



HAL
open science

Specified homogenization of a discrete traffic model leading to an effective junction condition

Nicolas Forcadel, Wilfredo Salazar, Mamdouh Zaydan

► **To cite this version:**

Nicolas Forcadel, Wilfredo Salazar, Mamdouh Zaydan. Specified homogenization of a discrete traffic model leading to an effective junction condition. *Communications on Pure and Applied Analysis*, 2018, 17 (5), pp.2173-2206. 10.3934/cpaa.2018104 . hal-01097085v3

HAL Id: hal-01097085

<https://hal.science/hal-01097085v3>

Submitted on 12 Jul 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1 A junction condition by specified homogenization
2 of a discrete model with a local perturbation
3 and application to traffic flow

4 N. Forcadel¹, W. Salazar¹, M. Zaydan¹

5 June 23, 2017

6 **Abstract**

7 In this paper, we focus on deriving traffic flow macroscopic models from microscopic models
8 containing a local perturbation such as a traffic light. At the microscopic scale, we consider
9 a first order model of the form "follow the leader" i.e. the velocity of each vehicle depends on
10 the distance to the vehicle in front of it. We consider a local perturbation located at the origin
11 that slows down the vehicles. At the macroscopic scale, we obtain an explicit Hamilton-Jacobi
12 equation left and right of the origin and a junction condition at the origin (in the sense of
13 [18]) which keeps the memory of the local perturbation. As it turns out, the macroscopic
14 model is equivalent to a LWR model, with a flux limiting condition at the junction. Finally,
15 we also present qualitative properties concerning the flux limiter at the junction.

16 **AMS Classification:** 35D40, 90B20, 35B27, 35F20, 45K05.

17 **Keywords:** specified homogenization, Hamilton-Jacobi equations, integro-differential operators,
18 Slepčev formulation, viscosity solutions, traffic flow, microscopic models, macroscopic models.

19 **1 Introduction**

20 The goal of this paper is to derive a macroscopic model for traffic flow problems from a microscopic
21 model. The idea is to rescale the microscopic model, which describes the dynamics of each vehicle
22 individually, in order to get a macroscopic model which describes the dynamics of density of vehi-
23 cles. The main motivation for deriving macroscopic models from microscopic models comes from
24 the fact that macroscopic models are more adapted to simulate traffic at large scales. Moreover,
25 microscopic models are based on assumptions that are easier to verify and therefore to derive a
26 macroscopic model allows to rigorously verify it.

27 The problem of deriving macroscopic models from microscopic ones has already been studied
28 for models of the type following the leader (i.e. the velocity or the acceleration of each vehicle
29 depends only on the distance to the vehicle in front of it). We refer for example to [4, 7, 16, 21]
30 where the authors rescaled the empirical measure and obtained a scalar conservation law (LWR
31 model [22, 25]). Recently, another approach has been introduced in [10] (see also [9, 11, 12])
32 where the authors work on the primitive of the empirical measure and, at the limit, obtain a
33 Hamilton-Jacobi equation which is the primitive of the LWR model.

34 The originality of our work is that we assume that there is a local perturbation that slows down
35 the vehicles and we want to understand how this local perturbation influences the macroscopic
36 dynamics. This local perturbation can be constant in time and represent a slowdown near a school
37 or due to a car crash near the road. It can also depend (periodically) in time and represent for

¹Normandie Univ, INSA de Rouen, LMI (EA 3226 - FR CNRS 3335), 76000 Rouen, France, 685 Avenue de l'Université, 76801 St Etienne du Rouvray cedex. France

1 example a traffic light. The schematic representation of the microscopic model is given in Figure
 2 1.

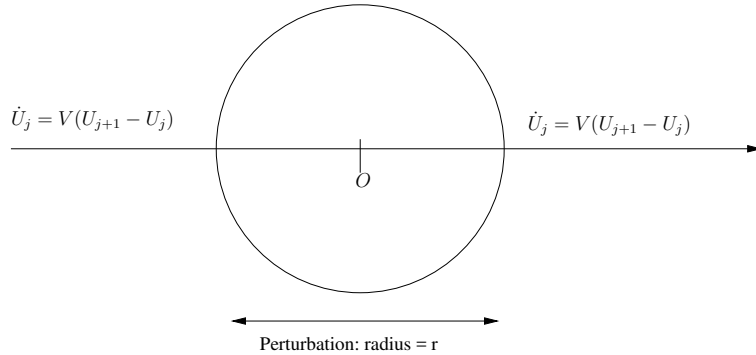


Figure 1: Schematic representation of the microscopic model.

3 We denote by $U_j(t)$ the position of the j th vehicle and we assume that the velocity of each
 4 vehicle is given by the function V . In order to obtain our homogenization result, we proceed as in
 5 [9, 10, 11, 12, 13] and rescale the microscopic model which describes the dynamics of each vehicle,
 6 to obtain a macroscopic model that describes the density of vehicles. If the local perturbation
 7 is located around zero, at the macroscopic scale it is natural to get an Hamilton-Jacobi equation
 8 with a junction condition at the origin (see Figure 2, u_x^0 is the primitive of the density of vehicles
 9 and the effective Hamiltonian \bar{H} is defined later in the paper), since the size of the perturbation
 10 goes to zero when we do the rescaling. This junction condition keeps the memory of the presence
 11 of the local perturbation.

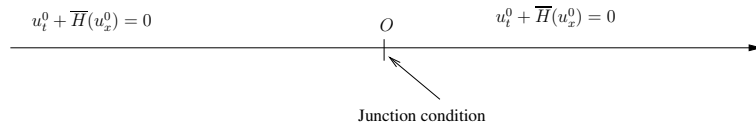


Figure 2: Schematic representation of the macroscopic model.

12 Recently, the theory of Hamilton-Jacobi equations with junction or more generally on networks
 13 has known important developments in particular since the works of Achdou, Camilli, Cutri, and
 14 Tchou [1] and Imbert, Monneau, and Zidani [20]. In this direction, we would like to mention the
 15 recent work of Imbert and Monneau [18] in which they give a suitable definition of (viscosity)
 16 solutions at the junction which allows to prove comparison principle, stability and so on.

17 In this paper, we will use the ideas developed in [10] in order to pass from microscopic models to
 18 macroscopic ones. In particular, we will show that this problem can be seen as an homogenization
 19 result. The difficulty here is that, due to the local perturbation, we are not in a periodic setting
 20 and so the construction of suitable correctors is more complicated. In particular, we will use the
 21 idea developed by Achdou and Tchou in [2], by Galise, Imbert, and Monneau in [14], and in
 22 the lectures of Lions at the "College de France" [23], which consists in constructing correctors on
 23 truncated domains.

24 2 Main results

25 2.1 General model: first order model with a single perturbation

26 In this paper, we are interested in a first order microscopic model of the form

$$\dot{U}_j(t) = V(U_{j+1}(t) - U_j(t)) \cdot \phi(t, U_j(t)), \quad (2.1)$$

1 where $U_j : [0, +\infty) \rightarrow \mathbb{R}$ denotes the position of the j -th vehicle and \dot{U}_j is its velocity. The
2 function $\phi : \mathbb{R} \times \mathbb{R} \rightarrow [0, 1]$ simulates the presence of a local perturbation around the origin. We
3 denote by r the radius of influence of the perturbation.

4 The function V is called the optimal velocity function and we make the following assumptions
5 on V and ϕ :

6 **Assumption (A)**

7 (A1) $V : \mathbb{R} \rightarrow \mathbb{R}^+$ is Lipschitz continuous, non-negative.

8 (A2) V is non-decreasing on \mathbb{R} .

9 (A3) There exists a $h_0 \in (0, +\infty)$ such that for all $h \leq h_0$, $V(h) = 0$.

10 (A4) There exists $h_{max} \in (h_0, +\infty)$ such that for all $h \geq h_{max}$, $V(h) = V(h_{max}) =: V_{max}$.

11 (A5) There exists a real $p_0 \in [-1/h_0, 0)$ such that the function $p \mapsto pV(-1/p)$ is decreasing
12 on $[-1/h_0, p_0)$ and increasing on $[p_0, 0)$.

13 (A6) The function $\phi : \mathbb{R} \times \mathbb{R} \rightarrow [0, 1]$ is Lipschitz continuous and there exists $r > 0$ such that
14 $\phi(t, x) = 1$ for $|x| \geq r$. We assume also that ϕ is \mathbb{Z} -periodic in time.

15 **Remark 2.1.** Assumptions (A1)-(A2)-(A3)-(A5) are satisfied by several classical optimal velocity
16 functions. To be more precise, since V gives the velocity of a vehicle it is normal to assume that
17 the function should be regular, continuous and non-negative (the vehicles only go forward) which
18 explains assumption (A1). Moreover, a vehicle should go faster if he has more space in front of
19 it, which explains assumption (A2). Assumption (A3) comes from the fact that we want to avoid
20 any collisions and we added a safety distance h_0 to our model: if a vehicles has less than h_0 in
21 front of it the vehicles should not advance. Assumption (A5) is used to simplify the definition
22 on the macroscopic model that we obtain later in this paper, however it is not a very restrictive
23 assumption. We have added assumption (A4) to work with V' with a bounded support. But by
24 modifying slightly the classical optimal velocity functions, we obtain a function that satisfies all
25 the assumptions. For instance, in the case of the Greenshields based models [15](see also [5]):

$$V(h) = \begin{cases} 0 & \text{for } h \leq h_0, \\ V_{max} \left(1 - \left(\frac{h_0}{h} \right)^n \right) & \text{for } h_0 < h \leq h_{max}, \\ V_{max} \left(1 - \left(\frac{h_0}{h_{max}} \right)^n \right) & \text{for } h > h_{max}, \end{cases}$$

26 with $n \in \mathbb{N} \setminus \{0\}$. Another optimal velocity function, based on the Newell model [24](see also [8]),
27 is given by:

$$V(h) = \begin{cases} 0 & \text{for } h \leq h_0, \\ V_{max} \left(1 - \exp \left(- \left(\frac{h - h_0}{b} \right)^n \right) \right) & \text{for } h_0 < h \leq h_{max}, \\ V_{max} \left(1 - \exp \left(- \left(\frac{h_{max} - h_0}{b} \right)^n \right) \right) & \text{for } h > h_{max}, \end{cases}$$

28 with $n \in \mathbb{N} \setminus \{0\}$ and $b \in [0, +\infty)$. See Figure 3 for a schematic representation of an optimal
29 velocity function satisfying assumption (A).

30 **Remark 2.2.** We will give an example of the function ϕ . We will define ϕ on the interval $[0, 1]$
31 since it's a \mathbb{Z} -periodic function. For $t \in [0, 1]$,

$$\phi(t, x) = \begin{cases} 1 & \text{if } |x| > r \\ \frac{(\phi_0(t) - 1)}{r}x + \phi_0(t) & \text{if } x \in [-r, 0] \\ \frac{(1 - \phi_0(t))}{r}x + \phi_0(t) & \text{if } x \in (0, r]. \end{cases}$$

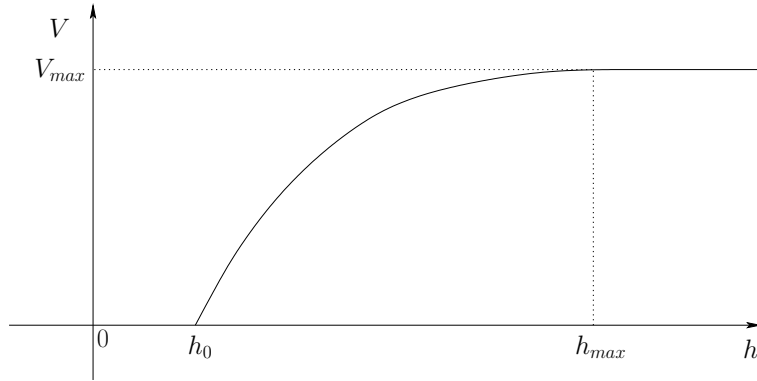


Figure 3: Schematic representation of the optimal velocity function V .

1 where ϕ_0 is defined in the following form

$$\phi_0(t) = \begin{cases} 4t & \text{if } 0 < t < \frac{1}{4}, \text{ The end of the red light time} \\ 1 & \text{if } \frac{1}{4} < t < \frac{1}{2}, \text{ Green light time} \\ -4t + 3 & \text{if } \frac{1}{2} < t < \frac{3}{4}, \text{ Orange light time} \\ 0 & \text{if } \frac{3}{4} < t < 1, \text{ Red light time.} \end{cases}$$

2 2.2 Injecting the system of ODEs into a single PDE

3 In this paper, we will study the traffic flow when the number of vehicles per unit length tends to
4 infinity by introducing the rescaled "cumulative distribution function" of vehicles, ρ^ε , defined by

$$\rho^\varepsilon(t, y) = -\varepsilon \left(\sum_{i \geq 0} H(y - \varepsilon U_i(t/\varepsilon)) + \sum_{i < 0} (-1 + H(y - \varepsilon U_i(t/\varepsilon))) \right), \quad (2.2)$$

5 with

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \quad (2.3)$$

6 Under assumption (A), the function ρ^ε satisfies in the viscosity sense (see Definition 3.1 and
7 Theorem 8.1 for the proof of this result) the following non-local equation

$$u_t^\varepsilon + M^\varepsilon \left[\frac{u^\varepsilon(t, \cdot)}{\varepsilon} \right] (x) \cdot \phi \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right) \cdot |u_x^\varepsilon| = 0 \quad \text{on } (0, +\infty) \times \mathbb{R}, \quad (2.4)$$

8 where M^ε is a non-local operator defined by

$$M^\varepsilon[U](x) = \int_{-\infty}^{+\infty} J(z) E(U(x + \varepsilon z) - U(x)) dz - \frac{3}{2} V_{max} \quad (2.5)$$

9 and with

$$E(z) = \begin{cases} 0 & \text{if } z \geq 0 \\ 1/2 & \text{if } -1 \leq z < 0 \\ 3/2 & \text{if } z < -1, \end{cases} \quad \text{and } J = V' \text{ on } \mathbb{R}. \quad (2.6)$$

10 **Remark 2.3** (Concerning the cumulative distribution function). *Let us consider $\varepsilon = 1$. First,*
11 *notice that the function ρ^1 is actually the primitive of the empirical measure of the position of the*

1 vehicles and we have for all $j \in \mathbb{Z}$, $\rho^1(t, U_j(t)) = -(j+1)$. This implies (see Section 8) that we
 2 have

$$M^1[\rho^1(t, \cdot)](U_j(t)) = -V(U_{j+1}(t) - U_j(t)).$$

3 This result helped us inject the system of ODEs into a single PDE. The function ρ^ε is simply the
 4 rescaling of the function ρ^1 .

5 In the rest of this paper, we couple equation (2.4) with the following initial condition

$$u^\varepsilon(0, x) = u_0(x) \quad \text{on } \mathbb{R}. \quad (2.7)$$

6 We also assume that the initial condition satisfies the following assumption:

7 **(A0) (Gradient bound)** The function u_0 is Lipschitz continuous and satisfies

$$-k_0 := -1/h_0 \leq (u_0)_x \leq 0 \quad \text{for all } x \in \mathbb{R}. \quad (2.8)$$

8 **Remark 2.4.** This condition ensures that initially the vehicles have a security distance between
 9 them and since we are working with a first order model, this security distance will be preserved.
 10 In fact, h_0 (from assumption (A3)) is called the safety distance. However, since we work with
 11 Eulerian coordinates, we use k_0 which is the inverse of the safety distance. We choose u_0 a regular
 12 function such that for all ε ,

$$|\rho^\varepsilon(0, x) - u_0(x)| \leq f(\varepsilon),$$

13 with $f(\varepsilon) \rightarrow 0$ as ε goes to 0. This is explain in (2.17).

14 **Remark 2.5** (Lagrangian formulation). Another way to treat this problem is to consider a La-
 15 grangian formulation, like in [12], considering the function,

$$v : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}, \quad v(t, y) = U_{\lfloor y \rfloor}(t).$$

16 This function satisfies for all $(t, y) \in [0, T] \times \mathbb{R}$

$$\begin{cases} v_t(t, y) = V(u(t, y+1) - u(t, y)) \cdot \phi(t, v(t, y)), \\ v(0, y) = v_0(y). \end{cases} \quad (2.9)$$

17 The difficulty with this formulation is that the function ϕ is evaluated at $v(t, y)$ and not at a
 18 physical point of the road. The notion of junction in this case is not well defined and this is why
 19 we use the formulation (2.4) (where the perturbation function is evaluated at a point of the road)
 20 instead of (2.9). This will allow us to use the results of Imbert and Monneau [18] concerning
 21 quasi-convex Hamiltonians with a junction condition.

22 2.3 Convergence result

23 We recall that $k_0 = 1/h_0$ and we define $\bar{H} : \mathbb{R} \rightarrow \mathbb{R}$, by

$$\bar{H}(p) = \begin{cases} -p - k_0 & \text{for } p < -k_0, \\ -V\left(\frac{-1}{p}\right) |p| & \text{for } -k_0 \leq p \leq 0, \\ p & \text{for } p > 0. \end{cases} \quad (2.10)$$

24 Note that such a \bar{H} is continuous, coercive $\left(\lim_{|p| \rightarrow +\infty} \bar{H}(p) = +\infty\right)$ and because of (A5), there
 25 exists a unique point $p_0 \in [-k_0, 0]$ such that

$$\begin{cases} \bar{H} \text{ is decreasing on } (-\infty, p_0), \\ \bar{H} \text{ is increasing on } (p_0, +\infty). \end{cases} \quad (2.11)$$

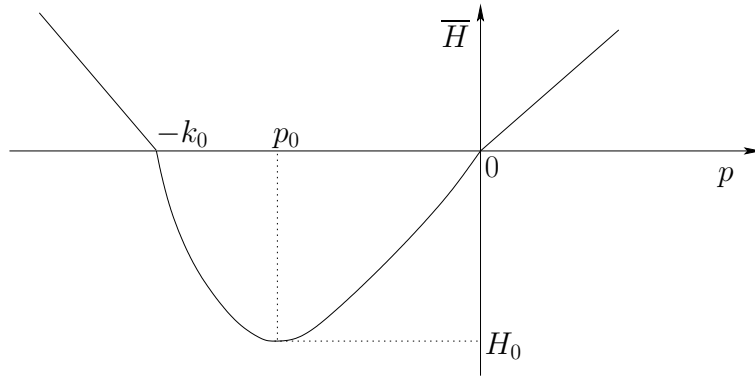


Figure 4: Schematic representation of \bar{H} .

1 We denote by

$$H_0 = \min_{p \in \mathbb{R}} \bar{H}(p) = \bar{H}(p_0) \quad (2.12)$$

2 and we refer to Figure 4 for a schematic representation of \bar{H} .

3 The main purpose of this article is to prove that the viscosity solution of (2.4)-(2.7) converges
 4 uniformly on compact subsets of $(0, +\infty) \times \mathbb{R}$ as ε goes to 0 to the unique viscosity solution of the
 5 following problem

$$\begin{cases} u_t^0 + \bar{H}(u_x^0) = 0 & \text{for } (t, x) \in (0, +\infty) \times (-\infty, 0) \\ u_t^0 + \bar{H}(u_x^0) = 0 & \text{for } (t, x) \in (0, +\infty) \times (0, +\infty) \\ u_t^0 + F_{\bar{A}}(u_x^0(t, 0^-), u_x^0(t, 0^+)) = 0 & \text{for } (t, x) \in (0, +\infty) \times \{0\} \\ u^0(0, x) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \quad (2.13)$$

6 where \bar{A} has to be determined and $F_{\bar{A}}$ is defined by

$$F_{\bar{A}}(p_-, p_+) = \max(\bar{A}, \bar{H}^+(p_-), \bar{H}^-(p_+)), \quad (2.14)$$

7 with

$$\bar{H}^-(p) = \begin{cases} \bar{H}(p) & \text{if } p \leq p_0, \\ \bar{H}(p_0) & \text{if } p \geq p_0, \end{cases} \quad \text{and} \quad \bar{H}^+(p) = \begin{cases} \bar{H}(p_0) & \text{if } p \leq p_0, \\ \bar{H}(p) & \text{if } p \geq p_0. \end{cases} \quad (2.15)$$

8 The following theorems are the main results of this paper, and their proof are postponed. The
 9 proofs of Theorem 2.6 and Theorem 2.10 are done in Section 5 and the proof of Theorem 2.7 is
 10 done in Section 8.

11 **Theorem 2.6** (Junction condition by homogenisation). *Assume (A) and (A0). For $\varepsilon > 0$, let*
 12 *u^ε be the solution of (2.4)-(2.7). Then there exists $\bar{A} \in [H_0, 0]$ such that u^ε converges locally*
 13 *uniformly to the unique viscosity solution u^0 of (2.13) (in the sense of Definition 3.4).*

14 **Theorem 2.7** (Junction condition by homogenisation: application to traffic flow). *Assume (A)*
 15 *and that at the initial time, we have, for all $i \in \mathbb{Z}$,*

$$U_i(0) \leq U_{i+1}(0) - h_0. \quad (2.16)$$

16 We also assume that there exists a constant $R > 0$ such that, for all $i \in \mathbb{Z}$, if $|U_i(0)| \geq R$

$$U_{i+1}(0) - U_i(0) = h, \quad (2.17)$$

17 with $h \geq h_0$. We define the function u_0 (satisfying (A0)) by $u_0(x) = -x/h$ for all $x \in \mathbb{R}$. Then
 18 there exists $\bar{A} \in [H_0, 0]$ such that the function ρ^ε defined by (2.2) converges towards the unique
 19 solution u^0 of (2.13).

1 **Remark 2.8.** Condition (2.17) means that the initial condition is well-prepared.

2 **Remark 2.9.** We notice that in the case of traffic flow, (2.13) is equivalent (deriving in space) to
 3 a LWR model (see [22, 25]) with a flux limiting condition at the origin. In fact, the fundamental
 4 diagram of the model is $pV(1/p)$ and u_x^0 corresponds to the density of vehicles.

5 The following theorem ensures that when we use (2.13) we only evaluate the function \overline{H} in
 6 $[-k_0, 0]$.

7 **Theorem 2.10.** Assume (A0)-(A). Let u^0 be the unique solution of (2.13), then we have for all
 8 $(t, x) \in [0, T] \times \mathbb{R}$,

$$-k_0 \leq u_x^0 \leq 0,$$

9 with k_0 defined in (A0).

10 **Remark 2.11** (Extension of the effective Hamiltonian). This theorem implies in particular that
 11 in the case of traffic flow, the effective Hamiltonian only needs to be computed for $p \in [-k_0, 0]$.
 12 However, for the construction of the correctors it is necessary to work with a coercive Hamiltonian
 13 in \mathbb{R} that is why we extend the function \overline{H} in (2.10).

14 **Remark 2.12** (Particular form of the effective Hamiltonian). The particular form of the effective
 15 Hamiltonian (2.10) comes from the classical Ansatz:

$$u^\varepsilon(t, x) = u^0(t, x) + \varepsilon v\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right). \quad (2.18)$$

16 In fact, in Proposition 2.13, we prove that $v \equiv 0$ is a suitable corrector which in turn will imply
 17 that for a fixed $p \in [-k_0, 0]$, the effective Hamiltonian is given by $-V(-1/p)|p| = \overline{H}(p)$.

18 2.4 Effective Hamiltonian and effective flux-limiter

19 We define the non-local operator M_p by

$$M_p[U](x) = \int_{-\infty}^{+\infty} J(z)E(U(x+z) - U(x) + p \cdot z) dz - \frac{3}{2}V_{max}. \quad (2.19)$$

20 We then have the following result

21 **Proposition 2.13** (Homogenization left and right of the perturbation). Assume (A). Then for
 22 every $p \in [-k_0, 0]$, there exists a unique $\lambda \in \mathbb{R}$, such that there exists a bounded solution v of

$$\begin{cases} M_p[v](x) \cdot |v_x + p| = \lambda, & x \in \mathbb{R}, \\ v \text{ is } \mathbb{Z}\text{-periodic}, \end{cases} \quad (2.20)$$

23 with M_p defined in (2.19). Moreover, for $p \in [-k_0, 0]$, we have $\lambda = \overline{H}(p)$.

Proof. Let us prove that $v = 0$ is an obvious solution of (2.20) with $\lambda = \overline{H}(p)$, for $p \in [-k_0, 0]$.
 First, let us notice that if $p = 0$ the result is obvious since by definition of \overline{H} , we have $\overline{H}(0) = 0$
 and $M_0[0](x)$ is finite (for all $x \in \mathbb{R}$) by definition (see (2.23)). Let us now consider $p > 0$, we have

for all $x \in \mathbb{R}$,

$$\begin{aligned}
M_p[0](x) &= \int_{-\infty}^{+\infty} J(z)E(pz)dz - \frac{3}{2}V_{max} \\
&= \int_0^{+\infty} J(z)E(pz)dz - \frac{3}{2}V_{max} \\
&= \int_0^{-1/p} \frac{1}{2}J(z)dz + \int_{-1/p}^{+\infty} \frac{3}{2}J(z)dz - \frac{3}{2}V_{max} \\
&= \frac{1}{2} \left(V \left(\frac{-1}{p} \right) - V(0) \right) + \frac{3}{2} \left(\left(\lim_{h \rightarrow +\infty} V(h) \right) - V \left(\frac{-1}{p} \right) \right) - \frac{3}{2}V_{max} \\
&= -V \left(\frac{-1}{p} \right),
\end{aligned}$$

1 where we have used assumption (A3) for the second line, the definition of E and J (see (2.6))
2 for the third and fourth line. Finally, using this result and the definition of \bar{H} , we notice that
3 $\bar{H}(p) = M_p[0](x)|p| = \lambda$. The uniqueness of λ is classical (see for instance [9, Proof of Proposition
4 4.6]) so we skip it.

5

□

6 To construct the effective flux-limiter \bar{A} , we consider the following cell problem: find $\lambda \in \mathbb{R}$
7 such that there exists a solution w of the following Hamilton-Jacobi equation

$$\begin{cases} w_t + M[w(t, \cdot)](x) \cdot \phi(t, x) \cdot |w_x| = \lambda & \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}. \\ w \text{ is } 1\text{-periodic in time} \end{cases} \quad (2.21)$$

8 More precisely, we have the following result, whose proof is postponed until Section 6.

9 **Theorem 2.14** (Effective flux limiter). *Assume (A). We define the following set of functions*

$$\mathcal{S} = \{w \text{ s.t. } \exists \text{ a Lipschitz continuous function } m \text{ and } C \geq 0 \text{ s.t. } \|w - m\|_{L^\infty(\mathbb{R})} \leq C\}.$$

10 Then we have

$$\bar{A} = \inf \{ \lambda \in \mathbb{R} : \exists w \in \mathcal{S} \text{ solution of (2.21)} \}.$$

11 **Remark 2.15.** *This theorem allows us to characterize and give uniqueness to the flux limiter that*
12 *we present in Section 4 whose construction is presented in Section 6.*

13 2.5 Qualitative properties of the effective flux limiter

14 We have the following qualitative properties on the effective flux limiter \bar{A} , the proof of this result
15 is postponed until Section 7.

16 **Proposition 2.16** (Monotonicity of the flux-limiter). *Assume (A) and let $\phi_1, \phi_2 : \mathbb{R}^+ \times \mathbb{R} \rightarrow [0, 1]$*
17 *be two functions satisfying (A6). Let \bar{A}_1 and \bar{A}_2 be their respective flux limiters given by Theorem*
18 *2.6. If, for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, we have*

$$\phi_1(t, x) \leq \phi_2(t, x),$$

19 then

$$\bar{A}_1 \geq \bar{A}_2.$$

2.6 Notations

We denote by M the non-local operator M^ε (defined in (2.5)) in the case $\varepsilon = 1$. To each operator M , we associate the operator \tilde{M} which is defined in the same way except that the function E is replaced by the function \tilde{E} , defined by

$$\tilde{E}(z) = \begin{cases} 0 & \text{if } z > 0 \\ 1/2 & \text{if } -1 < z \leq 0 \\ 3/2 & \text{if } z \leq -1. \end{cases} \quad (2.22)$$

Remark 2.17. Using the fact that E and V are bounded, we get that for every function U and every $x \in \mathbb{R}$, we have

$$-M_0 = -\frac{3}{2}V_{max} \leq M[U](x) \leq 0. \quad (2.23)$$

We also use the following notations for the upper and lower semi-continuous envelopes of a locally bounded function u :

$$u^*(t, x) = \limsup_{s \rightarrow t, y \rightarrow x} u(s, y) \quad \text{and} \quad u_*(t, x) = \liminf_{s \rightarrow t, y \rightarrow x} u(s, y).$$

2.7 Organization of the article

Section 3 contains the definition of the viscosity solutions for the problems we consider in the entire article and it also contains some results for those problems. In Section 4 we present some results on the correctors at the junction (Theorem 4.1) that will be used in Section 5 to prove Theorem 2.6. Section 6 contains the proof of Theorem 4.1. In Section 7 we give the proof of the qualitative properties of the flux-limiter. Finally, Section 8 details the link between the system of ODEs (2.1) and the PDE (2.4) (with $\varepsilon = 1$).

3 Viscosity solutions for (2.4) and (2.13)

3.1 Definitions

In order to give a general definition for all the non-local problems we consider, we will give the definition for the following equation, with $p \in \mathbb{R}$, for all $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$\begin{cases} u_t + \psi(x) \cdot M_p[u(t, \cdot)](x) \cdot \phi(t, x) \cdot |p + u_x| + (1 - \psi(x)) \cdot \bar{H}(u_x) = 0 \\ u(0, x) = u_0(x), \end{cases} \quad (3.1)$$

with $\psi : \mathbb{R} \rightarrow [0, 1]$ a Lipschitz continuous function.

Definition 3.1 (Viscosity solutions for (3.1)). *Let $T > 0$. An upper semi-continuous function (resp. lower semi-continuous) $u : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a viscosity sub-solution (resp. super-solution) of (3.1) on $[0, T] \times \mathbb{R}$, if $u(0, x) \leq u_0(x)$ (resp. $u(0, x) \geq u_0(x)$) and for all $(t, x) \in (0, T) \times \mathbb{R}$ and for all $\varphi \in C^2([0, T] \times \mathbb{R})$ such that $u - \varphi$ reaches a maximum (resp. a minimum) at the point (t, x) , we have*

$$\varphi_t(t, x) + \psi(x) \cdot \phi(t, x) \cdot M_p[u(t, \cdot)](x) \cdot |p + \varphi_x(t, x)| + (1 - \psi(x))\bar{H}(\varphi_x(t, x)) \leq 0$$

$$\text{(resp. } \varphi_t(t, x) + \psi(x) \cdot \phi(t, x) \cdot \tilde{M}_p[u(t, \cdot)](x) \cdot |p + \varphi_x(t, x)| + (1 - \psi(x))\bar{H}(\varphi_x(t, x)) \geq 0 \text{)}.$$

We say that a function u is a viscosity solution of (3.1) if u^* and u_* are respectively a sub-solution and a super-solution of (3.1).

Remark 3.2. We use this definition in order to have a stability result for the non-local term. We refer to [6, 26] for such kind of definition and to [10, Proposition 4.2] for the corresponding stability result.

1 **Definition 3.3** (Class of test functions for (2.13)). We denote by $J_\infty := (0, +\infty) \times \mathbb{R}$,
2 $J_\infty^+ := (0, +\infty) \times [0, +\infty)$ and $J_\infty^- := (0, \infty) \times (-\infty, 0]$. We define a class of test functions on J_∞
3 by

$$\mathcal{C}^1(J_\infty) = \{\varphi \in C(J_\infty), \text{ the restriction of } \varphi \text{ to } J_\infty^+ \text{ and to } J_\infty^- \text{ is } C^1\}.$$

4

5 **Definition 3.4** (Viscosity solutions for (2.13)). Let \bar{H} be given by (2.10) and $\bar{A} \in \mathbb{R}$. An upper
6 semi-continuous (resp. lower semi-continuous) function $u : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a viscosity sub-
7 solution (resp. super-solution) of (2.13) if $u(0, x) \leq u_0(x)$ (resp. $u(0, x) \geq u_0(x)$) and for all
8 $(t, x) \in J_\infty$ and for all $\varphi \in \mathcal{C}^1(J_\infty)$ such that

$$u \leq \varphi \text{ (resp. } u \geq \varphi) \text{ in a neighbourhood of } (t, x) \in J_\infty \text{ and } u(t, x) = \varphi(t, x),$$

9 we have

$$\begin{aligned} \varphi_t(t, x) + \bar{H}(\varphi_x(t, x)) &\leq 0 \quad (\text{resp. } \geq 0) && \text{if } x \neq 0, \\ \varphi_t(t, x) + F_{\bar{A}}(\varphi_x(t, 0^-), \varphi_x(t, 0^+)) &\leq 0 \quad (\text{resp. } \geq 0) && \text{if } x = 0. \end{aligned}$$

10 We say that a function u is a viscosity solution of (2.13) if u^* and u_* are respectively a sub-solution
11 and a super-solution of (2.13). We refer to this solution as an \bar{A} -flux limited solution.

12 3.2 Results for viscosity solutions of (3.1)

13 **Proposition 3.5** (Comparison principle for (3.1)). Assume (A0) and (A). Let u be a sub-solution
14 of (3.1) and v be a super-solution of (3.1). Let us also assume that there exists a constant $K > 0$
15 such that for all $(t, x) \in [0, T] \times \mathbb{R}$,

$$u(t, x) \leq u_0(x) + Kt \quad \text{and} \quad -v(t, x) \leq -u_0(x) + Kt. \quad (3.2)$$

16 Then we have $u(t, x) \leq v(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}$.

17 *Proof.* The only difficulty in proving the comparison principle comes from the non-local term, but
18 in our case the proof is similar to the proof of [10, Theorem 4.4] and we skip it.

19 □

20 We now give a comparison principle on bounded sets, to do this, we define for a given point
21 $(t_0, x_0) \in (0, T) \times \mathbb{R}$ and for $\bar{r}, \bar{R} > 0$, the set

$$\mathcal{Q}_{\bar{r}, \bar{R}}(t_0, x_0) = (t_0 - \bar{r}, t_0 + \bar{r}) \times (x_0 - \bar{R}, x_0 + \bar{R}).$$

22 **Theorem 3.6** (Comparison principle on bounded sets for (3.1)). Assume (A). Let u be a sub-
23 solution of (3.1) and let v be a super-solution of (3.1) on the open set $\mathcal{Q}_{\bar{r}, \bar{R}} \subset (0, T) \times \mathbb{R}$. We
24 assume that u (resp. v) is upper semi-continuous (resp. lower semi-continuous) on $\overline{\mathcal{Q}_{\bar{r}, \bar{R}}}$. Also
25 assume that

$$u \leq v \quad \text{outside } \mathcal{Q}_{\bar{r}, \bar{R}},$$

26 then

$$u \leq v \quad \text{on } \mathcal{Q}_{\bar{r}, \bar{R}}.$$

27

28 *Proof.* The proof of this theorem is similar to the one of Proposition 3.5, so we skip it.

29 □

1 **Lemma 3.7** (Existence of barriers for (3.1)). *Assume (A0) and (A). There exists a constant*
 2 $K_1 > 0$ such that

$$u^+(t, x) = K_1 t + u_0(x) \quad \text{and} \quad u^-(t, x) = u_0(x),$$

3 *are respectively super and sub-solutions of (3.1).*

4 *Proof.* We define $K_1 = M_0 \cdot (|p| + k_0) + |H_0|$. Let us prove that u^+ is a super-solution of (3.1).
 5 Using assumption (A0) and the form of the non-local operator and of \bar{H} , we have

$$\begin{aligned} \phi(t, x)\psi(x)M_p[u_0](x) \cdot |p + (u_0)_x| + (1 - \psi(x)) \cdot \bar{H}((u_0)_x) &\geq -M_0 \cdot |p + (u_0)_x| + H_0 \\ &\geq -M_0(|p| + k_0) - |H_0| = -K_1, \end{aligned}$$

6 where we used (2.23) and (2.12). The proof for u^- is simpler, it uses (2.23) and (2.12),

$$\phi(t, x)\psi(x)M_p[u_0](x) \cdot |p + (u_0)_x| + (1 - \psi(x)) \cdot \bar{H}((u_0)_x) \leq 0.$$

7 □

8 Applying Perron's method (see [19, Proof of Theorem 6], [3] or [17] to see how to apply
 9 Perron's method for problems with non-local terms), joint to the comparison principle, we obtain
 10 the following result.

11 **Theorem 3.8** (Existence and uniqueness of viscosity solutions for (3.1)). *Assume (A0) and (A).*
 12 *Then, there exists a unique continuous solution u of (3.1) which satisfies (for some constant K_1)*

$$u_0(x) \leq u(t, x) \leq u_0(x) + K_1 t$$

13 3.3 Results for viscosity solutions of (2.13)

14 Now we recall an equivalent definition (see [18, Theorem 2.5]) for sub and super solutions at the
 15 junction. We will also consider the following problem,

$$u_t + \bar{H}(u_x) = 0 \quad \text{for } t \in (0, T) \text{ and } x \in \mathbb{R} \setminus \{0\}. \quad (3.3)$$

16 **Theorem 3.9** (Equivalent definition for sub/super-solutions). *Let \bar{H} given by (2.10) and consider*
 17 *$A \in [H_0, +\infty)$ with H_0 defined in (2.12). Given arbitrary solutions $p_{\pm}^A \in \mathbb{R}$ of*

$$\bar{H}(p_+^A) = \bar{H}^+(p_+^A) = A = \bar{H}^-(p_-^A) = \bar{H}(p_-^A), \quad (3.4)$$

18 *let us fix any time independent test function $\phi^0(x)$ satisfying*

$$\phi_x^0(0^{\pm}) = p_{\pm}^A.$$

19 *Given a function $u : (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$, the following properties hold true.*

20 *i) If u is an upper semi-continuous sub-solution of (3.3), then u is a H_0 -flux limited sub-*
 21 *solution.*

22 *ii) Given $A > H_0$ and $t_0 \in (0, T)$, if u is an upper semi-continuous sub-solution of (3.3) and if*
 23 *for any test function φ touching u from above at $(t_0, 0)$ with*

$$\varphi(t, x) = \psi(t) + \phi^0(x), \quad (3.5)$$

24 *for some $\psi \in C^1(0, +\infty)$, we have*

$$\varphi_t + F_A(\varphi_x(t_0, 0^-), \varphi_x(t_0, 0^+)) \leq 0 \quad \text{at } (t_0, 0),$$

25 *then u is an A -flux limited sub-solution at $(t_0, 0)$.*

26 *iii) Given $t_0 \in (0, T)$, if u is a lower semi-continuous super-solution of (3.3) and if for any*
 27 *test function φ satisfying (3.5) touching u from above at $(t_0, 0)$ we have*

$$\varphi_t + F_A(\varphi_x(t_0, 0^-), \varphi_x(t_0, 0^+)) \geq 0 \quad \text{at } (t_0, 0),$$

28 *then u is an A -flux limited super-solution at $(t_0, 0)$.*

29 *Proof.* The proof of Theorem 3.9 can be founded in [18, Theorem 2.5].

30 □

1 3.4 Control of the oscillations for (2.4)-(2.7)

2 **Theorem 3.10** (Control of the oscillations). *Let $T > 0$. Assume (A0)-(A) and let u be a solution*
 3 *of (2.4)-(2.7), with $\varepsilon = 1$. Then for all $x, y \in \mathbb{R}$, $x \geq y$ and for all $t \in [0, T]$, we have*

$$-k_0(x - y) - 1 \leq u(t, x) - u(t, y) \leq 0, \quad (3.6)$$

4 *with k_0 defined in (2.8).*

5 *Proof.* In this proof we used the barriers given by Lemma 3.7 (with $p = 0$ and $\psi \equiv 1$), which
 6 means that the solution u of (2.4)-(2.7) with $\varepsilon = 1$ satisfies for all $(t, x) \in [0, +\infty) \times \mathbb{R}$,

$$0 \leq u(t, x) - u_0(x) \leq M_0 k_0 t. \quad (3.7)$$

7 In the rest of the proof we will use the following notation:

$$\Omega = \{(t, x, y) \in [0, T) \times \mathbb{R}^2 \text{ s.t. } x \geq y\}.$$

8 **Proof of the upper inequality for the control of the space oscillations.** We introduce,

$$M = \sup_{(t, x, y) \in \Omega} \{u(t, x) - u(t, y)\}.$$

9 We want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

10 **Step 1: the test function.** For $\eta, \alpha > 0$, small parameters, we define

$$\varphi(t, x, y) = u(t, x) - u(t, y) - \frac{\eta}{T-t} - \alpha x^2 - \alpha y^2.$$

11 Using (3.7), we have that

$$\varphi(t, x, y) \leq u_0(x) - u_0(y) + 2M_0 k_0 T - \alpha(x^2 + y^2) \leq -\alpha(x^2 + y^2) + 2M_0 k_0 T,$$

12 where we used assumption (A0) for the second inequality. Therefore we have

$$\lim_{|x|, |y| \rightarrow +\infty} \varphi(t, x, y) = -\infty.$$

13 Since φ is upper-semi continuous, it reaches a maximum at a point that we denote by $(\bar{t}, \bar{x}, \bar{y}) \in \Omega$.

14 Classically we have for η and α small enough,

$$\begin{cases} 0 < \frac{M}{2} \leq \varphi(\bar{t}, \bar{x}, \bar{y}), \\ \alpha|\bar{x}|, \alpha|\bar{y}| \rightarrow 0 \text{ as } \alpha \rightarrow 0. \end{cases}$$

15 **Step 2: $\bar{t} > 0$ and $\bar{x} > \bar{y}$.** By contradiction, assume first that $\bar{t} = 0$. Then we have

$$\frac{\eta}{T} < u_0(\bar{x}) - u_0(\bar{y}) \leq 0,$$

16 where we used that u_0 is non-increasing, and we get a contradiction. The fact that $\bar{x} > \bar{y}$, comes
 17 directly from the fact that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$.

18 **Step 3: utilisation of the equation.** By doing a duplication of the time variable and
 19 passing to the limit in this duplication parameter, we get that

$$\frac{\eta}{(T - \bar{t})^2} \leq \tilde{M}[u(\bar{t}, \cdot)](\bar{y}) \cdot |2\alpha\bar{y}| \cdot \phi(\bar{t}, \bar{y}) - M[u(\bar{t}, \cdot)](\bar{x}) \cdot \phi(\bar{t}, \bar{x}) \cdot |2\alpha\bar{x}| \leq 2M_0 \cdot \alpha(|\bar{x}| + |\bar{y}|),$$

20 passing to the limit as α goes to 0, we obtain a contradiction.

1 **Proof of the lower inequality for the control of the space oscillations** Let us introduce,

$$M = \sup_{(t,x,y) \in \Omega} \{u(t,y) - u(t,x) - 1 - k_0(x-y)\}.$$

2 We want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

3 **Step 1: the test function.** For $\alpha, \eta > 0$, small parameters we consider the function

$$\varphi(t,x,y) = u(t,y) - u(t,x) - 1 - k_0(x-y) - \alpha(x^2 + y^2) - \frac{\eta}{T-t}.$$

4 We have that

$$\begin{aligned} \varphi(t,x,y) &\leq u_0(y) - u_0(x) - \alpha(x^2 + y^2) + 2M_0k_0T - k_0(x-y) - 1 \\ &\leq -\alpha(x^2 + y^2) + 2M_0k_0T. \end{aligned}$$

5 Therefore, we have

$$\lim_{|x|,|y| \rightarrow +\infty} \varphi(t,x,y) = -\infty.$$

6 Using the fact that φ is upper-semi continuous we deduce that φ reaches a maximum at a finite
7 point that we denote $(\bar{t}, \bar{x}, \bar{y}) \in \Omega$. Classically we have for η and α small enough,

$$\begin{cases} 0 < \frac{M}{2} \leq \varphi(\bar{t}, \bar{x}, \bar{y}), \\ \alpha|\bar{x}|, \alpha|\bar{y}| \rightarrow 0 \text{ as } \alpha \rightarrow 0. \end{cases}$$

8 **Step 2: $\bar{t} > 0$ and $\bar{x} > \bar{y}$.** By contradiction, assume that $\bar{t} = 0$. Using the fact that
9 $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$ and (A0), we have

$$\frac{\eta}{T} < u(0, \bar{y}) - u(0, \bar{x}) - k_0(\bar{x} - \bar{y}) - 1 \leq -1,$$

10 which is a contradiction. Hence $\bar{t} > 0$. Using that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$, we also deduce that $\bar{x} > \bar{y}$.

11 **Step 3: Utilisation of the equation** By duplicating the time variable and passing to
12 the limit we have that there exists two real numbers a, b , such that $(a, -k_0 + 2\alpha\bar{y}) \in \overline{\mathcal{D}}^+ u(\bar{t}, \bar{y})$,
13 $(b, -k_0 + 2\alpha\bar{x}) \in \overline{\mathcal{D}}^- u(\bar{t}, \bar{x})$ and

$$a - b = \frac{\eta}{(T - \bar{t})^2}. \quad (3.8)$$

14 Using that u is a sub-solution of (2.4)-(2.7) (with $\varepsilon = 1$), we get

$$a + M[u(\bar{t}, \cdot)](\bar{y}) \cdot \phi(\bar{t}, \bar{y}) \cdot | -k_0 + 2\alpha\bar{y}| \leq 0. \quad (3.9)$$

15 We claim that

$$M[u(\bar{t}, \cdot)](\bar{y}) = \int_{\mathbb{R}} J(z) E(u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y})) dz - \frac{3}{2} V_{max} = 0.$$

16 Indeed, let $z \in (h_0, h_{max}]$. If $\bar{y} + z \geq \bar{x}$, using that u is non-increasing in space, we get

$$u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) \leq u(\bar{t}, \bar{x}) - u(\bar{t}, \bar{y}) \leq -k_0(\bar{x} - \bar{y}) - 1 < -1.$$

17 If $\bar{y} + z < \bar{x}$, using the fact that $\varphi(\bar{t}, \bar{x}, \bar{y} + z) \leq \varphi(\bar{t}, \bar{x}, \bar{y})$, for α small enough, we obtain

$$u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) \leq -k_0z + \alpha(2z\bar{y} + z^2) \leq -k_0z + \alpha(2h_{max}\bar{y} + h_{max}^2) < -1.$$

1 This implies that we have for all $z \in (h_0, h_{max}]$,

$$E(u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y})) = \frac{3}{2}.$$

2 Injecting this in the non-local term, we deduce the claim.

3 Finally, the fact that $u_t \geq 0$ implies that $a, b \geq 0$. Therefore, inequality (3.9) implies

$$a = 0.$$

4 Finally, using (3.8), we obtain

$$\frac{\eta}{T^2} \leq 0,$$

5 which is a contradiction. This ends the proof.

6

□

7 4 Correctors for the junction

8 The key ingredient to prove the convergence result is to construct correctors for the junction. The
 9 main result of this section is the existence of appropriate correctors. The proof of this theorem is
 10 presented in Section 6. Given $\bar{A} \in \mathbb{R}$, $\bar{A} \geq H_0$, we introduce two real numbers $\bar{p}_+, \bar{p}_- \in \mathbb{R}$, defined
 11 by $\bar{p}_\pm = p_\pm^{\bar{A}}$ (see (3.4)). Due to the form of \bar{H} (see (2.10)) these two real numbers exist and are
 12 unique.

13 **Theorem 4.1** (Existence of a global corrector for the junction). *Assume (A).*

14 *i) (General properties) There exists a constant $\bar{A} \in [H_0, 0]$ such that there exists a solution w*
 15 *of (2.21) with $\lambda = \bar{A}$ and such that there exists a constant C and a globally Lipschitz continuous*
 16 *function m such that for all $x \in \mathbb{R}$,*

$$|w(t, x) - m(x)| \leq C. \quad (4.1)$$

17 *ii) (Bound from below at infinity) If $\bar{A} > H_0$, then there exists a γ_0 such that for every*
 18 *$\gamma \in (0, \gamma_0)$, we have*

$$\begin{cases} w(t, x+h) - w(t, x) \geq (\bar{p}_+ - \gamma)h - C & \text{for } x \geq r \text{ and } h \geq 0, \\ w(t, x-h) - w(t, x) \geq (-\bar{p}_- - \gamma)h - C & \text{for } x \leq -r \text{ and } h \geq 0. \end{cases} \quad (4.2)$$

19 *iii) (Rescaling w) For $\varepsilon > 0$, we set*

$$w^\varepsilon(t, x) = \varepsilon w\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right),$$

20 *then (along a subsequence $\varepsilon_n \rightarrow 0$) we have that w^ε converges locally uniformly towards a function*
 21 *$W = W(x)$ which satisfies*

$$\begin{cases} |W(x) - W(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \\ \bar{H}(W_x) = \bar{A} & \text{for all } x \in \mathbb{R} \setminus \{0\}, \end{cases} \quad (4.3)$$

22 *In particular, we have (with $W(0) = 0$)*

$$W(x) = \bar{p}_+ x 1_{\{x > 0\}} + \bar{p}_- x 1_{\{x < 0\}}. \quad (4.4)$$

5 Proof of convergence

This section contains the proof of the main homogenization result (Theorem 2.6). This proof relies on the existences of correctors (Proposition 2.13 and Theorem 4.1).

We begin with two useful lemmas for the proof of Theorem 2.6. The first result is a direct consequence of Perron's method and Lemma 3.7.

Lemma 5.1 (Barriers uniform in ε). *Assume (A0) and (A). There exists a constant $C > 0$ (depending only on M_0 and k_0) such that for all $t > 0$ and $x \in \mathbb{R}$,*

$$|u^\varepsilon(t, x) - u_0(x)| \leq Ct.$$

The following lemma is a direct result of Theorem 3.10.

Lemma 5.2 (Uniform gradient bound). *Assume (A0) and (A). Then the solution u^ε of (2.4)-(2.7) satisfies for all $t > 0$, for all $x, y \in \mathbb{R}$, $x \geq y$,*

$$-k_0(x - y) - \varepsilon \leq u^\varepsilon(t, x) - u^\varepsilon(t, y) \leq 0. \quad (5.1)$$

Before passing to the proof of Theorem 2.6, let us show how it allows us to prove Theorem 2.10.

Proof of Theorem 2.10. We want to prove that for all $t \in [0, +\infty)$ and for all $x, y \in \mathbb{R}$, $x \geq y$,

$$-k_0(x - y) \leq u^0(t, x) - u^0(t, y) \leq 0. \quad (5.2)$$

Using Lemma 5.2, we have that the solution u^ε of (2.4)-(2.7), satisfies for all $(t, x, y) \in [0, +\infty) \times \mathbb{R} \times \mathbb{R}$, with $x \geq y$,

$$-k_0(x - y) - \varepsilon \leq u^\varepsilon(t, x) - u^\varepsilon(t, y) \leq 0.$$

Now using Theorem 2.6, passing to the limit as $\varepsilon \rightarrow 0$, we obtain the result. □

We now turn to the proof of Theorem 2.6.

Proof of Theorem 2.6. We introduce

$$\bar{u}(t, x) = \limsup_{\varepsilon \rightarrow 0}^* u^\varepsilon \quad \text{and} \quad \underline{u}(t, x) = \liminf_{\varepsilon \rightarrow 0}^* u^\varepsilon. \quad (5.3)$$

Thanks to Lemma 5.1, we know that these functions are well defined. We want to prove that \bar{u} and \underline{u} are respectively a sub-solution and a super-solution of (2.13). In this case, the comparison principle [18, Theorem 1.4] will imply that $\bar{u} \leq \underline{u}$. But, by construction, we have $\underline{u} \leq \bar{u}$, hence we will get $\underline{u} = \bar{u} = u^0$, the unique solution of (2.13).

Let us prove that \bar{u} is a sub-solution of (2.13) (the proof for \underline{u} is similar and we skip it). We argue by contradiction and assume that there exists a test function $\varphi \in \mathcal{C}^1(J_\infty)$ (in the sense of Definition 3.3), and a point $(\bar{t}, \bar{x}) \in (0, +\infty) \times \mathbb{R}$ such that

$$\begin{cases} \bar{u}(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x}) & \text{on } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}) & \text{with } \bar{r} > 0 \\ \bar{u} \leq \varphi & \text{outside } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}) & \text{with } \eta > 0 \\ \varphi_t(\bar{t}, \bar{x}) + \bar{H}(\bar{x}, \varphi_x(\bar{t}, \bar{x})) = \theta & \text{with } \theta > 0, \end{cases} \quad (5.4)$$

1 where

$$\bar{H}(\bar{x}, \varphi_x(\bar{t}, \bar{x})) := \begin{cases} \bar{H}(\varphi_x(\bar{t}, \bar{x})) & \text{if } \bar{x} \neq 0, \\ \bar{F}_{\bar{A}}(\varphi_x(\bar{t}, 0^-), \varphi_x(\bar{t}, 0^+)) & \text{if } \bar{x} = 0. \end{cases}$$

2 Given Lemma 5.2 and (5.3), we can assume (up to changing φ at infinity) that for ε small enough,
3 we have

$$u^\varepsilon \leq \varphi - \eta \quad \text{outside } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

4 Using the previous lemmas we get that the function \bar{u} satisfies for all $t > 0$ and $x, y \in \mathbb{R}$, $x \geq y$,

$$\begin{aligned} |\bar{u}(t, x) - u_0(x)| &\leq Ct, \\ -k_0(x - y) &\leq \bar{u}(t, x) - \bar{u}(t, y) \leq 0. \end{aligned} \quad (5.5)$$

5 **First case:** $\bar{x} \neq 0$. We only consider $\bar{x} > 0$, since the other case ($\bar{x} < 0$) is treated in the same
6 way. We define $p = \varphi_x(\bar{t}, \bar{x})$ that according to (5.5) satisfies

$$-k_0 \leq p \leq 0.$$

7 We choose \bar{r} small enough so that $\bar{x} - 2\bar{r} > 0$. Let us prove that the test function φ satisfies
8 in the viscosity sense, the inequality

$$\varphi_t + \tilde{M}^\varepsilon \left[\frac{\varphi}{\varepsilon}(t, \cdot) \right] (x) \cdot \phi \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right) \cdot |\varphi_x| \geq \frac{\theta}{2} \quad \text{for } (t, x) \in \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}). \quad (5.6)$$

9 Let us notice that for ε small enough we have

$$\phi \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right) = 1 \quad \text{for all } (t, x) \in \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

10 For all $(t, x) \in \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$, we have for \bar{r} small enough

$$\begin{aligned} \varphi_t(t, x) + \tilde{M}^\varepsilon \left[\frac{\varphi}{\varepsilon}(t, \cdot) \right] (x) \cdot |\varphi_x| &= \varphi_t(\bar{t}, \bar{x}) + o_{\bar{r}}(1) + \tilde{M}^\varepsilon \left[\frac{\varphi}{\varepsilon}(t, \cdot) \right] (x) \cdot |\varphi_x| \\ &= \theta + o_{\bar{r}}(1) + \tilde{M}^\varepsilon \left[\frac{\varphi}{\varepsilon}(t, \cdot) \right] (x) \cdot |p| - \bar{H}(p) \\ &=: \Delta, \end{aligned} \quad (5.7)$$

11 where we have used (5.4). We recall that for $-k_0 \leq p \leq 0$,

$$\bar{H}(p) = M_p0|p|.$$

12 Moreover, for all $z \in [h_0, h_{max}]$, and for ε and \bar{r} small enough we have that

$$\begin{aligned} \frac{\varphi(t, x + \varepsilon z) - \varphi(t, x)}{\varepsilon} &= z\varphi_x(t, y) + \varepsilon z^2 \varphi_{xx}(t, \xi(x, x + \varepsilon z)) \\ &\leq pz + o_{\bar{r}}(1) + c\varepsilon, \end{aligned}$$

13 where we have used the fact that $\varphi \in \mathcal{C}^2$ and that $z \in [h_0, h_{max}]$. Now using the fact that \tilde{E} is
14 decreasing we have

$$\tilde{E}(pz + c\varepsilon + o_{\bar{r}}(1)) \leq \tilde{E} \left(\frac{\varphi(t, x + \varepsilon z) - \varphi(t, x)}{\varepsilon} \right).$$

Using this result and replacing the non-local operators in (5.7) by their definition (see 2.19), we obtain

$$\begin{aligned} \Delta &\geq \theta + o_{\bar{r}}(1) + |p| \int_{h_0}^{h_{max}} J(z) \tilde{E}(pz + c\varepsilon + o_{\bar{r}}(1)) dz \\ &\quad - |p| \int_{h_0}^{h_{max}} J(z) \tilde{E}(pz) dz. \end{aligned} \quad (5.8)$$

We can see that if we have $p = 0$, we obtain directly our result. However, if $-k_0 \leq p < 0$,

$$\begin{aligned} \int_{\mathbb{R}} J(z) \tilde{E}(pz + c\varepsilon + o_{\bar{r}}(1)) dz &= -V \left(\frac{-1 - c\varepsilon + o_{\bar{r}}(1)}{p} \right) - \frac{1}{2} V \left(-\frac{c\varepsilon + o_{\bar{r}}(1)}{p} \right) + \frac{3}{2} V_{max}, \\ \int_{\mathbb{R}} J(z) \tilde{E}(pz) dz &= -V \left(\frac{-1}{p} \right) + \frac{3}{2} V_{max}. \end{aligned} \quad (5.9)$$

1 Injecting (5.9) in (5.8) and choosing ε and \bar{r} , we obtain

$$\begin{aligned} \Delta &\geq \theta + o_{\bar{r}}(1) + |p| \cdot \left[-V \left(\frac{-1 - c\varepsilon + o_{\bar{r}}(1)}{p} \right) + V \left(\frac{-1}{p} \right) \right] \\ &\geq \theta + o_{\bar{r}}(1) - \|V'\|_{\infty} \cdot (c\varepsilon + o_{\bar{r}}(1)) \\ &\geq \frac{\theta}{2}, \end{aligned}$$

2 where we have used assumption (A1) for the second line.

3 **Getting a contradiction.** By definition, we have for ε small enough,

$$u^\varepsilon \leq \varphi - \eta \quad \text{outside } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

4 Using the comparison principle on bounded subsets for (2.4) (Theorem 3.6), we get

$$u^\varepsilon \leq \varphi - \eta \quad \text{on } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

5 Passing to the limit as $\varepsilon \rightarrow 0$, we get $\bar{u} \leq \varphi - \eta$ on $\mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$ and this contradicts the fact that
6 $\bar{u}(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x})$.

7 **Second case:** $\bar{x} = 0$. Using Theorem 3.9, we may assume that the test function has the following
8 form

$$\varphi(t, x) = g(t) + \bar{p}_- x 1_{\{x < 0\}} + \bar{p}_+ x 1_{\{x > 0\}} \quad \text{on } \mathcal{Q}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0), \quad (5.10)$$

9 where g is a C^1 function defined in $(0, +\infty)$. The last line in condition (5.4) becomes

$$g'(t) + F_A(\bar{p}_-, \bar{p}_+) = g'(t) + \bar{A} = \theta \quad \text{at } (\bar{t}, 0). \quad (5.11)$$

10 Let us consider the solution w of (2.21) provided by Theorem 4.1, and let us denote by

$$\varphi^\varepsilon(t, x) = \begin{cases} g(t) + w^\varepsilon(t, x) & \text{on } \mathcal{Q}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0), \\ \varphi(t, x) & \text{outside } \mathcal{Q}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0). \end{cases} \quad (5.12)$$

11 We would like to prove that this function satisfies in the viscosity sense, for \bar{r} and ε small
12 enough,

$$\varphi^\varepsilon(t, x) + \tilde{M}^\varepsilon \left[\frac{\varphi^\varepsilon}{\varepsilon}(t, \cdot) \right] (x) \cdot \phi \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right) \cdot |\varphi_x^\varepsilon| \geq \frac{\theta}{2} \quad \text{on } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, 0).$$

13 Let h be a test function touching φ^ε from below at $(t_1, x_1) \in \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, 0)$, so we have

$$w \left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon} \right) = \frac{1}{\varepsilon} (h(t_1, x_1) - g(t_1)),$$

14 and

$$w(s, y) \geq \frac{1}{\varepsilon} (h(\varepsilon s, \varepsilon y) - g(\varepsilon s)),$$

1 for (s, y) in a neighbourhood of $\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}\right)$. Therefore, we have

$$h_t(t_1, x_1) - g'(t_1) + \tilde{M} \left[w \left(\frac{t_1}{\varepsilon}, \cdot \right) \right] \left(\frac{x_1}{\varepsilon} \right) \cdot \phi \left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon} \right) \cdot |h_x(t_1, x_1)| \geq \bar{A}.$$

2 This implies that (using (5.11) and taking \bar{r} small enough)

$$h_t(t_1, x_1) + \tilde{M} \left[w \left(\frac{t_1}{\varepsilon}, \cdot \right) \right] \left(\frac{x_1}{\varepsilon} \right) \cdot \phi \left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon} \right) \cdot |h_x(t_1, x_1)| \geq \bar{A} + g'(t_1) \geq \frac{\theta}{2}.$$

3 Now for ε small enough such that $\varepsilon h_{max} \leq \bar{r}$, we deduce from the previous inequality and using
4 the fact that \tilde{M} is a non-local operator with a bounded support, that we have

$$h_t(t_1, x_1) + \tilde{M}^\varepsilon \left[\frac{\varphi^\varepsilon(t_1, \cdot)}{\varepsilon} \right] (x_1) \cdot \phi \left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon} \right) \cdot |h_x(t_1, x_1)| \geq \frac{\theta}{2}.$$

5 **Getting the contradiction.** We have that for ε small enough

$$u^\varepsilon + \eta \leq \varphi = g(t) + \bar{p}_- x 1_{\{x < 0\}} + \bar{p}_+ x 1_{\{x > 0\}} \quad \text{on } \mathcal{Q}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0) \setminus \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, 0).$$

6 Using the fact that $w^\varepsilon \rightarrow W$, and using (4.4), we have for ε small enough

$$u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{on } \mathcal{Q}_{2\bar{r}, 2\bar{r}}(\bar{t}, 0) \setminus \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, 0).$$

7 Combining this with (5.12), we get that

$$u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{outside } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, 0),$$

8 By the comparison principle on bounded subsets (Theorem 3.6) the previous inequality holds in
9 $\mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, 0)$. Passing to the limit as $\varepsilon \rightarrow 0$ and evaluating the inequality in $(\bar{t}, 0)$, we obtain

$$\bar{u}(\bar{t}, 0) + \frac{\eta}{2} \leq \varphi(\bar{t}, 0) = \bar{u}(\bar{t}, 0),$$

10 which is a contradiction.

11

□

12 6 Truncated cell problems

13 This section contains the proof of Theorem 4.1. To do this, we will construct correctors on
14 truncated domains and then pass to the limit as the size of the domain goes to infinity. This idea
15 comes from [2] and [14]. The difficulty in our non-local case is that it is non-standard to well define
16 boundary conditions. In order to overcome this difficulty, we will replace the non-local operator
17 by a local one near the boundary. More precisely, for $l \in (r, +\infty)$, $r \ll l$ and $r \leq R \ll l$, we
18 want to find $\lambda_{l,R}$, such that there exists a solution $w^{l,R}$ of

$$\begin{cases} w_t^{l,R} + G_R(t, x, [w^{l,R}(t, \cdot)], w_x^{l,R}) = \lambda_{l,R} & \text{if } (t, x) \in \mathbb{R} \times (-l, l) \\ w_t^{l,R} + \bar{H}^-(w_x^{l,R}) = \lambda_{l,R} & \text{if } (t, x) \in \mathbb{R} \times \{-l\} \\ w_t^{l,R} + \bar{H}^+(w_x^{l,R}) = \lambda_{l,R} & \text{if } (t, x) \in \mathbb{R} \times \{l\} \\ w^{l,R} \text{ is 1-periodic in } t. \end{cases} \quad (6.1)$$

19 with

$$G_R(t, x, [U], q) = \psi_R(x) \phi(t, x) \cdot M[U](x) \cdot |q| + (1 - \psi_R(x)) \cdot \bar{H}(q), \quad (6.2)$$

20 and $\psi_R \in C^\infty$, $\psi_R : \mathbb{R} \rightarrow [0, 1]$, with

$$\psi_R \equiv \begin{cases} 1 & \text{on } [-R, R] \\ 0 & \text{outside } [-R-1, R+1], \end{cases} \quad \text{and } \psi_R(x) < 1 \quad \forall x \notin [-R, R]. \quad (6.3)$$

21 To G_R , we associate \tilde{G}_R which is defined in the same way but the operator M is replaced by \tilde{M} .

1 **Remark 6.1.** The operator G_R is used to have a local operator near the boundary and then to
 2 well define the boundary conditions.

3 6.1 Comparison principle for a truncated problem

4 **Proposition 6.2** (Comparison principle on truncated domains). *Let us consider the following*
 5 *problem for $r < l_1 < l_2$ and $\lambda \in \mathbb{R}$, with and $l_2 \gg R$.*

$$\begin{cases} v_t + \tilde{G}_R(t, x, [v(t, \cdot)], v_x) \geq \lambda & \text{for } (t, x) \in \mathbb{R} \times (l_1, l_2) \\ v_t + \overline{H}^+(v_x) \geq \lambda & \text{for } (t, x) \in \mathbb{R} \times \{l_2\} \\ v(t, x) \geq U_0(t) & \text{for } (t, x) \in \mathbb{R} \times \{l_1\} \\ v \text{ is 1-periodic in } t, \end{cases} \quad (6.4)$$

6 where U_0 is continuous, and for $\varepsilon_0 > 0$

$$\begin{cases} u_t + G_R(t, x, [u(t, \cdot)], u_x) \leq \lambda - \varepsilon_0 & \text{for } (t, x) \in \mathbb{R} \times (l_1, l_2) \\ u_t + \overline{H}^+(u_x) \leq \lambda - \varepsilon_0 & \text{for } (t, x) \in \mathbb{R} \times \{l_2\} \\ u(t, x) \leq U_0(t) & \text{for } (t, x) \in \mathbb{R} \times \{l_1\} \\ u \text{ is 1-periodic in } t, \end{cases} \quad (6.5)$$

7 Then we have $u \leq v$ in $\mathbb{R} \times [l_1, l_2]$.

8 *Proof.* The only difficulty in proving this result is the comparison at the boundary $\{l_2\}$. However,
 9 for x close to l_2 , the function G_R is actually the effective Hamiltonian \overline{H} . Therefore, we can
 10 proceed as in the proof of [14, Proposition 4.1] and so we skip the proof. □

12 **Remark 6.3.** We have a similar result for $l_1 < l_2 < -r$ and if for all $x \in [l_2, l_2 + h_{max}]$,
 13 $u(t, x) \leq v(t, x)$ and the following conditions are imposed at $x = l_1$:

$$\begin{cases} v_t + \overline{H}^-(v_x) \geq \lambda & \text{for } x = l_1, \\ u_t + \overline{H}^-(u_x) \leq \lambda - \varepsilon_0 & \text{for } x = l_1. \end{cases}$$

14 6.2 Existence of correctors on a truncated domain

15 **Proposition 6.4** (Existence of correctors on truncated domains). *There exists a unique $\lambda_{l,R} \in \mathbb{R}$*
 16 *such that there exists a solution $w^{l,R}$ of (6.1). Moreover, there exists a constant C (depending*
 17 *only on k_0, V_{max} and $|H_0|$), and a Lipschitz continuous function $m^{l,R}$, such that*

$$\begin{cases} H_0 \leq \lambda_{l,R} \leq 0, \\ |m^{l,R}(x) - m^{l,R}(y)| \leq C|x - y| & \text{for } x, y \in [-l, l], \\ |w^{l,R}(t, x) - m^{l,R}(x)| \leq C & \text{for } (t, x) \in \mathbb{R} \times [-l, l], \end{cases} \quad (6.6)$$

18 with $H_0 = \min \overline{H}$.

19 *Proof.* In order to construct a corrector on the truncated domain, we will classically consider the
 20 approximated problem

$$\begin{cases} \delta v^\delta + v_t^\delta + \psi_R(x)M[v^\delta(t, \cdot)](x) \cdot \phi(t, x) \cdot |v_x^\delta| + (1 - \psi_R(x))\overline{H}(v_x^\delta) = 0 & \text{for } (t, x) \in \mathbb{R} \times (-l, l) \\ \delta v^\delta + v_t^\delta + \overline{H}^-(v_x^\delta) = 0 & \text{for } (t, x) \in \mathbb{R} \times \{-l\} \\ \delta v^\delta + v_t^\delta + \overline{H}^+(v_x^\delta) = 0 & \text{for } (t, x) \in \mathbb{R} \times \{l\} \\ v^\delta \text{ is 1-periodic in } t \end{cases} \quad (6.7)$$

21 **Step 1: construction of barriers.** Using that 0 and $\delta^{-1}C_0$ are respectively sub and super-
 22 solution of (6.7) with $C_0 = |H_0|$, the comparison principle and Perron's method for 1-periodic
 23 solutions, we deduce that there exists a continuous viscosity solution, v^δ of (6.7) which satisfies

$$0 \leq v^\delta \leq \frac{C_0}{\delta}. \quad (6.8)$$

1 **Step 2: control of the space oscillations of v^δ .**

2 **Lemma 6.5.** *The function v^δ satisfies for all $t \in \mathbb{R}$ and for all $x, y \in [-l, l]$, $x \geq y$,*

$$-k_0(x - y) - 1 \leq v^\delta(t, x) - v^\delta(t, y) \leq 0,$$

3 *with k_0 defined in (A0).*

4 *Proof of Lemma 6.5.* In the rest of the proof we will use the following notation,

$$\Omega = \{(t, x, y) \in \mathbb{R} \times [-l, l]^2 \text{ such that } x \geq y\}.$$

5 **Step 2.1: proof of the upper inequality.** Let $\varepsilon > 0$. We want to prove that

$$M = \sup_{(t, x, y) \in \Omega} \{v^\delta(t, x) - v^\delta(t, y)\} \leq 0.$$

6 We argue by contradiction and assume that $M > 0$. We then consider

$$M_\nu = \sup_{t, s \in \mathbb{R}, x \geq y} \left\{ v^\delta(t, x) - v^\delta(s, y) - \frac{(t - s)^2}{2\nu} \right\}.$$

7 Since $M > 0$, we deduce that $M_\nu > 0$. Remark also that we consider the supremum of a continuous,
8 1-periodic in t and s function, so we deduce that M_ν is reached at a point $(\bar{t}, \bar{s}, \bar{x}, \bar{y})$. Given that
9 $M_\nu > 0$, we deduce that $\bar{x} \neq \bar{y}$ if ν is small enough (classically we have that $|t - s| \rightarrow 0$ as $\nu \rightarrow 0$
10). Therefore, we can use the viscosity inequalities for (6.7).

11 -If $(\bar{x}, \bar{y}) \in (-l, l)^2$, we have

$$\begin{aligned} \delta v^\delta(\bar{t}, \bar{x}) + \frac{\bar{t} - \bar{s}}{\nu} + G_R(\bar{x}, [v^\delta(\bar{t}, \cdot)], 0) &\leq 0 \\ \delta v^\delta(\bar{s}, \bar{y}) + \frac{\bar{t} - \bar{s}}{\nu} + G_R(\bar{y}, [v^\delta(\bar{s}, \cdot)], 0) &\geq 0, \end{aligned}$$

12 combining these two inequalities with the fact that $G_R(x, [U], 0) = 0$, we obtain

$$\delta (v^\delta(\bar{t}, \bar{x}) - v^\delta(\bar{s}, \bar{y})) \leq 0.$$

13 -If $\bar{x} = l$ and $\bar{y} \in (-l, l)$, similarly we obtain

$$\delta (v^\delta(\bar{t}, \bar{x}) - v^\delta(\bar{s}, \bar{y})) \leq 0,$$

14 where we have used the fact that $\overline{H}^+(0) = 0$.

15 -If $\bar{x} \in (-l, l)$ and $\bar{y} = -l$, we obtain

$$\delta (v^\delta(\bar{t}, \bar{x}) - v^\delta(\bar{s}, \bar{y})) \leq H_0 \leq 0,$$

16 where we used the fact that $\overline{H}^-(0) = H_0$.

17 -If $\bar{x} = l$ and $\bar{y} = -l$, we obtain

$$\delta (v^\delta(\bar{t}, \bar{x}) - v^\delta(\bar{s}, \bar{y})) \leq H_0 \leq 0.$$

18 For every value of \bar{x} and \bar{y} we obtain a contradiction, therefore we have $M \leq 0$.

1 **Step 2.2: proof of the lower inequality.** We want to prove that

$$M = \sup_{(t,x,y) \in \Omega} \{v^\delta(t,y) - v^\delta(t,x) - k_0(x-y) - 1\} \leq 0.$$

2 We argue by contradiction and assume that $M > 0$. We then consider

$$M_\nu = \sup_{t,s \in \mathbb{R}, x \geq y} \left\{ v^\delta(t,y) - v^\delta(s,x) - k_0(x-y) - 1 - \frac{(t-s)^2}{2\nu} \right\}.$$

3 Since $M > 0$, we get $M_\nu > 0$. Remark also that we consider the supremum of a continuous,
4 1-periodic in t and s function, so we deduce that M_ν is reached at a point $(\bar{t}, \bar{s}, \bar{x}, \bar{y})$. Given that
5 $M_\nu > 0$, we deduce that $\bar{x} \neq \bar{y}$ if ν is small enough (classicaly we have that $|\bar{t} - \bar{s}| \rightarrow 0$ as $\nu \rightarrow 0$).
6 Therefore, we can use the viscosity inequalities for (6.7).

7 **Case 1:** $\bar{y} \in (-l, l)$. If $\bar{y} \in (-l, l)$, we have

$$\delta v^\delta(\bar{t}, \bar{y}) + \frac{\bar{t} - \bar{s}}{\nu} + \psi_R(\bar{y})M[v^\delta(\bar{t}, \cdot)](\bar{y}) \cdot \phi(\bar{t}, \bar{y}) \cdot |-k_0| + (1 - \psi_R(\bar{y}))\overline{H}(-k_0) \leq 0. \quad (6.9)$$

8 We claim that $M[v^\delta(\bar{t}, \cdot)](\bar{y}) = 0$.

9 Indeed, for all $z > h_0$, if $\bar{x} > \bar{y} + z$ using the fact that the maximum is reached for $(\bar{t}, \bar{s}, \bar{x}, \bar{y})$,
10 we deduce that

$$v^\delta(\bar{t}, \bar{y} + z) - v^\delta(\bar{t}, \bar{y}) \leq -k_0 z < -1.$$

11 On the contrary, if $\bar{x} \leq \bar{y} + z$, using the fact that v^δ is continuous, non-increasing in space, and
12 the fact that $v^\delta(\bar{s}, \bar{x}) - v^\delta(\bar{t}, \bar{y}) < -1$, we deduce that

$$v^\delta(\bar{t}, \bar{y} + z) - v^\delta(\bar{t}, \bar{y}) \leq v^\delta(\bar{t}, \bar{x}) - v^\delta(\bar{t}, \bar{y}) < -1.$$

13 We can therefore, conclude that for all $z \in (h_0, +\infty)$, $E(v^\delta(\bar{t}, \bar{y} + z) - v^\delta(\bar{t}, \bar{y})) = -\frac{3}{2}$ and so we
14 get $M[v^\delta(\bar{t}, \cdot)](\bar{y}) = 0$. Using also that $\overline{H}(-k_0) = 0$, equation (6.9) becomes

$$\delta v^\delta(\bar{t}, \bar{y}) + \frac{\bar{t} - \bar{s}}{\nu} \leq 0.$$

15 Moreover, whether $\bar{x} \in (-l, l)$ or $\bar{x} = l$, since the non-local operator is negative and $H^+(-k_0) < 0$,
16 we have that

$$-\delta v^\delta(\bar{s}, \bar{x}) - \frac{\bar{t} - \bar{s}}{\nu} \leq 0.$$

17 We deduce that

$$\delta(v^\delta(\bar{t}, \bar{y}) - v^\delta(\bar{s}, \bar{x})) \leq 0,$$

18 which is a contradiction.

19 **Case 2:** $\bar{y} = -l$. In this situation, the viscosity inequality becomes

$$\delta v^\delta(\bar{t}, \bar{y}) + \frac{\bar{t} - \bar{s}}{\nu} + \overline{H}^-(-k_0) \leq 0.$$

20 Using the fact that $\overline{H}^-(-k_0) = \overline{H}(-k_0) = 0$, and as in the previous case, we obtain a contradiction.
21 This ends the proof of the lemma.

22 □

1 **Step 3: control of the time oscillations of v^δ .**

2 **Lemma 6.6.** *The function v^δ satisfies for all $x \in [-l, l]$ and for all $t, s \in \mathbb{R}$,*

$$|v^\delta(t, x) - v^\delta(s, x)| \leq C_1$$

3 with $C_1 = \frac{3}{2}V_{max}k_0 + |H_0| + 1$.

4 *Proof.* Since v^δ is 1-periodic in t , it is sufficient to show that for all $x \in [-l, l]$ and for all $t, s \in \mathbb{R}$
5 such that $t \geq s$, we have that

$$v^\delta(t, x) - v^\delta(s, x) \leq C_2(t - s) + 1. \quad (6.10)$$

6 with $C_2 = C_1 - 1$. In order to prove that, we will fix $x_0 \in (-l, l)$ and $s_0 \in \mathbb{R}$, and we will prove
7 that if $t \geq s_0$, then

$$v^\delta(t, x_0) \leq v^\delta(s_0, x_0) + C_2(t - s_0) + 1. \quad (6.11)$$

We define

$$w^\delta(t, x) = v^\delta(s_0, x_0) + C_2(t - s_0) + k_0|x - x_0| + 1.$$

8 Using the space oscillation of v^δ , we have that $v^\delta(s_0, x) \leq w^\delta(s_0, x)$. On the other hand, we can
9 check that w^δ is a super solution of (6.7) on $(s_0, +\infty) \times [-l, l]$ using that

$$\begin{cases} w^\delta(t, x) \geq 0 \\ |H_0| \geq -\overline{H}, -\overline{H}^+, -\overline{H}^- \\ \frac{3}{2}V_{max} \geq -M[U](x) \\ 1 \geq \phi. \end{cases}$$

10 Finally, using the comparison principle on $[s_0, +\infty) \times [-l, l]$, we deduce that

$$v^\delta(t, x) \leq w^\delta(t, x).$$

11 In particular, for $x = x_0$, we obtain (6.11). We deduce that (6.10) is true even if $x = \pm l$ because
12 v^δ is continuous. The proof is now complete. \square

13 **Step 4: construction of a Lipschitz estimate.**

14 **Lemma 6.7.** *There exists a Lipschitz continuous function m^δ , such that there exists a constant*
15 *C , (independent of l, R and δ) such that*

$$\begin{cases} |m^\delta(x) - m^\delta(y)| \leq C|x - y| & \text{for all } x, y \in [-l, l], \\ |v^\delta(t, x) - m^\delta(x)| \leq C & \text{for all } (t, x) \in \mathbb{R} \times [-l, l]. \end{cases} \quad (6.12)$$

16

17 *Proof of Lemma 6.7.* Let us define m^δ as an affine function in each interval of the form
18 $[ih_0, (i+1)h_0]$, with $i \in \mathbb{Z}$, such that

$$m^\delta(ih_0) = v^\delta(0, ih_0) \quad \text{and} \quad m^\delta((i+1)h_0) = v^\delta(0, (i+1)h_0).$$

19 Since $m^\delta, v^\delta(0, \cdot)$ are non-increasing and $|v^\delta(0, (i+1)h_0) - v^\delta(0, ih_0)| \leq k_0h_0 + 1 = 2$, we deduce
20 that $\forall x \in [ih_0, (i+1)h_0]$,

$$-2 \leq v^\delta(0, (i+1)h_0) - m^\delta(ih_0) \leq v^\delta(0, x) - m^\delta(x) \leq v^\delta(0, ih_0) - m^\delta((i+1)h_0) \leq 2,$$

21 and for all $x, y \in [-l, l]$,

$$|m^\delta(x) - m^\delta(y)| \leq 2k_0|x - y|.$$

22 Using the time oscillations of v^δ , we deduce that

$$|v^\delta(t, x) - m^\delta(x)| \leq C \quad \text{for all } (t, x) \in \mathbb{R} \times [-l, l]$$

23 with $C = \frac{3}{2}V_{max}k_0 + |H_0| + 3$. \square

1 **Step 4: passing to the limit as δ goes to 0.** Using (6.8) and (6.12), we deduce that there
 2 exists $\delta_n \rightarrow 0$ such that

$$\begin{aligned} \delta_n v^{\delta_n}(0, 0) &\rightarrow -\lambda_{l,R} && \text{as } n \rightarrow +\infty, \\ m^{\delta_n} - m^{\delta_n}(0) &\rightarrow m^{l,R} && \text{as } n \rightarrow +\infty, \end{aligned}$$

3 the second convergence being locally uniform. Let us consider,

$$\bar{w}^{l,R}(t, x) = \limsup_{\delta_n \rightarrow 0}^* (v^{\delta_n} - v^{\delta_n}(0, 0)) \quad \text{and} \quad \underline{w}^{l,R} = \liminf_{\delta_n \rightarrow 0} (v^{\delta_n} - v^{\delta_n}(0, 0)).$$

4 Therefore, we have that $\lambda_{l,R}, m^{l,R}, \bar{w}^{l,R}$ and $\underline{w}^{l,R}$ satisfy

$$\begin{aligned} H_0 &\leq \lambda_{l,R} \leq 0, \\ |\bar{w}^{l,R} - m^{l,R}| &\leq C, \\ |\underline{w}^{l,R} - m^{l,R}| &\leq C, \\ |m_x^{l,R}| &\leq C. \end{aligned} \tag{6.13}$$

5 By stability of the solutions we have that $\bar{w}^{l,R} - 2C$ and $\underline{w}^{l,R}$ are respectively a sub-solution and
 6 a super-solution of (6.1) and

$$\bar{w}^{l,R} - 2C \leq \underline{w}^{l,R}.$$

7 By Perron's method we can construct a solution $w^{l,R}$ of (6.1) and thanks to (6.8) and (6.13), $m^{l,R}$,
 8 $\lambda_{l,R}$ and $w^{l,R}$ satisfy (6.6).

9 The uniqueness of $\lambda_{l,R}$ is classical so we skip it. This ends the proof of Proposition 6.4.
 10 □

11 **Proposition 6.8** (First definition of the flux limiter). *The following limits exist (up to a subse-*
 12 *quence)*

$$\begin{cases} \bar{A}_R = \lim_{l \rightarrow +\infty} \lambda_{l,R} \\ \bar{A} = \lim_{R \rightarrow +\infty} \bar{A}_R. \end{cases} \tag{6.14}$$

13 Moreover, we have

$$H_0 \leq \bar{A}_R, \bar{A} \leq 0.$$

14

15 *Proof.* This results comes from the fact that we have the following bound on $\lambda_{l,R}$ which is inde-
 16 pendent of l and R (see Proposition 6.4),

$$H_0 \leq \lambda_{l,R} \leq 0.$$

17 □

18 **Remark 6.9.** *This proposition does not ensure the uniqueness of the flux limiter \bar{A} . However,*
 19 *since we know that such a limit exists, we can obtain the converge result. The uniqueness of \bar{A} is*
 20 *given in Theorem 2.14.*

21 **Proposition 6.10** (Control of the slopes on a truncated domain). *Assume that l and R are big*
 22 *enough. Let $w^{l,R}$ be the solution of (6.1) given by Proposition 6.4. We also assume that up to a*
 23 *sub-sequence $\bar{A} = \lim_{R \rightarrow +\infty} \lim_{l \rightarrow +\infty} \lambda_{l,R} > H_0$. Then there exists $\gamma_0 > 0$ such that for all $\gamma \in (0, \gamma_0)$,*
 24 *there exists a constant C (independent of l and R) such that for all $x \geq r$ and $h \geq 0$*

$$w^{l,R}(t, x+h) - w^{l,R}(t, x) \geq (\bar{p}_+ - \gamma)h - C. \tag{6.15}$$

25 Similarly, for all $x \leq -r$ and $h \geq 0$,

$$w^{l,R}(t, x-h) - w^{l,R}(t, x) \geq (-\bar{p}_- - \gamma)h - C. \tag{6.16}$$

26

1 *Proof.* We only prove (6.15) since the proof for (6.16) is similar. For $\mu > 0$ small enough, we
 2 denote by p_+^μ the real number such that

$$\overline{H}(p_+^\mu) = \overline{H}^+(p_+^\mu) = \lambda_{l,R} - \mu.$$

3 Using that

$$H_0 < \lambda_{l,R} \leq 0,$$

4 we deduce that p_+^μ exists, is unique and satisfies $-k_0 \leq p_+^\mu \leq 0$ for μ small enough.

5 Let us now consider the function $w^+ = p_+^\mu x$ that satisfies

$$\overline{H}(w_x^+) = \lambda_{l,R} - \mu \quad \text{for } x \in \mathbb{R}.$$

We also have

$$\begin{aligned} M[w^+](x) &= \int_{\mathbb{R}} J(z)E(p_+^\mu(x+z) - p_+^\mu x)dz - \frac{3}{2}V_{max} \\ &= \int_0^{\frac{-1}{p_+^\mu}} \frac{1}{2}J(z)dz + \int_{\frac{-1}{p_+^\mu}}^{+\infty} \frac{3}{2}J(z)dz - \frac{3}{2}V_{max} \\ &= -V\left(\frac{-1}{p_+^\mu}\right). \end{aligned}$$

6 For all $x \in (r, l)$, using that $\phi(t, x) = 1$, we deduce that

$$M[w^+](x) \cdot \phi(t, x) \cdot |w_x^+| = -V\left(\frac{-1}{p_+^\mu}\right) \cdot |p_+^\mu| = \overline{H}(p_+^\mu) = \lambda_{l,R} - \mu,$$

7 and so the restriction of w^+ to $(r, l]$ satisfies

$$\begin{cases} w_t^+ + G_R(t, x, [w^+], w_x^+) = \lambda_{l,R} - \mu & \text{for } (t, x) \in \mathbb{R} \times (r, l) \\ w_t^+ + \overline{H}^+(w_x^+) = \lambda_{l,R} - \mu & \text{for } (t, x) \in \mathbb{R} \times \{l\}. \end{cases}$$

8 Let us denote by $g(t, x) = w^{l,R}(t, x) - w^{l,R}(0, x_0)$ and $u(x) = w^+(x) - w^+(x_0) - C$, for some
 9 $x_0 \in (r, l)$ and C defined as in Proposition 6.4. Then we have

$$g(t, x_0) \geq -C = u(x_0).$$

10 Using that g is a solution of (6.4) and u is a solution of (6.5) (with $\varepsilon_0 = \mu$) joint to the comparison
 11 principle (Proposition 6.2) we get that

$$w^{l,R}(t, x) - w^{l,R}(t, x_0) = g(t, x) \geq u(x) = p_+^\mu(x - x_0) - C.$$

12 This implies that for all $h \geq 0$ and for all $x \in (r, l)$,

$$w^{l,R}(t, x+h) - w^{l,R}(t, x) \geq p_+^\mu h - C.$$

13 Finally, if we choose $\gamma_0 < |p_0 - \bar{p}_+|$ (with p_0 defined in (2.12)), then

$$\overline{H}(\bar{p}_+ - \gamma) = \overline{H}^+(\bar{p}_+ - \gamma),$$

14 and we can choose $\mu > 0$ such that

$$p_+^\mu = \bar{p}_+ - \gamma.$$

15 This implies inequality (6.15).

16

□

17 *Proof of Theorem 4.1.* The proof is performed two steps.

- 1 **Step 1: proof of i) and ii).** The goal is to pass to the limit as $l \rightarrow +\infty$ and then as $R \rightarrow +\infty$.
2 Using Proposition 6.4, there exists $l_n \rightarrow +\infty$, such that

$$m^{l_n, R} - m^{l_n, R}(0) \rightarrow m^R \quad \text{as } n \rightarrow +\infty,$$

- 3 the convergence being locally uniform. We also define

$$\begin{aligned} \bar{w}^R(t, x) &= \limsup_{l_n \rightarrow +\infty}^* (w^{l_n, R} - w^{l_n, R}(0, 0)), \\ \underline{w}^R(t, x) &= \liminf_{l_n \rightarrow +\infty} (w^{l_n, R} - w^{l_n, R}(0, 0)). \end{aligned}$$

- 4 Thanks to (6.6), we know that \bar{w}^R and \underline{w}^R are finite and satisfy

$$m^R - C \leq \underline{w}^R \leq \bar{w}^R \leq m^R + C.$$

- 5 By stability of viscosity solutions, $\bar{w}^R - 2C$ and \underline{w}^R are respectively a sub and a super-solution of

$$w_t^R + G_R(x, [w^R(t, \cdot)], w_x^R) = \bar{A}_R \quad \text{for } (t, x) \in \mathbb{R} \times \mathbb{R} \quad (6.17)$$

- 6 Therefore, using Perron's method, we can construct a solution w^R of (6.17) with m^R, \bar{A}^R and w^R
7 satisfying

$$\begin{cases} |m^R(x) - m^R(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \\ |w^R(t, x) - m^R(x)| \leq C & \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}, \\ H_0 \leq \bar{A}_R \leq 0. \end{cases} \quad (6.18)$$

- 8 Using Proposition 6.10, if $\bar{A} > H_0$, we know that there exists a γ_0 and a constant C , such that for
9 all $\gamma \in (0, \gamma_0)$,

$$\begin{cases} w^R(t, x+h) - w^R(t, x) \geq (\bar{p}_+ - \gamma)h - C & \text{for all } x \geq r, h \geq 0, \\ w^R(t, x-h) - w^R(t, x) \geq (-\bar{p}_- - \gamma)h - C & \text{for all } x \leq -r, h \geq 0. \end{cases} \quad (6.19)$$

- 10 We now pass to the limit as $R \rightarrow +\infty$. We consider (up to some subsequence)

$$\begin{cases} \bar{w}(t, x) = \limsup_{R \rightarrow +\infty}^* (w^R - w^R(0, 0)), \\ \underline{w}(t, x) = \liminf_{R \rightarrow +\infty} (w^R - w^R(0, 0)), \\ \bar{A} = \lim_{R \rightarrow +\infty} \bar{A}_R, \\ m = \lim_{R \rightarrow +\infty} (m^R - m^R(0)). \end{cases}$$

- 11 The last convergence being locally uniform. Thanks to (6.18), we know that \bar{w} and \underline{w} are finite
12 and satisfy

$$m - C \leq \underline{w} \leq \bar{w} \leq m + C.$$

- 13 By stability of viscosity solutions, $\bar{w} - 2C$ and \underline{w} are respectively a sub and a super-solution of
14 (2.21) with $\lambda = \bar{A}$. Using Perron's method, we can then construct a solution w of (2.21) with
15 $\lambda = \bar{A}$ that satisfies (4.1) and (4.2).

1 **Step 2: proof of iii).** We are now interested in the rescaled function $w^\varepsilon(t, x) = \varepsilon w\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$.
 2 Using (4.2), we have that

$$w^\varepsilon(t, x) = \varepsilon m\left(\frac{x}{\varepsilon}\right) + O(\varepsilon).$$

3 Therefore, we can find a sequence $\varepsilon_n \rightarrow 0$, such that

$$w^{\varepsilon_n} \rightarrow W \quad \text{locally uniformly as } n \rightarrow +\infty,$$

4 with $W(0) = 0$. Like in [18], arguing as in the proof of convergence away from the junction point,
 5 we have that W satisfies

$$\overline{H}(W_x) = \overline{A} \quad \text{for } x \neq 0.$$

6 For all $\gamma \in (0, \gamma_0)$, we have that if $\overline{A} > H_0$ and $x > 0$,

$$W_x \geq \overline{p}_+ - \gamma,$$

7 where we have used (4.2). Therefore we get

$$W_x = \overline{p}_+ \quad \text{for } x > 0,$$

8 this result remains valid even if $\overline{A} = H_0$ (in this particular case $W_x = p_0$). Similarly, we get

$$W_x = \overline{p}_- \quad \text{for } x < 0.$$

9 which implies (4.3) and (4.4). This ends the proof of Theorem 4.1.

10

□

11 6.3 Proof of Theorem 2.14

12 *Proof of Theorem 2.14.* Up to a sub-sequence, we assume that $\overline{A} = \lim_{R \rightarrow +\infty} \lim_{l \rightarrow +\infty} \lambda_{l,R}$. We want
 13 to prove that $\overline{A} = \inf E$, where

$$E = \{\lambda \in [h_0, 0] : \exists w \in \mathcal{S} \text{ solution of (2.21)}\},$$

14 with

$$\mathcal{S} = \{w \text{ s.t. } \exists \text{ a Lipschitz continuous function } m \text{ and a } C > 0 \text{ s.t. } |w(t, x) - m(x)| \leq C\}.$$

15 We argue by contradiction and assume that there exists a $\lambda < \overline{A}$ and a function $w^\lambda \in \mathcal{S}$ solution
 16 of (2.21). We assume that $w^\lambda(0, 0) = 0$ (if we are not in this situation, we do a translation since
 17 we have $w^\lambda - w^\lambda(0, 0) \in \mathcal{S}$). Arguing as in the proof of Theorem 4.1, we deduce that the function

$$w_\lambda^\varepsilon(t, x) = \varepsilon w^\lambda\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$

18 has a limit W^λ (with $W^\lambda(0) = 0$) which satisfies

$$\overline{H}(W_x^\lambda) = \lambda \quad \text{for } x > 0,$$

19 which means that for all $x > 0$,

$$W_x^\lambda \leq p_+^\lambda < \overline{p}_+ \quad \text{with } \overline{H}(p_+^\lambda) = \overline{H}^+(p_+^\lambda) = \lambda. \quad (6.20)$$

20 Similarly we have for all $x < 0$,

$$W_x^\lambda \geq p_-^\lambda > \overline{p}_- \quad \text{with } \overline{H}(p_-^\lambda) = \overline{H}^-(p_-^\lambda) = \lambda. \quad (6.21)$$

1 These inequalities imply that for all $\gamma > 0$, there exists a constant $\tilde{C}_\gamma > 0$ such that

$$w^\lambda(t, x) \leq \begin{cases} (p_+^\lambda + \gamma)x + \tilde{C}_\gamma & \text{for } x > 0, \\ (p_-^\lambda - \gamma)x + \tilde{C}_\gamma & \text{for } x < 0, \end{cases} \quad (6.22)$$

2 In fact, if w^λ does not satisfies (6.22), we cannot have (6.20) and (6.21). Using Theorem 4.1, we
3 get

$$w^\lambda < w \quad \text{for } |x| \geq \tilde{R}$$

4 if γ is small enough and \tilde{R} big enough. This implies that there exists a constant $C_{\tilde{R}} > 0$ such that
5 for all $x \in \mathbb{R}$, we have

$$w^\lambda(t, x) < w(t, x) + C_{\tilde{R}}.$$

6 Let us now introduce, $u(t, x) = w(t, x) + C_{\tilde{R}} - \bar{A}t$ and $u_\lambda(t, x) = w^\lambda(t, x) - \lambda t$ both solutions of
7 (2.4) with $\varepsilon = 1$ and $u_\lambda(0, x) \leq u(0, x)$. Therefore, the comparison principle implies

$$w^\lambda(t, x) - \lambda t \leq w(t, x) + C_{\tilde{R}} - \bar{A}t$$

8 Dividing by t and passing to the limit as t goes to infinity, we get

$$\bar{A} \leq \lambda,$$

9 which is a contradiction.

10

□

11 7 Qualitative properties of the flux limiter

12 This section is devoted to the proof of Proposition 2.16.

13 *Proof of Proposition 2.16.* In order to establish the monotonicity, we have to consider the ap-
14 proximated truncated cell problem (6.7). Let us consider v_1^δ and v_2^δ viscosity solutions of (6.7),
15 respectively for ϕ_1 and ϕ_2 , with $0 \leq \phi_1 \leq \phi_2$. First, using the fact that the non-local operator is
16 negative, we have

$$G_R^2(x, [U], q) \leq G_R^1(x, [U], q),$$

17 with

$$G_R^i(x, [U], q) = \phi_i(t, x) \cdot M[U](x) \cdot \psi_R(x) \cdot |q| + (1 - \psi_R(x))\bar{H}(q), \quad \text{for } i = 1, 2.$$

18 Therefore, we have

$$0 = \delta v_1^\delta + (v_1^\delta)_t + G_R^1(x, [v_1^\delta(t, \cdot)], (v_1^\delta)_x) \geq \delta v_1^\delta + (v_1^\delta)_t + G_R^2(x, [v_1^\delta(t, \cdot)], (v_1^\delta)_x),$$

19 meaning that v_1^δ is a sub-solution of (6.7) with ϕ_2 . The comparison principle and (6.8) imply that

$$0 \leq \delta v_1^\delta \leq \delta v_2^\delta \leq |H_0|.$$

20 Passing to the limit as $\delta \rightarrow 0$, we obtain

$$0 \geq \lambda_{l,R}^1 \geq \lambda_{l,R}^2 \geq H_0.$$

21 Passing to the limit as $l, R \rightarrow +\infty$, we get the result.

22

□

8 Link between the system of ODEs and the PDE

This section is devoted to the proof of Theorem 2.7, which is a direct application of our convergence result, Theorem 2.6.

Theorem 8.1. *For $\varepsilon = 1$, the cumulative distribution function ρ defined by (2.2) is a discontinuous viscosity solution of*

$$\rho_t + M[\rho(t, \cdot)](x) \cdot \phi(t, x) \cdot |\rho_x| = 0 \quad \text{for } (t, x) \in [0, +\infty) \times \mathbb{R}. \quad (8.1)$$

Conversely, if u is a bounded and continuous viscosity solution of (8.1) satisfying for some time $T > 0$, and for all $t \in (0, T)$

$$u(t, x) \text{ is decreasing in } x,$$

then the points $U_j(t)$, defined by $u(t, U_j(t)) = -(j+1)$ for $j \in \mathbb{Z}$, satisfy the system (2.1) on $(0, T)$.

Before giving the proof of Theorem 8.1, let us do the proof of Theorem 2.7.

Proof of Theorem 2.7. We recall that in Theorem 2.7, we have $u_0(x) = -x/h$, with $h \geq h_0$. First, we would like to prove that for all $\varepsilon > 0$, we have

$$|\rho^\varepsilon(0, x) - u_0(x)| \leq f(\varepsilon) \quad \text{for all } x \in \mathbb{R}, \quad (8.2)$$

with $f(\varepsilon) \rightarrow 0$ as ε goes to 0. To do this, we define a piece-wise affine function v satisfying

$$\rho^1(0, x) = v(x) \quad \text{for } x = U_i(0), \text{ for all } i \in \mathbb{Z}.$$

Given that for all $U_{i+1}(0) - U_i(0) \geq h_0$, we notice that v is k_0 -Lipschitz continuous and by definition of $\rho^1(0, x)$, we have

$$|\rho^1(0, x) - v(x)| \leq 1 \quad \text{for all } x \in \mathbb{R}.$$

Let us consider the integer $i_0 \in \mathbb{N}$ defined by

$$i_0 = \sup \{i \in \mathbb{Z}, \text{ s.t. } U_i(0) \leq -R\}.$$

Using the assumption that for all $i \in \mathbb{Z}$ such that $U_i(0) \leq -R$ we have $U_{i+1}(0) - U_i(0) = h$, we deduce that for all $x \leq U_{i_0}(0)$

$$v(x) = -\frac{x}{h} + \frac{U_{i_0}(0)}{h} + \rho^1(0, U_{i_0}(0)) = -\frac{x}{h} + \frac{U_{i_0}(0)}{h} - i_0 - 1.$$

Let us now consider the integer $i_1 \in \mathbb{N}$ defined by

$$i_1 = \inf \{i \in \mathbb{Z}, \text{ s.t. } U_i(0) \geq R\}.$$

Now using the assumption that for all $i \in \mathbb{Z}$ such that $U_i(0) \geq R$ we have $U_{i+1}(0) - U_i(0) = h$, we deduce that for all $x \geq U_{i_1}(0)$

$$v(x) = -\frac{x}{h} + \frac{U_{i_1}(0)}{h} + \rho^1(0, U_{i_1}(0)) = -\frac{x}{h} + \frac{U_{i_1}(0)}{h} - i_1 - 1.$$

Moreover, we recall that for all $\varepsilon > 0$, we have $\rho^\varepsilon(0, x) = \varepsilon \rho^1(0, x/\varepsilon)$, this implies that for all $x \notin [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$,

$$\begin{aligned} |\rho^\varepsilon(0, x) - u_0(x)| &\leq \left| \rho^\varepsilon(0, x) - \varepsilon v\left(\frac{x}{\varepsilon}\right) \right| + \left| \varepsilon v\left(\frac{x}{\varepsilon}\right) - u_0(x) \right| \\ &\leq \varepsilon + \varepsilon \max \left(\left| \frac{U_{i_1}(0)}{h} - i_1 - 1 \right|, \left| \frac{U_{i_0}(0)}{h} - i_0 - 1 \right| \right). \end{aligned} \quad (8.3)$$

Similarly, we have for all $x \in [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$,

$$\begin{aligned} |\rho^\varepsilon(0, x) - u_0(x)| &\leq \left| \rho^\varepsilon(0, x) - \varepsilon v\left(\frac{x}{\varepsilon}\right) \right| + \left| \varepsilon v\left(\frac{x}{\varepsilon}\right) - \varepsilon u_0\left(\frac{x}{\varepsilon}\right) \right| \\ &\leq \varepsilon + \varepsilon \max_{y \in [U_{i_0}(0), U_{i_1}(0)]} (|v(y) - u_0(y)|), \end{aligned} \quad (8.4)$$

where we have used the fact that $\varepsilon u_0(x/\varepsilon) = u_0(x)$. Combining (8.3) and (8.4) and choosing

$$f(\varepsilon) = \varepsilon + \varepsilon \max \left(\left| \frac{U_{i_0}(0)}{h} - i_0 - 1 \right|, \max_{y \in [U_{i_0}(0), U_{i_1}(0)]} (|v(y) - u_0(y)|), \left| \frac{U_{i_1}(0)}{h} - i_1 - 1 \right| \right)$$

1 we deduce (8.2). Notice also that thanks to (8.2), we have

$$|(\rho^\varepsilon)^*(0, x) - u_0(x)| \leq f(\varepsilon) + \varepsilon. \quad (8.5)$$

2 Therefore, we have

$$u_0(x) - f(\varepsilon) \leq \rho^\varepsilon(0, x) \leq (\rho^\varepsilon)^*(0, x) \leq u_0(x) + f(\varepsilon) + \varepsilon.$$

3 Using the fact that ρ^ε is a viscosity solution of (2.4) and the comparison principle (Proposition
4 3.5) we deduce that (with u^ε the continuous solution of (2.4) associated to the initial condition
5 $u_0(x) = -x/h$)

$$u^\varepsilon(t, x) - f(\varepsilon) \leq \rho^\varepsilon(t, x) \leq (\rho^\varepsilon)^*(t, x) \leq u^\varepsilon(t, x) + f(\varepsilon) + \varepsilon,$$

6 where we have used the fact that (2.4) is invariant by addition of constants to the solutions.
7 Passing to the limit as $\varepsilon \rightarrow 0$ and using Theorem 2.6 we get that $\rho^\varepsilon \rightarrow u^0$, which ends the proof
8 of Theorem 2.7.

9 □

10 *Proof of Theorem 8.1.* Theorem 8.1 is a consequence of the following lemma.

11 **Lemma 8.2** (Link between the velocities). *Assume (A). Let $((U_j)_j)$ be the solution of (2.1) with*

$$U_{j+1}(0) - U_j(0) > h_0. \quad (8.6)$$

12 *Then we have*

$$\dot{U}_j(t) = -M[u(t, \cdot)](U_j(t)) \cdot \phi(t, U_j(t)), \quad (8.7)$$

13 *where E and J are defined in (2.6) and $u(t, x)$ is a continuous function such that*

$$\begin{cases} u(t, x) = \rho_*(t, x) = \rho(t, x) \text{ for } x = U_j(t), j \in \mathbb{Z}, \\ u \text{ is decreasing in } x, \end{cases} \quad (8.8)$$

14 *with ρ defined in (2.2) (with $\varepsilon = 1$).*

15 *Proof.* We drop the time dependence to simplify the presentation. Let $j \in \mathbb{Z}$. Using the fact that
16 $u(U_j) = -(j+1)$ and (8.8), we have for all $z \in [0, +\infty)$,

$$\begin{cases} 0 \geq u(U_j + z) - u(U_j) > u(U_{j+1}) - u(U_j) = -1 & \text{if } z \in [0, U_{j+1} - U_j) \\ -1 \geq u(U_j + z) - u(U_j) & \text{if } z \in [U_{j+1} - U_j, +\infty). \end{cases}$$

17 Given that u is continuous, this implies that

$$M[u](U_j) = \int_0^{U_{j+1} - U_j} \frac{1}{2} J(z) dz + \int_{U_{j+1} - U_j}^{+\infty} \frac{3}{2} J(z) dz - \frac{3}{2} V_{max} = -V(U_{j+1} - U_j).$$

18 Combining this result with (2.1), we obtain (8.7).

19 □

1 Noticing that because of (8.8), we have for $x = U_j(t)$, $j \in \mathbb{Z}$,

$$\tilde{M}[\rho_*(t, \cdot)](x) = \tilde{M}[u(t, \cdot)](x) = M[u(t, \cdot)](x),$$

2 and using Lemma 8.2, and Definition 3.1, we can see that ρ_* is a discontinuous viscosity super-
3 solution of (8.1). We obtain a similar result for ρ^* , therefore, ρ is a discontinuous viscosity solution
4 of (8.1).

5 We prove the converse. For the readers convenience we recall Proposition 4.8 from [10] that
6 we will use later. The proof of this proposition remains almost the same in our case the only
7 difference being the definition of the functions E and \tilde{E} .

8 **Lemma 8.3.** *Assume that $\theta : \mathbb{R} \rightarrow \mathbb{R}$ is a non-decreasing and upper semi-continuous (resp. lower*
9 *semi-continuous). Assume also that*

$$\theta(v) - v \text{ is } 1\text{-periodic in } v.$$

10 *Assume that $\varepsilon = 1$ in (2.4). Consider also a sub-solution (resp. a super-solution) u of (2.4). Then*
11 *$\theta(u)$ is also a sub-solution (resp. a super-solution) of (2.4).*

12 Using Lemma 8.3 we can conclude that $\rho_* = \lceil u \rceil$ (resp. $\rho^* = \lfloor u \rfloor$) is a viscosity super-solution
13 (resp. sub-solution) of

$$\partial_t \rho - \tilde{c}(t, x) \partial_x \rho = 0 \quad \text{with } \tilde{c}(t, x) = M[u(t, \cdot)](x) \cdot \phi(t, x) = \tilde{M}[u(t, \cdot)](x) \cdot \phi(t, x).$$

14 Using the fact that u is decreasing in space, we define

$$U_i(t) = \inf\{x, u(t, x) \leq -(i+1)\} = (u(t, \cdot))^{-1}(-i-1)$$

15 and we consider the functions $t \mapsto U_i(t)$. They are continuous because u is decreasing in x and is
16 continuous in (t, x) .

17 We now prove that the functions U_i are viscosity solutions of (2.1). Let φ be a test function
18 such that $\varphi(t) \leq U_i(t)$ and $\varphi(t_0) = U_i(t_0)$. Let us now define $\hat{\varphi}(t, x) = -(i+1) + \varphi(t) - x$. It
19 satisfies

$$\hat{\varphi}(t_0, U_i(t_0)) = \rho_*(t_0, U_i(t_0)),$$

20 and

$$\hat{\varphi}(t, x) \leq \rho_*(t, x) \quad \text{for } U_i(t) - 1 < x < U_{i+1}(t).$$

21 This implies that

$$\begin{aligned} \varphi_i(t_0) + \tilde{c}(t_0, U_i(t_0)) &\geq 0 \\ \Leftrightarrow \varphi_i(t_0) &\geq -\tilde{c}(t_0, U_i(t_0)) = -\bar{c}_i(t_0) = V(U_{i+1}(t_0) - U_i(t_0)) \cdot \phi(t, U_i(t_0)). \end{aligned}$$

22 This proves that U_i are viscosity super-solutions of (2.1). The proof for sub-solutions is similar
23 and we skip it. Moreover, since \bar{c}_i is continuous, we deduce that $U_i \in C^1$ and it is therefore a
24 classical solution of (2.1). □

25

26 ACKNOWLEDGMENTS

27 The authors would like to thank R. Monneau for fruitful discussion during the preparation
28 of this paper. This project was co-financed by the European Union with the European regional
29 development fund (ERDF, HN0002137) and by the Normandie Regional Council via the M2NUM
30 project and by ANR HJNet (ANR-12-BS01-0008-01).

References

- [1] Y. ACHDOU, F. CAMILLI, A. CUTRÌ, AND N. TCHOU, *Hamilton-jacobi equations constrained on networks*, Nonlinear Differential Equations and Applications NoDEA, 20 (2013), pp. 413–445.
- [2] Y. ACHDOU AND N. TCHOU, *Hamilton-jacobi equations on networks as limits of singularly perturbed problems in optimal control: dimension reduction*, NETCO, (2014).
- [3] O. ALVAREZ AND A. TOURIN, *Viscosity solutions of nonlinear integro-differential equations*, in Annales de l’Institut Henri Poincaré. Analyse non linéaire, vol. 13, Elsevier, 1996, pp. 293–317.
- [4] A. AW, A. KLAR, M. RASCLE, AND T. MATERNE, *Derivation of continuum traffic flow models from microscopic follow-the-leader models*, SIAM Journal on Applied Mathematics, 63 (2002), pp. 259–278.
- [5] M. BATISTA AND E. TWRDY, *Optimal velocity functions for car-following models*, Journal of Zhejiang University SCIENCE A, 11 (2010), pp. 520–529.
- [6] F. DA LIO, C. I. KIM, AND D. SLEPČEV, *Nonlocal front propagation problems in bounded domains with neumann-type boundary conditions and applications*, Asymptotic Analysis, 37 (2004), pp. 257–292.
- [7] M. DI FRANCESCO AND M. D. ROSINI, *Rigorous derivation of the lighthill-whitham-richards model from the follow-the-leader model as many particle limit*, arXiv preprint arXiv:1404.7062, (2014).
- [8] L. C. EDIE, *Car-following and steady-state theory for noncongested traffic*, Operations Research, 9 (1961), pp. 66–76.
- [9] N. FORCADEL, C. IMBERT, AND R. MONNEAU, *Homogenization of fully overdamped frenkel-kontorova models*, Journal of Differential Equations, 246 (2009), pp. 1057–1097.
- [10] ———, *Homogenization of some particle systems with two-body interactions and of the dislocation dynamics*, Discrete and Continuous Dynamical Systems-Series A, 23 (2009).
- [11] ———, *Homogenization of accelerated frenkel-kontorova models with n types of particles*, Transactions of the American Mathematical Society, 364 (2012), pp. 6187–6227.
- [12] N. FORCADEL AND W. SALAZAR, *Homogenization of second order discrete model and application to traffic flow*, Differential and Integral Equations, 28 (2015), pp. 1039–1068.
- [13] N. FORCADEL, W. SALAZAR, AND M. ZAYDAN, *Homogenization of second order discrete model with local perturbation and application to traffic flow*, (2016).
- [14] G. GALISE, C. IMBERT, AND R. MONNEAU, *A junction condition by specified homogenization and application to traffic lights*, Analysis & PDE, 8 (2015), pp. 1891–1929.
- [15] B. GREENSHIELDS, W. CHANNING, H. MILLER, ET AL., *A study of traffic capacity*, in Highway research board proceedings, vol. 1935, National Research Council (USA), Highway Research Board, 1935.
- [16] D. HELBING, *From microscopic to macroscopic traffic models*, in A perspective look at nonlinear media, Springer, 1998, pp. 122–139.
- [17] C. IMBERT, *A non-local regularization of first order hamilton-jacobi equations*, Journal of Differential Equations, 211 (2005), pp. 218–246.

- 1 [18] C. IMBERT AND R. MONNEAU, *Flux-limited solutions for quasi-convex hamilton-jacobi equa-*
2 *tions on networks*, arXiv preprint arXiv:1306.2428, (2013).
- 3 [19] C. IMBERT, R. MONNEAU, AND E. ROUY, *Homogenization of first order equations with*
4 *(u/ε)-periodic hamiltonians part ii: Application to dislocations dynamics*, *Communications in*
5 *Partial Differential Equations*, 33 (2008), pp. 479–516.
- 6 [20] C. IMBERT, R. MONNEAU, AND H. ZIDANI, *A hamilton-jacobi approach to junction problems*
7 *and application to traffic flows*, *ESAIM: Control, Optimisation and Calculus of Variations*,
8 19 (2013), pp. 129–166.
- 9 [21] H. LEE, H.-W. LEE, AND D. KIM, *Macroscopic traffic models from microscopic car-following*
10 *models*, *Physical Review E*, 64 (2001), p. 056126.
- 11 [22] M. J. LIGHTHILL AND G. B. WHITHAM, *On kinematic waves. ii. a theory of traffic flow on*
12 *long crowded roads*, *Proceedings of the Royal Society of London. Series A. Mathematical and*
13 *Physical Sciences*, 229 (1955), pp. 317–345.
- 14 [23] P. L. LIONS, *Lectures at collège de france*, 2013-2014.
- 15 [24] G. F. NEWELL, *Nonlinear effects in the dynamics of car following*, *Operations Research*, 9
16 (1961), pp. 209–229.
- 17 [25] P. I. RICHARDS, *Shock waves on the highway*, *Operations research*, 4 (1956), pp. 42–51.
- 18 [26] D. SLEPČEV, *Approximation schemes for propagation of fronts with nonlocal velocities and*
19 *neumann boundary conditions*, *Nonlinear Analysis: Theory, Methods & Applications*, 52
20 (2003), pp. 79–115.