, the loss of a country's reputation.

The first implication of our results stands in contrast with the limit case where there is no output drop. Assuming that credit exclusion is the only consequence of default, Hellwig and Lorenzoni (2009) observed that the lack of commitment implies that the debt limits typically bind and interest rates may be lower than agents' growth rates. Their key finding is that, with sufficiently low interest rates, debt limits allow agents to roll-over existing debt period by period. This turns out to be a necessary and sufficient condition for debt repayment. The implied connection of repayment incentives to the ability to roll over debt makes possible to interpret their characterisation result as "rational bubbles" on equilibrium debt limits. Exploiting this characterization they subsequently constructed an example where positive debt levels are sustained at equilibrium. Our first result shows that the presence of "rational bubbles" on equilibrium debt limits is not robust to a realistic strengthening of the default punishment. Indeed, we show that, no matter how small is the output drop parameter, interest rates must be higher than growth rates. This, in turn, prevents agents to roll-over their debt limits.

The second implication of our results stands in contrast to Bulow and Rogoff (1989)'s analysis of repayment incentives. They argued that the threat of enjoining from ever borrowing in the future in no way enhances a country's ability to borrow.[3] More precisely, when output drop costs are present, they showed that self-enforcing debt limits coincide with the present value of output losses and claimed that these debt limits reflect solely the loss in utility due to threat of output drop. In other words, according to their interpretation, "reputation debt"–defined as the maximum amount of debt supported by the mere threat of credit exclusion–is never sustained. We instead argue that, even if the "bubble mechanism" exhibited by Hellwig and Lorenzoni (2009) is not robust to output costs, their reputation debt sustainability result is. Formally, we show that, when default induces both exclusion from asset markets and output losses, then part of the ability to

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[2]We have chosen to represent output loss as an exogenous fraction of income only for the sake of simplifying the analysis. In that respect, we follow Cole and Kehoe (2000), Aguiar and Gopinath (2006), Tomz and Wright (2007), Arellano (2008), Abraham and Carceles-Poveda (2010) and Bai and Zhang (2010, 2012).

[3]Bulow and Rogoff (1989) led to a vast literature studying alternative mechanisms to answer why countries repay their debts. It includes, among others, Cole and Kehoe (1995), Cole and Kehoe (1998), Kletzer and Wright (2000), Kehoe and Perri (2002), Gul and Pesendorfer (2004), Krueger and Uhlig (2006) and Amador (2012). We refer to Aguiar and Amador (2014), Wright (2011) and Tomz and Wright (2013) for a thorough discussion of this literature.

borrow may reflect the threat of loss of access to international credit markets.

To achieve our goals, we first characterize not-too-tight debt limits when there is output drop. The novel feature of our characterization result, which has no analogue in the absence of output costs, is its implication about the level of interests rates. We do not impose a priori any restriction on interest rates. We instead show that no matter how small is the output drop parameter, interest rates must be higher than agents' growth rates if debt is self-enforcing. This, in turn, allows us to use the characterization result in Hellwig and Lorenzoni (2009) to prove that any process of self-enforcing debt limits is the sum of the present value of output losses and a "bubble" component. Taking into account market clearing conditions, we derive that, at equilibrium, self-enforcing debt limits coincide with the present value of future output losses (i.e., equilibrium debt limits do not have a bubble component).

The fact that with direct output costs low interest rates are not compatible with self-enforcing debt has perhaps some value, but it says nothing on how each punishment (exclusion versus output drop) affects agents' ability to borrow. We would like to go further and understand whether some amount of borrowing can be attributed solely to the threat of credit exclusion as in the limit case (with no output drop) analyzed by Hellwig and Lorenzoni (2009). Bulow and Rogoff (1989) argued that the present value of future output losses represents exactly the amount of resources an agent is ready to give up in order to avoid loosing output in the future. According to this interpretation, our characterization result would mean that the threat of credit exclusion does not provide by itself enough incentives for repayment when default also entails some loss of output. We argue that disentangling repayment incentives in this way is questionable. Actually, it is true that with full commitment, the present value of future output losses does represent the utility loss in terms of current resources. This is because debt limits are set to avoid Ponzi schemes and never bind. However, in our environment without commitment, the self-enforcing debt limits typically bind and the present value of future output losses may not anymore represent the debt level sustained exclusively by the threat of output drop.

To support our claim we modify the stationary example of Hellwig and Lorenzoni (2009) by introducing output drop and analyze the asymptotic behavior of equilibria when the output drop parameter converges to zero. According to Bulow and Rogoff (1989)'s way of disentangling repayment incentives, when the output drop parameter vanishes, we should expect that the debt level–if it reflects only the output costs–also vanishes. Interestingly enough, our analysis shows that this is not the case: the equilibrium self-enforcing debt limits of the modified economy do not vanish, and more importantly, they converge to the positive debt limits of the economy in Hellwig and Lorenzoni (2009) where exclusion is the only consequence of default. Therefore, by a continuity argument, it sounds reasonable to consider that a fraction of self-enforcing debt limits must reflect the loss of consumption smoothing (and risk-sharing) due to the exclusion from credit markets.

Our analysis motivates the study of an alternative way of disentangling repayment

incentives which is not in contradiction with the aforementioned asymptotic result. We address this issue by proposing a new definition for the part of the self-enforcing debt limits that reflects the loss of utility merely due to the output drop. We show that our definition delivers some intuitive properties. More precisely, the debt level attributed to output costs never exceeds the present value of output losses and, more importantly, it converges to zero when the drop parameter vanishes.[4] In addition, since the overall debt in our modified example converges to the reputation debt sustained in the original example of Hellwig and Lorenzoni (2009), we obtain the following two results: (a) credit exclusion is strong enough to sustain positive levels of debt in an environment with output costs and high interest rates; (b) the reputation debt sustained through a bubble in an equilibrium with low interest rates can be approximated by the reputation debt sustained in an equilibrium with high interest rates. We interpret this as a robustness property of the positive result in Hellwig and Lorenzoni (2009). Indeed, even if bubbles are not robust to a small (but realistic) change in the default punishment, sustained reputation debt levels are.

The paper is organized as follows: Section 2 describes the stochastic dynamic economy with lack of commitment. Section 3 characterizes self-enforcing debt when default consequences consist of both exclusion from international borrowing and direct output costs. Section 4 contains our main findings. It illustrates the implications of our characterization result for reputation debt sustainability in the presence of output costs and proposes a new way to disentangle repayment incentives. Section 5 concludes. The proofs of the main results are detailed in Section 6 while technical proofs are presented in an appendix.

2. The Model

Here we present an infinite horizon general equilibrium model with lack of commitment and self-enforcing debt limits, along the lines of Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2009). Time and uncertainty are both discrete and there is a single non-storable consumption good. The economy consists of a finite set I of infinitely lived agents sharing risks in an environment where debtors cannot commit to their promises.

2.1. Uncertainty

We use an event tree Σ to describe time, uncertainty and the revelation of information over an infinite horizon. There is a unique initial date-0 event $s^0 \in \Sigma$ and for each date $t \in \{0, 1, 2, \ldots\}$ there is a finite set $S^t \subset \Sigma$ of date-t events s^t. Each s^t has a unique predecessor $\sigma(s^t)$ in S^{t-1} and a finite number of successors s^{t+1} in S^{t+1} for which $\sigma(s^{t+1}) = s^t$. We use the notation $s^{t+1} \succ s^t$ to specify that s^{t+1} is a successor of s^t.

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[4]Recall that the present value of output losses reflects the overall ability to borrow due to both default punishments.

Event $s^{t+\tau}$ is said to follow event s^t, also denoted $s^{t+\tau} \succ s^t$, if $\sigma^{(\tau)}(s^{t+\tau}) = s^t$. The set $S^{t+\tau}(s^t) := \{s^{t+\tau} \in S^{t+\tau} : s^{t+\tau} \succ s^t\}$ denotes the collection of all date-$(t+\tau)$ events following s^t. Abusing notation, we let $S^t(s^t) := \{s^t\}$. The subtree of all events starting from s^t is then

$$\Sigma(s^t) := \bigcup_{\tau \geq 0} S^{t+\tau}(s^t).$$

We use the notation $s^\tau \succeq s^t$ when $s^\tau \succ s^t$ or $s^\tau = s^t$. In particular, we have $\Sigma(s^t) = \{s^\tau \in \Sigma : s^\tau \succeq s^t\}$.

2.2. Endowments and Preferences

Agents' endowments are subject to random shocks. We denote by $y^i = (y^i(s^t))_{s^t \in \Sigma}$ agent i's process of positive endowments $y^i(s^t) > 0$ of the consumption good contingent to event s^t. Preferences over (non-negative) consumption processes $c = (c(s^t))_{s^t \in \Sigma}$ are represented by the lifetime expected and discounted utility functional

$$U(c) := \sum_{t \geq 0} \beta^t \sum_{s^t \in S^t} \pi(s^t) u(c(s^t))$$

where $\beta \in (0,1)$ is the discount factor, $\pi(s^t)$ is the unconditional probability of s^t and $u : \mathbb{R}_+ \to [-\infty, \infty)$ is a Bernoulli function assumed to be strictly increasing, concave, continuous on $\mathbb{R}_+$, differentiable on $(0,\infty)$, bounded from above and satisfying Inada's condition at the origin.[5]

Given a date-t event s^t, we denote by $U(c|s^t)$ the lifetime continuation utility conditional to event s^t, defined by

$$U(c|s^t) := u(c(s^t)) + \sum_{\tau \geq 1} \beta^\tau \sum_{s^{t+\tau} \succ s^t} \pi(s^{t+\tau}|s^t) u(c(s^{t+\tau}))$$

where $\pi(s^{t+\tau}|s^t) := \pi(s^{t+\tau})/\pi(s^t)$ is the conditional probability of $s^{t+\tau}$ given s^t. We assume that $U(y^i|s^0) > -\infty$ for every agent i.[6] Since the bernoulli function is bounded from above, we then get that $U(y^i|s^t) > -\infty$ for every event s^t.

A collection $(c^i)_{i \in I}$ of consumption processes is called an allocation. It is said to be resource feasible if $\sum_{i \in I} c^i = \sum_{i \in I} y^i$.

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[5]The function u is said to satisfy the Inada's condition at the origin if $\lim_{\varepsilon \to 0}[u(\varepsilon) - u(0)]/\varepsilon = \infty$. This property is automatically satisfied if $u(0) = -\infty$. We assume that agents' preferences are homogenous. This is only for the sake of simplicity. All arguments can be adapted to handle the heterogenous case where the preference parameters (β, π, u) differ among agents.

[6]This assumption is automatically satisfied if either $u(0) > -\infty$ or the allocation $(y^i)_{i \in I}$ is uniformly bounded away from zero, in the sense that there exists $\varepsilon > 0$ such that $y^i(s^t) \geq \varepsilon$ for each agent i and event s^t.

2.3. Markets

At every date-t event s^t, agents can issue and trade a complete set of one-period contingent bonds, which promise to pay one unit of the consumption good contingent on the realization of any successor event $s^{t+1} \succ s^t$. Let $q(s^{t+1}) > 0$ denote the price at event s^t of the s^{t+1}-contingent bond. Agent i's holding of this bond is $a^i(s^{t+1})$. The amount of state-contingent debt agent i can issue is observable and subject to state-contingent (non-negative) upper bounds (or debt limits) $D^i = (D^i(s^t))_{s^t \in \Sigma}$. Given an initial financial claim $a^i(s^0)$, we denote by $B^i(D^i, a^i(s^0))$ the budget set of an agent who never defaults.[7] It consists of all pairs (c^i, a^i) of consumption and bond holdings satisfying the following constraints: for every event s^t,

$$(2.1) \qquad c^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a^i(s^{t+1}) \leq y^i(s^t) + a^i(s^t)$$

and

$$(2.2) \qquad a^i(s^{t+1}) \geq -D^i(s^{t+1}).$$

Given some initial claim $b \in \mathbb{R}$ at event s^τ, we denote by $J^i(D^i, b|s^\tau)$ the largest continuation utility defined by

$$J^i(D^i, b|s^\tau) := \sup\{U(c^i|s^\tau) : (c^i, a^i) \in B^i(D^i, b|s^\tau)\}$$

where $B^i(D^i, b|s^\tau)$ is the set of all plans (c^i, a^i) satisfying $a^i(s^\tau) = b$, together with Equations (2.1) and (2.2) for every successor event $s^t \succeq s^\tau$. The demand set contingent to event s^τ, denoted by $d^i(D^i, b|s^\tau)$, consists of all optimal consumption and bond holdings plans in the budget set $B^i(D^i, b|s^\tau)$. Formally, $d^i(D^i, b|s^\tau) := \operatorname{argmax}\{U(c^i|s^\tau) : (c^i, a^i) \in B^i(D^i, b|s^\tau)\}$.

2.4. Default Costs

We consider an environment where there is no commitment. Agent i might not honor his debt obligations and decide to default if it is optimal for him. His decision depends on the consequences of default. Following Bulow and Rogoff (1989) (see also Hellwig and Lorenzoni (2009)), we assume that a defaulting agent starts with neither assets nor liabilities, is excluded from future credit but retains the ability to purchase bonds, and suffers a drop in output.[8]

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[7]The initial financial claim $a^i(s^0)$ can be interpreted as an initial transfer (by a social planner) or as the consequence of (un-modeled) past transactions.

[8]Bulow and Rogoff (1989) used the terminology "direct sanctions" for the output drop (see their Section III.B).

It has been extensively documented in the sovereign debt literature that output falls during sovereign default.[9] Disruption of international trade or domestic financial systems can lead to output drops if either trade or banking credit is essential for production. Mendoza and Yue (2012) and Gennaioli et al. (2014) propose explicit models of the mechanisms by which sovereign default causes an efficiency loss in production. We follow the tradition in the sovereign debt literature (see for instance Cohen and Sachs (1986), Cole and Kehoe (2000), Aguiar and Gopinath (2006), Arellano (2008), Abraham and Carceles-Poveda (2010), and Bai and Zhang (2010, 2012)) and model the negative implications on output by the loss of an exogenous fraction $\lambda \in [0, 1)$ of income. Formally, we denote by $B^i_\lambda(D, b|s^t)$ and $d^i_\lambda(D, b|s^t)$ the budget and demand sets corresponding to $B^i(D, b|s^t)$ and $d^i(D, b|s^t)$ where the endowment $y^i(s^\tau)$ at every successor event $s^\tau \succeq s^t$ is replaced by $(1 - \lambda)y^i(s^\tau)$. Given these notations, country i's default option at event s^t is

$$V^i_\lambda(s^t) := J^i_\lambda(0, 0|s^t)$$

where $J^i_\lambda(0, 0|s^t) := \sup\{U(c^i|s^t) : (c^i, a^i) \in B^i_\lambda(0, 0|s^t)\}$ is the largest continuation utility at event s^t when agent i starts with zero financial claim, cannot borrow forever and endows income $(1 - \lambda)y^i(s^\tau)$ at every $s^\tau \succeq s^t$. The fraction $\lambda \in [0, 1)$ is called the output drop parameter.[10]

2.5. Self-enforcing Debt Limits

We should now incorporate the fact that agents have the option to default. Since borrowers issue contingent bonds, lenders have no incentives to provide credit contingent to some event if they anticipate that the borrower will default.[11] The maximum amount of debt $D^i(s^t)$ at any event $s^t \succ s^0$ should reflect this property. If agent i's initial financial claim at event s^t corresponds to the maximum debt $-D^i(s^t)$, then he prefers to repay his debt if, and only if, $J^i(D^i, -D^i(s^t)|s^t) \geq V^i_\lambda(s^t)$. When a process of bounds satisfies the above inequality for every event $s^t \succ s^0$, it is called self-enforcing.[12] Competition among lenders naturally leads to consider the largest self-enforcing bound $D^i_\lambda(s^t)$ defined by the equation

$$(2.3) \qquad J^i(D^i_\lambda, -D^i_\lambda(s^t)|s^t) = V^i_\lambda(s^t).$$

We follow Alvarez and Jermann (2000) and refer to such debt limits as not-too-tight.

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[9]We refer to Cohen and Sachs (1986), Cohen (1992) and Tomz and Wright (2007, 2013) for details.

[10]We assume that the output loss fraction is homogenous among agents. This is only for the sake of simplicity. All the arguments follow almost verbatim in the heterogenous case.

[11]Since the default punishment is independent of the default level, an agent either fully repays his debt or defaults totally. There is no partial default.

[12]Indeed, since the function $J^i(D^i, \cdot|s^t)$ is increasing, for any bond holding $a^i(s^t)$ satisfying the restriction $a^i(s^t) \geq -D^i(s^t)$, agent i prefers honoring his obligation than defaulting on $a^i(s^t)$.

2.6. Competitive Equilibrium

Fix an allocation $(a^i(s^0))_{i \in I}$ of initial financial claims that satisfies market clearing, i.e., $\sum_{i \in I} a^i(s^0) = 0$. A competitive equilibrium with self-enforcing debt is a list $(q, (c^i, a^i, D^i_\lambda)_{i \in I})$ which consists of state-contingent bond prices q, a resource feasible consumption allocation $(c^i)_{i \in I}$, a market clearing allocation of bond holdings $(a^i)_{i \in I}$ and an allocation of debt limits $(D^i_\lambda)_{i \in I}$ such that (c^i, a^i) belongs to the demand set $d^i(D^i_\lambda, a^i(s^0)|s^0)$ and debt limits are not-too-tight.[13]

2.7. Natural Debt Limits

Given state-contingent bond prices $q = (q(s^t))_{s^t \succ s^0}$ we denote by $p(s^t)$ the associated date-0 price of consumption at s^t defined recursively by $p(s^0) = 1$ and $p(s^{t+1}) = q(s^{t+1})p(s^t)$ for every $s^{t+1} \succ s^t$. We use $\text{PV}(q; x|s^t)$ to denote the present value at date-t event s^t of a process x restricted to the subtree $\Sigma(s^t)$ and defined by

$$\text{PV}(q; x|s^t) := \frac{1}{p(s^t)} \sum_{s^{t+\tau} \in \Sigma(s^t)} p(s^{t+\tau})x(s^{t+\tau}).$$

When there is no ambiguity regarding asset prices, we use the simpler notation $\text{PV}(x|s^t)$. We say that interest rates are higher than agent i's growth rates when the present value $\text{PV}(y^i|s^0)$ of his future endowments is finite, and lower (or equal) when it is infinite.[14] When interest rates are higher than agent i's growth rates, his natural debt limits are then defined by $N^i(s^t) := \text{PV}(y^i|s^t)$.

3. Characterizing Self-enforcing Debt

Bulow and Rogoff (1989) analyzed initially a special case of the model where default entails solely credit exclusion (i.e., $\lambda = 0$). They proved a striking characterization result: self-enforcing debt is never sustained if interest rates are higher than growth rates and debt limits are tighter than natural debt limits.[15]

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[13]Since bonds are in zero net supply, market clearing means $\sum_{i \in I} a^i = 0$.

[14]The choice of this terminology is driven by the following particular case. Assume that interest rates and bounds on growth rates are time and state invariant, i.e., $q(s^{t+1}) = \pi(s^{t+1}|s^t)(1 + r)^{-1}$ and $y^i(s^{t+1}) = (1 + g^i(s^{t+1}))y^i(s^t)$ for every $s^{t+1} \succ s^t$, where the stochastic growth rate is such that $m^i g^i \le g^i(s^{t+1}) \le M^i g^i$ for some parameter g^i with $0 < m^i \le M^i < \infty$. Then, the present value of agent i's endowments is finite if, and only if, $r > g^i$, and infinite if, and only if, $r \le g^i$.

[15]The model studied by Bulow and Rogoff (1989) is slightly different than the one presented here. They analyzed repayment incentives of a small open economy borrowing from competitive, risk neutral foreign investors. The sovereign country trades at the initial event a complete set of state-contingent contracts that specify the net transfers to foreign investors in all future periods and events. Contracts are restricted to be compatible with repayment incentives and to allow investors to break even in present value terms (this environment is in the spirit of Kehoe and Levine (1993), but with a different default option). We show in Martins-da-Rocha and Vailakis (2014) that the First BR Theorem is a corollary of Theorem 1 in Bulow and Rogoff (1989).

THEOREM 3.1 (First BR Theorem) *Assume that the output drop parameter is zero (i.e., $\lambda = 0$), and that interest rates are higher than agent i's growth rates. If D_0^i is a process of not-too-tight debt limits tighter than natural debt limits,*[16] *then for every event s^t, we have $D_0^i(s^t) = 0$.*

Bulow and Rogoff (1989) also studied the case where there is a loss in production after default. They found that self-enforcing debt can be sustained but up to a level that does not exceed the present value of the loss in output. Formally, their second characterization result is as follows.[17]

THEOREM 3.2 (Second BR Theorem) *Assume that output drop is positive after default (i.e., $\lambda > 0$), and that interest rates are higher than agent i's growth rates. If D_λ^i is a process of not-too-tight debt limits tighter than natural debt limits,*[18] *then for every event s^t, we have $D_\lambda^i(s^t) = \mathrm{PV}(\lambda y^i | s^t)$.*

The First and Second BR Theorem should be interpreted as partial equilibrium results. Indeed, in the context of a general equilibrium model, interest rates are determined endogenously to clear bond markets. Therefore, assuming a priori that interest rates are higher than growth rates is an ad-hoc assumption on endogenous variables. In an environment with full commitment, there is no loss of generality in assuming that equilibrium interest rates are higher than growth rates. Indeed, in such an environment, debt limits are only set to avoid Ponzi schemes and do not bind. In contrast, Hellwig and Lorenzoni (2009) observed that without commitment, debt limits typically bind and interest rates are not necessarily higher than growth rates. They asked subsequently whether the First BR Theorem is valid without imposing ad-hoc restrictions on interest rates.[19] Working in this direction, they provided the following characterization of not-too-tight debt limits.

THEOREM 3.3 (HL Theorem) *Assume that the output drop parameter is zero (i.e., $\lambda = 0$). If D_0^i is a process of not-too-tight debt limits, then D_0^i allows for exact roll-over, in the sense that*

$$(3.1) \qquad D_0^i(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D_0^i(s^{t+1}), \quad \text{for all } s^t \in \Sigma.$$

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[16]That is, $D_0^i \leq N^i$ and $J^i(D_0^i, -D_0^i(s^t)|s^t) = V_0^i(s^t)$ for every $s^t \succ s^0$.

[17]Again, the model in Bulow and Rogoff (1989) is not exactly the same as ours. However, we show in Martins-da-Rocha and Vailakis (2014) that the Second BR Theorem follows as a corollary of Theorem 2 in Bulow and Rogoff (1989).

[18]That is, $D_\lambda^i \leq N^i$ and $J^i(D_\lambda^i, -D_\lambda^i(s^t)|s^t) = V_\lambda^i(s^t)$ for every $s^t \succ s^0$.

[19]When interest rates are lower than agent i's growth rates, his natural debt limits are infinite, therefore imposing that not-too-tight debt limits are tighter than natural debt limits is an innocuous assumption.

HL Theorem is a complete characterization of not-too-tight debt limits since any process of debt limits allowing for exact roll-over is not-too-tight.[20] Hellwig and Lorenzoni (2009) build on this result to construct a simple stationary economy with two agents where, at equilibrium, debt limits are positive and interest rates are lower than each agent's growth rates (zero non-contingent interest rate). That is, their example implies that the First BR Theorem does not have a general equilibrium counterpart.

The first contribution of this paper is to explore whether the Second BR Theorem holds true in a general equilibrium setting. To achieve our goal, we extend the HL Theorem and characterize not-too-tight debt limits allowing for the possibility of interest rates to be lower than growth rates, but without assuming that the output drop parameter is zero. We obtain the following characterization result.

THEOREM 3.4   *Assume that output drop is positive after default (i.e., $\lambda > 0$). If there exists a process D_λ^i of not-too-tight debt limits, then interest rates are necessarily higher than agent i's growth rates and there exists a non-negative process M^i allowing for exact roll-over such that $D_\lambda^i = \mathrm{PV}(\lambda y^i) + M^i$.*

Some remarks are in order. As expected, more severe punishments, by reducing the risk of default, enlarge insurance opportunities. However, the striking observation concerns with the implications of Theorem 3.4 about interest rates: no matter how small is the output drop parameter, interest rates must be higher than agent i's growth rates. This is because the not-too-tight debt limit $D_\lambda^i(s^t)$ must be larger that the present value $\mathrm{PV}(\lambda y^i | s^t)$ of the future output losses.

The detailed proof of Theorem 3.4 is postponed to Section 6.1. Here we only comment on the key steps. The novel step, which has no counterpart when the output drop parameter is zero, amounts to show that the not-too-tight debt limit $D_\lambda^i(s^t)$ is necessarily larger than the present value $\mathrm{PV}(\lambda y^i | s^t)$ of output drop. The natural approach to prove this is to show that $D_\lambda^i(s^t) \geq \lambda y^i(s^t) + \tilde{D}_\lambda^i(s^t)$ where $\tilde{D}_\lambda^i(s^t) := \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D_\lambda^i(s^{t+1})$, and then use a standard iteration argument. Given the definition of the not-too-tight debt limit $D_\lambda^i(s^t)$, this is equivalent to proving that

$$(3.2) \qquad J^i(D_\lambda^i, -\lambda y^i(s^t) - \tilde{D}_\lambda^i(s^t) | s^t) \geq V_\lambda^i(s^t).$$

Observe that we always have

$$(3.3) \qquad V_\lambda^i(s^t) \geq u((1-\lambda)y^i(s^t)) + \beta \sum_{s^{t+1} \succ s^t} \pi(s^{t+1} | s^t) V_\lambda^i(s^{t+1}).$$

If we had an equality in (3.3), then the inequality (3.2) would be straightforward.[21] Indeed, consuming $(1-\lambda)y^i(s^t)$ and borrowing up to the debt limits $D_\lambda^i(s^{t+1})$ at event s^t

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[20]This follows from a simple translation invariance of the budget restrictions (2.1) and (2.2).

[21]In the simpler environment where saving is not possible after default (as it is the case in Alvarez and Jermann (2000)) we always have an equality in (3.3).

leads to the right hand side continuation utility in (3.3), and satisfies the solvency constraint at event s^t in the budget set defining the left hand side of (3.2). However, in our environment where a country can save after default, (3.3) need not be satisfied with an equality. Dealing with this problem constitutes the technical contribution in the proof of Theorem 3.4.

The second step of the proof consists in showing that the difference between the debt limit and the present value of output drop involves a bubble (in the sense that it allows for exact roll-over). The proof of this step is based on a weaker but technically less involved version of the HL Theorem we propose (see Section 6.2), for which interest rates are known to be higher than the agents' growth rates.

We know that the First BR Theorem is not valid in a general equilibrium environment since, due to Hellwig and Lorenzoni (2009)'s example, positive levels of debt can be sustained at equilibrium. Interestingly enough, we show below that the Second BR Theorem does extend to a general equilibrium setting.

COROLLARY 3.1 *Assume that output drop is positive after default (i.e., $\lambda > 0$). At any competitive equilibrium with self-enforcing debt $(q, (c^i, a^i, D^i_\lambda)_{i \in I})$, interest rates are higher than any agent's growth rates and the debt limits coincide with the present value of future output losses, i.e., $D^i_\lambda(s^t) = \mathrm{PV}(\lambda y^i | s^t)$ for all $s^t \in \Sigma$.*

If we assume that equilibrium debt limits are tighter than natural debt limits then the above result is a simple corollary of Theorem 3.4. Indeed, we know that equilibrium interest rates must be higher than growth rates. Moreover, a non-negative and non-zero process allowing for exact roll-over cannot be bounded from above by the natural debt limits (see the proof of Proposition 3 in Hellwig and Lorenzoni (2009)). However, not-too-tight debt limits, like prices, are endogenously determined. Therefore, we cannot assume a priori that they are tighter than the natural debt limits. We should instead prove this is necessarily the case at equilibrium. One may argue (see for instance the discussion following Proposition 3 in Hellwig and Lorenzoni (2009)) that if debt limits are not tighter than the natural debt limits, then the aggregate stock of debt, and thus the savings of some lender, will eventually exceed the value of aggregate endowments. This way of reasoning is not as simple as it may appear since the debt limits are associated to savings of some lender only when they bind. To complete the argument and remove the bubble component from the equilibrium self-enforcing debt limits we prove the necessity of a "market transversality" condition.

PROOF OF COROLLARY 3.1: Let $(q, (c^i, a^i, D^i_\lambda)_{i \in I})$ be a competitive equilibrium with self-enforcing debt. It follows from Theorem 3.4 that interest rates are higher than every agent's growth rates, i.e., $\mathrm{PV}(y^i | s^0) < \infty$ for each i. Since consumption markets clear, we have $\sum_{i \in I} c^i = \sum_{i \in I} y^i$ which implies that each consumption process c^i has finite present

value. Applying Lemma 6.3, we deduce the following market transversality condition[22]

$$\lim_{t\to\infty} \sum_{s^t \in S^t} p(s^t)[a^i(s^t) + D_\lambda^i(s^t)] = 0.$$

Market clearing of bond markets then implies

$$(3.4) \qquad \lim_{t\to\infty} \sum_{s^t \in S^t} \sum_{i \in I} p(s^t) D_\lambda^i(s^t) = 0.$$

Since $D_\lambda^i = \text{PV}(\lambda y^i) + M^i$ where M^i is non-negative and satisfies exact roll-over (Theorem 3.4), we have

$$(3.5) \qquad \sum_{s^t \in S^t} \sum_{i \in I} p(s^t) D_\lambda^i(s^t) = \sum_{i \in I} M^i(s^0) + \lambda \sum_{s^t \in S^t} \sum_{i \in I} p(s^t) \text{PV}(y^i | s^t).$$

Combining Equations (3.4) and (3.5), and using the fact that $\text{PV}(y^i | s^0) < \infty$, we deduce that $\sum_{i \in I} M^i(s^0) = 0$. Non-negativity of each $M^i(s^0)$ implies that $M^i(s^0) = 0$ for every i. Since M^i is non-negative and allows for exact roll-over, we get the desired result: $M^i = 0$ for every i. *Q.E.D.*

Corollary 3.1 reveals that with output drop, some risk sharing is always possible at equilibrium. The fact that this result is obtained without assuming a priori that interest rates are higher than growth rates is an improvement to Bulow and Rogoff (1989) and an extension of Hellwig and Lorenzoni (2009)'s analysis. However, the result itself says nothing on how each punishment (exclusion versus output drop) affects agents' ability to borrow. In particular, it does not reveal whether some amount of borrowing is attributed to the threat of credit exclusion. To answer this question we need to study and disentangle borrowers' repayment incentives. This is the subject of the following section.

4. Repayment Incentives

Following the terminology introduced in Bulow and Rogoff (1989), we call *reputation debt* the maximum amount of debt supported by the mere threat of credit exclusion in case of default. When credit exclusion is the only punishment for default (i.e., $\lambda = 0$), the not-too-tight debt limits D_0^i defined by the equations

$$J^i(D_0^i, -D_0^i(s^t) | s^t) = J^i(0, 0 | s^t), \quad \text{for all } s^t \in \Sigma$$

represent exactly the reputation debt. In that respect, the First BR Theorem can be interpreted as a negative result: no reputation debt is sustainable if interest rates are higher than growth rates and debt limits are not allowed to exceed the natural debt

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[22]Our market transversality condition differs from the standard transversality condition. Indeed, due to the lack of commitment, debt limits may bind for some agent i in which case we do not necessarily have that $p(s^t) = \beta^t \pi(s^t) u'(c^i(s^t)) / u'(c^i(s^0))$.

limits. When the output drop parameter is positive (i.e., $\lambda > 0$), the not-too-tight debt limits D_λ^i defined by the equations

$$J^i(D_\lambda^i, -D_\lambda^i(s^t)|s^t) = J_\lambda^i(0,0|s^t), \quad \text{for all } s^t \in \Sigma$$

represent instead the maximum debt level sustained by the threat of two punishments: the credit exclusion and the loss of the amount λy^i of future endowments.

Bulow and Rogoff (1989) propose to disentangle the two sources of repayment incentives. They identify the present value $\mathrm{PV}(\lambda y^i|s^t)$ as the level of debt merely due to the output drop costs. The reputation debt is then defined as the difference $D_\lambda^i(s^t) - \mathrm{PV}(\lambda y^i|s^t)$. Given this terminology, the Second BR Theorem can also be interpreted as a negative result: if interest rates are higher than growth rates and, in addition to the credit exclusion, there is some loss of output after default, then debt can be sustained ($D_\lambda^i(s^t) > 0$) but only on the basis of the output drop sanction (since $D_\lambda^i(s^t) = \mathrm{PV}(\lambda y^i|s^t)$). In other words, with interest rates being higher than agents' growth rates, even if there is a loss of output after default, reputation debt cannot be sustained.

When there are no direct sanctions (i.e., $\lambda = 0$), the HL Theorem connects borrowers' repayment incentives to the ability to roll-over their debt. This feature can be interpreted as a rational bubble on debt limits which is compatible with equilibrium when the present value of the agents' endowments is infinite, or equivalently when interest rates are lower than growth rates. But with output drop (i.e., $\lambda > 0$), Corollary 3.1 says that rolling over debt is impossible since equilibrium interest rates must necessarily be higher than agents' growth rates. Moreover, the debt limit $D_\lambda^i(s^t)$ coincides with the present value $\mathrm{PV}(\lambda y^i|s^t)$, implying that reputation debt as defined by Bulow and Rogoff (1989) cannot be sustained even in a general equilibrium setting.

If we accept Bulow and Rogoff (1989)'s disentanglement of repayments incentives, Corollary 3.1 suggests that sustaining reputation debt in Hellwig and Lorenzoni (2009) is not robust to a more realistic modeling of the default consequences. This is because we obtain the following discontinuity result: reputation debt can never be sustained when $\lambda > 0$, but it does in the limit case where $\lambda = 0$. However, the discontinuity may not arise, if a part of the sustained debt $\mathrm{PV}(\lambda y^i|s^t)$ does actually reflect the threat of exclusion from credit markets. In this case, Corollary 3.1 would suggest three things. First, sustaining reputation debt is robust to a realistic strengthening of the default punishment. Second, reputation debt cannot be attributed to the "rational bubble mechanism" of Hellwig and Lorenzoni (2009). Third, the disentanglement of repayments incentives proposed by Bulow and Rogoff (1989) is not the appropriate one. The next session is devoted to the investigation of those issues. Working in this direction, we modify the example in Hellwig and Lorenzoni (2009) by introducing output drop ($\lambda > 0$), and we analyze the asymptotic behavior of equilibria when λ converges to 0.

4.1. Adding Output Drop in Hellwig and Lorenzoni (2009)

Denote by $\mathcal{E}^{\text{HL}}$ the economy considered in Hellwig and Lorenzoni (2009). There are two agents $I = \{a, b\}$ and uncertainty is captured by a Markov process with state space $Z = \{z^a, z^b\}$. The state z^i corresponds to the situation where agent i's endowment is $\bar{e} \in (1/2, 1)$ and agent j's endowment, with $j \neq i$, is $\underline{e} := 1 - \bar{e} < \bar{e}$. The transition probabilities are symmetric where $\alpha \in (0, 1)$ is the probability of switching states, i.e., $\text{Prob}(z^i | z^j) = \alpha$ for $i \neq j$. The Bernoulli function $u : [0, \infty) \to \mathbb{R}$ is strictly concave, strictly increasing, continuously differentiable and bounded; and the discount factor β belongs to $(0, 1)$. The parameters $(u, \beta, \alpha, \bar{e})$ are chosen such that there exists $\bar{c} > 0$ satisfying

$$(4.1) \qquad \bar{c} < \bar{e} \quad \text{and} \quad 1 - \beta(1 - \alpha) = \beta\alpha \frac{u'(1 - \bar{c})}{u'(\bar{c})}.$$

The corresponding event tree Σ can be defined as follows. The initial event is $s^0 := z^a$ and a date-t event is a history of state realizations $s^t = (s^0, s_1, \ldots, s_t)$ with $s_r \in Z$ for each $1 \leq r \leq t$. The transition probabilities are defined by $\pi(s^{t+1} | s^t) = \alpha$ if $s_{t+1} \neq s_t$ and $\pi(s^{t+1} | s^t) = 1 - \alpha$ if $s_{t+1} = s_t$. The endowment process y^i is defined by $y^i(s^{t+1}) := \bar{e}$ if $s_{t+1} = z^i$ and $y^i(s^{t+1}) := \underline{e}$ if $s_{t+1} \neq z^i$. Since the initial state is z^a, agent a begins with the high endowment $\bar{e}$ while agent b's initial endowment is $\underline{e}$.

4.2. Credit Exclusion without Output Drop

We now recall the equilibrium with positive levels of reputation debt proposed by Hellwig and Lorenzoni (2009). The state-contingent bond prices q are defined by

$$q(s^{t+1}) := \begin{cases} q^c := \beta\alpha \dfrac{u'(1 - \bar{c})}{u'(\bar{c})} & \text{if} \quad s_{t+1} \neq s_t \\ q^{nc} := \beta(1 - \alpha) & \text{if} \quad s_{t+1} = s_t. \end{cases}$$

The consumption c^i satisfies $c^i(s^{t+1}) := \bar{c}$ if $s_{t+1} = z^i$ and $c^i(s^{t+1}) := 1 - \bar{c}$ if $s_{t+1} \neq z^i$. Bond holdings a^i are given by $a^i(s^{t+1}) := -\omega$ if $s_{t+1} = z^i$ and $a^i(s^{t+1}) := \omega$ if $s_{t+1} \neq z^i$, where ω is defined by the equation $(1 - q^{nc} + q^c)\omega = \bar{e} - \bar{c}$. Initial asset positions $(a^i(s^0))_{i \in I}$ are $a^a(s^0) := -\omega$ and $a^b(s^0) := \omega$.

Hellwig and Lorenzoni (2009) proved that $(q, (c^i, a^i, D_0^i)_{i \in I})$ is a competitive equilibrium with self-enforcing debt where the not-too-tight debt limits $(D_0^i)_{i \in I}$ are given by $D_0^i(s^t) := \omega$. Observe that the state-contingent bond prices are such that $q^c + q^{nc} = 1$, that is, the non-contingent interest rate is zero. In particular, interest rates are lower than each agent's growth rates. Trivially, D_0^i allows for exact roll-over and is therefore not-too-tight.

4.3. Credit Exclusion with Output Drop

We propose to modify the above example by incorporating a positive output drop after default. For any small enough output drop parameter $\lambda > 0$, we show that there exists a symmetric Markov equilibrium where the two equilibrium consumption levels $\bar{c}_\lambda$ and $\underline{c}_\lambda := 1 - \bar{c}_\lambda$ are defined as follows.[23]

LEMMA 4.1   There exists $\bar{\lambda} > 0$ small enough such that for every $\lambda \in (0, \bar{\lambda})$, there exists $\bar{c}_\lambda \in (1/2, \bar{c})$ satisfying

$$(4.2) \qquad \left[1 - q^{nc} + q^c(\bar{c}_\lambda)\right] \lambda w(\bar{c}_\lambda) = \bar{e} - \bar{c}_\lambda$$

where for any $x \in (0, \bar{c})$,

$$q^c(x) := \beta \alpha \frac{u'(1-x)}{u'(x)} \quad \text{and} \quad w(x) := \frac{(1 - q^{nc})\bar{e} + q^c(x)\underline{e}}{(1 - q^{nc})^2 - (q^c(x))^2}.$$

Moreover, we have

$$(4.3) \qquad \lim_{\lambda \to 0} \bar{c}_\lambda = \bar{c}.$$

We denote by $\mathcal{E}_\lambda^{\mathrm{HL}}$ the economy where default entails output drop with parameter $\lambda \in (0, \bar{\lambda})$ and where agents' initial asset holdings are defined by $a_\lambda^a(s^0) := -\lambda w(\bar{c}_\lambda)$ and $a_\lambda^b(s^0) := \lambda w(\bar{c}_\lambda)$. All the other primitives remain the same as in $\mathcal{E}^{\mathrm{HL}}$.[24] We now describe a symmetric Markovian equilibrium of the economy $\mathcal{E}_\lambda^{\mathrm{HL}}$.

Let q_λ be the price process defined by $q_\lambda(s^{t+1}) := q^c(\bar{c}_\lambda)$ if $s_{t+1} \neq s_t$ and $q_\lambda(s^{t+1}) := q^{nc}$ if $s_{t+1} = s_t$. We claim that interest rates are higher than each agent's growth rates. Indeed, since endowments are uniformly bounded from above, it is sufficient to show that

$$q^c(\bar{c}_\lambda) + q^{nc} < 1.$$

This follows from the fact that $x \mapsto q^c(x)$ is increasing, $\bar{c}_\lambda < \bar{c}$ and $q^c(\bar{c}) + q^{nc} = 1$ (by construction of $\bar{c}$). Observe that the present value of future endowments y^i at any event s^t computed with prices q_λ is given by

$$(4.4) \qquad \mathrm{PV}(q_\lambda, y^i | s^t) = \begin{cases} w(\bar{c}_\lambda) & \text{if } s_t = z^i \\ w(\bar{c}_\lambda) - \dfrac{\bar{e} - \underline{e}}{1 - q^{nc} + q^c(\bar{c}_\lambda)} & \text{if } s_t \neq z^i. \end{cases}$$

Let c_λ^i be the consumption process defined by $c_\lambda^i(s^{t+1}) := \bar{c}_\lambda$ if $s_{t+1} = z^i$ and $c_\lambda^i(s^{t+1}) := 1 - \bar{c}_\lambda$ if $s_{t+1} \neq z^i$; and let a_λ^i be the bond holdings process defined by $a_\lambda^i(s^{t+1}) := -\lambda w(\bar{c}_\lambda)$ if $s_{t+1} = z^i$ and $a_\lambda^i(s^{t+1}) := \lambda w(\bar{c}_\lambda)$ if $s_{t+1} \neq z^i$.

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[23]The proof of Lemma 4.1 is postponed to Section 6.3.1.

[24]We modify the initial endowments in order to get existence of a symmetric Markovian equilibrium and simplify the asymptotic analysis. In particular, we will show that $a_\lambda^i(s^0) \longrightarrow a^i(s^0)$ when $\lambda \to 0$.

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We can show that $(q_\lambda, (c_\lambda^i, a_\lambda^i, D_\lambda^i)_{i \in I})$ is a competitive equilibrium of the economy $\mathcal{E}_\lambda^{\mathrm{HL}}$ where the not-too-tight debt limits $(D_\lambda^i)_{i \in I}$ are given by

$$D_\lambda^i(s^t) := \lambda \, \mathrm{PV}(q_\lambda; y^i | s^t) = \mathrm{PV}(q_\lambda; \lambda y^i | s^t), \quad \text{for all } s^t \in \Sigma.$$

We postpone the detailed arguments to Section 6.3.2. The important issue we want to analyze is the asymptotic behavior of the equilibrium $(q_\lambda, (c_\lambda^i, a_\lambda^i, D_\lambda^i)_{i \in I})$ when $\lambda \to 0$.

PROPOSITION 4.1 When the output drop parameter converges to zero, i.e., $\lambda \to 0$, the competitive equilibrium $(q_\lambda, (c_\lambda^i, a_\lambda^i, D_\lambda^i)_{i \in I})$ of the economy $\mathcal{E}_\lambda^{\mathrm{HL}}$ converges (for the product topology) to the equilibrium $(q, (c^i, a^i, D_0^i)_{i \in I})$ of the economy $\mathcal{E}^{\mathrm{HL}}$. In particular,

$$\lim_{\lambda \to 0} D_\lambda^i(s^t) = \lim_{\lambda \to 0} \mathrm{PV}(q_\lambda; \lambda y^i | s^t) = \omega = D_0^i(s^t) > 0.$$

PROOF OF PROPOSITION 4.1: The convergence of (q_λ) and (c_λ^i) follows from (4.3). Recall that $\bar{c}_\lambda$ satisfies

$$[1 - q^{nc} + q^c(\bar{c}_\lambda)] \, \lambda w(\bar{c}_\lambda) = \bar{e} - \bar{c}_\lambda.$$

Passing to the limit, we get

$$\lim_{\lambda \to 0} \lambda w(\bar{c}_\lambda) = \frac{\bar{e} - \bar{c}}{1 - q^{nc} + q^c} = \omega.$$

This implies that (a_λ^i) converges to a^i.

Recall that $D_\lambda^i(s^t) = \lambda \, \mathrm{PV}(q_\lambda; y^i | s^t)$ where $\mathrm{PV}(q_\lambda; y^i | s^t)$ is specified in (4.4). If the current state is the high endowment, i.e., $s_t = z^i$, then

$$D_\lambda^i(s^t) = \lambda w(\bar{c}_\lambda) \xrightarrow[\lambda \to 0]{} \omega.$$

If the current state is the low endowment, i.e., $s_t \neq z^i$, then

$$D_\lambda^i(s^t) = \lambda w(\bar{c}_\lambda) - \frac{\lambda(\bar{e} - \underline{e})}{1 - q^{nc} + q^c(\bar{c}_\lambda)} \xrightarrow[\lambda \to 0]{} \omega.$$

That is, when the output drop λ converges to 0, the non-contingent interest rates also converge to zero $(q^{nc} + q^c(\bar{c}_\lambda) \longrightarrow q^{nc} + q^c = 1)$ and the present value $\mathrm{PV}(q_\lambda; y^i | s^t)$ of future endowments converges to infinite. The decrease in the output drop is compensated by the increase in the present value of income in a way that the debt limit $D_\lambda^i(s^t)$, being the product of the two effects, converges to ω. Q.E.D.

This result is intriguing. Given the interpretation proposed by Bulow and Rogoff (1989), the debt level $D_\lambda^i(s^t)$ should only reflect the threat of loss of output. When λ vanishes, the output drop λy^i vanishes and the outside option $J_\lambda^i(0, 0 | s^t)$ converges to $J^i(0, 0 | s^t)$. It is then intuitive to expect that the debt limit $D_\lambda^i(s^t)$ converges to zero too. However, this level of debt does not vanish, and more importantly, it converges to the

quantity $D_0^i = \omega$–which is the debt limit in Hellwig and Lorenzoni (2009)'s economy– that reflects only the loss in reputation. It is then reasonable to conclude that a fraction of D_λ^i must reflect the exclusion from credit markets and the implied loss in consumption smoothing (and risk-sharing) opportunities. In other words, some level of reputation debt must be sustained in the economy $\mathcal{E}_\lambda^{\mathrm{HL}}$ even if interest rates are higher than agents' growth rates.

The analysis so far shows that the way Bulow and Rogoff (1989) disentangle repayment incentives is questionable. This, in turn, motivates the study of an alternative way of disentangling repayment incentives which is not in contradiction with the aforementioned asymptotic result. The next section proposes such a way by introducing an alternative definition for the part of debt supported merely by the output drop punishment (and, consequently, for the part that is related only to the loss in reputation).

4.4. Disentangling Repayment Incentives

Consider a general economy $\mathcal{E}_\lambda$ where the output drop parameter λ is positive. Fix a process q_λ of bond prices such that interest rates are higher than agent i's growth rates, i.e., $\mathrm{PV}(q_\lambda; y^i | s^0) < \infty$. The question at issue is to propose a meaningful disentanglement

$$D_\lambda^i = \Delta_\lambda^i + R_\lambda^i$$

where we refer to Δ_λ^i as the *output drop debt* and to R_λ^i as the *reputation debt*. The amount $\Delta_\lambda^i(s^t)$ should represent solely the loss of utility due to the contraction of income. This means that, in order to define $\Delta_\lambda^i(s^t)$, we shall consider an environment where the agent keeps access to credit markets after default. Formally, the bound $\Delta_\lambda^i(s^t)$ is defined by the equation

$$(4.5) \qquad J^i(D_\lambda^i, -\Delta_\lambda^i(s^t) | s^t) = J_\lambda^i(\widetilde{D}_\lambda^i, 0 | s^t).$$

The left-hand side represents the largest continuation utility agent i obtains if he starts with debt $\Delta_\lambda^i(s^t)$, faces the debt limits $D_\lambda^i(s^\tau)$ for $s^\tau \succ s^t$ and does not default. The right-hand side represents the default option when output drop is the only punishment: if agent i defaults, then he looses the fraction $\lambda y^i(s^\tau)$ at all successor events $s^\tau \succeq s^t$ but keeps access to credit markets represented by the debt limits $\widetilde{D}_\lambda^i(s^\tau)$ for all $s^\tau \succ s^t$. These debt limits, defined on the out-of-equilibrium path associated to default, should also reflect the agent's repayment incentives. Recall that the not-too-tight self-enforcing debt limits are given by $D_\lambda^i = \lambda \mathrm{PV}(q_\lambda; y^i)$ along the equilibrium path where income is represented by the process y^i. Since the income of the defaulting agent suffers the multiplicative shock $(1 - \lambda)$ after default, the corresponding debt limits $\widetilde{D}_\lambda^i$ should also reflect this negative shock. We therefore pose

$$\widetilde{D}_\lambda^i := \lambda \mathrm{PV}(q_\lambda; (1 - \lambda)y^i) = (1 - \lambda)D_\lambda^i.$$

We next provide some properties of the output drop debt process Δ_λ^i. It follows from the Intermediate Value Theorem that $\Delta_\lambda^i(s^t)$ exists, is strictly larger than the current output drop $\lambda y^i(s^t)$ but lower than $D_\lambda^i(s^t)$, or equivalently, the present value $\mathrm{PV}(q_\lambda; \lambda y^i | s^t)$ of future output drop.

LEMMA 4.2 For every event s^t, the output drop debt $\Delta_\lambda^i(s^t)$ exists and satisfies $\lambda y^i(s^t) < \Delta_\lambda^i(s^t) \leq \mathrm{PV}(q_\lambda; \lambda y^i | s^t)$.

PROOF: Let $\Phi : [-D_\lambda^i(s^t), \infty) \longrightarrow \mathbb{R}$ be the function defined by $\Phi(b) := J^i(D_\lambda^i, b | s^t)$. It is continuous and satisfies

$$\Phi(-D_\lambda^i(s^t)) = J_\lambda^i(0, 0 | s^t) \leq J_\lambda^i(\widetilde{D}_\lambda^i, 0 | s^t) < J^i(D_\lambda^i, -\lambda y^i(s^t) | s^t) = \Phi(-\lambda y^i(s^t)).$$

Applying the Intermediate Value Theorem, we get the existence of $\Delta_\lambda^i(s^t)$ such that $\Phi(-\Delta_\lambda^i(s^t)) = J_\lambda^i(D_\lambda^i, 0 | s^t)$. Moreover, since Φ is strictly increasing, we also have $\lambda y^i(s^t) < \Delta_\lambda^i(s^t) \leq D_\lambda^i(s^t) = \mathrm{PV}(q_\lambda; \lambda y^i | s^t)$. Q.E.D.

We derive from the above argument that the output drop debt $\Delta_\lambda^i(s^t)$ coincides with the sustained debt $D_\lambda^i(s^t)$ if, and only if,

$$J_\lambda^i(0, 0 | s^t) = J_\lambda^i(\widetilde{D}_\lambda^i, 0 | s^t).$$

This occurs only when, after default, the borrower does not benefit from keeping his access to credit markets. We then obtain the following intuitive result: according to our definition, the reputation debt $R_\lambda^i(s^t) := D_\lambda^i(s^t) - \Delta_\lambda^i(s^t)$ is zero if, and only if, the defaulting borrower is not hurt by loosing access to credit markets. This property provides ground to our proposed disentanglement of repayment incentives.

The important issue we want to address is whether the reputation debt $R_\lambda^i(s^t)$ can be non-zero even if interest rates are higher than growth rates. The answer is yes and it is a consequence of the following important property: under some reasonable conditions on asset prices, the output drop debt $\Delta_\lambda^i(s^t)$ must converge to 0 when the output drop parameter converges to 0. This property gives an additional account for our proposed disentanglement of repayment incentives. Indeed, the fact that the outside option $J_\lambda^i(0, 0 | s^t)$ converges to $J^i(0, 0 | s^t)$ when $\lambda \to 0$, should be reflected on the asymptotic behavior of the output drop debt limits.

PROPOSITION 4.2 Assume that the Bernoulli function u is bounded and that for each event s^t the prices $q_\lambda(s^t)$ and the present values $\mathrm{PV}(q_\lambda, \lambda y^i | s^t)$ are uniformly bounded from above and away from zero, for λ small enough.[25] Then, $\lim_{\lambda \to 0} \Delta_\lambda^i(s^t) = 0$, for all $s^t \in \Sigma$.

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[25]In the sense that for each s^t, there exists $\bar{\lambda} \in (0, 1)$ and $0 < m < M$ such that $m \leq q_\lambda(s^t) \leq M$ and $m \leq \mathrm{PV}(q_\lambda, \lambda y^i | s^t) \leq M$ for each $\lambda \in (0, \bar{\lambda}]$.

PROOF: Let $(\bar{c}^i_\lambda, \bar{a}^i_\lambda) \in d^i(D^i_\lambda, -\Delta^i_\lambda(s^t)|s^t)$. Define the consumption process $\tilde{c}^i_\lambda$ as follows

$$\tilde{c}^i_\lambda(s^t) := (1-\lambda)(\bar{c}^i_\lambda(s^t) + \Delta^i_\lambda(s^t)) \quad \text{and} \quad \forall s^r \succ s^t, \quad \tilde{c}^i_\lambda(s^r) := (1-\lambda)\bar{c}^i_\lambda(s^r).$$

Consider the asset holdings $\tilde{a}^i_\lambda$ defined by $\tilde{a}^i_\lambda(s^r) := (1-\lambda)\bar{a}^i_\lambda(s^r)$ for all $s^r \succ s^t$. Observe that $(\tilde{c}^i_\lambda, \tilde{a}^i_\lambda)$ belongs to $B^i_\lambda(\widetilde{D}^i_\lambda, 0|s^t)$. This implies that

$$U(\tilde{c}^i_\lambda|s^t) \le J^i_\lambda(\widetilde{D}^i_\lambda, 0|s^t) = J^i(D^i_\lambda, -\Delta^i_\lambda(s^t)|s^t) = U(\bar{c}^i_\lambda|s^t).$$

Assume, by way of contradiction, that $\Delta^i_\lambda(s^t)$ does not converge to 0 when $\lambda \to 0$. This means that there exist $\delta > 0$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \lambda_n = 0$ and $\Delta^i_{\lambda_n}(s^t) > \delta$ for every $n \in \mathbb{N}$. Passing to a subsequence if necessary, we can assume that the sequence $(\bar{c}^i_{\lambda_n}, \bar{a}^i_{\lambda_n})_{n\in\mathbb{N}}$ converges to some pair $(\bar{c}^i, \bar{a}^i)$ for the product topology.[26]

Since u is bounded, we then get that

$$
\begin{aligned}
u(\bar{c}^i(s^t) + \delta) + \sum_{s^r \succ s^t} \beta^{r-t}\pi(s^r|s^t)u(\bar{c}^i(s^r)) &\le \lim_{n\to\infty} U(\tilde{c}^i_{\lambda_n}|s^t) \\
&\le \lim_{n\to\infty} U(\bar{c}^i_{\lambda_n}|s^t) \\
&\le u(\bar{c}^i(s^t)) + \sum_{s^r \succ s^t} \beta^{r-t}\pi(s^r|s^t)u(\bar{c}^i(s^r))
\end{aligned}
$$

which contradicts the strict monotonicity of u. Q.E.D.

One can easily verify that the assumptions in Proposition 4.2 hold true for the "modified" Hellwig-Lorenzoni economy $\mathcal{E}^{\mathrm{HL}}_\lambda$ analyzed in the previous section. Indeed, we do have

$$q_\lambda(s^t) \xrightarrow[\lambda\to 0]{} \begin{cases} q^c(\bar{c}) & \text{if } s_t \ne s_{t-1} \\ q^{nc} & \text{if } s_t = s_{t-1}. \end{cases}$$

Moreover, we have $D^i_\lambda(s^t) \to \omega$ when $\lambda \to 0$. We can therefore conclude that the reputation debt satisfies

$$R^i_\lambda(s^t) := D^i_\lambda(s^t) - \Delta^i_\lambda(s^t) \xrightarrow[\lambda\to 0]{} D^i_0(s^t) = \omega > 0.$$

Following our disentanglement of repayment incentives one reaches the opposite conclusion with respect to Bulow and Rogoff (1989): positive levels of reputation debt are sustained even if interest rates are higher than growth rates. In addition, the way repayment incentives are disentangled is consistent with the asymptotic exercise of the previous

---

[26]This is because under our assumptions, Lemma 4.2 implies that, for n small enough, $\Delta^i_{\lambda_n}(s^t)$ belongs to a compact set. Passing to a subsequence if necessary, we can assume that for every $s^\tau \succeq s^t$, the sequences $(q_{\lambda_n}(s^\tau))$ and $(\Delta^i_{\lambda_n}(s^t))$ converge to some point in $(0,\infty)$. This implies that there exists a compact set K (for the product topology) such that each set $B^i(D^i_{\lambda_n}, -\Delta^i_{\lambda_n}(s^t)|s^t)$ is a subset of K. Since $(\bar{c}^i_{\lambda_n}, \bar{a}^i_{\lambda_n}) \in B^i(D^i_{\lambda_n}, -\Delta^i_{\lambda_n}(s^t)|s^t)$, passing to a subsequence if necessary, we can assume that for every $s^\tau \succeq s^t$, the sequence $(\bar{c}^i_{\lambda_n}(s^\tau)), \bar{a}^i_{\lambda_n}(s^\tau))$ converges to some point in $(0,\infty) \times (0,\infty)$.

section: there is no discontinuity, the reputation debt sustained through bubbles in the limit case ($\lambda = 0$) can be approximated by the reputation debt that is sustained in an environment without bubbles (where interest rates are higher than growth rates). Our interpretation is that this provides robustness to the debt-sustainability results of Hellwig and Lorenzoni (2009). Indeed, even if bubbles are not robust to a small (but realistic) change in the default punishment, sustained reputation debt levels are.

REMARK 4.1   The disentanglement proposed by Bulow and Rogoff (1989) is more appropriate to models with full commitment. Indeed, under full commitment, debt limits are imposed only to prevent Ponzi schemes and should be non-binding. In particular, we can arbitrarily set them to be the natural debt limits N^i.[27] We then have that $\mathrm{PV}(\lambda y^i|s^t)$ represents the consumption at event s^t that agent i is willing to give up in order to avoid loosing the income $\lambda y^i(s^\tau)$ at any successor event $s^\tau \succeq s^t$. Formally, if we denote by $\widetilde{N}^i := \mathrm{PV}((1-\lambda)y^i) = (1-\lambda)N^i$ the process of natural debt limits associated to agent i's income after output drop, then we have

$$(4.6) \qquad J^i(N^i, -\mathrm{PV}(\lambda y^i|s^t)|s^t) = J_\lambda^i(\widetilde{N}^i, 0|s^t).$$

Indeed, if $(\tilde{c}^i, \tilde{a}^i)$ belongs to $d_\lambda^i(\widetilde{N}^i, 0|s^t)$, then we must have $\tilde{a}^i(s^\tau) \geq -\widetilde{N}^i(s^\tau)$. We can then set $a^i := \tilde{a}^i - \mathrm{PV}(\lambda y^i)$ and show that $(\tilde{c}^i, a^i)$ belongs to $d^i(N^i, -\mathrm{PV}(\lambda y^i|s^t))$, which implies Equation (4.6).[28]

In our environment with lack of commitment, debt limits D_λ^i typically bind and cannot be arbitrarily set to be the natural debt limits N^i. In particular, the standard argument presented above to prove Equation (4.6) cannot be applied. This explains why the output drop debt level $\Delta_\lambda^i(s^t)$ does not necessarily coincide with the not-too-tight debt limit $D_\lambda^i(s^t) = \mathrm{PV}(\lambda y^i|s^t)$, or equivalently, why positive levels of reputation debt ($R_\lambda^i(s^t) > 0$) can be sustained even when interest rates are higher than growth rates (in particular, in the absence of a bubble component in debt limits).

5. CONCLUSION

Clarifying the impact of different default punishments on creditors' motives and debtors' repayment incentives is fundamental for understanding sovereign contractual arrangements, international interest rates and market-driven debt dynamics. This paper explores this issue by studying the interplay between reputational factors and direct costs on the domestic economy in determining interest rates and sovereign debt sustainability.

We show that part of the ability to borrow must necessarily reflect the loss of access to international borrowing despite the fact that equilibrium debt limits are bubble free. We also propose a new way to disentangle repayment incentives by quantifying the part of the self-enforcing debt limits that reflects the loss of utility merely due to the domestic costs.

---

[27]Recall that $N^i(s^t) := \mathrm{PV}(y^i|s^t)$ for all $s^t \succeq s^0$

[28]Indeed, we have $a^i = \tilde{a}^i - \mathrm{PV}(\lambda y^i) \geq -\widetilde{N}^i - \mathrm{PV}(\lambda y^i) = -N^i$.

6. Proofs of the Main Results

Section 6.1 is devoted to the proof of Theorem 3.4. Section 6.2 proves a weaker version of the HL Theorem under the additional assumption that interest rates are higher than an agent's growth rates. The details regarding the construction and properties of the competitive equilibrium of $\mathcal{E}_\lambda^{\text{HL}}$ are presented in Section 6.3. Proofs of some technical results are collected in appendix.

6.1. Not-too-tight Debt Limits with Output Drop

This section proves Theorem 3.4. Since we are exclusively concerned with the single-agent problem, we simplify notation by dropping the superscript i.

The first part of the proof (Lemma 6.1) constitutes an original technical contribution that has no analogue in the limit case where $\lambda = 0$. It formally shows that a necessary condition for the existence of not-too-tight debt limits is that they exceed the present value of the output losses.[29] This, in turn, implies that interest rates must necessarily be higher than the agent's growth rates.[30]

LEMMA 6.1 If D_λ is a process of not-too-tight debt limits, then interest rates are higher than the agent's growth rates and $D_\lambda \geq \text{PV}(\lambda y)$.

PROOF OF LEMMA 6.1: Let D_λ be a process of not-too-tight bounds. We first show that there exists a non-negative process $\underline{D}$ satisfying

$$(6.1) \qquad \underline{D}(s^t) = \lambda y(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\{D_\lambda(s^{t+1}), \underline{D}(s^{t+1})\}, \quad \text{for all } s^t \in \Sigma.$$

Indeed, let Φ be the mapping $B \in \mathbb{R}^\Sigma \longmapsto \Phi B \in \mathbb{R}^\Sigma$ defined by

$$(\Phi B)(s^t) := \lambda y(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\{D_\lambda(s^{t+1}), B(s^{t+1})\}, \quad \text{for all } s^t \in \Sigma.$$

Denote by $[0, \bar{D}]$ the set of all processes $B \in \mathbb{R}^\Sigma$ satisfying $0 \leq B \leq \bar{D}$ where

$$\bar{D}(s^t) := \lambda y(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D_\lambda(s^{t+1}), \quad \text{for all } s^t \in \Sigma.$$

The mapping Φ is continuous (for the product topology) and we have $\Phi[0, \bar{D}] \subset [0, \bar{D}]$. Since $[0, \bar{D}]$ is convex and compact (for the product topology), it follows that Φ admits a fixed point $\underline{D}$ in $[0, \bar{D}]$.

---

[29]This implies that if interest rates are lower than the agent's growth rates, then the equations $J(D_\lambda, -D_\lambda(s^t)|s^t) = V_\lambda(s^t)$ for all $s^t \in \Sigma$, have no solution.

[30]It is important to point out that this follows from the existence of not-too-tight debt limits. We do not assume a priori that interest rates are higher than growth rates.

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CLAIM 6.1 The limits $\underline{D}$ are tighter than D_λ, i.e., $\underline{D} \leq D_\lambda$.

PROOF: Fix a event s^t. It is sufficient to show that $J(D_\lambda, -\underline{D}(s^t)|s^t) \geq V_\lambda(s^t)$.[31] Denote by $(\tilde{c}, \tilde{a})$ the optimal consumption and bond holdings associated to the default option at s^t, i.e., $(\tilde{c}, \tilde{a}) \in d_\lambda(0, 0|s^t)$.[32] We let $\widehat{D}$ be the process defined by $\widehat{D}(s^t) := \min\{D_\lambda(s^t), \underline{D}(s^t)\}$ for every s^t. Observe that

$$
\begin{aligned}
y(s^t) - \underline{D}(s^t) &= (1-\lambda)y(s^t) - \sum_{s^{t+1} \succ s^t} q(s^{t+1})\widehat{D}(s^{t+1}) \\
&= \tilde{c}(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})[\tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1})] \\
&= \tilde{c}(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})a(s^{t+1})
\end{aligned}
$$

where $a(s^{t+1}) := \tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1})$. Since $\widehat{D} \leq D_\lambda$ we have $a(s^{t+1}) \geq -D_\lambda(s^{t+1})$. At any successor event $s^{t+1} \succ s^t$, we have

$$
\begin{aligned}
y(s^{t+1}) + a(s^{t+1}) &= y(s^{t+1}) + \tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1}) \\
&\geq y(s^{t+1}) + \tilde{a}(s^{t+1}) - \underline{D}(s^{t+1}) \\
&\geq (1-\lambda)y(s^{t+1}) + \tilde{a}(s^{t+1}) - \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2})\widehat{D}(s^{t+2}) \\
&\geq \tilde{c}(s^{t+2}) + \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2})[\tilde{a}(s^{t+2}) - \widehat{D}(s^{t+2})] \\
&\geq \tilde{c}(s^{t+2}) + \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2})a(s^{t+2})
\end{aligned}
$$

where $a(s^{t+2}) := \tilde{a}(s^{t+2}) - \widehat{D}(s^{t+2})$.[33] Observe that $a(s^{t+2}) \geq -D_\lambda(s^{t+2})$ (since $\widehat{D} \leq D_\lambda$).

Defining $a(s^\tau) := \tilde{a}(s^\tau) - \widehat{D}(s^\tau)$ for any successor $s^\tau \succ s^t$ and iterating the above argument, we can show that $(\tilde{c}, a)$ belongs to the budget set $B(D_\lambda, -\underline{D}(s^t)|s^t)$. It follows that

$$
J(D_\lambda, -\underline{D}(s^t)|s^t) \geq U(\tilde{c}|s^t) = V_\lambda(s^t)
$$

implying the desired result: $\underline{D}(s^t) \leq D_\lambda(s^t)$.   Q.E.D.

It follows from Claim 6.1 that $\underline{D}$ satisfies

$$
(6.2) \qquad \underline{D}(s^t) = \lambda y(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})\underline{D}(s^{t+1}), \quad \text{for all } s^t \in \Sigma.
$$

---

[31]Recall that $V_\lambda(s^t) = J(D_\lambda, -D_\lambda(s^t)|s^t)$ and $J(D_\lambda, \cdot|s^t)$ is strictly increasing.
[32]Equivalently, $(\tilde{c}, \tilde{a})$ satisfies $U(\tilde{c}|s^t) = V_\lambda(s^t) := J_\lambda(0, 0|s^t)$ and belongs to $B_\lambda(0, 0|s^t)$.
[33]To get the second weak inequality we make use of equation (6.1).

Applying Equation (6.2) recursively we get

$$p(s^t)\underline{D}(s^t) = \lambda p(s^t)y(s^t) + \sum_{s^{t+1}\in S^{t+1}(s^t)} \lambda p(s^{t+1})y(s^{t+1}) + \dots$$

$$\dots + \sum_{s^T\in S^T(s^t)} \lambda p(s^T)y(s^T) + \sum_{s^{T+1}\in S^{T+1}(s^t)} p(s^{T+1})\underline{D}(s^{T+1})$$

for any $T > t$. Since $\underline{D}$ is non-negative, it follows that

$$p(s^t)\underline{D}(s^t) \geq \lambda \sum_{r=0}^{T-t} \sum_{s^{t+r}\in S^{t+r}(s^t)} p(s^{t+r})y(s^{t+r}).$$

Passing to the limit when T goes to infinite we get that $\mathrm{PV}(y|s^t)$ is finite for any event s^t (in particular for s^0). This means that interest rates are higher than the agent's growth rates. Recalling that $D_\lambda \geq \underline{D}$, we also get that $D_\lambda(s^t) \geq \mathrm{PV}(\lambda y|s^t)$. $\qquad$ Q.E.D.

Once we know that interest rates are higher than growth rates, we can use the weaker version of the HL Theorem we prove in Appendix 6.2 to characterize not-too-tight debt limits as follows.

LEMMA 6.2   Let D_λ be a process of not-too-tight debt limits. If interest rates are higher than the agent's growth rates, then there exists a non-negative process M that allows for exact roll-over and satsifies $D_\lambda = \mathrm{PV}(\lambda y) + M$.

PROOF: Let D_λ be a process of not-too-tight bounds. Fix an event s^t and let (c,a) be a plan in the demand set $d(D_\lambda, -D_\lambda(s^t)|s^t)$. Denote by $\tilde{\mathcal{E}}$ the economy where the endowment process y is replaced by $\tilde{y} := (1-\lambda)y$ and for which the default punishment amounts solely to exclusion from borrowing, that is, there is no output drop after default in the economy $\tilde{\mathcal{E}}$. Let $\tilde{a}$ be the bond-holding process defined by

$$\tilde{a}(s^\tau) := a(s^\tau) + \mathrm{PV}(\lambda y|s^t), \quad \text{for all } s^\tau \succ s^t.$$

It is straightforward to check that $(c,\tilde{a})$ belongs to the demand set $\tilde{d}(\tilde{D}_0, -\tilde{D}_0(s^t)|s^t)$ where $\tilde{d}$ is the demand associated to the economy $\tilde{\mathcal{E}}$ and $\tilde{D}_0 := D_\lambda - \mathrm{PV}(\lambda y)$. It follows that

$$\tilde{J}(\tilde{D}_0, -\tilde{D}_0(s^t)|s^t) = U(c|s^t) = J(D_\lambda, -D_\lambda(s^t)|s^t).$$

Let $\tilde{V}(s^t) := \tilde{J}(0,0|s^t)$ be the corresponding default option in the economy $\tilde{\mathcal{E}}$. The fact that $V_\lambda(s^t) := J_\lambda(0,0|s^t) = \tilde{J}(0,0|s^t)$ and D_λ is not too tight gives that

$$\tilde{J}(\tilde{D}_0, -\tilde{D}_0(s^t)|s^t) = J(D_\lambda, -D_\lambda(s^t)|s^t) = J_\lambda(0,0|s^t) = \tilde{V}(s^t).$$

The above equations imply that $\tilde{D}_0$ is a not-too-tight process in the economy $\tilde{\mathcal{E}}$. We can then apply Proposition 6.1 (see Section 6.2) to ensure the existence of a non-negative process M allowing for exact roll-over such that $\tilde{D}_0 = M$. $\qquad$ Q.E.D.

Combining Lemma 6.1 and Lemma 6.2, completes the proof of Theorem 3.4.

6.2. A Weaker Version of the HL Theorem

Assume that the output drop parameter is zero, i.e., $\lambda = 0$. It is straightforward to see that if debt limits allow for exact roll-over (as defined by Eq. (3.1)), then they are not-too-tight. This follows from a translational invariance of the constraints (2.1) and (2.2). The HL Theorem states that the converse is also true without imposing any restriction on interest rates.[34] We present below an alternative (and technically less involved) way of proving this result at the expense of assuming that interest rates are higher than the agent's growth rates. The proof exploits two intermediate results which are of independent interest: (a) the necessity of a "market transversality" condition (Lemma 6.3), and (b) a decentralization property (Lemma 6.4). The proofs of these intermediate results are postponed to Appendix A.

Since we are exclusively concerned with the single-agent problem, we simplify notation by dropping the superscript i.

LEMMA 6.3 Let D_0 be a process of not-too-tight debt limits when there is no output drop ($\lambda = 0$). Given an event s^t and an initial claim b, let (c, a) be an optimal plan, i.e., $(c, a) \in d(D_0, b|s^t)$. If c has finite present value, then the following market transversality condition is satisfied

$$(6.3) \qquad \lim_{\tau \to \infty} \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau)[a(s^\tau) + D_0(s^\tau)] = 0.$$

LEMMA 6.4 Assume that interest rates are higher than the agent's growth rates and let D_0 be a process of not-too-tight debt limits when there is no output drop ($\lambda = 0$). Fix an even s^t and let $(c, a) \in d(D_0, b|s^t)$ where $b \geq -D_0(s^t)$. Then, c has finite present value and $\mathrm{PV}(c|s^t) \leq \mathrm{PV}(y|s^t)$. If $b = -D_0(s^t)$, then $\mathrm{PV}(c|s^t) = \mathrm{PV}(y|s^t)$.

We can now prove the following characterization result.

PROPOSITION 6.1 Assume that interest rates are higher than the agent's growth rates. If D_0 is a process of not-too-tight debt limits when the output drop parameter is zero, then D_0 allows for exact roll-over.

PROOF: Fix an event s^t, and let $(c, a) \in d(D_0, -D_0(s^t)|s^t)$. Summing the flow budget constraints in $B(D_0, -D_0(s^t)|s^t)$ we get, for every $\xi > t$,

$$(6.4) \qquad \sum_{\tau=t}^{\xi-1} \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau)[c(s^\tau) - y(s^\tau)] + \sum_{s^\xi \in S^\xi(s^t)} p(s^\xi)a(s^\xi) = -p(s^t)D_0(s^t).$$

---

[34]Recently, Bidian and Bejan (2014) have identified that the arguments in Hellwig and Lorenzoni (2009) implicitly make use of some boundedness assumptions on the agent's discounted debt limits.

Let $(\alpha_\xi)_{\xi \geq t}$ be the sequence defined by

$$\alpha_\xi := \frac{1}{p(s^t)} \sum_{s^\xi \in S^\xi(s^t)} p(s^\xi)[a(s^\xi) + D_0(s^\xi)].$$

Since the plan (c, a) satisfies the transversality condition of Lemma 6.3, we have that $\lim_{\xi \to \infty} \alpha_\xi = 0$. It follows from Equation (6.4) that

$$M(s^t) := \lim_{\xi \to \infty} \frac{1}{p(s^t)} \sum_{s^\xi \in S^\xi(s^t)} p(s^\xi) D_0(s^\xi)$$

exists and is finite. Passing to the limit in Equation (6.4) we have

$$M(s^t) - D_0(s^t) = \mathrm{PV}(c - y | s^t).$$

Since $(c, a) \in d(D_0, -D_0(s^t)|s^t)$ we can then apply Lemma 6.4 to get that $\mathrm{PV}(c|s^t) = \mathrm{PV}(y|s^t)$. It follows that $D_0 = M$. Since, by construction, the process M allows for exact roll-over, we get the desired result.   Q.E.D.

6.3. Credit Exclusion with Output Drop

6.3.1. *Proof of Lemma 4.1*

Let $x \in (1/2, \bar{c})$ represent the consumption level contingent to the high income state ($1 - x$ then represents consumption contingent to the low income). Define

$$q^c(x) := \beta \alpha \frac{u'(1 - x)}{u'(x)}.$$

This corresponds to the price of the bond contingent to a switch of endowment from the high to the low level. Observe that $1 > \bar{c} > x > 1 - x > 1 - \bar{c} > 0$. We also let

$$w(x) := \frac{(1 - q^{nc})\bar{e} + q^c(x)\underline{e}}{(1 - q^{nc})^2 - (q^c(x))^2}.$$

Strict concavity of u implies that the function $q^c(\cdot)$ is strictly increasing. It follows that $q^c(x) < q^c(\bar{c})$ and therefore $q^{nc} + q^c(x) < 1$. In particular, we have $w(x) > 0$ and we can define the function $\Lambda : (1/2, \bar{c}) \longrightarrow (0, \infty)$ by posing

$$\Lambda(x) := \frac{\bar{e} - x}{[1 - q^{nc} + q^c(x)]w(x)} = \frac{[1 - q^{nc} - q^c(x)](\bar{e} - x)}{(1 - q^{nc})\bar{e} + q^c(x)\underline{e}}.$$

Since Λ is continuous, its range is an interval. Moreover, $\lim_{x \to \bar{c}} q^c(x) = q^c$ implies that

(6.5) $$\lim_{x \to \bar{c}} \Lambda(x) = 0.$$

We can then apply the Intermediate Value Theorem and deduce that there exists $\bar{\lambda} > 0$ such that the interval $(0, \bar{\lambda})$ belongs to the range of Λ, i.e.,

$$(0, \bar{\lambda}) \subset \{\Lambda(x) \ : \ x \in (1/2, \bar{c})\}.$$

If we fix an arbitrary $\lambda \in (0, \bar{\lambda})$, then there exists some $\bar{c}_\lambda \in (1/2, \bar{c})$ such that $\Lambda(\bar{c}_\lambda) = \lambda$, i.e., Equation (4.2) is satisfied. Since Λ is strictly decreasing, Equation (4.3) follows from Equation (6.5).

6.3.2. *Symmetric Markov Equilibrium of the Economy $\mathcal{E}_\lambda^{\mathrm{HL}}$*

We show that $(q_\lambda, (c_\lambda^i, a_\lambda^i, D_\lambda^i)_{i \in I})$ is a competitive equilibrium of the economy $\mathcal{E}_\lambda^{\mathrm{HL}}$ where the not-too-tight debt limits $(D_\lambda^i)_{i \in I}$ are defined by $D_\lambda^i(s^t) := \mathrm{PV}(q_\lambda; \lambda y^i | s^t)$ for all $s^t \in \Sigma$. The proof follows in a series of claims.

CLAIM 6.2   The consumption allocation $(c_\lambda^i)_{i \in I}$ and the bond holdings allocation $(a_\lambda^i)_{i \in I}$ satisfy the market clearing conditions.

PROOF: Given an event s^t, there exists an agent $i \in I$ such that $s_t = z^i$. Denote by j the other agent. We then have

$$c_\lambda^i(s^t) + c_\lambda^j(s^t) = \bar{c}_\lambda + (1 - \bar{c}_\lambda) = 1 = \bar{e} + \underline{e}$$

and

$$a_\lambda^i(s^t) + a_\lambda^j(s^t) = -\lambda w(\bar{c}_\lambda) + \lambda w(\bar{c}_\lambda) = 0 = a_\lambda^i(s^0) + a_\lambda^j(s^0).$$

<div align="right">Q.E.D.</div>

CLAIM 6.3   For each agent i, the plan $(c_\lambda^i, a_\lambda^i)$ belongs to $B^i(D_\lambda^i, a_\lambda^i(s^0) | s^0)$.

PROOF: Fix an arbitrary event s^t. If $s_t = z^i$ then

$$
\begin{aligned}
c_\lambda^i(s^t) + \sum_{s^{t+1} \succ s^t} q_\lambda(s^{t+1}) a_\lambda^i(s^{t+1}) &= \bar{c}_\lambda + q^c(\bar{c}_\lambda) \lambda w(\bar{c}_\lambda) - q^{nc} \lambda w(\bar{c}_\lambda) \\
&= \bar{c}_\lambda + [q^c(\bar{c}_\lambda) - q^{nc}] \lambda w(\bar{c}_\lambda) \\
&= \bar{e} - \lambda w(\bar{c}_\lambda) \\
&= \bar{e} + a_\lambda^i(s^t)
\end{aligned}
$$

where the third equality follows from the fact that λ and $\bar{c}_\lambda$ satisfy Equation (4.2). If $s_t \neq z^i$ then

$$
\begin{aligned}
c_\lambda^i(s^t) + \sum_{s^{t+1} \succ s^t} q_\lambda(s^{t+1}) a_\lambda^i(s^{t+1}) &= (1 - \bar{c}_\lambda) - q^c(\bar{c}_\lambda) \lambda w(\bar{c}_\lambda) + q^{nc} \lambda w(\bar{c}_\lambda) \\
&= (1 - \bar{c}_\lambda) - [q^c(\bar{c}_\lambda) - q^{nc}] \lambda w(\bar{c}_\lambda) \\
&= 1 - \bar{e} + \lambda w(\bar{c}_\lambda) \\
&= \underline{e} + a_\lambda^i(s^t)
\end{aligned}
$$

where the third equality follows from the fact that λ and $\bar{c}_\lambda$ satisfy Equation (4.2) and the fourth because $\underline{e} = 1 - \bar{e}$. To conclude, we still have to show that debt constraints are satisfied. If $s_{t+1} = z^i$, then Equation (4.4) implies

$$a_\lambda^i(s^{t+1}) = -\lambda w(\bar{c}_\lambda) = -\text{PV}(q_\lambda; \lambda y^i | s^{t+1}) = -D_\lambda^i(s^{t+1})$$

and the debt constraint binds. If $s_{t+1} \neq z^i$, then Equation (4.4) implies

$$a_\lambda^i(s^{t+1}) = \lambda w(\bar{c}_\lambda) > 0 > -\text{PV}(q_\lambda; \lambda y^i | s^{t+1}) = -D_\lambda^i(s^{t+1}).$$

We have thus proved that $(c_\lambda^i, a_\lambda^i) \in B^i(D_\lambda^i, a_\lambda^i(s^0) | s^0)$.   Q.E.D.

CLAIM 6.4   Euler equations are satisfied.

PROOF: If $s_t = s_{t+1}$ then

$$q_\lambda(s^{t+1}) = q^{nc} = \beta(1 - \alpha) = \beta\pi(s^{t+1} | s^t)\frac{u'(c_\lambda^i(s^{t+1}))}{u'(c_\lambda^i(s^t))}$$

since $c_\lambda^i(s^{t+1}) = c_\lambda^i(s^t)$. If $(s_t, s_{t+1}) = (z^j, z^i)$ with $j \neq i$, then agent's j debt constraint is not-binding at the event s^{t+1} and

$$q_\lambda(s^{t+1}) = q^c(\bar{c}_\lambda) = \beta\alpha\frac{u'(1 - \bar{c}_\lambda)}{u'(\bar{c}_\lambda)} = \beta\alpha\frac{u'(c_\lambda^j(s^{t+1}))}{u'(c^j(s^t)))} > \beta\pi(s^{t+1} | s^t)\frac{u'(c_\lambda^i(s^{t+1}))}{u'(c_\lambda^i(s^t))}$$

since $(c_\lambda^j(s^{t+1}), c_\lambda^i(s^{t+1})) = (1 - \bar{c}_\lambda, \bar{c}_\lambda)$ and $(c_\lambda^j(s^t), c_\lambda^i(s^t)) = (\bar{c}_\lambda, 1 - \bar{c}_\lambda)$.   Q.E.D.

CLAIM 6.5   The transversality conditions are satisfied.

PROOF: We should prove that for each i,

$$\liminf_{t \to \infty} \beta^t \sum_{s^t \in S^t} \pi(s^t)u'(c_\lambda^i(s^t))[a_\lambda^i(s^t) + D_\lambda^i(s^t)] \leq 0.$$

If $s_t = z^i$ then $a_\lambda^i(s^t) + D_\lambda^i(s^t) = 0$. If $s_t \neq z^i$, Equation (4.4) implies that

$$a_\lambda^i(s^t) = \lambda w(\bar{c}_\lambda) \quad \text{and} \quad D_\lambda^i(s^t) = \text{PV}(q_\lambda; y^i | s^t) < \lambda w(\bar{c}_\lambda).$$

It follows that the process $a_\lambda^i + D_\lambda^i$ is uniformly bounded from above by some $M > 0$. In particular,

$$\liminf_{t \to \infty} \beta^t \sum_{s^t \in S^t} \pi(s^t)u'(c_\lambda^i(s^t))[a_\lambda^i(s^t) + D_\lambda^i(s^t)] \leq \liminf_{t \to \infty} \beta^t u'(1 - \bar{c}_\lambda)M = 0.$$

          Q.E.D.

The desired result follows from Claims 6.2, 6.3, 6.4 and 6.5.

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Appendix A: SUPPLEMENTARY MATERIAL

In this appendix we provide the detailed arguments of some technical results. Since we are exclusively concerned with single-agent problems, we simplify notation by dropping the superscript i. All results are presented assuming that the output drop parameter is zero, i.e., $\lambda = 0$. This is without loss of generality since for $\lambda > 0$ everything is the same provided that we replace $V(s^t) = V_0(s^t)$ by $V_\lambda(s^t)$.

Let us next introduce some notations. If c is a strictly positive consumption process (in the sense that $c(s^t) > 0$ for every event s^t), then the agent's marginal rate of substitution at event s^t is defined by

$$\mathrm{MRS}(c|s^t) := \beta\pi(s^t|\sigma(s^t))\frac{u'(c(s^t))}{u'(c(\sigma(s^t)))}.$$

If a pair (c, a) belongs to the demand $d(D, b|s^t)$ at event s^t where D is a self-enforcing debt limit (not necessarily not-too-tight), then c^i satisfies the participation constraint $U(c|s^\tau) \geq V_0(s^\tau)$ for every successor event $s^\tau \succ s^t$, and the consumption process is said self-enforcing at s^t (with respect to the default option V_0).

A.1. Strictly Positive Allocations

LEMMA A.1   Let D_0 be a process of not-too-tight debt constraints and consider (c, a) in the demand $d(D_0, b|s^\tau)$ at some event s^τ, for some initial claim b. For every successor event $s^t \succ s^\tau$ we have $c(s^t) > 0$. If moreover $U(c|s^\tau) \geq V(s^\tau)$, then $c(s^\tau) > 0$.

PROOF: For simplicity, we only prove the result for $s^\tau = s^0$. Let (c, a) in the demand set $d(D_0, b|s^0)$. Because D_0 is not-too-tight, the consumption process c is self-enforcing. Assume by way of contradiction that $c(s^1) = 0$ for some $s^1 \succ s^0$. We let $\Delta U(c|s^t) = U(c|s^t) - V(s^t)$ for any event s^t. Observe that

$$\Delta U(c|s^1) = \big[u(0) - u(\hat{c}_{s^1}(s^1))\big] + \beta\sum_{s^2 \in S^2(s^1)}\pi(s^2|s^1)[U(c|s^2) - U(\hat{c}_{s^1}|s^2)]$$

where $(\hat{c}_{s^1}, \hat{a}_{s^1})$ belongs to $d(0, 0|s^1)$. In particular, we have $U(\hat{c}_{s^1}|s^2) \geq V(s^2)$ implying that

$$(\mathrm{A.1}) \qquad \Delta U(c|s^1) \leq \big[u(0) - u(\hat{c}_{s^1}(s^1))\big] + \beta\sum_{s^2 \in S^2_{\mathrm{NB}}(s^1)}\pi(s^2|s^1)\Delta U(c|s^2)$$

where $S^2_{\mathrm{NB}}(s^1)$ is the set of date-2 events following s^1 where the participation constraint is not binding, i.e., for every $s^2 \succ s^1$, we have $U(c|s^2) > V(s^2)$ if, and only if, $s^2 \in S^2_{\mathrm{NB}}(s^1)$.

Following almost verbatim the argument presented above, we can show that for every non-binding event $s^2 \in S^2_{\mathrm{NB}}(s^1)$

$$(\mathrm{A.2}) \qquad \Delta U(c|s^2) \leq \big[u(c(s^2)) - u(\hat{c}_{s^2}(s^2))\big] + \beta\sum_{s^3 \in S^3_{\mathrm{NB}}(s^2)}\pi(s^3|s^2)\Delta U(c|s^3)$$

where $(\hat{c}_{s^2}, \hat{a}_{s^2}) \in d(0,0|s^2)$ and $S_{\mathrm{NB}}^3(s^2)$ is the set of following events $s^3 \in S^3(s^2)$ where the participation constraint is not binding.[35] Combining (A.1) and (A.2) we have

$$\Delta U(c|s^1) \leq \left[u(0) - u(\hat{c}_{s^1}(s^1))\right] \quad + \quad \beta \sum_{s^2 \in S_{\mathrm{NB}}^2(s^1)} \pi(s^2|s^1)\left[u(c(s^2)) - u(\hat{c}_{s^2}(s^2))\right]$$

$$+ \quad \beta^3 \sum_{s^3 \in S_{\mathrm{NB}}^3(s^1)} \pi(s^3|s^1)\Delta U(c|s^3)$$

where $S_{\mathrm{NB}}^3(s^1)$ is the set of all date-3 events $s^3 \succ s^1$ such that participation constraints are non-binding at event s^3 and its predecessor $\sigma(s^3)$, i.e.,

$$S_{\mathrm{NB}}^3(s^1) = \bigcup_{s^2 \in S_{\mathrm{NB}}^2(s^1)} S_{\mathrm{NB}}^3(s^2).$$

Repeating the above argument, we can prove that for every $t \geq 2$

$$\begin{aligned}
\Delta U(c|s^1) \quad \leq \quad & \left[u(0) - u(\hat{c}_{s^1}(s^1))\right] \\
+ \quad & \beta \sum_{s^2 \in S_{\mathrm{NB}}^2(s^1)} \pi(s^2|s^1)\left[u(c(s^2)) - u(\hat{c}_{s^2}(s^2))\right] \\
& \vdots \\
+ \quad & \beta^{t-1} \sum_{s^{t-1} \in S_{\mathrm{NB}}^{t-1}(s^1)} \pi(s^{t-1}|s^1)\left[u(c(s^{t-1}) - u(\hat{c}_{s^{t-1}}(s^{t-1}))\right] \\
+ \quad & \beta^t \sum_{s^t \in S_{\mathrm{NB}}^t(s^1)} \pi(s^t|s^1)\Delta U(c|s^t)
\end{aligned}$$

where $S_{\mathrm{NB}}^t(s^1)$ is the set of all date-t events $s^t \succ s^1$ such that for every predecessor event s^r (i.e., satisfying $s^t \succeq s^r \succ s^1$), the participation constraint is not binding, i.e., $U(c|s^r) > V_0(s^r)$. Passing to the limit we get

$$\beta \pi(s^1)\Delta U(c|s^1) = \beta \pi(s^1)[u(0) - u(\hat{c}_{s^1}(s^1))]$$

$$+ \sum_{t \geq 2} \beta^t \sum_{s^t \in S_{\mathrm{NB}}^t(s^1)} \pi(s^t)[u(c(s^t)) - u(\hat{c}_{s^t}(s^t))].$$

Since $\Delta U(c|s^1) \geq 0$, for some $t \geq 2$, there must exist $s^t \in S_{\mathrm{NB}}^t(s^1)$ such that $c(s^t) > 0$. We propose to replace $(c(s^t), a(s^t))$ by

$$(\tilde{c}(s^t), \tilde{a}(s^t)) := (c(s^t) - \varepsilon, a(s^t) - \varepsilon)$$

for some $\varepsilon > 0$ small enough (the way we chose ε is explained below). At the predecessor event $s^{t-1} = \sigma(s^t)$, we replace $a(s^{t-1})$ by $\tilde{a}(s^{t-1}) = a(s^{t-1}) - q(s^t)\varepsilon$ and we let

---

[35]Observe that, given $s^2 \in S_{\mathrm{NB}}^2(s^1)$, for every following event $s^3 \in S_{\mathrm{NB}}^3(s^2)$, the participation constraint is non-binding not only at event s^3 but also at the predecessor event s^2.

the consumption unchanged, i.e., $\tilde{c}(s^{t-1}) = c(s^{t-1})$. For any other predecessor event s^r satisfying $s^{t-1} \succ s^r \succ s^1$ we pose

$$\tilde{a}(s^r) := a(s^r) - q(s^{r+1})\ldots q(s^t)\varepsilon$$

and $\tilde{c}(s^r) := c(s^r)$. For event s^1 the pair $(c(s^1), a(s^1))$ is replaced by $(\tilde{c}(s^1), \tilde{a}(s^1))$ defined by

$$\tilde{c}(s^1) := c(s^1) + q(s^2)\ldots q(s^t)\varepsilon \quad \text{and} \quad \tilde{a}(s^1) := a(s^1).$$

Observe that for every event s^r satisfying $s^t \succeq s^r \succ s^1$ we have $a(s^r) > -D_0(s^r)$.[36] This implies that we can choose $\varepsilon > 0$ small enough such that $\tilde{a}(s^r) \geq -D_0(s^r)$. In particular, we have $(\tilde{c}, \tilde{a}) \in B(D_0, b|s^0)$.[37] Since u satisfies Inada's property at the origin, we can choose ε small enough such the marginal gain at event s^1 compensates the marginal loss at event s^t. We then get the contradiction: $U(\tilde{c}|s^0) > U(c|s^0)$.

We have proved that $c(s^1) > 0$ for every event $s^1 \succ s^0$. We can adapt in a straightforward manner the above arguments to show that: $c(s^t) > 0$ for every event $s^t \succ s^0$; and $c(s^0) > 0$ if $U(c|s^0) \geq V(s^0)$. Q.E.D.

A.2. Finite Present Value Under Personalized Prices

LEMMA A.2 Consider an event s^t, a strictly positive consumption process c and a strictly positive process $q = (q(s^t))_{s^t \in \Sigma}$ of state-contingent bond prices such that $\mathrm{PV}(q; y|s^t)$ is finite. Assume that, for every successor event $s^\tau \succ s^t$, the participation constraint is satisfied (i.e., $U(c|s^\tau) \geq V(s^\tau)$), and $\mathrm{MRS}(c|s^\tau) \leq q(s^\tau)$. Then

$$\mathrm{PV}(\hat{q}; c|s^\tau) < \infty$$

where $\hat{q}$ is the process of individual state-contingent bond prices defined by $\hat{q}(s^\tau) := \mathrm{MRS}(c|s^\tau)$ for every event s^τ.

PROOF: We provide a proof for $s^t = s^0$. The general case obtains replacing the tree Σ by the subtree $\Sigma(s^t)$. Denote by $\hat{p}$ the process of individual Arrow–Debreu prices defined recursively by $\hat{p}(s^0) := 1$ and $\hat{p}(s^t) := \mathrm{MRS}(c|s^t)\hat{p}(\sigma(s^t))$.[38] By concavity of the Bernoulli function we have

$$\frac{U(y|s^0) - U(c|s^0)}{u'(c(s^0))} \leq \sum_{t=0}^{r-1} \sum_{s^t \in S^t} \hat{p}(s^t)[y(s^t) - c(s^t)]$$
$$+ \beta^r \sum_{s^r \in S^r} \pi(s^r) \frac{U(y|s^r) - U(c|s^r)}{u'(c(s^0))}.$$

---

[36] Observe that for any event $s^t \succ s^0$, the plan (c, a) belongs to the demand set $d(D_0, a(s^t)|s^t)$. Since the debt process D_0 is not-too-tight, it then follows that $U(c|s^t) > V(s^t)$ if, and only if, $a(s^\tau) > -D_0(s^\tau)$, for any event s^τ.

[37] For events s^r that do not satisfy $s^t \succeq s^r \succeq s^1$, we pose $(\tilde{c}(s^r), \tilde{a}(s^r)) := (c(s^r), a(s^r))$.

[38] In particular, we have $\hat{p}(s^t) = \beta^t \pi(s^t) u'(c(s^t))/u'(c(s^0))$.

Denote by p the process of Arrow–Debreu prices defined recursively by $p(s^0) := 1$ and $p(s^t) := q(s^t)p(\sigma(s^t))$. Since $U(c|s^r) \geq V(s^r) \geq U(y|s^r)$ we get that

$$
\begin{aligned}
\sum_{t=0}^{r-1} \sum_{s^t \in S^t} \hat{p}(s^t)c(s^t) &\leq \sum_{t=0}^{r-1} \sum_{s^t \in S^t} \hat{p}(s^t)y(s^t) + \frac{U(c|s^0) - U(y|s^0)}{u'(c(s^0))} \\
&\leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p(s^t)y(s^t) + \frac{U(c|s^0) - U(y|s^0)}{u'(c(s^0))}
\end{aligned}
$$

where the last inequality follows from $\hat{p} \leq p$. The desired result holds because u is bounded from above, $U(y|s^0) > -\infty$, and $\mathrm{PV}(p; y|s^0)$ is finite. $\hfill Q.E.D.$

A.3. Proof of Lemma 6.3

The result follows if we prove that $a(s^\tau) + D_0(s^\tau) \leq \mathrm{PV}(c|s^\tau)$ for every $s^\tau \succ s^t$. Assume, by way of contradiction, that there exists $s^\tau \succ s^t$ such that

$$(A.3) \qquad a(s^\tau) + D_0(s^\tau) > \mathrm{PV}(c|s^\tau).$$

Let $\theta(s^r) := \mathrm{PV}(c|s^r)$ for every event $s^r \succeq s^\tau$. By construction we have

$$(A.4) \qquad c(s^r) + \sum_{s^{r+1} \succ s^r} q(s^{r+1})\theta(s^{r+1}) = \theta(s^r), \quad \text{for all } s^r \succeq s^\tau.$$

CLAIM A.1   For every $s^r \succeq s^\tau$, we have $D_0(s^r) \leq y(s^r) + \sum_{s^{r+1} \succ s^r} q(s^{r+1})D_0(s^{r+1})$.

PROOF: Let $b = -y(s^r) - \sum_{s^{r+1} \succ s^r} q(s^{r+1})D_0(s^{r+1})$ and choose $(c, a) \in d(D_0, b|s^r)$. It is straightforward to see that we must have $c(s^r) = 0$ and $a(s^{r+1}) = -D_0(s^{r+1})$ implying that

$$
U(c|s^r) = u(0) + \beta \sum_{s^{r+1} \succ s^r} \pi(s^{r+1}|s^r)V(s^{r+1}).
$$

We know that $V(s^r) = J(0, 0|s^t) = U(\hat{c}|s^r)$ for a consumption process $\hat{c}$ satisfying participation constraints at all successor events, i.e., $U(\hat{c}|s^{r+1}) \geq V(s^{r+1})$. In particular, we have

$$
V(s^r) \geq u(\hat{c}(s^r)) + \beta \sum_{s^{r+1} \succ s^r} \pi(s^{r+1}|s^r)V(s^{r+1}) \geq U(c|s^r)
$$

which implies that $b \leq -D_0(s^r)$. $\hfill Q.E.D.$

Posing $\widetilde{a} := \theta - D_0$, it follows from Claim A.1 that

$$(A.5) \qquad c(s^r) + \sum_{s^{r+1} \succ s^r} q(s^{r+1})\widetilde{a}(s^{r+1}) \leq y(s^r) + \widetilde{a}(s^r), \quad \text{for all } s^r \succeq s^\tau.$$

Since $\widetilde{a}(s^r) \geq -D_0(s^r)$, we get that $(c, \widetilde{a}) \in B(D_0, \widetilde{a}(s^\tau)|s^\tau)$. The bond holding $\widetilde{a}$ finances the consumption c on the subtree $\Sigma(s^\tau)$ with the initial claim $\widetilde{a}(s^\tau)$. Following Equation (A.3) we have $a(s^\tau) > \widetilde{a}(s^\tau)$. This contradicts the optimality of a. Indeed, we can increase the consumption at the predecessor event $\sigma(s^\tau)$ by replacing $(a(s^r))_{s^r \succeq s^\tau}$ with $(\widetilde{a}(s^r))_{s^r \succeq s^\tau}$.

A.4. Proof of Lemma 6.4

We need an intermediate step before proving Lemma 6.4. Denote by $\mathrm{PC}(s^t)$ the set of consumption processes c satisfying the participation constraint $U(c|s^\tau) \geq V_0(s^\tau)$ for every successor event $s^\tau \succ s^t$. Recall that c is said to be self-enforcing at s^t when it belongs to $\mathrm{PC}(s^t)$.

LEMMA A.3   Consider an event s^t and a strictly positive consumption process c. Assume that, for all successors $s^\tau \succ s^t$, the participation constraint is satisfied (i.e., $U(c|s^\tau) \geq V(s^\tau)$) and $\mathrm{MRS}(c|s^\tau) \leq q(s^\tau)$ with equality if $U(c|s^\tau) > V(s^\tau)$. If there is a self-enforcing consumption process $\tilde{c} \in \mathrm{PC}(s^t)$ with finite present value, then c also has finite present value and

$$\frac{1}{u'(c(s^t))} \left[U(\tilde{c}|s^t) - U(c|s^t) \right] \leq \mathrm{PV}(\tilde{c} - c|s^t).$$

PROOF OF LEMMA A.3:   To simplify the presentation we provide a proof for $s^t = s^0$. The general case follows if we replace Σ by $\Sigma(s^t)$. For every event $s^\tau \succ s^0$, we denote by $\hat{q}(s^\tau)$ the marginal rate of substitution $\mathrm{MRS}(c|s^\tau)$ and we let $\hat{p}$ be the associated *personalized* Arrow–Debreu price process.[39] Since we now have two price processes p and $\hat{p}$, we use the following notations $\mathrm{PV}(p, x|s^t)$ and $\mathrm{PV}(\hat{p}, x|s^t)$ to represent the present value of x at s^t under, respectively, the prices p and $\hat{p}$.

Assume there is a self-enforcing consumption process $\tilde{c} \in \mathrm{PC}(s^t)$ with finite present value under p. For every event s^t, we let $b(s^t) := \mathrm{PV}(\hat{p}, \tilde{c} - c|s^t)$.[40] A.2. Concavity of the Bernoulli function implies that for every $s^\tau \in \Sigma$,

$$(A.6) \qquad \frac{\beta^\tau \pi(s^\tau)}{u'(c(s^0))} \left[U(\tilde{c}|s^\tau) - U(c|s^\tau) \right] \leq \sum_{s^r \in \Sigma(s^\tau)} \hat{p}(s^r)(\tilde{c}(s^r) - c(s^r)) = \hat{p}(s^\tau) b(s^\tau).$$

By definition of the process b, we get for every event s^τ,

$$b(s^\tau) = (\tilde{c}(s^\tau) - c(s^\tau)) + \sum_{s^{\tau+1} \succ s^\tau} \hat{q}(s^{\tau+1}) b(s^{\tau+1}).$$

---

[39] That is, $\hat{p}(s^\tau) = \hat{q}(s^\tau) \hat{p}(\sigma(s^\tau))$, or equivalently $\hat{p}(s^\tau) = \beta^\tau \pi(s^\tau) u'(c(s^\tau))/u'(c(s^0))$.

[40] To define $b(s^t)$ we need to show that $\mathrm{PV}(\hat{p}, \tilde{c}|s^0)$ and $\mathrm{PV}(\hat{p}, c|s^0)$ are both finite. Since $\hat{p} \leq p$ and $\tilde{c}$ has a finite present value, we have that $\mathrm{PV}(\hat{p}, \tilde{c}|s^0)$ is finite. The fact that $\mathrm{PV}(\hat{p}, c|s^0)$ is finite follows from Lemma

If $U(c|s^{\tau+1}) > V(s^{\tau+1})$, then $\hat{q}(s^{\tau+1}) = q(s^{\tau+1})$. If $U(c|s^{\tau+1}) = V(s^{\tau+1})$, then we have $U(\tilde{c}|s^{\tau+1}) \geq U(c|s^{\tau+1})$ since $\tilde{c}$ is self-enforcing. Equation (A.6) then implies $b(s^{\tau+1}) \geq 0$. In both cases we have $\hat{q}(s^{\tau+1})b(s^{\tau+1}) \leq q(s^{\tau+1})b(s^{\tau+1})$, implying that

$$b(s^\tau) \leq (\tilde{c}(s^\tau) - c(s^\tau)) + \sum_{s^{\tau+1} \succ s^\tau} q(s^{\tau+1})b(s^{\tau+1}).$$

Multiplying by $p(s^\tau)$ and summing the inequalities over all events s^τ up to date $\xi \geq 1$, we get

$$(\text{A.7}) \qquad p(s^0)b(s^0) \leq \sum_{\tau=0}^{\xi-1} \sum_{s^\tau \in S^\tau} p(s^\tau)(\tilde{c}(s^\tau) - c(s^\tau)) + \sum_{s^\xi \in S^\xi} p(s^\xi)b(s^\xi).$$

To finish the proof, we need to prove the following result.

CLAIM A.2   For any process x with finite present value under p, i.e., $\mathrm{PV}(p, x|s^0) < \infty$, we have

$$(\text{A.8}) \qquad \lim_{\xi \to \infty} \sum_{s^\xi \in S^\xi} p(s^\xi)\,\mathrm{PV}(\hat{p}, x|s^\xi) = 0.$$

PROOF OF CLAIM A.2:   Observe that

$$p(s^\xi)\,\mathrm{PV}(\hat{p}, x|s^\xi) = p(s^\xi) \sum_{s^r \in \Sigma(s^\xi)} \frac{\hat{p}(s^r)}{\hat{p}(s^\xi)} x(s^r).$$

For every $s^r \in \Sigma(s^\xi)$ there exists a finite family of events $(s^{\xi+1}, \ldots, s^{r-1})$ such that

$$s^\xi \prec s^{\xi+1} \prec s^{\xi+2} \prec \ldots \prec s^{r-1} \prec s^r.$$

In particular

$$\frac{\hat{p}(s^r)}{\hat{p}(s^\xi)} = \hat{q}(s^{\xi+1}) \ldots \hat{q}(s^r) \leq q(s^{\xi+1}) \ldots q(s^r) = \frac{p(s^r)}{p(s^\xi)}.$$

It follows that $p(s^\xi)\,\mathrm{PV}(\hat{p}, x|s^\xi) \leq p(s^\xi)\,\mathrm{PV}(p, x|s^\xi)$ and consequently

$$\sum_{s^\xi \in S^\xi} p(s^\xi)\,\mathrm{PV}(\hat{p}, x|s^\xi) \leq \sum_{r \geq \xi} \sum_{s^r \in S^r} p(s^r)x(s^r).$$

The desired result follows from the fact that $\mathrm{PV}(p, x|s^0) < \infty$. *Q.E.D.*

From Equation (A.7) we get that

$$\forall \xi > 0, \quad p(s^0)b(s^0) + \sum_{s^\tau \in \Sigma^{\xi-1}} p(s^\tau)c(s^\tau) \leq \mathrm{PV}(p, \tilde{c}|s^0) + \sum_{s^\xi \in S^\xi} p(s^\xi)\,\mathrm{PV}(\hat{p}, \tilde{c}|s^\xi).$$

Since $\tilde{c}$ has finite present value under p, we can combine the above inequality with Claim A.2 to get that c has finite present value. Now Combining Equations (A.6), (A.7) and Claim A.2, we get the desired result. *Q.E.D.*

Combining Lemma 6.3 and the above Lemma A.3, we can provide a simple proof of the decentralization result Lemma 6.4.

PROOF OF LEMMA 6.4: Since D_0 is not-too-tight and $(c, a) \in d(D_0, b|s^t)$ with $b \geq D_0(s^t)$, we have that c is self-enforcing at s^t (i.e., $U(c|s^\tau) \geq V(s^\tau)$ for all $s^\tau \succ s^t$) and that $U(c|s^t) \geq V(s^t)$. We can then apply Lemma A.1 (see Appendix A) to get that c is strictly positive, i.e., $c(s^\tau) > 0$ for every event $s^\tau \succeq s^t$. Following standard variational arguments, we can conclude that c satisfies the Euler equations: for all $s^\tau \succ s^t$, we have $\text{MRS}(c|s^\tau) \leq q(s^\tau)$ with equality if $U(c|s^\tau) > V(s^\tau)$.

Let $(\hat{c}, \hat{a})$ be the optimal path associated with the default option, i.e., $(\hat{c}, \hat{a}) \in d(0, 0|s^t)$. We know that the process of zero bounds is not-too-tight. As a result, we have that $\hat{c}$ is self-enforcing, i.e., $\hat{c} \in \text{PC}(s^t)$. Summing budget restrictions in $B(0, 0|s^t)$ we get that

$$(A.9) \quad \sum_{\tau=t}^{\xi-1} \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau)[\hat{c}(s^\tau) - y(s^\tau)] + \sum_{s^\xi \in S^\xi(s^t)} p(s^\xi)\hat{a}(s^\xi) = 0.$$

Since the present value of the agent's endowments is finite, we get that $\hat{c}$ also has finite present value (recall that $\hat{a}(s^\xi) \geq 0$). In particular, $(\hat{c}, \hat{a})$ satisfies the transversality condition presented in Lemma 6.3, i.e.,

$$\lim_{\xi \to \infty} \sum_{s^\xi \in S^\xi(s^t)} p(s^\xi)\hat{a}(s^\xi) = 0.$$

Letting ξ go to infinite in (A.9) we can conclude that $\text{PV}(\hat{c}|s^t) = \text{PV}(y|s^t)$. We can now apply Lemma A.3 to get that $\text{PV}(c|s^t)$ is finite and

$$(A.10) \quad U(\hat{c}|s^t) - U(c|s^t) \leq u'(c(s^t)) \times \text{PV}(\hat{c} - c|s^t) = u'(c(s^t)) \times \text{PV}(y - c|s^t).$$

This implies that $\text{PV}(c|s^t) \leq \text{PV}(y|s^t)$.

Assume now that $b = -D_0(s^t)$. We shall prove that $\text{PV}(c|s^t) \geq \text{PV}(y|s^t)$. We have seen that $\hat{c}$ is self-enforcing, i.e., $\hat{c} \in \text{PC}(s^t)$. By definition, we also have that $V(s^t) = U(\hat{c}|s^t)$. Lemma A.1 then implies that $\hat{c}(s^\tau) > 0$ for every event $s^\tau \succeq s^t$, in which case one can show that $\hat{c}$ satisfies the Euler equations: for all $s^\tau \succ s^t$, we have $\text{MRS}(\hat{c}|s^\tau) \leq q(s^\tau)$ with equality if $U(\hat{c}|s^\tau) > V(s^\tau)$. We can subsequently apply Lemma A.3 by interchanging the roles of $\hat{c}$ and c to get that

$$(A.11) \quad U(c|s^t) - U(\hat{c}|s^t) \leq u'(\hat{c}(s^t)) \times \text{PV}(c - y|s^t).$$

Since $b = -D_0(s^t)$, we have by construction,

$$(A.12) \quad U(\hat{c}|s^t) = J(0, 0|s^t) = V(s^t) = J(D_0, -D_0(s^t)|s^t) = U(c|s^t).$$

Combining (A.10), (A.11) and (A.12) gives the desired result, i.e., $\text{PV}(c - y|s^t) = 0$. Q.E.D.