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# Robust control of an inclined cable using a partial differential equation model

Lucie Baudouin, Aude Rondepierre, Simon Neild.

**Abstract**—This paper presents a detailed study of the robust control of an inclined cable’s vibrations, with emphasis on considering a model of infinite dimension. Indeed, considering the partial differential equation model of the vibrations of an inclined cable with sag, we are interested in illustrating the application of  $\mathcal{H}_\infty$ -robust feedback control to this infinite dimensional system. The approach relies on Riccati equations to stabilize the system under measurement feedback when it is submitted to external disturbances. Henceforth, our study focuses on the construction of a standard linear infinite dimensional state space description of the inclined cable under consideration before writing its approximation of finite dimension and studying the  $\mathcal{H}_\infty$  feedback control of vibrations with partial observation of the state in both cases. The closed loop system is numerically simulated to demonstrate the effectiveness of the resulting control law.

**Keyword:** robust control, inclined cable, partial differential equations, state-space model, measurement feedback.

## I. INTRODUCTION

Inclined cables are common and critical components in a lot of civil engineering’s structures and a large range of applications, from cable stayed bridges to telescopes and spacecraft [SBP11]. Since cables are very flexible and lightly damped, one of the major issues related to such structures involving cables is the control of vibrations induced by any exterior perturbation [Reg05]. Their modeling is therefore very important in predicting and controlling the response to excitation. Many cable models exist, see for instance [Irv81]. Of interest here is the modal formulation developed in [WFS95]. This model has been partly validated experimentally in [GBNWM08] and [MDN<sup>+</sup>10]. Vibration suppression in civil structures is also well documented, see [SSL06] and [Pre97] and reference therein. Passive dampers are the usual devices in civil structures but active control is potentially more effective and adaptive [FS94].

In this paper we study the design of robust control laws for a vibrating system composed of an inclined cable connected at its bottom end to an active control device in the framework of distributed parameter systems. More precisely, we work on a linearized model using partial differential equations (PDE) and choose a model-based feedback approach to disturbance rejection, namely the  $\mathcal{H}_\infty$  measurement feedback control of the

vibrating inclined cable. We can mention here [BD90] which consider the suppression of vibrations in flexible structures with similar  $\mathcal{H}_\infty$ -approaches, but in the finite dimensional setting.

The contribution of our article is first to illustrate with a specific application, namely the control of an inclined cable, a theoretical result presented in [BB93] and [vK93] that gives the  $\mathcal{H}_\infty$ -robust control of infinite dimensional systems in terms of solvability of two coupled Riccati equations. Adopting this approach, we detail the PDE modeling of the system so that it fits into the appropriate state-space framework. Then, recalling the key aspects of the control theorem, we demonstrate that the required assumptions are met. Secondly, we approach the infinite-dimensional robust control problem by appropriate finite-dimensional robust control problems and perform numerical simulations.

Before treating any robust control issues, we focus in Section II on the modeling of the inclined cable in the state-space framework. The first step is the construction of a mechanical model of the inclined cable, subject to gravitational effects (hence termed a *cable* rather than a *string*, a designation corresponding to a situation without sag). Among the numerous possibilities for modeling the motion of inclined cables with small sag, we adopt an approach largely inspired from that presented in [WN10], which is based on the derivation of [WFS95]. In this model, the cable response is decomposed into a modal component which captures the dynamic response of the cable with fixed ends, and a quasi-static component which satisfies the boundary conditions but has no significant dynamic response. In a second step we describe how to control the inclined cable system by the mean of an active tendon, bringing active damping into the cable structure [BP01]. Lastly, we reformulate the robust control problem into an appropriate state-space framework. Section III is devoted to robust control issues. We first recall the  $\mathcal{H}_\infty$  robust control theorem for infinite dimensional systems stated in [vK93] and [BB93]. Then, this is applied to the inclined cable control system once we prove the required assumptions in terms of stabilizability and detectability of the system. In Section IV, we explore an early lumping approach: the infinite-dimensional control system is approximated by a sequence of finite dimensional systems (each corresponding to a finite number of modes of the inclined cable) that will be robustly controlled using usual  $\mathcal{H}_\infty$  tools of automatic control. This section is then dedicated to show some numerical experiments.

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## II. INFINITE DIMENSIONAL MODEL

As described in Figure 1, we consider a cable of length  $\ell$ , supported at end points  $a$  and  $b$ , such that the direction of the chord line from  $a$  to  $b$  is defined as  $x$ , and the angle of inclination relative to the horizontal is denoted  $\theta$ .

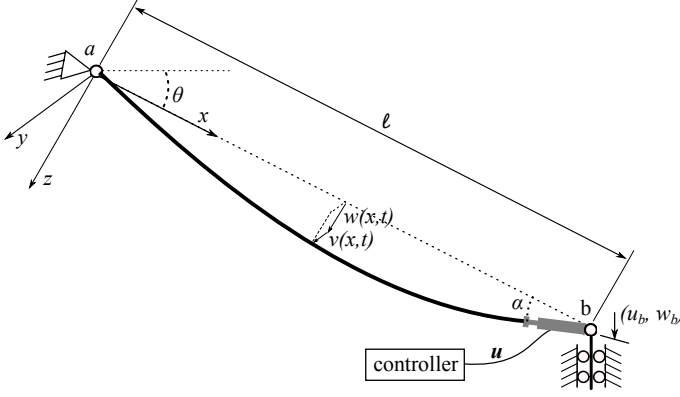


Fig. 1. Inclined Cable

Moreover we set  $\rho$  to be the density of the cable,  $A$  the cross-sectional area,  $E$  Young's modulus and  $g$  the gravity. We then define  $\varrho = \rho g \cos \theta$  as the distributed weight perpendicular to the cable chord. The cable equilibrium sag position and the chord line both lie in the gravity plane, namely the  $xz$ -plane.

### A. Modeling of an inclined cable

The modeling of an inclined cable presented hereafter is largely inspired from those detailed in [WN10, sections 7.2 and 7.3], but the final equations of the motion are not exactly the same. Indeed, using a similar approach to [WN10], but with an emphasis on the perturbed dynamics, rather than nonlinearity, we now derive a model that satisfies the time dependent boundary constraints.

Here, we will only recall the main outlines of the modeling hereafter and highlight the main differences from what presented in [WN10]. Let us start by introducing the notations:

- $u(x, t)$  is the dynamic axial displacement of the cable (in  $x$ -direction).
- $v(x, t)$  is the dynamic out-of-plane transverse displacement (in  $y$ -direction).
- $w(x, t)$  is the dynamic in-plane transverse displacement (in  $z$ -direction).
- $T_s$  is the static tension of the cable and is assumed to be constant (w.r.t.  $x$  and  $t$ ).
- $w_s(x) = \varrho A (\ell x - x^2) / 2T_s$  is the static in-plane displaced shape of the cable.
- $T(x, t)$  is the dynamic tension of the cable. As long as the cable remains within its elastic range, we have the nonlinear equation:

$$T = AE \left[ \partial_x u + \frac{1}{2} (\partial_x v)^2 + \frac{1}{2} (\partial_x w)^2 + \frac{dw_s}{dx} \partial_x w \right].$$

Note that the sag is assumed small in comparison to the length of the cable. Still, the sag affects the static deflexion of the cable so that  $w_s$  can be calculated precisely [WN10].

In our study the cable is excited vertically at its lower end. This yields the following boundary conditions corresponding to the support motion: for all  $t > 0$

$$\begin{cases} u(0, t) = 0, & u(\ell, t) = u_b(t), \\ v(0, t) = 0, & v(\ell, t) = 0, \\ w(0, t) = 0, & w(\ell, t) = w_b(t). \end{cases} \quad (1)$$

To satisfy these time-varying conditions, we assume that the cable response can be decomposed into a *modal* component (denoted by the subscript  $m$ , as in  $u_m, w_m, T_m$ ) which captures the dynamic response of the cable with fixed ends (boundary conditions equals to zero), and a *quasi-static* component (denoted by the subscript  $q$ , as in  $u_q, w_q, T_q$ ) which corresponds to the displacements of the cable moving as an elastic tendon due to support movement, and satisfies the boundary conditions (1).

In this paper we are interested in studying the robust control of the linear PDE approximation. Therefore, we will ignore the non-linearities of the cable dynamics that are often included in the literature, see for instance [NP04], [WFS95]. In [WN10, section 7.2], linearizing the equations enables the authors to completely decouple the quasi-static and modal terms under the assumption that quasi-static and modal motions are small compared with the static sag. Here we choose a slightly different approach: we first write and solve the quasi-static equations of motion as done in [WN10, section 7.2], and then we reinject these solutions in the complete equations of motion to obtain the modal PDE.

Let us first linearize the dynamic tension:  $T$  is then given by, for all  $(x, t)$  in  $(0, \ell) \times (0, \infty)$ ,

$$T(x, t) = AE \left[ \partial_x u(x, t) + \frac{dw_s}{dx}(x) \partial_x w(x, t) \right]. \quad (2)$$

As with [WFS95] and others, we assume that there is no significant dynamic response along the  $x$ -axis (meaning that we do not consider any evolution equation on the variable  $u$ , i.e. in particular  $u_m = 0$ ) as the axial vibrations are usually excluded from models since the frequency of oscillations is much faster and of smaller amplitude than that in the other directions. Assuming a small linearized dynamic tension compared to the static tension (i.e.  $T \ll T_s$ ), the equations of motion for the dynamic analysis of the inclined cable are given, for all  $(x, t)$  in  $(0, \ell) \times (0, \infty)$ , by:

$$\begin{cases} \rho A \partial_{tt} v(x, t) = T_s \partial_{xx} v(x, t), \\ \rho A \partial_{tt} w(x, t) = T_s \partial_{xx} w(x, t) + T(x, t) \frac{d^2 w_s}{dx^2}. \end{cases}$$

Observe that when linearizing the dynamic tension of the cable, we lose the sole coupling between  $v$  and  $w$ . The out-of-plane motion  $v$  satisfies a conservative wave equation that could only be influenced by coupling nonlinearities we don't consider here. Since the control and the perturbations will only act in the gravity plane ( $xz$ ), we can not consider the out-of-plane motion  $v$  as a part of our control system anymore. Therefore  $v$  will not appear in the construction of our state space model.

We now focus on the in-plane motion for the dynamic analysis of the inclined cable:

$$\rho A \partial_{tt} w(x, t) = T_s \partial_{xx} w(x, t) + T(x, t) \frac{d^2 w_s}{dx^2} \quad (3)$$

along with the boundary conditions (1) corresponding to the support motion and some appropriate initial data. On the one hand, we will calculate the quasi-static components  $u_q$  and  $w_q$  corresponding to the motion of the cable without taking into account any dynamic response, but using the boundary conditions (1). On the other hand, homogeneous Dirichlet boundary conditions along with the initial data allow to write a well-posed Cauchy problem for the modal components.

As detailed in [WN10], we first have to solve the quasi-static equations with time dependent boundary conditions, i.e. precisely, for all  $(x, t)$  in  $(0, \ell) \times (0, \infty)$ :

$$\begin{cases} T_q = AE \left[ \partial_x u_q + \frac{dw_s}{dx}(x) \partial_x w_q \right], \\ T_s \partial_{xx} w_q + T_q \frac{d^2 w_s}{dx^2} = 0, \\ u_q(0, t) = 0, \quad u_q(\ell, t) = u_b(t), \\ w_q(0, t) = 0, \quad w_q(\ell, t) = w_b(t), \end{cases} \quad (4)$$

whose solutions are:

$$\begin{cases} u_q(x, t) = \frac{E_q}{E} u_b(t) \frac{x}{\ell} - \frac{\rho A \ell}{2T_s} w_b(t) \left[ \frac{x}{\ell} - \left( \frac{x}{\ell} \right)^2 \right] \\ \quad + \frac{\lambda^2 E_q}{4E} u_b(t) \left[ \frac{x}{\ell} - 2 \left( \frac{x}{\ell} \right)^2 + \frac{4}{3} \left( \frac{x}{\ell} \right)^3 \right], \\ w_q(x, t) = w_b(t) \frac{x}{\ell} - \frac{\rho E_q \ell A^2}{2T_s^2} u_b(t) \left[ \frac{x}{\ell} - \left( \frac{x}{\ell} \right)^2 \right], \\ T_q(t) = \frac{AE_q}{\ell} u_b(t), \end{cases} \quad (5)$$

where  $E_q = E/(1 + \lambda^2/12)$  is the equivalent modulus of the cable and  $\lambda^2 = E \rho^2 \ell^2 A^3 / T_s^3$  the Irvine's parameter.

Then, because  $u_m = 0$ , the modal dynamic tension satisfies

$$T_m = AE \frac{dw_s}{dx} \partial_x w_m = \frac{\rho A^2 E}{2T_s} (\ell - 2x) \partial_x w_m$$

and from (3) and (4), the in-plane modal displacement is solution of the following PDE on  $(0, \ell) \times (0, \infty)$ :

$$\rho A \partial_{tt} (w_q + w_m) = T_s \partial_{xx} w_m + T_m \frac{d^2 w_s}{dx^2},$$

submitted to homogeneous Dirichlet boundary conditions  $w_m(0, t) = 0$ ,  $w_m(\ell, t) = 0$  for all  $t \in (0, \infty)$  and initial conditions equal to zero.

Since one can calculate easily  $\partial_{tt} w_q$  from (5) and since  $d^2 w_s / dx^2 = -\rho A / T_s$ , we obtain the self-contained equation on  $(0, \ell) \times (0, \infty)$

$$\begin{aligned} \partial_{tt} w_m &= \frac{T_s}{\rho A} \partial_{xx} w_m - \frac{\rho^2 A^2 E}{2\rho T_s^2} (\ell - 2x) \partial_x w_m \\ &\quad - \xi \partial_t w_m - \frac{x}{\ell} w_b'' + \frac{\rho E_q \ell A^2}{2T_s^2} \left[ \frac{x}{\ell} - \left( \frac{x}{\ell} \right)^2 \right] u_b''. \end{aligned} \quad (6)$$

One can notice that we added to our hyperbolic equation a realistic viscous damping term  $\xi \partial_t w_m$ ,  $\xi$  being a positive

diagonal bounded operator, that will take the shape of a *modal damping* when translated in the finite dimensional system constructed in Section IV.

### B. Modeling of the measurement and control terms

The inclined cable device pictured in Figure 1, is perturbed by in-plane oscillations  $(u_b, w_b)$  and connected at its bottom end with an active tendon. Using a support motion at the cable's anchorage is a natural choice of active control since the installation of the proper device can be done with small modifications of the lower end of the cable [FS94]. Moreover, we aim to obtain good results when considering robust control with partial observation using an active tendon since the collocation of actuator and sensor has proved great effectiveness in active damping of cables, as presented in [BP01] and [SSL06].

An active tendon can be described as a displacement actuator collocated with a force sensor (see e.g. [PB00]). Therefore, on the one hand, the force sensor allows to define the dynamic tension at the location of the tendon  $T(\ell, t)$  as the observation we can measure to build our feedback. On the other hand, even if the action of a tendon is principally meant to be an axial movement [Pre97], a careful consideration of the projection of the tendon's displacement on the  $x$  and  $z$ -axis shows that its action can be written in terms of the angle  $\alpha$  it makes with the chord line (see Fig. 1). It gives easily a control of coordinates  $(\mathbf{u} \cos \alpha, \mathbf{u} \sin \alpha)$ . Approximating  $\cos \alpha$  and  $\sin \alpha$  since  $\alpha$  should be very small (as the sag is small), this means we have to consider two different contributions of the control of intensity  $\mathbf{u}$  in equation (6): one is an additive displacement term to the perturbation  $u_b''$  and writes  $\alpha_c (1 - \frac{\alpha^2}{2}) \mathbf{u}''$  with  $\alpha_c = \frac{\rho E_q \ell A^2}{2T_s^2} \left[ \frac{x}{\ell} - \left( \frac{x}{\ell} \right)^2 \right]$ ; the other one takes the shape  $\alpha \mathbf{u}''$  added to the perturbation  $w_b''$ .

*Remark 1:* Using the denominations given in [FS94] or [Pre97] for instance, the axial part of the control we use is actually an *inertial control*  $\alpha_c \mathbf{u}''$  and if we have this sole contribution, we only have access to half of the modes of vibration (the symmetric ones). A *parametric control*, that would take the shape  $\mathbf{u} w_m$  in the equation if it hasn't been linearized, usually gives access to the control of all the vibration modes. But the linear framework we work with has lost track of this bilinear control. Therefore, to overcome this, we consider that the active tendon also acts through the in-plane bottom displacement as a small proportion of  $\mathbf{u}''$  added to the perturbation  $w_b$  as explained above. It has the additional advantage of illustrating the alignment defect of the active tendon with the cable's chord.

Let us now translate this information into the equations. We now consider the following state equation on  $(0, \ell) \times (0, \infty)$ :

$$\begin{aligned} \partial_{tt} w_m &= \frac{T_s}{\rho A} \partial_{xx} w_m - \frac{\rho^2 A^2 E}{2\rho T_s^2} (\ell - 2x) \partial_x w_m \\ &\quad + \frac{\rho E_q \ell A^2}{2T_s^2} \left[ \frac{x}{\ell} - \left( \frac{x}{\ell} \right)^2 \right] (u_b'' + (1 - \frac{\alpha^2}{2}) \mathbf{u}'') \\ &\quad - \xi \partial_t w_m - \frac{x}{\ell} (w_b'' + \alpha \mathbf{u}'') \end{aligned} \quad (7)$$

with the localized measurement output

$$\begin{aligned} T(\ell, t) &= T_q(t) + T_m(\ell, t) \\ &= \frac{\mathcal{A}E_q}{\ell} u_b(t) - \frac{\rho \mathcal{A}^2 E \ell}{2T_s} \partial_x w_m(\ell, t). \end{aligned} \quad (8)$$

### C. State space model of the robust control system

In order to fit in the classical state space formalism, we now introduce the following notations: the state is  $X = (w_m, \partial_t w_m)^\top$ ; the exogenous disturbance, which include the time dependent boundary conditions, is  $W = (W_{\text{mod}}, u_b, w_b'')^\top$  where  $W_{\text{mod}}$  gathers uncertainty on the model (e.g. the neglected nonlinearities) and we assume that  $u_b'' = -\omega_u u_b$ ; the control input  $U = \mathbf{u}''$  is the acceleration of the displacement actuator; the measurement output  $Y = T(\ell, \cdot)$  is given by the force sensor (and is the sum of the modal tension  $T_m(\ell)$  - captured by  $C_2 X$  - and of the quasi-static tension  $T_q$  - captured by  $D_{21} W$ ); the “to be controlled” output  $Z$  will gather  $w_m$  and  $\mathbf{u}''$  in appropriate ways according to the robust control objectives.

The linear infinite-dimensional state-space model takes the usual shape (see [DZG96]), for all  $t > 0$

$$\begin{cases} X'(t) = AX(t) + B_1 W(t) + B_2 U(t), \\ Z(t) = C_1 X(t) + D_{12} U(t), \\ Y(t) = C_2 X(t) + D_{21} W(t), \end{cases} \quad (9)$$

with  $X(0) = 0$ , and is also formally described by the closed loop system sketched by the standard Figure 2.

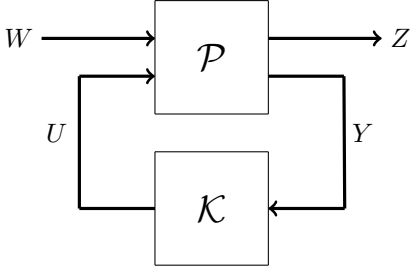


Fig. 2. Closed-loop system, Plant  $\mathcal{P}$ , controller  $\mathcal{K}$ .

Mainly based on equations (7)-(8), the operator matrices involved in (9) are therefore given by:

$$\begin{aligned} A &= \begin{pmatrix} 0 & I \\ \frac{T_s}{\rho \mathcal{A}} \partial_{xx} - \frac{\rho^2 \mathcal{A}^2 E}{2\rho T_s^2} (\ell - 2x) \partial_x & -\xi \end{pmatrix}, \\ B_1 &= \begin{pmatrix} 0 & 0 & 0 \\ d^1 & -\omega_u^2 \frac{\rho E_q \ell \mathcal{A}^2}{2T_s^2} \left[ \frac{x}{\ell} - \left( \frac{x}{\ell} \right)^2 \right] & -\frac{x}{\ell} \end{pmatrix}, \\ B_2 &= \begin{pmatrix} 0 \\ \left( 1 - \frac{\alpha^2}{2} \right) \frac{\rho E_q \ell \mathcal{A}^2}{2T_s^2} \left[ \frac{x}{\ell} - \left( \frac{x}{\ell} \right)^2 \right] - \alpha \frac{x}{\ell} \end{pmatrix}, \\ C_2 &= \begin{pmatrix} -\frac{\rho \mathcal{A}^2 E \ell}{2T_s} \partial_x \cdot |_{x=\ell} & 0 \end{pmatrix}, \\ D_{21} &= \begin{pmatrix} d^2 & \frac{\mathcal{A} E_q}{\ell} & 0 \end{pmatrix}, \end{aligned}$$

where the linear application  $d^1$  and the real number  $d^2$  are tuning parameters and  $\xi$  is the modal damping operator. Then, depending on the control objectives of performance, we can choose for instance

$$C_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad D_{12} = \begin{pmatrix} 0 \\ I \end{pmatrix}$$

that describes the objective of reducing the in-plane movement of the cable, while limiting the amplitude of the control, since here,  $Z = (w_m, \mathbf{u}'')^\top$ . But other objectives can and will be studied in numerical experiments later on.

The appropriate functional Hilbert spaces associated with the infinite-dimensional model are now precisely defined. We consider the state space

$$\mathcal{X} = H_0^1(0, \ell) \times L^2(0, \ell)$$

and the input or output spaces  $\mathcal{U} = \mathbb{R}$ ,  $\mathcal{W} = \mathbb{R}^3$ ,  $\mathcal{Y} = \mathbb{R}$ ,  $\mathcal{Z} = H_0^1(0, \ell) \times \mathbb{R}^2$ .

The operator  $A$  of domain  $\mathcal{D}(A) = (H^2 \cap H_0^1)(0, \ell) \times H_0^1(0, \ell)$  is the infinitesimal generator of a  $C_0$ -semigroup  $T(t) = e^{At}$  on the space  $\mathcal{X}$ , since  $-\partial_{xx}$  is a self-adjoint, non-negative and coercive operator, and  $(\ell - 2x)\partial_x$  is a linear bounded perturbation of it. One can rely on the classical theory of semi-groups as in [Paz83] or refer to the book [TW09]. Indeed, let us briefly study the well-posedness of equation (7), giving the regularity of its solutions. Under any initial data  $w_m(t=0) = w_0 \in H_0^1(0, \ell)$  and  $\partial_t w_m(t=0) = w_1 \in L^2(0, \ell)$ , assuming that  $u_b, w_b$  and  $u$  belong to  $W^{2,\infty}(\mathbb{R}_+)$  and that  $\xi \in \mathcal{L}(L^2(0, \ell); \mathbb{R}_+^*)$ , there exists a unique solution

$$w_m \in C(\mathbb{R}_+; H_0^1(0, \ell)) \cap C^1(\mathbb{R}_+; L^2(0, \ell)).$$

One can also prove that operators  $B_1 \in \mathcal{L}(\mathcal{W}, \mathcal{X})$ ,  $B_2 \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ ,  $C_1 \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ ,  $D_{12} \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$  and  $D_{21} \in \mathcal{L}(\mathcal{W}, \mathcal{Y})$  are bounded. We have to make a specific comment about the measurement output operator  $C_2$ . As long as we decide to rely only on a boundary observation (in  $x = \ell$ ) of the cable's tension,  $C_2$  does not belong to  $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ . Instead, one can confirm that  $C_2 \in \mathcal{L}(\mathcal{D}(A), \mathcal{Y})$ .

### III. ROBUST CONTROL ISSUES

In this section, we first recall the  $\mathcal{H}_\infty$ -robust control theorem for infinite-dimensional systems proved in [vK93] and revisited in [BB93] and [vK94] that we will apply in a second step to the PDE model of the inclined cable derived in Section II.

As its usual finite-dimension counterpart presented in a standard state-space approach (see for instance [DZG96] and [Tad90]), this result gives an equivalence between the  $\mathcal{H}_\infty$ -robust control with measurement-feedback of a PDE system and the solvability of two Riccati equations. A survey of the  $\mathcal{H}_\infty$ -control theory with state-feedback in the infinite-dimensional case can also be read [vKPC93] and we specifically refer to [vK94] for the case of unbounded observation operator as it is our situation here.

### A. $\mathcal{H}_\infty$ -control with measurement feedback

Let us first give some details about what we meant by  $\mathcal{H}_\infty$ -optimal control (or robust control) with measurement feedback. We assume that  $A$  is the infinitesimal generator of a  $C_0$ -semigroup on the space  $\mathcal{X}$  and  $B_1, B_2, C_1, C_2, D_{12}$  and  $D_{21}$  are bounded (or even unbounded, e.g. [vK94]) operators in the appropriate functional spaces. We can also make the normalization assumptions  $D_{12}^*[C_1 \ D_{12}] = [\mathbf{0} \ I]$  and  $D_{21}[D_{21}^* \ B_1^*] = [I \ \mathbf{0}]$  in order to simplify the formulation of the problem, but it is not mandatory (a change of variables can deal with the situation).

The state-space description (9) of the system, as implied by Figure 2, allows the plant  $\mathcal{P}$  to be controlled with the knowledge of the partial observation  $Y = C_2X + D_{21}W$  and under the cost function (corresponding to the norm of output  $Z$ )

$$J_0(U, W) = \int_0^\infty (\|C_1X(t)\|_{\mathcal{Z}}^2 + \|U(t)\|_{\mathcal{U}}^2) dt.$$

The objective is to construct a feedback controller  $\mathcal{K} = (A_K, B_K, C_K, D_K)$  of shape, for all  $t > 0$

$$\begin{cases} \Phi'(t) &= (A + A_K)\Phi(t) + B_KY(t), \\ U(t) &= C_K\Phi(t) + D_KY(t), \end{cases} \quad (10)$$

with  $\Phi(0) = 0$  where  $\Phi$ , the adjoint state, depends on the measurement  $Y$  and leads to the control  $U$ . Thus, the coupled system is written as follows:

$$\begin{cases} X' = (A + B_2D_KC_2)X + B_2C_K\Phi \\ \quad + (B_1 + B_2D_KD_{21})W \\ \Phi' = B_KC_2X + (A + A_K)C_2\Phi + B_KD_{21}W \end{cases}$$

and introduces an operator

$$\Lambda = \begin{pmatrix} A + B_2D_KC_2 & B_2C_K \\ B_KC_2 & A + A_K \end{pmatrix}.$$

The goal is to find a dynamic measurement-feedback controller  $\mathcal{K}$  that exponentially stabilizes this system (meaning that  $\Lambda$  is exponentially stable, and also yield to a finite cost  $J_0(C_K\Phi + D_KY, W)$ ) and ensures that the influence of the disturbances on the “to be controlled output”  $Z$ , i.e. the ratio

$$\rho(\mathcal{K}) = \sup_{W \in \mathcal{W}} \frac{J_0(C_K\Phi + D_KY, W)}{\|W\|_{\mathcal{W}}^2}$$

is smaller than some specific bound.

The result we will apply to the dynamic measurement feedback control of an inclined cable is the following.

*Theorem 1:* (Proof in [BB93] or [vK93])

Let  $\gamma > 0$  and assume that the pair  $(A, B_1)$  is stabilisable and the pair  $(A, C_1)$  is detectable. The following assertions are equivalent:

- (i) The  $\gamma^2$ -robustness property with partial observation hold for the system (9) : there exists an exponentially stabilising dynamic output-feedback controller  $\mathcal{K}$  of the form (10) such that  $\Lambda$  is exponentially stable and  $\rho(\mathcal{K}) < \gamma^2$ .
- (ii) There exist two nonnegative definite symmetric operators  $P, \Sigma \in \mathcal{L}(\mathcal{X})$  solutions of the Riccati and compatibility equations:

$$\bullet \forall X \in D(A), \quad PX \in D(A^*),$$

$$(PA + A^*P + P(B_2B_2^* - \gamma^{-2}B_1B_1^*)P + C_1^*C_1)X = 0$$

and  $A - (B_2B_2^* - \gamma^{-2}B_1B_1^*)P$  generates an exponentially stable semigroup ;

$$\bullet \forall X \in D(A^*), \quad PX \in D(A),$$

$$(\Sigma A^* + A\Sigma + \Sigma(C_2^*C_2 - \gamma^{-2}C_1^*C_1)\Sigma + B_1B_1^*)X = 0$$

and  $A^* - (C_2^*C_2 - \gamma^{-2}C_1^*C_1)\Sigma$  generates an exponentially stable semigroup ;

$$\bullet I - \gamma^{-2}P\Sigma \text{ is invertible and}$$

$$\Pi = \Sigma (I - \gamma^{-2}P\Sigma)^{-1} \geq 0.$$

Moreover, if these three conditions hold, then the feedback controller  $\mathcal{K}$  specified by

$$\begin{aligned} A_K &= -(B_2B_2^* - \gamma^{-2}B_1B_1^*)P - \Pi C_2^*C_2 \\ B_K &= \Pi C_2^*, \quad C_K = -B_2^*P, \quad D_K = 0 \end{aligned} \quad (11)$$

gives an exponentially stable operator  $\Lambda$  and guarantees that  $\rho(\mathcal{K}) < \gamma^2$ . Finally, if the solutions to the Riccati equations exists, then they are unique.

We will not discuss the technical assumptions given in [vK93] (or [vKPC93], [vK94]), leading specially to simplify the formulas. One will also find a slightly different formulation of this result in [Mor01]. Finally, another detailed proof is given in [Bar95], for the state-feedback case, allowing to consider the boundary control (thus unbounded) of hyperbolic equations.

One can notice that the feedback controller  $\mathcal{K}$  given in (11) is actually sub-optimal and known as the central controller.

### B. Controllability and observability assumptions

This subsection is devoted to the verification of the assumptions of Theorem 1 in the context where we wish to apply it, namely our inclined cable. We are therefore interested in proving the stabilizability of the pair  $(A, B_1)$  and the detectability of the pair  $(A, C_1)$  given in Section II-C. Since it is well-known that regarding PDE's (see [CZ95]), *exact controllability* implies exponential stabilizability, as well as for the dual properties we have *exact observability* implying exponential detectability, we actually focus instead on these specific properties.

In fact, using a result proved in [Rus67], we will precisely obtain that the wave equation  $X' = AX + B_1W$  is exactly controllable through  $W = (W_{\text{mod}}, u_b, w_b'')^T$ , which will imply the exponential stabilizability of the pair  $(A, B_1)$ . The specificity of the controllability result we need relies on the force distribution functions (e.g.  $x \mapsto -x/\ell$ ) through which the controls (e.g.  $w_b''$ ) are acting on the cable. On the other hand, the exponential detectability of the pair  $(A, C_1)$  will stem from the exact observability property easily proved through the method described in [TW09].

a) *Observability of the pair*  $(A, C_1)$ : According to the ideas about perturbed operators or semi-groups presented in [TW09], (see also [Kom89] or [LTY99]), the observability of  $(A, C_1)$  defined in Section II-C can be deduced for instance from the observability of the simplified (undamped and unperturbed) pair  $(A_0, C_0)$  defined by

$$A_0 = \begin{pmatrix} 0 & I \\ \partial_{xx} & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} I & 0 \end{pmatrix}.$$

The Hilbert space  $\mathcal{X} = H_0^1(0, \ell) \times L^2(0, \ell)$  is given with the scalar product

$$\begin{aligned} \left\langle \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right\rangle_{\mathcal{X}} &= \langle \partial_x f_1, \partial_x f_2 \rangle_{L^2} + \langle g_1, g_2 \rangle_{L^2} \\ &= \int_0^\ell \partial_x f_1(x) \overline{\partial_x f_2(x)} dx + \int_0^\ell g_1(x) \overline{g_2(x)} dx. \end{aligned}$$

On the one hand, defining  $\mathcal{D}(A) = H^2(0, \ell) \cap H_0^1(0, \ell) \times H_0^1(0, \ell)$ , we have

$$A_0 : (f, g) \in \mathcal{D}(A) \mapsto (g, \partial_{xx} f) \in \mathcal{X}$$

whose eigenvalues are  $\lambda_n = in\pi/\ell$  and eigenvectors take the shape,  $\forall n \in \mathbb{Z}^*$ :

$$\phi_n = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\ell}{in\pi} \varphi_n \\ \varphi_n \end{pmatrix}, \quad \varphi_n = \sqrt{\frac{2}{\ell}} \sin\left(\frac{n\pi x}{\ell}\right). \quad (12)$$

The operator  $A_0$  is well known to generate a unitary group  $\mathcal{T}_0$  on  $\mathcal{X}$  (e.g. semi-group theory or separation principle) given by,  $\forall (f, g) \in \mathcal{X}$ :

$$\begin{aligned} \mathcal{T}_0(t) \begin{pmatrix} f \\ g \end{pmatrix} &= \sum_{n \in \mathbb{Z}^*} e^{\lambda_n t} \left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \phi_n \right\rangle_{\mathcal{X}} \phi_n \\ &= \frac{1}{2} \sum_{n \in \mathbb{Z}^*} e^{\frac{in\pi}{\ell} t} (i \langle \partial_x f, \psi_n \rangle_{L^2} + \langle g, \varphi_n \rangle_{L^2}) \begin{pmatrix} \frac{\ell}{in\pi} \varphi_n \\ \varphi_n \end{pmatrix}. \end{aligned}$$

where  $\psi_n = \sqrt{\frac{2}{\ell}} \cos\left(\frac{n\pi x}{\ell}\right)$  for all  $n \in \mathbb{Z}$ .

On the other hand,  $C_0 \in \mathcal{L}(\mathcal{X}, H_0^1(0, \ell))$  is defined by

$$C_0 : (f, g) \in \mathcal{X} \mapsto f \in H_0^1(0, \ell)$$

and it is easy to prove that the pair  $(A_0, C_0)$  is exactly observable in time  $T > 2\ell$ . It requires for instance to prove there exists  $k > 0$  such that for all  $(f, g) \in \mathcal{D}(A)$ ,

$$\int_0^{2\ell} \left\| C_0 \mathcal{T}_0(t) \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{H_0^1(0, \ell)}^2 dt \geq k \left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{\mathcal{X}}^2. \quad (13)$$

Following [TW09], first we have

$$\begin{aligned} 4\mathcal{C} &:= 4 \int_0^{2\ell} \left\| C_0 \mathcal{T}_0(t) \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{H_0^1}^2 dt = \\ &= \int_0^{2\ell} \left\| \sum_{n \in \mathbb{Z}^*} e^{\frac{in\pi}{\ell} t} (i \langle \partial_x f, \psi_n \rangle_{L^2} + \langle g, \varphi_n \rangle_{L^2}) \frac{\ell \varphi_n}{in\pi} \right\|_{H_0^1}^2 dt \end{aligned}$$

and then, using the orthonormality of the family  $\left\{ t \mapsto \frac{\exp(\frac{in\pi}{\ell} t)}{\sqrt{2\ell}}, n \in \mathbb{Z}^* \right\}$  in  $L^2(0, 2\ell)$ , and the fact that  $\left\| \frac{\ell}{in\pi} \varphi_n \right\|_{H_0^1(0, \ell)} = 1$ , we can write

$$\mathcal{C} = \frac{\ell}{2} \sum_{n \in \mathbb{Z}^*} |i \langle \partial_x f, \psi_n \rangle_{L^2} + \langle g, \varphi_n \rangle_{L^2}|^2.$$

Using that  $\varphi_{-n} = -\varphi_n$ ,  $\psi_{-n} = \psi_n$  and the parallelogram identity  $(|a+b|^2 + |a-b|^2 = 2|a|^2 + 2|b|^2)$ , it follows

$$\mathcal{C} = \ell \sum_{n \in \mathbb{N}} (|\langle \partial_x f, \psi_n \rangle_{L^2}|^2 + |\langle g, \varphi_n \rangle_{L^2}|^2).$$

This last relation, together with the fact that the families  $\{\psi_n, n \in \mathbb{N}\}$  and  $\{\varphi_n, n \in \mathbb{N}^*\}$  are hilbertian (orthonormal) basis of  $L^2(0, \ell)$ , implies (13) since

$$\mathcal{C} = \int_0^{2\ell} \left\| C_0 \mathcal{T}_0(t) \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{H_0^1(0, \ell)}^2 dt = \ell \left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{\mathcal{X}}^2.$$

b) *Stabilizability of the pair*  $(A, B_1)$ : If we decide to apply the same method in order to prove the stabilizability of the pair  $(A, B_1)$ , the best we obtain is strong stabilizability, instead of exponential stabilizability.

First, as mentioned before, the exact observability of the dual pair  $(A^*, B_1^*)$  can be deduced from the exact observability of the simplified (undamped and unperturbed) pair  $(A_0, B_0^*)$ ,  $A_0$  being skew-adjoint, and  $B_0$  satisfying

$$B_0 = \begin{pmatrix} 0 & 0 \\ 1 & \mu(x) = \left[ \frac{x}{\ell} - \left( \frac{x}{\ell} \right)^2 \right] \end{pmatrix} \quad \nu(x) = -\frac{x}{\ell}.$$

One can easily calculate that for all  $W \in \mathbb{R}^3$ ,  $\forall (f, g) \in \mathcal{X}$ ,

$$\begin{aligned} \left\langle B_0 W, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathcal{X}} &= \langle W_{\text{mod}} + \mu u_b + \nu w_b'', g \rangle_{L^2} \\ &= \left\langle W, B_0^* \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathbb{R}^3} \end{aligned}$$

where  $B_0^* = \begin{pmatrix} 0 & \langle 1, \cdot \rangle_{L^2} \\ 0 & \langle \mu(x), \cdot \rangle_{L^2} \\ 0 & \langle \nu(x), \cdot \rangle_{L^2} \end{pmatrix}.$

We are seeking for exponential stabilizability of the pair  $(A_0, B_0^*)$  and one can prove it following [Rus67]. This reference, specifically dealing with the control theory of hyperbolic PDEs, is concerned with the case of control parameters which are function of time only, and proves null-controllability results through resolution of moments problems in  $L^2(0, T)$ . Relying on the result of exact controllability in time  $T > 2\ell$  the article [Rus67] presents, we only have to check the assumption that the control, written  $v(x)u(t)$ , has to satisfy:

$$\begin{aligned} \liminf_{n \rightarrow \infty} n |\langle v, \varphi_n \rangle_{L^2(0, \ell)}| &> 0, \\ \text{and } \langle v, \varphi_n \rangle_{L^2(0, \ell)} &\neq 0, \forall n \in \mathbb{N}^*. \end{aligned}$$

Since we have indeed, even with the simple control term  $\nu(x)w_b''$ :

$$\begin{aligned} \liminf_{n \rightarrow \infty} n |\langle \nu, \varphi_n \rangle_{L^2}| &= \frac{\sqrt{2\ell}}{\pi}, \\ \text{and } \langle \nu, \varphi_n \rangle_{L^2} &\neq 0, \forall n \in \mathbb{N}^*, \end{aligned}$$

the assumption on the pair  $(A_0, B_0)$ , thus on the pair  $(A, B_1)$  is proved, thanks to this  $w_b''$  control contribution.

*Remark 2:* In order to re-emphasize Remark 1, since the calculation of  $\langle \mu, \varphi_n \rangle_{L^2}$  gives  $2\sqrt{2\ell}(1 - (-1)^n)/n^3\pi^3$ , we can prove that the  $u_b$  control contribution has no influence on even-indexed modes.

*Remark 3:* The very same argument can also prove the stabilizability of the pair  $(A, B_2)$  for instance. This is one of the assumptions in [Mor01] to obtain convergence of the solution of the robust control - a subject that we will not tackle here.

*Remark 4:* One should notice that since we are considering a lightly damped wave equation (term  $-\xi \partial_t w_m$ ), it gives an intrinsically stable system. But as commonly acknowledged (e.g. in [LGM99, chapter 4.3]) one shouldn't rely on this natural damping property of a lot of mechanical systems to study their stabilizability properties through active control.

We have now verified the main assumptions of Theorem 1 for the inclined cable model system under study. The theoretical analysis of the robust control of our infinite dimensional system then ends with the application of Theorem 1. Nevertheless, the difficulty of solving the operator Riccati equations is well known. Since we aim also at performing numerical simulations of the robust control, we will now consider a truncated model built on the PDE system we have described and theoretically controlled as a disturbance-attenuation problem.

#### IV. TOWARDS NUMERICAL SIMULATIONS

We demonstrated in the previous sections that a state-space based controller for the infinite-dimensional  $\mathcal{H}_\infty$ -control problem may be calculated by solving two Riccati equations. However, these operator Riccati equations can rarely be solved exactly [Mor01]. Therefore, we choose to approximate the original infinite-dimensional system by a sequence of finite-dimensional systems that we will robustly control with the usual tools of automatic control ( $\mathcal{H}_\infty$ -synthesis). We will perform a modal decomposition of our linear partial differential equation model, so that system (9) of infinite dimension becomes a classical state space system suitable for Matlab© simulations.

*Remark 5:* One can also mention other studies of inclined cables in finite dimension. For instance, in [WFS95], a non-linear PDE model is decomposed into the two first vibration modes from which the precise non-linear coupling between in-plane  $w$  and out-of-plane  $v$  vibrations can be seen and studied. Besides, a finite element modeling approach has been used in [Pre97], where one can also find an introduction to active tendon control of cables. It won't be our approach here. We finally refer to [BBC07] for a detailed study of several computational methods for the stabilization in a flexible structure modeled by the beam equation.

##### A. Finite dimensional model, by modal truncation

We consider the hermitian base  $(\varphi_n)_{n \in \mathbb{N}^*}$  of  $L^2(0, \ell)$  defined in (12) and given by the eigenfunctions of the (compact self-adjoint) operator  $\frac{T_s}{\rho A} \partial_{xx}$ :

$$\frac{T_s}{\rho A} \partial_{xx} \varphi_n(x) = -\omega_n^2 \varphi_n(x),$$

for all  $x \in (0, \ell)$  and  $n \in \mathbb{N}^*$  and where  $\omega_n = \frac{n\pi}{\ell} \sqrt{\frac{T_s}{\rho A}}$  denote the eigenvalues of the operator. We choose here to achieve the modal decomposition through the separation of variables, which meets the Galerkin method used in [WN10, chap 7].

The modal in-plane movement  $w_m$  can be decomposed in the base  $(\varphi_k)_{k \in \mathbb{N}^*}$  i.e.:

$$w_m(x, t) = \sum_{k=1}^{+\infty} z_k(t) \varphi_k(x),$$

where  $z_k(t) = \langle w_m(\cdot, t), \varphi_k \rangle_{L^2}$  and since we assume initial conditions equal to zero, we have  $z_n(0) = z'_n(0) = 0, \forall n \geq 1$ .

The first step is to rewrite the modal equation (7) as a linear system of ordinary differential equations in  $(z_n)_{n \geq 1}$ . One should note in advance that the viscous damping term  $\xi \partial_t w_m$  will be translated in a modal damping shape in the process:

$$\xi \partial_t w_m = \sum_{k=1}^{\infty} 2\omega_k \xi_k z'_k(t) \varphi_k(x)$$

where  $\xi_k < 1$  is the ratio of the actual damping over the critical damping. In an unperturbed hyperbolic system, the critical damping represents the smallest amount of damping for which no oscillations occurs in the free vibration response. Therefore, by projection on the chosen hermitian base, we obtain:  $\forall n \geq 1$ ,

$$\begin{aligned} z''_n(t) &= -\omega_n^2 z_n(t) - 2\omega_n \xi_n z'_n(t) \\ &\quad - \frac{\rho^2 A^2 E}{2\rho T_s^2} \sum_{k=1}^{+\infty} \langle (\ell - 2x) \partial_x \varphi_k, \varphi_n \rangle_{L^2} z_k(t) \\ &\quad + \alpha_n u''_b + \beta_n w''_b + \left( \left(1 - \frac{\alpha^2}{2}\right) \alpha_n + \alpha \beta_n \right) \mathbf{u}'', \end{aligned} \quad (14)$$

where

$$\begin{aligned} \alpha_n &= \frac{\rho E_q \ell A^2}{2T_s^2} \left\langle \frac{x}{\ell} - \left(\frac{x}{\ell}\right)^2, \varphi_n \right\rangle_{L^2} \\ \text{and } \beta_n &= \left\langle -\frac{x}{\ell}, \varphi_n \right\rangle_{L^2} \end{aligned} \quad (15)$$

The measurement output  $Y$  then becomes:

$$Y(t) = \frac{A E_q}{\ell} u_b(t) - \frac{\rho A^2 E \ell}{2T_s} \sum_{k=1}^{+\infty} z_k(t) \partial_x \varphi_k(\ell).$$

Given  $N \in \mathbb{N}^*$ , we can thus construct a finite dimensional model using the truncated basis  $(\varphi_n)_{n=1, \dots, N}$  of the  $N$  first modes. As in part II-C, the control input is the acceleration of the displacement actuator:

$$U = \mathbf{u}'' \in \mathbb{R}.$$

The choice of the state variables is not unique but numerically it is convenient to choose:

$$X_N = (z'_1, \omega_1 z_1, \dots, z'_N, \omega_N z_N)^\top \in \mathbb{R}^{2N}. \quad (16)$$

Beware of the difference with  $X = (w_m, \partial_t w_m)^\top$ . The finite dimensional model takes the usual shape:

$$\begin{cases} X'_N = A_N X_N + B_{1,N} U + B_{2,N} U, \\ Z_N = C_{1,N} X_N + D_{12,N} U, \\ Y_N = C_{2,N} X_N + D_{21} W, \end{cases} \quad (17)$$



with  $X_N(0) = 0$ , and where the operators of system (9) are replaced by real-valued matrices  $A_N$ ,  $B_{1,N}$ ,  $B_{2,N}$ ,  $C_{1,N}$ ,  $C_{2,N}$ , and  $D_{12,N}$  computed on a truncated basis  $(\varphi_n)_{n=1,\dots,N}$  as detailed hereafter. The measurement output  $Y_N$  is obtained by truncation of  $Y$  on the first  $N$  vibration modes. The controlled output vector  $Z_N$  will be defined accordingly to the expected performance objectives. Note that the exogenous perturbation vector  $W = (W_{mod}, u_b, w_b'') \in \mathbb{R}^3$  remains unchanged.

The advantage of this representation where  $X_N$  is defined by (16) is that all the state variables express a velocity. Thus, in (17),  $A_N$  is dimensionally homogeneous, which should improve the conditioning of the system.

Let us now define precisely the matrices involved in (17). The dynamic matrix  $A_N$ , of size  $2N \times 2N$ , is given by:

$$A_N = \text{block}_{n,k} \left( \begin{bmatrix} -2\omega_n \xi_n \delta_{nk} & a_{nk} \\ \omega_n \delta_{nk} & 0 \end{bmatrix} \right),$$

where  $\delta_{nk}$  is the Kronecker symbol,  $\xi_n$  are the modal damping ratios and

$$a_{nk} = -\omega_n \delta_{nk} - \frac{\varrho^2 \mathcal{A}^2 E}{2\omega_k \rho T_s^2} \langle (\ell - 2x) \partial_x \varphi_k, \varphi_n \rangle$$

Moreover, we assume that  $d_n^1, d^2 \in \mathbb{R}$  are tuning parameters defining the respective weights of the disturbance signals. The choice of diagonal matrices corresponds to a decoupling assumption of the different modes. They could refer for instance to the neglected non-linearities. Therefore, we can gather the following expressions.

By construction,  $B_{1,N}$  is a matrix of size  $2N \times 3$  and  $B_{2,N}$  is a column of size  $2N$ :

$$B_{1,N} = \text{vect}_n \left( \begin{bmatrix} d_n^1 & -\omega_u^2 \alpha_n & \beta_n \\ 0 & 0 & 0 \end{bmatrix} \right),$$

$$B_{2,N} = \text{vect}_n \left( \begin{bmatrix} \left(1 - \frac{\alpha^2}{2}\right) \alpha_n + \alpha \beta_n \\ 0 \end{bmatrix} \right),$$

with  $\alpha_n$  and  $\beta_n$  defined in (15). The line matrix  $C_{2,N}$  is of size  $2N$ :

$$C_{2,N} = \left( \text{vect}_k \left( \begin{bmatrix} c_k \\ 0 \end{bmatrix} \right) \right)^\top,$$

where

$$c_k = -\frac{\varrho \mathcal{A}^2 E \ell}{2\omega_k T_s} \partial_x \varphi_k(\ell) = (-1)^{k+1} \frac{\varrho \mathcal{A}^2 E}{T_s} \sqrt{\frac{\rho \ell \mathcal{A}}{2T_s}}.$$

Finally, depending on the control objectives of performance, we can choose for instance to stabilize each of the  $N$  first modes of vibration and the amplitude of the control, i.e.:

$$Z_N = (z_1, \dots, z_N, \mathbf{u}'')^\top.$$

With this choice,  $C_{1,N}$  is a  $(N+1) \times 2N$  matrix and  $D_{12,N}$  is a column matrix of size  $(N+1)$ :

$$C_{1,N} = \left( \text{diag}_n \left( \begin{bmatrix} 0 & \omega_n^{-1} \end{bmatrix} \right) \right), \quad D_{12,N} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

## B. First $\mathcal{H}_\infty$ -control numerical simulations

We simulate a  $\ell = 1.98\text{m}$  long steel cable inclined at  $\theta = 20^\circ$  to the horizontal. The cable has a diameter of  $0.8\text{ mm}$  and a mass of  $0.67\text{ kg.m}^{-1}$ . We also have  $\rho = 1.34 \times 10^6\text{ kg.m}^{-3}$ ,  $\mathcal{A} = 0.5 \times 10^{-6}\text{ m}^3$ ,  $T_s = 205\text{ N}$  and  $E = 200 \times 10^9\text{ N.m}^{-2}$ .

This gives the following parameters:  $E_q = 174 \times 10^9\text{ N.m}^{-2}$ ,  $\lambda^2 = 1.74$  and  $\varrho = \rho g \cos \theta = 12.35 \times 10^6\text{ kg.s}^{-2}\text{.m}^{-2}$ . These values were chosen to match at best a typical full-scale bridge cable of length  $400\text{ m}$ , mass per unit length  $130\text{ kg.m}^{-1}$  and tension  $8000\text{ kN}$  [GBNWM08].

The theoretical natural frequencies of the cable are the following for the 3 first vibration in-plane modes (in  $\text{rad.s}^{-1}$ ):  $\omega_1 = 27.7$ ,  $\omega_2 = 55.5$ ,  $\omega_3 = 83.3$ .

A realistic mean value of the cable's damping ratio for each mode can be taken as  $\xi_n = 0.2\%$  and as explained in [GBNWM08], if needed, natural frequencies and damping ratios could be identified numerically using free vibration tests. Moreover, we choose for instance  $\alpha = 0.1\text{ rad}$ , which is a reasonable estimation of the tendon's angle.

In the numerical experiments, we take  $\omega_u \simeq \omega_1$  in order to ensure disturbance rejection near the vibration mode we want to dampen the most, i.e. the first in-plane mode frequency. Besides, we take the respective weights of the disturbance signals such that:  $d_n^1 = 10^{-3}$ ,  $n \in \{1, \dots, N\}$  and  $d^2 = 10^{-3}$ .

The cable is excited vertically at its bottom end (point  $b$  in Figure 1). In the experiments we will consider two different excitations: first we will focus on the response of the system to a step excitation, and then to a sinusoidal excitation of amplitude 1 (our framework is linear) and angular frequency  $\Omega$  which can be expressed as:

$$u_b(t) = \cos(\Omega t) \sin(\theta), \quad w_b(t) = \cos(\Omega t) \cos(\theta).$$

No external forces are applied along the cable.

All our computations will be done with `hinfstruct` from the Matlab© Robust Control Toolbox.

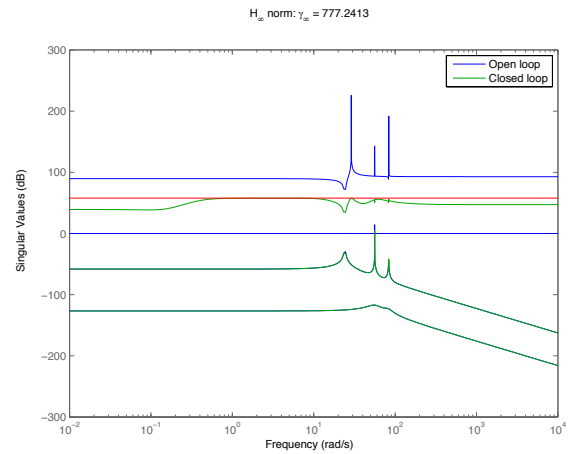


Fig. 3. Singular values: open-loop and closed-loop with no damping ( $\xi_n = 0$ ).

We obtain the results presented in Figures 3, 4 and 5. We observe in Figure 3 that the first and most important mode is well attenuated. The effect in closed-loop of the synthesized

controller is shown in time domain (and compared to the open-loop) considering the step excitation (Figure 4) or a sinusoidal excitation (Figure 5): the vibration reduction is clearly visible from the beginning of the control action.

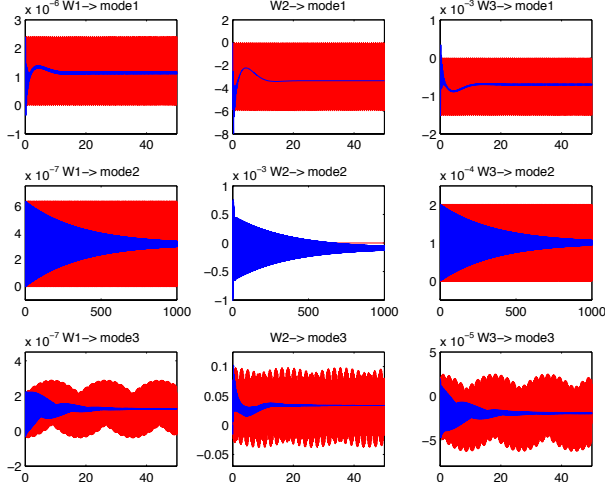


Fig. 4. Open-loop and closed-loop time response of the first three modes to a step excitation on each perturbation input  $W_i$ ,  $i = 1, 2, 3$ , with no damping ( $\xi_n = 0$ ).

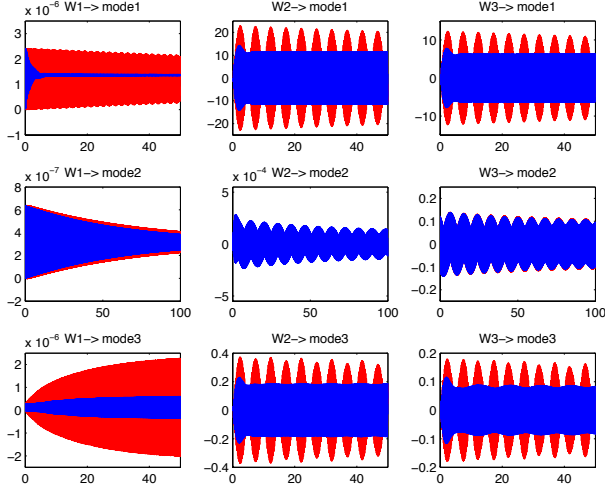


Fig. 5. Open-loop and closed-loop time response of the first three modes to a sinusoidal excitation with damping.

We also draw the reader's attention, in Figure 4 specifically, to the damping time scale of each mode, which is very different between even and odd modes. Following the parallel we made in Remark 1, this corroborates the comparison between the effect of inertial and parametric control in [Pre97]. As expected due to the definition of the system (17) and the content of Remark 2, we also observe that the perturbation  $W_2 = u_b$  has no influence on even-indexed modes: the closed loop result shows only the control action.

One can finally observe in Figure 6 that increasing the angle  $\alpha$  between the actuator inclination and the chord line, the

control exerted on the system increases and the stabilization time of both symmetric and anti-symmetric modes shortens. Nevertheless we have to make a careful evaluation of the angle  $\alpha$  since, as explain in Section II, the modeling of the inclined cable presented here is only valid for a small sag of the cable, i.e. small values of  $\alpha$ .

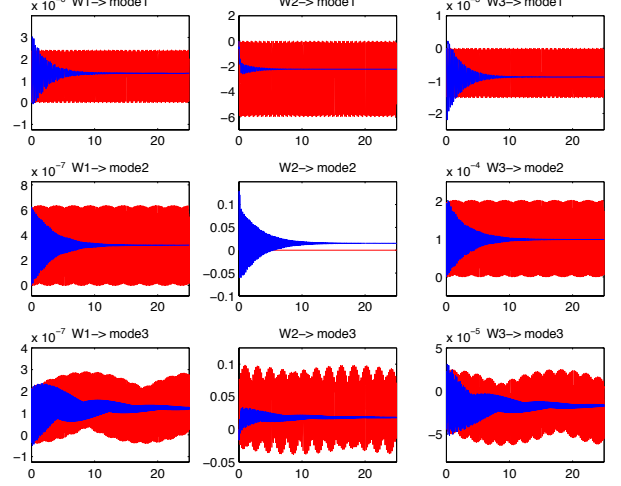


Fig. 6. Open and closed loop time response of the three first modes to a step excitation with no damping.  $\alpha = 0.5$ .

We conclude this first numerical experiments by observing that in this approach we do not have access to the real physical control, namely the displacement  $u$  of the actuator, but only to its acceleration (see Figure 7). In the next subsection we will slightly change the model in order to have access to the actuator displacement and to evaluate its movements.

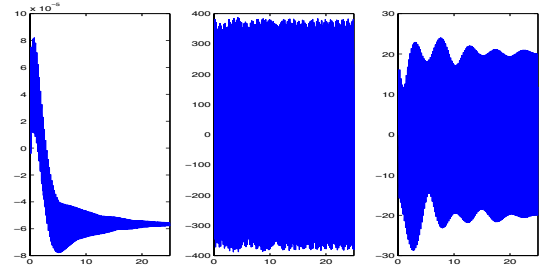


Fig. 7. Time response of the robust controller when applying a step excitation on the first input  $W_1$ , and sinusoidal excitations on the second and third inputs,  $W_2$  and  $W_3$ , for  $N = 3$ , with damping.

### C. Mixed PI/strain-control simulations

In practice we want to control not only the vibrations of the cable, but also reasonably limit the actuator displacement. For this purpose, we can seek an output feedback controller  $(k_p, k_i, K)$  such that:

$$K: \begin{bmatrix} \dot{X}_K \\ V \end{bmatrix} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} X_K \\ Y \end{bmatrix} \quad (18)$$

where  $X_K \in \mathbb{R}^{n_K}$  denotes the controller state, and

$$U(t) = \mathbf{u}''(t) = -k_p \mathbf{u}'(t) - k_i \mathbf{u}(t) + V(t), \quad (19)$$

where  $k_p, k_i > 0$  are feedback gains constants and  $V(t)$  denotes the strain exerted on the cable by the active tendon. This is a Proportional and Integral (PI) + a strain feedback control law.

To deal with the chosen control structure, we introduce the augmented state variable:

$$\tilde{X}_N = (X_N^\top, \mathbf{u}, \mathbf{u}')^\top.$$

The finite dimensional model can be written as:

$$\begin{cases} \tilde{X}'_N = \begin{pmatrix} A_N & -k_i B_{2,N} & -k_p B_{2,N} \\ \mathbf{0} & 0 & 1 \\ \mathbf{0} & -k_i & -k_p \end{pmatrix} \tilde{X}_N \\ \quad + \begin{pmatrix} B_{1,N} \\ 0 \\ 0 \end{pmatrix} W + \begin{pmatrix} B_{2,N} \\ 0 \\ 1 \end{pmatrix} V, \\ Z_N = \tilde{C}_{1,N} \tilde{X}_N + \tilde{D}_{12,N} V, \\ Y_N = [C_{2,N} \ \mathbf{0}] \tilde{X}_N + D_{21,N} W. \end{cases} \quad (20)$$

The “to be controlled output” is now chosen as:

$$Z_N = (z_1, \dots, z_N, \mathbf{u}, \mathbf{u}'')^\top.$$

With this choice, the matrices  $\tilde{C}_{1,N}$  and  $\tilde{C}_{2,N}$  are respectively of size  $(N+2) \times (N+2)$  and  $(N+2) \times 1$  and are given by:

$$\tilde{C}_{1,N} = \begin{pmatrix} \text{diag}_n([0 \ \omega_n^{-1}]) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 0 \end{pmatrix},$$

$$\tilde{D}_{12,N} = (\mathbf{0} \ 0 \ 1)^\top.$$

The use of `hinstruct` from the Matlab© Robust Control Toolbox specifically enables us to deal with the best tuning of parameters  $k_i, k_p$  inside this new state space model (20).

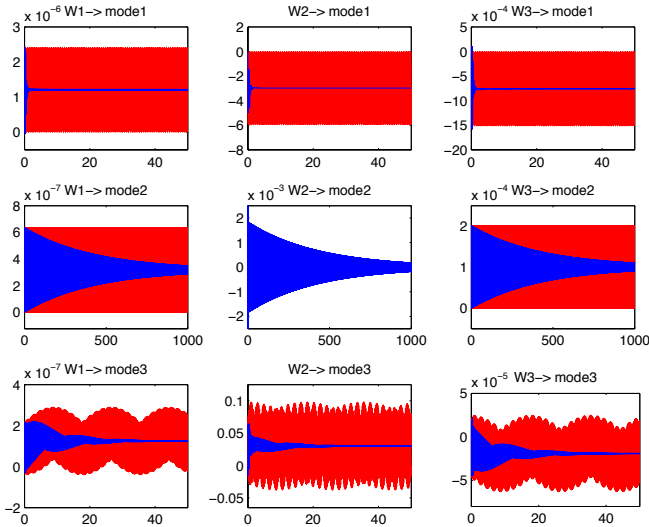


Fig. 8. Open-loop and closed-loop time response of the first three modes to a step excitation on each perturbation input  $W_i$ ,  $i = 1, 2, 3$ , with no damping ( $\xi_n = 0$ ).  $k_p = 15.625$ ,  $k_i = 17.809$ .

Figures 8 and 9 show that the temporal response of the new model is satisfactory for each kind of excitation. Moreover, as expected, the actuator displacement is now under control and stays bounded in reasonable limits, see Figures 10 and 11.

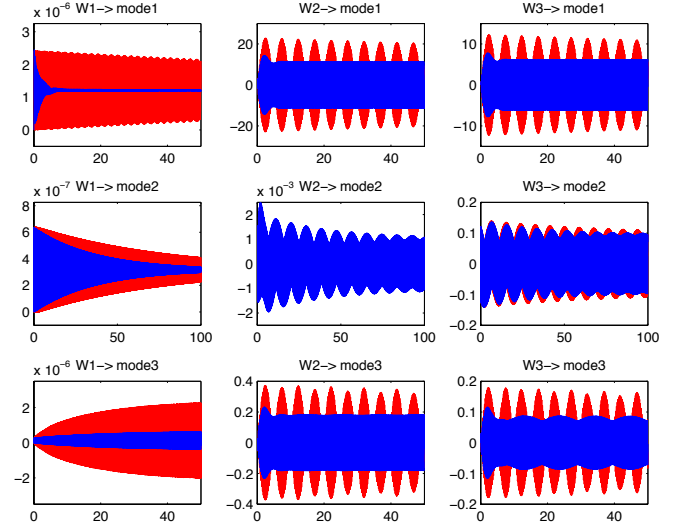


Fig. 9. Open-loop and closed-loop time response to a sinusoidal excitation for  $N = 3$  with damping.  $k_p = 15.625$ ,  $k_i = 17.809$ .

In this second numerical approach, we now have access the displacement of the actuator and, as expected, the amplitude of the actuator displacement remains bounded in physically reasonable limits. One can have a look to the closed-loop time response of the actuator displacement  $\mathbf{u}$  to a step excitation without damping (see Figure 10) and to a sinusoidal excitation with damping (see Figure 11).

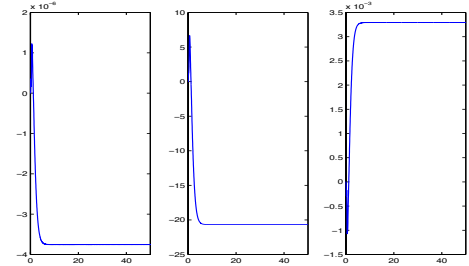


Fig. 10. Closed-loop time response of the actuator displacement to a step excitation for  $N = 3$  with no damping.  $k_p = 15.625$ ,  $k_i = 17.809$ .

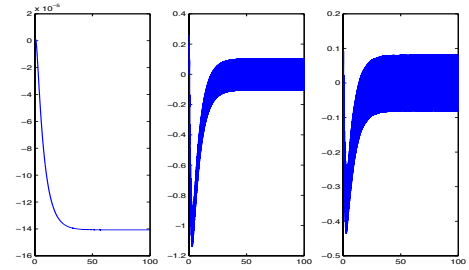


Fig. 11. Closed-loop time response of the actuator displacement to a sinusoidal excitation for  $N = 3$  with damping.  $k_p = 15.625$ ,  $k_i = 17.809$ .

We conclude this numerical section by some observation about the spillover effect. Indeed, with the lack of damping, controlling the  $N$  first modes of the inclined cable system

does not guarantee the control of the next modes. It is well-known that for vibration systems (covered by wave or plate partial differential equations), at least the first neglected mode is actually excited by the controller of all the previous ones (see [BD90]). Here, for example, we can numerically observe that the control synthesized for  $N = 3$  modes fails to stabilize the 4th mode as shown on Figure 12. In practice, this effect is easily avoided as soon as a small damping ( $\xi$ ) is included in the system.

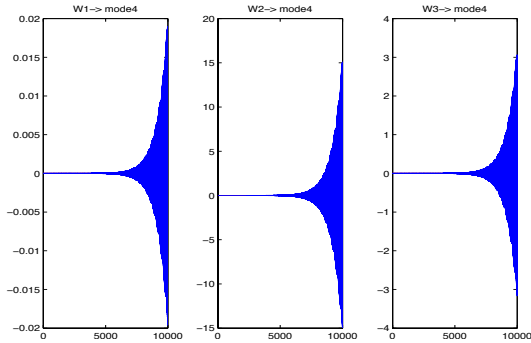


Fig. 12. Closed-loop time response of the 4th mode when applying the robust controller synthesized for  $N = 3$  modes. Spillover phenomenon.

Another strength of the present approach, using the infinite dimensional system modeling of the situation, is that it can deal with as many modes as needed. In practice, civil engineers typically deal with two or three modes (often to be able to keep track of the nonlinear couplings, which are not considered here). As illustrated on Figure 13, we can, for example, robustly control the ten first modes of the inclined cable.

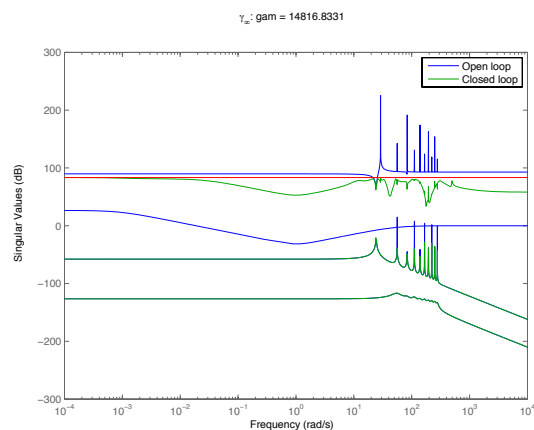


Fig. 13. Singular values: open-loop and closed-loop for  $N = 10$  modes, with no damping.

#### D. Conclusion

In this article, based on an infinite dimensional PDE modeling of an inclined cable, we were able to perform a robust control analysis of the vibration reduction of a highly flexible system. Taking advantage of our specific approach, based on the truncation of our PDE model, the numerical simulations we carried out allowed us to deal either with the first few modes

(for instance in order, later, to be able to be compared with results from other approaches, e.g. [GBNWM08], [SBP11], [WFS95]), or with a lot of modes, which is not usually possible when considering non-linearities for instance. In both cases, the numerical illustrations shows the efficiency of the robust control performed on the system, from localized measurements and control actions.

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