



**HAL**  
open science

## A linear dimensionless bound for the weighted Riesz vector

Komla Domelevo, Stefanie Petermichl, Janine Wittwer

► **To cite this version:**

Komla Domelevo, Stefanie Petermichl, Janine Wittwer. A linear dimensionless bound for the weighted Riesz vector. *Bulletin des Sciences Mathématiques*, 2017, 141 (5), pp.385-407. 10.1016/j.bulsci.2017.05.004 . hal-01097113

**HAL Id: hal-01097113**

**<https://hal.science/hal-01097113>**

Submitted on 18 Dec 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A linear dimensionless bound for the weighted Riesz vector

Komla Domelevo<sup>a</sup> Stefanie Petermichl<sup>a,1,2</sup> Janine Wittwer<sup>b</sup>

<sup>a</sup>*Université Paul Sabatier,  
Institut de Mathématiques de Toulouse,  
118 route de Narbonne,  
F-31062 Toulouse, France*

<sup>b</sup>*Westminster College,  
1840 South 1300 East,  
Salt Lake City, UT 84105, USA*

---

## Abstract

We show that the norm of the vector of Riesz transforms as operator in the weighted Lebesgue space  $L_\omega^2$  is bounded by a constant multiple of the first power of the Poisson- $A_2$  characteristic of  $\omega$ . The bound is free of dimension. We also show that for  $n > 1$ , the Poisson- $A_2$  class is properly included in the classical  $A_2$  class.

*Key words:* Bellman function, Riesz transforms, weighted estimates

---

## 1 Introduction

A weight is a positive  $L_{loc}^1$  function. Muckenhoupt proved in [11] that for  $1 < p < \infty$  the maximal function is bounded on  $L^p(\omega)$  iff the weight  $\omega$  belongs to the class  $A_p$ , where

$$\omega \in A_p \text{ iff } Q_p(\omega) := \sup_B \langle \omega \rangle_B \langle \omega^{-1/(p-1)} \rangle_B^{p-1} < \infty.$$

---

*Email addresses:* `komla.domelevo@math.univ-toulouse.fr` (Komla Domelevo), `stefanie.petermichl@math.univ-toulouse.fr` (Stefanie Petermichl), `jjwittwer@westminstercollege.edu` (Janine Wittwer).

<sup>1</sup> Research supported in part by ANR-12-BS01-0013-02. The author is a member of IUF

<sup>2</sup> Corresponding author, Tel:+33 5 61 55 76 59, Fax: +33 5 61 55 83 85

Here the notation  $\langle \cdot \rangle_B$  denotes the average over the ball  $B$  and the supremum runs over all balls  $B$ . Hunt, Muckenhoupt, Wheeden proved in [10] that the  $A_p$  condition also characterizes the boundedness of the Hilbert transform

$$Hf(x) = \frac{1}{\pi} \int \frac{f(y)}{x-y} dy$$

in  $L^p(\omega)$ . Coifman and Fefferman in [1] extended the theory to general Calderón-Zygmund operators.

The focus in this note is on the Riesz transforms in weighted spaces. The linear and optimal bound in terms of the classical  $A_2$  characteristic has been established by one of the authors in [13]. The same estimate holds for all Calderón-Zygmund operators by Hytonen's theorem [5]. This remarkable result has been reproven by Lerner [9]. The bound depends upon the dimension in all these proofs, as one should expect.

We are interested in a version of  $A_2$  which is particularly well-suited for working with the Riesz transforms in  $\mathbb{R}^n$ , where we exploit the intimate connection of Riesz transforms and harmonic functions. We use the Poisson- $A_2$  class, which considers Poisson averages instead of box averages in the definition of  $A_2$  and obtain a bound free of dimension for the Riesz vector. This way of measuring the characteristic of the weight arises naturally when working with Bellman functions, when convexity is replaced by harmonicity. This was also the approach in [12] as well as [15] for the weighted Hilbert transform and [3] for unweighted Riesz transforms. From a probabilistic angle, the Poisson- $A_2$  characteristic also arises naturally as a consequence of the use of martingales driven by space-time Brownian motion as in Gundy-Varopoulos [4].

An altered version of one-dimensional Poisson extensions of weights made a reappearance in the works concerned with the famous two-weight problem for the Hilbert transform, see [6] and [7]. It enjoys its interpretation as a 'tamed' Hilbert transform, a feature that appears to be lost in higher dimensions. In the one-dimensional case, we see a quadratic relation between the Poisson characteristic and the classical characteristic, but the classes themselves are the same. Interestingly, these different  $A_2$  classes are not identical when the dimension is larger. We will show examples of  $A_2$  weights whose Poisson integral diverges when the dimension is at least two. Such weights belong to  $A_2$  but not to Poisson- $A_2$ . This shows that the Poisson characteristic used on a pair of weights such as for the two-weight problems, is not necessary in higher dimension. This is one of several obstacles when considering the two-weight question for the Riesz transforms, that is currently under investigation. We mention the recent advance [8] where the Poisson characteristic is modified.

## 2 Notation

The Riesz transforms  $R_k$  in  $\mathbb{R}^n$  are the component operators of the Riesz vector  $\vec{R}$ , defined on the Schwartz class by

$$(R_k \hat{f})(\xi) = i \frac{\xi_k}{\|\xi\|} \hat{f}(\xi).$$

We consider the space  $L_\omega^2$ , where  $\omega$  is a positive scalar valued  $L_{\text{loc}}^1$  function, called a weight. More specifically, the space  $L_\omega^2(\mathbb{R}^n; \mathbb{C})$  consists of all measurable functions  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  so that the quantity

$$\|f\|_\omega := \left( \int_{\mathbb{R}^n} |f(x)|^2 \omega(x) dx \right)^{1/2}$$

is finite, where  $dx$  denotes the Lebesgue measure on  $\mathbb{R}^n$ . For the space of vector valued functions  $L_\omega^2(\mathbb{R}^n; \mathbb{C}^n)$ , we replace  $|\cdot|$  by the  $\ell^2$  norm  $\|\cdot\|$ .

We are concerned with a special class of weights, called Poisson- $A_2$ . We say  $\omega \in \tilde{A}_2$  if

$$\tilde{Q}_2(\omega) := \sup_{(x,t) \in \mathbb{R}^n \times \mathbb{R}_+} P_t(\omega)(x) P_t(\omega^{-1})(x) < \infty \quad (1)$$

where  $P_t$  denotes the Poisson extension operator into the upper half space defined by

$$P_t = e^{-tA}$$

where we define  $A := \sqrt{-\Delta}$  and  $\Delta$  be the Laplacian in  $\mathbb{R}^n$ . The scalar Riesz transforms can be written as

$$R_k = \partial_k \circ A^{-1}.$$

The Poisson kernel has the form

$$P_t(y) = c_n \frac{t}{(t^2 + |y|^2)^{\frac{n+1}{2}}}$$

where  $c_n$  is its normalizing factor. The extension operator becomes

$$P_t f(x) = c_n \int_{\mathbb{R}^n} f(y) P_t(x-y) dy.$$

## 3 Main results

The main purpose of this text is to provide the dimensionless estimate:

**Theorem 3.1** *There exists a constant  $c$  that does not depend on the dimension  $n$  or the weight  $\omega$  so that for all weights  $\omega \in \tilde{A}_2$  the Riesz vector as an operator in weighted space  $L_\omega^2 \rightarrow L_\omega^2$  has operator norm  $\|\vec{R}\|_{L_\omega^2 \rightarrow L_\omega^2} \leq c\tilde{Q}_2(\omega)$ .*

We also investigate the relationship between different Muckenhoupt classes. Notably, their relation changes with dimension:

**Theorem 3.2** *Poisson- $A_2$  and classical  $A_2$  only define the same classes of weights when the dimension is one:  $\tilde{A}_2 = A_2$  if and only if  $n = 1$ .*

## 4 The dimension-free estimate

Since

$$\|\vec{R}\|_{L_\omega^2(\mathbb{R}^n; \mathbb{C}) \rightarrow L_\omega^2(\mathbb{R}^n; \mathbb{C}^n)} = \|\omega^{1/2} \vec{R} \omega^{-1/2}\|_{L^2(\mathbb{R}^n; \mathbb{C}) \rightarrow L^2(\mathbb{R}^n; \mathbb{C}^n)}$$

where the outer multiplication by  $\omega^{1/2}$  is a scalar multiplication. We can estimate  $\|\vec{R}\|_{L_\omega^2 \rightarrow L_\omega^2}$  via  $L^2$  duality. It is sufficient to estimate

$$|(\vec{g}, \omega^{1/2} \vec{R} \omega^{-1/2} f)| \leq c\tilde{Q}_2(\omega) \|f\| \|\vec{g}\|$$

for test functions (smooth and compactly supported)  $f, \vec{g}$ , where  $f$  is scalar valued and  $\vec{g}$  vector valued. Or (considering  $\omega^{-1/2} f$  instead of  $f$  and  $\omega^{1/2} \vec{g}$  instead of  $\vec{g}$ ):

$$|(\vec{g}, \vec{R} f)| \leq c\tilde{Q}_2(\omega) \|\vec{g}\|_{\omega^{-1}} \|f\|_\omega.$$

To prove this estimate, we prove the following theorem:

**Theorem 4.1** *For test functions  $f, \vec{g}$  and  $\omega \in \tilde{A}_2$  we have the following estimate:*

$$|(\vec{g}, \vec{R} f)| \leq c\tilde{Q}_2(\omega) (\|\vec{g}\|_{\omega^{-1}}^2 + \|f\|_\omega^2); \quad (2)$$

here  $c$  does not depend on  $f, \vec{g}, n, k$  or  $\omega$ .

Considering  $\lambda f$  and  $\lambda^{-1} \vec{g}$  for appropriate  $\lambda$ , with the considerations above yields Theorem 3.1.

Before we turn to the proof of Theorem 4.1, let us formulate several useful lemmata.

### 4.1 Three useful Lemmata

The following is a well known fact. It is, for example, stated in [4].

**Lemma 4.2**

$$(g, R_k f) = 4 \int_0^\infty \left( \frac{d}{dt} P_t g, \partial_k P_t f \right) t dt.$$

The proof using semigroups is very simple and concise, so we include it for the convenience of the reader. Instead of using semigroups, the same result can be obtained by the use of the Fourier transform.

*Proof.* Observe that  $F(0) = \int_0^\infty F''(t) t dt$  for sufficiently fast decaying  $F$ . So

$$(g, R_k f) = (P_0 g, P_0 R_k f) = \int_0^\infty \frac{d^2}{dt^2} (P_t g, P_t R_k f) t dt.$$

The right hand side becomes

$$\int_0^\infty \left( \left( \frac{d^2}{dt^2} P_t g, P_t R_k f \right) + 2 \left( \frac{d}{dt} P_t g, \frac{d}{dt} P_t R_k f \right) + \left( P_t g, \frac{d^2}{dt^2} P_t R_k f \right) \right) t dt.$$

Now we use the fact that  $\frac{d}{dt} P_t = -A P_t$  and  $\frac{d^2}{dt^2} P_t = A^2 P_t$  and symmetry of  $A$  to see that the above equals

$$4 \int_0^\infty (A P_t g, A P_t R_k f) t dt.$$

Using the fact that  $P_t g$  is harmonic we can replace  $A$  by  $-\partial_t$  in this expression. Furthermore, observe that  $A$  commutes with  $P_t$  and  $\partial_k$  and that  $R_k = \partial_k \circ A^{-1}$ . We obtain

$$(g, R_k f) = 4 \int_0^\infty \left( \frac{d}{dt} P_t g, \partial_k P_t f \right) t dt.$$

For function  $f$  and vector function  $\vec{g}$  this becomes

$$(\vec{g}, \vec{R} f) = 4 \int_0^\infty \left( \frac{d}{dt} P_t \vec{g}, \nabla P_t f \right) t dt.$$

*QED*

We will need the following form of a Lemma that has been proven in [3] and generalised in [2], the so-called ‘ellipse lemma’.

**Lemma 4.3** *Let  $m, l, k \in \mathbb{N}$ . Denote  $d = m + l + k$ . For arbitrary  $u \in \mathbb{R}^{m+l+k}$  write  $u = u_m \oplus u_l \oplus u_k$ , where  $u_i \in \mathbb{R}^i$  for  $i = m, l, k$ . Let  $r_m = \|u_m\|, r_l = \|u_l\|$ . Suppose the matrix  $A \in \mathbb{R}^{d \times d}$  is such that*

$$(A u, u) \geq 2 r_m r_l$$

*for all  $u \in \mathbb{R}^d$ . Then there exists  $\tau > 0$  so that*

$$(A u, u) \geq \tau r_m^2 + \tau^{-1} r_l^2$$

for all  $u \in \mathbb{R}^d$ .

We will be using this lemma for  $m = 2$ ,  $l = 2n$  and  $k = 4$ .

Finally we crucially need the Lemma below, that has been proven in [14]. It was deduced from the sharp weighted inequality for martingales from [16].

**Lemma 4.4** *For any  $Q > 1$  let  $\mathcal{D}$  be a subset of  $\mathbb{R} \times \mathbb{R} \times \mathbb{C} \times \mathbb{C}^n \times \mathbb{R} \times \mathbb{R}$*

$$\mathcal{D}_Q = \{(\mathbf{X}, \mathbf{Y}, \mathbf{x}, \mathbf{y}, \mathbf{r}, \mathbf{s}) : |\mathbf{x}|^2 < \mathbf{X}\mathbf{s}, \|\mathbf{y}\|^2 < \mathbf{Y}\mathbf{r}, 1 < \mathbf{r}\mathbf{s} < Q\}.$$

*For any compact  $K \subset \mathcal{D}_Q$  there exists an infinitely differentiable function  $B_{K,Q}$  defined in a small neighborhood of  $K$  that still lies inside  $\mathcal{D}_Q$  so that the following estimates hold in  $K$ .*

$$0 \leq B_{K,Q} \leq 5Q(\mathbf{X} + \mathbf{Y}), \quad (3)$$

$$-d^2 B_{K,Q} \geq 2|d\mathbf{x}||d\mathbf{y}|. \quad (4)$$

The last inequality describes an operator inequality where the left hand side is the negative Hessian of  $B$ . This function is a mollified version of the Bellman function one gets by knowing that the equivalent weighted estimate holds for a certain dyadic model operator.

#### 4.2 Proof of the main theorem

Recall the inequality of theorem (4.1). We want to show for test functions  $f$  and  $g$  and  $\omega \in \tilde{A}_2$ :

$$|(\vec{g}, \vec{R}f)| \leq c\tilde{Q}_2(\omega)(\|\vec{g}\|_{\omega^{-1}}^2 + \|f\|_{\omega}^2).$$

For a fixed weight  $\omega$  we let  $Q = \tilde{Q}_2(\omega)$ . This gives rise to the set  $\mathcal{D}_Q$ . We define

$$b_{K,Q}(x, t) = B_{K,Q}(v(x, t))$$

where

$$v(x, t) = \left( P_t(|f|^2\omega), P_t(\|\vec{g}\|^2\omega^{-1}), P_t(f), P_t(\vec{g}), P_t(\omega), P_t(\omega^{-1}) \right) (x)$$

Here  $K$  is a compact subset of  $\mathcal{D}_Q$  to be chosen later.

Note that the vector  $v \in \mathcal{D}_Q$  for any choice of  $(x, t)$ . This is ensured by  $Q = \tilde{Q}_2(\omega)$  and several applications of Jensen's inequality. Notice also that the vector  $v$  takes compacts inside the interior of  $\mathbb{R}_+^{n+1}$  to compacts  $K$  inside  $\mathcal{D}_Q$

for fixed  $f, \vec{g}, \omega$ . By elementary application of the chain rule (using harmonicity of the components of  $v$ ) one shows that

$$\Delta_{x,t} b(x, t) = \sum_{i=1}^n \left( d^2 B(v) \frac{\partial}{\partial x_i} v, \frac{\partial}{\partial x_i} v \right) + \left( d^2 B(v) \frac{\partial}{\partial t} v, \frac{\partial}{\partial t} v \right).$$

Here  $\Delta_{x,t}$  is the full Laplacian in the upper half space

$$\Delta_{x,t} = \sum_{i=1}^n \partial_{x_i}^2 + \partial_t^2.$$

Notice that condition 4 means that at any  $v = (\mathbf{X}, \mathbf{Y}, \mathbf{x}, \mathbf{y}, \mathbf{r}, \mathbf{s}) \in K \subset \mathcal{D}_Q$  we have the operator inequality for any  $u \in \mathbb{R} \times \mathbb{R} \times \mathbb{C} \times \mathbb{C}^n \times \mathbb{R} \times \mathbb{R}$

$$\left( -d^2 B_{K,Q}(v) u, u \right) \geq 2(\|d\mathbf{x}\| \|d\mathbf{y}\| u, u) = 2|u_3| \|u_4\|.$$

In our situation,  $f, \vec{g}, \omega$  and  $Q$  are fixed, but we have varying  $K, x, t$ . So Lemma 4.3 guarantees us existence of  $\tau_{x,t,K}$  so that

$$\left( -d^2 B(v) \frac{\partial}{\partial x_i} v, \frac{\partial}{\partial x_i} v \right) \geq \tau_{x,t,K} \left| \frac{\partial}{\partial x_i} P_t f \right|^2 + \tau_{x,t,K}^{-1} \left\| \frac{\partial}{\partial x_i} P_t \vec{g} \right\|^2$$

for all  $i$  and

$$\left( -d^2 B(v) \frac{\partial}{\partial t} v, \frac{\partial}{\partial t} v \right) \geq \tau_{x,t,K} \left| \frac{\partial}{\partial t} P_t f \right|^2 + \tau_{x,t,K}^{-1} \left\| \frac{\partial}{\partial t} P_t \vec{g} \right\|^2.$$

So

$$\begin{aligned} & -\Delta_{x,t} b_{K,Q}(x, t) \\ & \geq \tau_{x,t,K} \left( \sum_{i=1}^n \left| \frac{\partial}{\partial x_i} P_t f \right|^2 + \left| \frac{\partial}{\partial t} P_t f \right|^2 \right) \\ & \quad + \tau_{x,t,K}^{-1} \left( \sum_{i=1}^n \left\| \frac{\partial}{\partial x_i} P_t \vec{g} \right\|^2 + \left\| \frac{\partial}{\partial t} P_t \vec{g} \right\|^2 \right) \\ & \geq 2 \left( \sum_{i=1}^n \left| \frac{\partial}{\partial x_i} P_t f \right|^2 + \left| \frac{\partial}{\partial t} P_t f \right|^2 \right)^{1/2} \left( \sum_{i=1}^n \left\| \frac{\partial}{\partial x_i} P_t \vec{g} \right\|^2 + \left\| \frac{\partial}{\partial t} P_t \vec{g} \right\|^2 \right)^{1/2} \\ & \geq 2 \left( \sum_{i=1}^n \left| \frac{\partial}{\partial x_i} P_t f \right|^2 \right)^{1/2} \left\| \frac{\partial}{\partial t} P_t \vec{g} \right\| \\ & = 2 \|\nabla P_t f\| \left\| \frac{\partial}{\partial t} P_t \vec{g} \right\| \end{aligned}$$

In what follows we first use formula 4.2, then the estimate for the Laplacian we just proved:

$$\begin{aligned}
& |(\vec{g}, \vec{R}f)| \\
& \leq 4 \int_0^\infty \left| \left( \frac{\partial}{\partial t} P_t \vec{g}, \nabla P_t f \right) \right| t dt \\
& \leq 4 \int_0^\infty \int_{\mathbb{R}^n} \left\| \frac{\partial}{\partial t} P_t \vec{g} \right\| \|\nabla P_t f\| dx dt \\
& \leq 2 \int_0^\infty \int_{\mathbb{R}^n} -\Delta_{x,t} b_{K,Q}(x,t) dx dt.
\end{aligned}$$

It remains to see that

$$-\int_0^\infty \int_{\mathbb{R}^n} \Delta_{x,t} b_{K,Q}(x,t) dx dt \leq C\tilde{Q}_2(\|f\|_{\omega^{-1}}^2 + \|\vec{g}\|_{\omega}^2) \quad (5)$$

with  $C$  independent of  $n$ . In order to obtain this last estimate, we will apply Green's formula as well as some properties of our Bellman function. We are going to pass through values of the function  $b$ .

Recall the statement of Green's formula:

**Theorem 4.5**

$$\int_{\Omega} f(x) \Delta g(x) - g(x) \Delta f(x) dA(x) = \int_{\partial\Omega} \left( f(t) \frac{\partial g}{\partial n}(t) - g(t) \frac{\partial f}{\partial n}(t) \right) dS(t)$$

where  $n$  is the outward normal and  $dS$  surface measure on  $\partial\Omega$ .

In order to be accurate, we are obliged to take care of a few technicalities first.

Let  $T_R$  be a cylinder with square base in upper half space  $[-R, R]^n \times [0, 2R]$ . For  $R$  not too small, the point  $(0, 1)$  lies inside  $T_R$ . Let  $T_{R,\epsilon} = T_R + (0, \epsilon)$ . For any interior point  $(\xi, \tau)$ , let  $G^{R,\epsilon}[(x, t), (\xi, \tau)]$  be its Green function, meaning that

$$\Delta_{x,t} G^{R,\epsilon}[(x, t), (\xi, \tau)] = -\delta_{(\xi, \tau)} \quad \text{and} \quad G^{R,\epsilon} = 0 \quad \text{on} \quad \partial T_{R,\epsilon}.$$

Notice that  $RT_{1,0} = T_{R,\epsilon} - (0, \epsilon)$  and the Green's functions relate as follows:

**Lemma 4.6** *The Green's function has the following scaling property:*

$$R^{n-1} G^{R,\epsilon}[(x, t), (\xi, \tau)] = G^{1,0}[(R^{-1}(x, t - \epsilon), R^{-1}(\xi, \tau - \epsilon))]. \quad (6)$$

*Proof.* By uniqueness it suffices to see that  $R^{-(n-1)} G^{1,0}[R^{-1}(x, t - \epsilon), R^{-1}(\xi, \tau - \epsilon)]$  is indeed the Green function for the region  $T_{R,\epsilon}$  at the point  $(\xi, \tau)$ . It is clear that it equals zero on  $\partial T_{R,\epsilon}$ . Furthermore for test function  $f$  we have

$$\begin{aligned}
& \int \int_{T_{R,\epsilon}} \Delta_{x,t} R^{-(n-1)} G^{1,0}[R^{-1}(x, t - \epsilon), R^{-1}(\xi, \tau - \epsilon)] f(x, t) dx dt \\
&= \int \int_{T_{1,0}} \Delta_{y,s} G^{1,0}[(y, s), R^{-1}(\xi, \tau - \epsilon)] f(Ry, Rs + \epsilon) dy ds \\
&= -f(\xi, \tau)
\end{aligned}$$

We did a substitution  $(x, t) = (Ry, Rs + \epsilon)$ . Note that there is  $R^{-2}$  term arising from the switch of  $\Delta_{x,t}$  to  $\Delta_{y,s}$  and  $R^{n+1}$  from the determinant. *QED*

Recall that the vector  $v$  maps each  $T_{R,\epsilon}$  into a compact  $K = K_{R,\epsilon} \subset \mathcal{D}_Q$ . For technical reasons we have to exhaust the upper half space by compacts. We fix any compact set  $M$  in the open upper half space and consider  $R$  large enough and  $\epsilon$  small enough so that  $M \subset T_{R,\epsilon}$ .

Let us start to use the size estimate of our Bellman function to obtain an estimate of the function value  $b_{K,Q}(0, R + \epsilon)$  from above:

$$\begin{aligned}
& b_{K,Q}(0, R + \epsilon) \\
&\leq C\tilde{Q}_2(P_{R+\epsilon}|f|^2\omega^{-1}(0) + P_{R+\epsilon}\|\vec{g}\|^2\omega(0)) \\
&= c_n C\tilde{Q}_2 \int_{\mathbb{R}^n} |f|^2(y)\omega^{-1}(y) \frac{R + \epsilon}{((R + \epsilon)^2 + |y|^2)^{\frac{n+1}{2}}} dy \\
&\quad + \int_{\mathbb{R}^n} \|\vec{g}\|^2(y)\omega(y) \frac{R + \epsilon}{((R + \epsilon)^2 + |y|^2)^{\frac{n+1}{2}}} dy \\
&\leq c_n (R + \epsilon)^{-n} C\tilde{Q}_2 (\|f\|_{\omega^{-1}}^2 + \|\vec{g}\|_{\omega}^2).
\end{aligned}$$

For an estimate from below, Green's formula applied to our situation gives:

$$\begin{aligned}
& b_{K,Q}(0, R + \epsilon) \\
&= - \int \int_{T_{R,\epsilon}} G^{R,\epsilon}((x, t), (0, R + \epsilon)) \Delta_{x,t} b_{K,Q}(x, t) dx dt \\
&\quad - \int_{\partial T_{R,\epsilon}} b_{K,Q}(x, t) \frac{\partial G^{R,\epsilon}((x, t), (0, R + \epsilon))}{\partial n} dx dt \\
&\quad + \int_{\partial T_{R,\epsilon}} G^{R,\epsilon}((x, t), (0, R + \epsilon)) \frac{\partial b_{K,Q}((x, t))}{\partial n} dx dt
\end{aligned}$$

The first boundary term is negative because  $b$  is non-negative and the outward normal of the Green's function is negative on the boundary of  $T_{R,\epsilon}$ . The second boundary term vanishes because  $G^{R,\epsilon} = 0$  on the boundary. So we have the following estimate:

$$\begin{aligned}
& b_{K,Q}(0, R + \epsilon) \\
& \geq - \int \int_{T_{R,\epsilon}} G^{R,\epsilon}((x, t), (0, R + \epsilon)) \Delta_{x,t} b_{K,Q}(x, t) dx dt. \\
& \geq - \int \int_M G^{R,\epsilon}((x, t), (0, R + \epsilon)) \Delta_{x,t} b_{K,Q}(x, t) dx dt.
\end{aligned}$$

since  $-\Delta b \geq 0$  and where we recall that  $M \subset T_{R,\epsilon}$ . We continue the estimate using the scaling properties of the Green functions (6).

$$b_{K,Q}(0, R + \epsilon) \geq - \int \int_M R^{-(n-1)} G^{1,0}(R^{-1}(x, t - \epsilon), (0, 1)) \Delta_{x,t} b(x, t) dx dt.$$

Since  $G^{1,0}((R^{-1}x, 0), (0, 1)) = 0$  we have

$$\begin{aligned}
& b_{K,Q}(0, R + \epsilon) \\
& \geq - \int \int_M R^{-(n-1)} \left\{ G^{1,0}((R^{-1}x, R^{-1}(t - \epsilon)), (0, 1)) - \right. \\
& \quad \left. G^{1,0}((R^{-1}x, 0), (0, 1)) \right\} \Delta_{x,t} b(x, t) dx dt \\
& = - \int \int_M R^{-(n-1)} \frac{\partial G^{1,0}}{\partial t}(R^{-1}x, \tau) R^{-1}(t - \epsilon) \Delta_{x,t} b_{K,Q}(x, t) dx dt \\
& = - \int \int_M R^{-n} \frac{\partial G^{1,0}}{\partial t}(R^{-1}x, \tau) \Delta_{x,t} b_{K,Q}(x, t) dx (t - \epsilon) dt,
\end{aligned}$$

where  $0 \leq \tau \leq R^{-1}(t - \epsilon)$ . Pulling this all together with the estimate from above,

$$\begin{aligned}
& - \int \int_M R^{-n} \frac{\partial G^{1,0}}{\partial t}(R^{-1}x, \tau) \Delta_{x,t} b_{K,Q}(x, t) dx (t - \epsilon) dt \\
& \leq b_{K,Q}(0, R + \epsilon) \leq c_n (R + \epsilon)^{-n} C \tilde{Q}_2 (\|f\|_{\omega^{-1}}^2 + \|\vec{g}\|_{\omega}^2),
\end{aligned}$$

hence

$$- \int \int_M \frac{\partial G^{1,0}}{\partial t}(R^{-1}x, \tau) \Delta_{x,t} b_{K,Q}(x, t) dx (t - \epsilon) dt \leq c_n C \tilde{Q}_2 (\|f\|_{\omega^{-1}}^2 + \|\vec{g}\|_{\omega}^2)$$

uniformly with respect to  $R$  and  $\epsilon$ , for all given  $M$ . When  $R$  goes to infinity, the normal derivative  $\frac{\partial G^{1,0}}{\partial t}(R^{-1}x, \tau)$  tends to  $\frac{\partial G^{1,0}}{\partial t}(0, 0)$  uniformly with respect to  $(x, t) \in M$ . But we know that the normal derivative  $\frac{\partial G^{1,0}}{\partial t}(0, 0)$  is exactly the normalizing factor  $c_n$  of the Poisson kernel (see [3] and the references therein Couhlon-Duong). Letting  $R$  go to infinity and  $\epsilon$  go to zero yields for all compact  $M$  of the upper half space:

$$- \int \int_M \Delta_{x,t} b_{K,Q}(x, t) dx t dt \leq C \tilde{Q}_2 (\|f\|_{\omega^{-1}}^2 + \|\vec{g}\|_{\omega}^2).$$

Finally, letting  $M$  exhaust the upper half space establishes (5).

## 5 The comparison of classical and Poisson characteristic.

In this section we prove Theorem 3.2. We provide an example that demonstrates that  $\tilde{A}_2 \neq A_2$  if  $n > 1$ . For the case  $n = 1$ , it is known that the two classes are the same. Infact, for  $n = 1$  the estimates

$$\tilde{Q}_2(\omega) \lesssim Q_2(\omega) \lesssim \tilde{Q}_2(\omega)^2$$

are proven in [5]. If  $n > 1$  however, an easy example shows that the Poisson integral of a simple power weight diverges, although the weight belongs to classical  $A_2$ . Consider  $\omega_\alpha(x) = |x|^\alpha$ . It is well known and straightforward to check that  $\omega_\alpha \in A_2 \leftrightarrow |\alpha| < n$ . Also  $Q_2(\omega_\alpha) \sim \frac{1}{n^2 - \alpha^2}$ . We show that the Poisson integral  $P_t \omega_\alpha(0)$  diverges for  $\alpha$  close to  $n$ .

$$\begin{aligned} & P_t(\omega_\alpha)(0) \\ & \sim \int_{\mathbb{R}^n} \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}} |x|^\alpha dx \\ & = |S| \int_0^\infty \frac{t}{(t^2 + r^2)^{\frac{n+1}{2}}} r^{\alpha+n-1} dr \\ & = |S| \sum_{k=1}^\infty \int_{2^{k-1}t}^{2^k t} \frac{t}{(t^2 + r^2)^{\frac{n+1}{2}}} r^{\alpha+n-1} dr \\ & \gtrsim |S| \sum_{k=1}^\infty 2^{k-1} t \frac{t}{(t^2 + 2^{2k} t^2)^{\frac{n+1}{2}}} (2^{k-1} t)^{\alpha+n-1} \\ & \gtrsim t^\alpha |S| 2^{-\alpha-n-1} \sum_{k=1}^\infty 2^{(\alpha-1)k} \end{aligned}$$

We see that this sum converges if and only if  $\alpha - 1 < 0$ . If  $n \geq 2$  we have  $\omega_\alpha \in A_2$  if and only if  $|\alpha| < n$  so we can easily pick a valid  $\alpha$  for which the above sum diverges.

Thus not every weight in  $A_2$  is in  $\tilde{A}_2$ . The converse is still true, though. Let  $w \in \tilde{A}_2$ , and let  $B$  be a ball with center  $a$  and radius  $r$ . Then for  $y \in B$ ,  $|a - y| < r$ , and so

$$\frac{1}{r^n} \leq 2^{\frac{(n+1)}{2}} \frac{r}{(r^2 + |a - y|^2)^{\frac{n+1}{2}}}$$

and so

$$\langle \omega \rangle_B \leq C \int_B \frac{r w(y)}{(r^2 + |a - y|^2)^{\frac{n+1}{2}}} dy \leq C' P_r(\omega)(a),$$

and similarly for  $\langle \omega^{-1} \rangle_B$ . Thus  $Q_2(\omega) \leq \tilde{C} \tilde{Q}_2(\omega)$ .

## References

- [1] R. R. COIFMAN, C. FEFFERMAN, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. , 51, (1974), pp. 241-250.
- [2] O. DRAGIČEVIĆ, S. TREIL, A. VOLBERG, *A theorem about three quadratic forms*. Int. Math. Res. Not. IMRN, (2008), pp. Art. ID rnn 072, 9
- [3] O. DRAGIČEVIĆ, A. VOLBERG, *Bellman Functions and Dimensionless Estimates Of Littlewood-Paley Type*. J. Operator Theory, 56, (2006), pp.167-198.
- [4] R. F. GUNDY, N. T. VAROPOULOS, *Les transformations de Riesz et les intégrales stochastiques*. C. R. Acad. Sci. Paris Sér. A-B, 289(1), (1979), pp. A13-A16.
- [5] S. HUKOVIĆ, *Thesis: Singular integral operators in weighted spaces and Bellman functions*, Brown University, (1998).
- [6] M. T. LACEY, *Two weight inequality for the Hilbert transform: A real variable characterization, ii*. preprint, <http://arxiv.org/abs/1301.4663>
- [7] M. T. LACEY, E. T. SAWYER, C.-Y. SHEN, I. URIARTE-TUERO, *Two weight inequality for the Hilbert transform: A real variable characterization, i*. preprint, <http://arxiv.org/abs/1201.4319>
- [8] M. T. LACEY, B. D. WICK, *Two weight inequalities for Riesz transforms: Uniformly full dimension weights*. preprint, <http://arxiv.org/abs/1312.6163>
- [9] A. K. LERNER, *A simple proof of the  $A_2$  conjecture*. Int. Math. Res. Not.,14, (2013), pp. 3159-3170.
- [10] R. A. HUNT, B. MUCKENHOUP, R. L. WHEEDEN, *Weighted norm inequalities for the conjugate function and the Hilbert transform*. Trans. Amer. Math. Soc., 176, (1973), pp. 227-251.
- [11] B. MUCKENHOUP, *Weighted norm inequalities for the Hardy maximal function*. Trans. Amer. Math. Soc., 165, (1972), pp. 207-226.
- [12] F. NAZAROV, S. TREIL, *The weighted norm inequalities for Hilbert transform are now trivial*. C. R. Acad. Sci. Paris, Ser. I, 323, (1996), pp. 717-722.
- [13] S. PETERMICHL, *The sharp weighted bound for the Riesz transforms*. Proc. Amer. Math. Soc., 136, (2008), pp. 1237-1249
- [14] S. PETERMICHL, A. VOLBERG, *Heating of the Beurling operator: weakly quasiregular maps on the plane are quasiregular*. Duke Math. J., 112, 2, (2002), pp. 281-305.
- [15] S. PETERMICHL, J. WITWER , *A sharp estimate for the weighted Hilbert transform via Bellman functions*. Mich. Math. J., 50, (2002), pp. 71-87.
- [16] J. WITWER, *A sharp estimate on the norm of the martingale transform*. Math. Res. Lett., 7, 1, (2000), pp. 1-12.