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**RESONANCES FOR THE LAPLACIAN
ON RIEMANNIAN SYMMETRIC SPACES:
THE CASE OF $SL(3, \mathbb{R})/SO(3)$**

J. HILGERT, A. PASQUALE, AND T. PRZEBINDA

ABSTRACT. We show that the resolvent of the Laplacian on $SL(3, \mathbb{R})/SO(3)$ can be lifted to a meromorphic function on a Riemann surface which is a branched covering of \mathbb{C} . The poles of this function are called the resonances of the Laplacian. We determine all resonances and show that the corresponding residue operators are given by convolution with spherical functions parameterized by the resonances. The ranges of these operators are infinite dimensional irreducible $SL(3, \mathbb{R})$ -representations whose Langlands parameters can also be read off from the corresponding resonances. Alternatively, they are given by the differential equations which determine the image of the Poisson transform associated with the resonance.

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INTRODUCTION

The notion of resonance was introduced in quantum mechanics to study metastable states of a system, that is long-lived states from which the system deviates only with sufficiently strong disturbances. Mathematically, resonances replace discrete eigenvalues of linear operators on non-compact domains and appear as poles of the meromorphic continuation of their resolvents.

The mathematical study of resonances initiated for Schrödinger operators on \mathbb{R}^n . Later, it was extended to more geometric situations, such as the Laplacian on hyperbolic and asymptotically hyperbolic manifolds, symmetric or locally symmetric spaces, and Damek-Ricci spaces. In a typical situation, one works on a complete Riemannian manifold \mathbf{X} , for which the positive Laplacian Δ is an essentially self-adjoint operator on the Hilbert space $L^2(\mathbf{X})$ of square integrable functions on \mathbf{X} . Suppose that Δ has a continuous spectrum $[\rho_{\mathbf{X}}^2, +\infty)$, with $\rho_{\mathbf{X}}^2 \geq 0$. The resolvent $R(z) = (\Delta_{\mathbf{X}} - \rho_{\mathbf{X}}^2 - z^2)^{-1}$ of the shifted Laplacian (or Helmholtz operator) $\Delta - \rho_{\mathbf{X}}^2$ is then a holomorphic function of z on the upper (and on the lower) complex halfplane, with values in the space of bounded linear operators on $L^2(\mathbf{X})$. Let the resolvent act, not on the entire $L^2(\mathbf{X})$, but on a dense subspace of $L^2(\mathbf{X})$, for instance the space $C_c^\infty(\mathbf{X})$ of compactly supported smooth functions on \mathbf{X} or on some suitable weighted L^2 space. Then the map $z \mapsto R(z)$ might admit a meromorphic extension across \mathbb{R} to a larger domain in \mathbb{C} or to a cover of such a domain. The poles, if they exist, are the resonances of $\Delta - \rho_{\mathbf{X}}^2$. Sometimes also the name scattering poles is used, but the two concepts are not completely synonymous (see e.g. [4]). The basic questions concern the existence of the meromorphic extension of the resolvent, the distribution and counting properties of the resonances, the rank and interpretation of the residue operators associated with the resonances.

Let G be a connected noncompact real semisimple Lie group with finite center and let K be a maximal compact subgroup of G . Then the homogeneous space $\mathbf{X} = G/K$ is a Riemannian symmetric space of the noncompact type. It is a complete Riemannian manifold with respect to its canonical G -invariant Riemannian structure. The positive Laplacian is the opposite of the Laplace-Beltrami operator. An important example of such spaces are the real hyperbolic spaces $\mathbb{H}^n = \mathrm{SO}_0(n, 1)/\mathrm{SO}(n)$. The resonances of the positive Laplacian on \mathbb{H}^n have been studied by Guillopé and Zworski, [5]; see also [25]. They proved that there are no resonances for n even; for n odd, there are resonances (which are explicitly determined) and the corresponding residue operators have finite rank.

The study of the analytic extension of the resolvent of the Laplacian for general Riemannian symmetric spaces of the noncompact type $\mathbf{X} = G/K$ was started by Mazzeo and Vasy, [13]. The motivations were of different nature. First of all, these spaces form a natural class of complete Riemannian manifolds for which the geometric properties are well understood. Moreover, the analytic properties of their Laplace-Beltrami operator play an important role in representation theory and number theory. Furthermore, the radial component of the Laplace-Beltrami operator on a maximal flat subspace is a many-body type Hamiltonian, with the walls of Weyl chambers of the maximal flat corresponding to the collision planes. This suggested that many-body methods of geometric scattering theory could have been appropriate to this setting. More precisely, the analysis carried out in [13] combines microlocal techniques and an adaptation of the scattering method of complex scaling of Aguilar-Balslev-Combes, see e.g. [9]. A different point of view, using the Helgason-Fourier

analysis, was employed by A. Strohmaier, [22], and by Hilgert and Pasquale, [8]. A further approach, using asymptotics of solutions the Laplacian on Damek-Ricci spaces, was employed by Miatello and Will in [16].

For a general Riemannian symmetric space of the noncompact type $X = G/K$, Mazzeo and Vasy [13] and Strohmaier [22] independently proved that the resolvent $z \mapsto R(z^2)$ of the shifted Laplacian $\Delta_X - \rho_X^2$ admits a holomorphic extension across \mathbb{R} . The domain of the extension depends on the parity of the real rank of the symmetric space X (i.e. the dimension of the maximal flat subspace of X). This dependence on the parity of the dimension parallels the case of the Laplacian on \mathbb{R}^n (see e.g. [15, §1,6]).

Despite the many articles studying resonances on complete Riemannian manifolds, detailed information on the existence and nature of the resonances for the Laplacian on X is so far available only in the so-called even multiplicity and real rank one cases. The even multiplicity case corresponds to the situation in which the Lie algebra of G has a unique class of Cartan subalgebras. This happens for instance when G possesses a complex structure. In the even multiplicity case, the resolvent has an entire extension to a suitable covering of the complex plane, see [22, Theorem 3.3]. So there are no resonances in this case. The general rank-one case was considered, with different approaches, in [16] and [8]: unless the symmetric space has even multiplicities (in which case there are no resonances), the map $z \mapsto R(z^2)$ admits a meromorphic extension to \mathbb{C} with simple poles along the negative imaginary axis. The poles are at the points $\zeta_k = |\alpha|\lambda_k$, $k \in \mathbb{N}$, where $|\alpha|$ is a constant depending on the normalization of the Riemannian measure and the λ_k 's range among the spectral parameters of the spherical functions φ_{λ_k} on X which are matrix coefficients of finite dimensional spherical representations of G . The resolvent residue operator at ζ_k is a constant multiple of the convolution operator by φ_{λ_k} and its image is the space of the corresponding finite dimensional spherical representation. In particular, the rank of the residue operators is finite. See [8, Theorem 3.8].

In [12] and [14], Mazzeo and Vasy considered the specific case of $X = \mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$, to exemplify their microlocal and complex-scaling methods of analytic extension of the resolvent of the positive Laplacian. The space $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ is a symmetric space of real rank-two which can be realized as the space of symmetric 3-by-3 positive definite matrices with determinant 1. Restricted to a maximal flat, its Laplace-Beltrami operator is a Calogero-Moser-Sutherland 3-body Hamiltonian of type II associated with the root system A_2 , see e.g. [18, (3.1.14) and (3.8.3)]. The analysis of [12] and [14] left nevertheless open the basic questions on the existence and nature of resonances and resolvent residue operators.

In this paper we provide answers to these questions. In a first step one obtains that for fixed $f \in C_c^\infty(X)$ and $y \in X$ the resolvent function $z \mapsto (R(z)f)(y)$ extends holomorphically to $\mathbb{C} \setminus ((-\infty, 0] \cup i(-\infty, -\frac{1}{2}\rho_X])$. The cut $(-\infty, 0]$ leads to a logarithmic Riemann surface covering $\mathbb{C} \setminus (i(-\infty, -\frac{1}{2}\rho_X] \cup \{0\} \cup i[\frac{1}{2}\rho_X, +\infty))$ to which $z \mapsto (R(z)f)(y)$ can be lifted holomorphically (see Corollary 4). This narrows down the location of the resonances to the negative imaginary axis. Our main result, Theorem 20, then says that for each $N \in \mathbb{N}$ there is an open neighborhood of $i(0, N+1)$ together with a branched cover $M_{(\gamma_N)}$ to which $z \mapsto (R(z)f)(y)$ can be lifted meromorphically. Poles can occur only above the points $-i(N+\frac{1}{2})$, and they are of order at most one. For special f and y the lift may be holomorphic at some of these points. Obviously this is the case for $f = 0$. The residues of the lifted functions may be calculated and they are given as convolution of f with the spherical function

whose spectral parameter is the resonance. Unlike the rank one case, the residue resolvent operators are not of finite rank (see Proposition 21). On the other hand, as in the rank one case the range of each of the residue operator is a G -representation which can be identified explicitly. More precisely, it is the unique irreducible subquotient of a (non-unitary) principal series whose Langlands parameter can be read off from the resonance (see Proposition 23). Alternatively, the range of the residue operator at some resonance is given by the differential equations which determine the image of the Poisson transform associated with this resonance (see Remark 6).

Notation. We shall use the standard notation \mathbb{Z} , \mathbb{N} , \mathbb{R} , \mathbb{R}^+ , \mathbb{C} and \mathbb{C}^\times for the integers, the nonnegative integers, the reals, the positive reals, the complex numbers and the non-zero complex numbers, respectively. We also set $\mathbb{R}^- = -\mathbb{R}^+$. If X is a manifold, then $C^\infty(X)$ and $C_c^\infty(X)$ respectively denote the space of smooth functions and the space of smooth compactly supported functions on X .

1. PRELIMINARIES

1.1. **Structure of $X = \mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$.** Let $G = \mathrm{SL}(3, \mathbb{R})$ be the Lie group of 3-by-3 real matrices of determinant 1. The Lie algebra $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ of G consists of the 3-by-3 matrices with real coefficients and trace equal to 0. The ± 1 -eigenspace decomposition of \mathfrak{g} with respect to the Cartan involution $\theta(x) = -x^t$, where \cdot^t denote transposition, yields the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Here $\mathfrak{k} = \mathfrak{so}(3)$ is the Lie algebra of skew-symmetric 3-by-3 matrices and \mathfrak{p} is the vector subspace of symmetric matrices in \mathfrak{g} .

We consider $X = G/K$ as a symmetric space endowed with the G -invariant Riemannian metric associated with the Cartan-Killing form of \mathfrak{g} . It can be realized as the space \tilde{X} of the 3-by-3 symmetric positive definite matrices with determinant 1. Indeed, G acts transitively on \tilde{X} , the element $g \in G$ acting as the isometry $x \mapsto gxg^t$ of \tilde{X} , and K is the isotropy subgroup of the identity matrix.

Choose

$$\mathfrak{a} = \left\{ \mathrm{diag}(a_1, a_2, a_3); a_j \in \mathbb{R}, \sum_{j=1}^3 a_j = 0 \right\}$$

as a maximal abelian subalgebra in \mathfrak{p} . For $j = 1, 2, 3$ define $\varepsilon_j \in \mathfrak{a}^*$ by

$$\varepsilon_j(\mathrm{diag}(a_1, a_2, a_3)) = a_j.$$

Then the set Σ of (restricted) roots of $(\mathfrak{g}, \mathfrak{a})$ is

$$\Sigma = \{ \alpha_{i,j} = \varepsilon_i - \varepsilon_j : 1 \leq i \neq j \leq 3 \}. \quad (1)$$

It is a root system of type A_2 . All root multiplicities m_α are equal to 1. Take $\Sigma^+ = \{ \alpha_{i,j} : 1 \leq i < j \leq 3 \}$ as a set of positive roots. The corresponding system of simple roots is $\Pi = \{ \alpha_{1,2}, \alpha_{2,3} \}$. The element $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha \in \mathfrak{a}^*$ is equal to

$$\rho = \alpha_{1,2} + \alpha_{2,3} = \alpha_{1,3}.$$

We denote by \mathfrak{a}^+ the positive Weyl chamber associated with Σ^+ and by \mathfrak{a}_+^* the elements in the real dual space \mathfrak{a}^* of \mathfrak{a} which are positive on \mathfrak{a}^+ . The Weyl group W of Σ is S_3 acting as the group of permutations of the three elements $\varepsilon_1, \varepsilon_2, \varepsilon_3$.

In the following, we denote by the same symbol $\langle \cdot, \cdot \rangle$ the restriction of the Cartan-Killing form of \mathfrak{g} to $\mathfrak{a} \times \mathfrak{a}$, the dual inner product on \mathfrak{a}^* and their \mathbb{C} -bilinear extensions to the

complexifications $\mathfrak{a}_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}}^*$, respectively. Explicitly, $\langle \cdot, \cdot \rangle$ is given by $\langle \varepsilon_i, \varepsilon_j \rangle = 6\delta_{i,j}$ for all $i, j = 1, 2, 3$. Hence, $\langle \alpha, \alpha \rangle = 12$ for every $\alpha \in \Sigma$. Moreover, $\langle \alpha_{1,2}, \alpha_{2,3} \rangle = -6$.

In later sections of this paper we will find it convenient to identify \mathfrak{a}^* with \mathbb{C} by choosing a suitable basis. To distinguish the resulting complex structure in \mathfrak{a}^* from the natural complex structure of $\mathfrak{a}_{\mathbb{C}}^*$, we shall indicate the complex units in $\mathfrak{a}^* \cong \mathbb{C}$ and $\mathfrak{a}_{\mathbb{C}}^*$ by i and \mathbf{i} , respectively. So $\mathfrak{a}^* \cong \mathbb{C} = \mathbb{R} + i\mathbb{R}$, whereas $\mathfrak{a}_{\mathbb{C}}^* = \mathfrak{a}^* + \mathbf{i}\mathfrak{a}^*$. For $r, s \in \mathbb{R}$ and $\lambda, \nu \in \mathfrak{a}^*$ we have $(r + is)(\lambda + \mathbf{i}\nu) = (r\lambda - s\nu) + \mathbf{i}(r\nu + s\lambda) \in \mathfrak{a}_{\mathbb{C}}^*$.

Let $\exp : \mathfrak{g} \rightarrow G$ be the exponential map of G . Then $A := \exp \mathfrak{a}$ is the abelian subgroup of G consisting of diagonal matrices with positive components. Let N denote the group formed by the upper triangular matrices with diagonal entries equal to 1, and set $K = \mathrm{SO}(3)$. Then the map $(k, a, n) \mapsto kan$ is an analytic diffeomorphism of the product manifold $K \times A \times N$ onto G . The resulting decomposition $G = KAN$ is called the Iwasawa decomposition of G . Thus, for $g \in G$ we have $g = \kappa(g)a(g)n(g) = \kappa(g)\exp H(g)n(g)$ for uniquely determined $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$, $a(g) \in A$ and $n(g) \in G$.

Let M be the centralizer of A in K . Then M consists of the diagonal matrices of determinant 1 with ± 1 on the diagonal. The space $B = K/M$ is the Furstenberg boundary of X . We define the map $A : X \times B \rightarrow \mathfrak{a}$ by

$$A(gK, kM) := -H(g^{-1}k) \quad (g \in G, k \in K). \quad (2)$$

1.2. Spherical harmonic analysis on X . References for this section are [3, Ch. 3], [6, Ch. II] and [7, Ch. III].

Let Δ be the nonnegative Laplacian on X . It is defined for $f \in C^\infty(X)$ by $\Delta f = -\mathrm{div} \circ \mathrm{grad} f$. Then Δ is an essentially self-adjoint G -invariant differential operator on $L^2(X)$.

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $b \in B$ the function $e_{\lambda,b} : X \rightarrow \mathbb{C}$ defined by

$$e_{\lambda,b}(x) = e^{(\lambda+\rho)(A(x,b))} \quad (x \in X), \quad (3)$$

is a joint eigenfunction of the commutative algebra $\mathbb{D}(X)$ of G -invariant differential operators on X . In particular, it is an eigenfunction of Δ , and

$$\Delta e_{\lambda,b} = (\langle \rho, \rho \rangle - \langle \lambda, \lambda \rangle) e_{\lambda,b}. \quad (4)$$

Let db denote the K -invariant probability measure on B . The function

$$\varphi_\lambda(x) = \int_B e_{\lambda,b}(x) db \quad (x \in X), \quad (5)$$

is Harish-Chandra's (elementary) spherical function of spectral parameter λ . It is the unique joint eigenfunction of $\mathbb{D}(X)$ which is K -invariant and satisfying the normalization condition $\varphi_\lambda(o) = 1$, where $o = K$ is the base point in X . The function $\varphi_\lambda(x)$ is real analytic and K -invariant in $x \in X$, and entire and W -invariant in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

The Helgason-Fourier transform of a sufficiently regular function $f : X \rightarrow \mathbb{C}$ is the function $\mathcal{F}f$ defined by

$$\mathcal{F}f(\lambda, b) = \int_X f(x) e_{-\lambda,b}(x) dx \quad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*, b \in B), \quad (6)$$

whenever this integral exists. Here dx is a fixed normalization of the G -invariant measure on X . If f is K -invariant, then

$$\mathcal{F}f(\lambda, b) = \int_X f(x) \varphi_{-\lambda}(x) dx \quad (7)$$

is independent of $b \in B$ and agrees with the spherical Fourier transform of f at λ .

Let $\mathcal{H}(\mathfrak{a}_{\mathbb{C}}^* \times B)_W$ denote the space of C^∞ functions $F : \mathfrak{a}_{\mathbb{C}}^* \times B \rightarrow \mathbb{C}$ satisfying the following conditions:

- (1) F is entire of uniform exponential type. This means that for every fixed $b \in B$ the function $F(\cdot, b)$ is entire in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, and there exists a constant $R \geq 0$ so that for each $N \in \mathbb{N}$

$$\sup_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*, b \in B} e^{-R|\operatorname{Re}\lambda|} (1 + |\lambda|)^N |F(\lambda, b)| < \infty. \quad (8)$$

- (2) For all $w \in W$ and $x \in X$

$$\int_B e_{w\lambda, b}(x) F(w\lambda, b) db = \int_B e_{\lambda, b}(x) F(\lambda, b) db.$$

By the Paley-Wiener Theorem (see [7, Theorem 5.1, p. 260]), the Helgason-Fourier transform \mathcal{F} is a bijection of the space $C_c^\infty(X)$ onto the space $\mathcal{H}(\mathfrak{a}_{\mathbb{C}}^* \times B)_W$. The constant R appearing in the estimate (8) for $F = \mathcal{F}f$ is related to the size of the support of $f \in C_c^\infty(X)$.

The Plancherel Theorem [7, Theorem 1.5, p. 202] states that, for a suitable normalization of the Lebesgue measure $d\lambda$ on \mathfrak{a}^* , the Helgason-Fourier transform, defined by (6) on $C_c^\infty(X)$, extends to an isometry of $L^2(X)$ onto the space $L^2(\mathfrak{a}_+^* \times B, |c_{\text{HC}}(\mathbf{i}\lambda)|^{-2} d\lambda db)$. Moreover, for all $f, h \in L^2(X)$

$$\langle f, h \rangle_{L^2(X)} = \langle \mathcal{F}f, \mathcal{F}h \rangle_{L_{\text{spec}}^2} \quad (9)$$

where we have set

$$\begin{aligned} \langle f, h \rangle_{L^2(X)} &= \int_X f(x) \overline{h(x)} dx \\ \langle F, H \rangle_{L_{\text{spec}}^2} &= \frac{1}{|W|} \int_{\mathfrak{a}^* \times B} F(\mathbf{i}\lambda, b) \overline{H(\mathbf{i}\lambda, b)} \frac{d\lambda db}{|c_{\text{HC}}(\mathbf{i}\lambda)|^2} \end{aligned}$$

and $|W|$ denotes the cardinality of the Weyl group W . The function $c_{\text{HC}}(\lambda)$ occurring in the Plancherel density is Harish-Chandra's c -function; see e.g. [3, Theorem 4.7.5]. It satisfies $|c_{\text{HC}}(\mathbf{i}\lambda)|^2 = c_{\text{HC}}(\mathbf{i}\lambda)c_{\text{HC}}(-\mathbf{i}\lambda)$ for $\lambda \in \mathfrak{a}^*$. The Plancherel density is a meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$, given by

$$[c_{\text{HC}}(\mathbf{i}\lambda)c_{\text{HC}}(-\mathbf{i}\lambda)]^{-1} = c_0 \prod_{\alpha \in \Sigma^+} \lambda_\alpha \operatorname{th}(\pi\lambda_\alpha) \quad (10)$$

where c_0 is a normalizing constant and for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\alpha \in \Sigma$ we have set

$$\lambda_\alpha = \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}. \quad (11)$$

Notice that the λ_α do not depend of the normalization of the inner product $\langle \cdot, \cdot \rangle$.

If $\lambda = 6 \sum_{j=1}^3 \lambda_j \varepsilon_j$, then $\lambda_{\alpha_{j,k}} = \frac{\lambda_j - \lambda_k}{2}$ for all $j, k = 1, 2, 3$, and one can rewrite (10) as

$$[c_{\text{HC}}(\mathbf{i}\lambda)c_{\text{HC}}(-\mathbf{i}\lambda)]^{-1} = c_0 \prod_{1 \leq j \leq k \leq 3} \frac{\lambda_j - \lambda_k}{2} \operatorname{th}(\pi \frac{\lambda_j - \lambda_k}{2}), \quad (12)$$

which is Bhanu Murty's formula; see [1] or [10, p. 204]. It is equivalent to Gindikin-Karpelevic formula

$$[c_{\text{HC}}(\mathbf{i}\lambda)c_{\text{HC}}(-\mathbf{i}\lambda)]^{-1} = c_{00} \prod_{1 \leq j \leq k \leq 3} \frac{\Gamma(i\lambda_{\alpha_{j,k}} + \frac{1}{2})\Gamma(-i\lambda_{\alpha_{j,k}} + \frac{1}{2})}{\Gamma(i\lambda_{\alpha_{j,k}})\Gamma(-i\lambda_{\alpha_{j,k}})},$$

where c_{00} is a suitable constant.

1.3. Spherical representations. Let $G_{\mathbb{C}} = \mathrm{SL}(3, \mathbb{C})$. Then $G_{\mathbb{C}}$ is the simply connected complexification of G , and it contains G . Notice that \mathfrak{a} is a Cartan subalgebra of \mathfrak{g} . Its complexification $\mathfrak{a}_{\mathbb{C}}$ consists of the 3-by-3 complex diagonal matrices with 0 trace. We consider the finite dimensional holomorphic representations π_{μ} of $G_{\mathbb{C}}$ with highest weight μ relative to $\mathfrak{a}_{\mathbb{C}}$, with dominance defined by Σ^{+} . Let μ be the restriction to \mathfrak{a} of such a weight. Then $\mu \in \mathfrak{a}^{*}$. Recall that π_{μ} is said to be spherical, if there exists a non-zero vector v_{μ} in the representation space of π_{μ} which is K -fixed, i.e. so that $\pi_{\mu}(k)v_{\mu} = v_{\mu}$ for all $k \in K$. According to the Cartan-Helgason's theorem, π_{μ} is spherical if and only if $\mu_{\alpha} \in \mathbb{N}$ for all $\alpha \in \Sigma^{+}$. In this case, the vector v_{μ} is unique up to constant multiples.

Thus $\mu \in \mathfrak{a}_{\mathbb{C}}^{*}$ is the highest restricted weight of a finite-dimensional spherical representation of G if and only if there exists $n_1, n_2 \in \mathbb{Z}_{+}$ so that $\mu = n_1 w_{1,2} + n_2 w_{2,3}$. Here $w_{1,2}, w_{2,3} \in \mathfrak{a}^{*}$ are the fundamental restricted weights, which are defined by the conditions

$$\frac{\langle w_{i,j}, \alpha_{k,l} \rangle}{\langle \alpha_{k,l}, \alpha_{k,l} \rangle} = \delta_{(i,k),(j,l)}$$

for $(i,j), (k,l) \in \{(1,2), (2,3)\}$. Hence

$$w_{1,2} = \frac{2}{3}(2\alpha_{1,2} + \alpha_{2,3}) \quad \text{and} \quad w_{2,3} = \frac{2}{3}(\alpha_{1,2} + 2\alpha_{2,3}). \quad (13)$$

Observe that if $\lambda \in \mathfrak{a}_{\mathbb{C}}^{*}$ is written as

$$\lambda = \lambda_{1,2} w_{1,2} + \lambda_{2,3} w_{2,3}, \quad (14)$$

then $\lambda_{i,j} = \lambda_{\alpha_{i,j}}$, where for $\alpha \in \Sigma$ the numbers λ_{α} are defined by (11). For instance, for $\lambda = \rho$, we have $\rho = \frac{1}{2}(w_{1,2} + w_{2,3})$ as $\rho_{1,2} = \rho_{2,3} = \frac{1}{2}$.

Let $Y = U/K$ be the simply connected Riemannian symmetric space of compact type which is dual to $X = G/K$. Then $U = \mathrm{SU}(3)$. Let π_{μ} be the finite-dimensional spherical representation of $G_{\mathbb{C}}$ of highest restricted weight μ , and let v_{μ} be a K -fixed vector in the space of π_{μ} having norm one in the inner product (\cdot, \cdot) making the restriction of π_{μ} to U unitary. Then the matrix coefficient $g \mapsto (\pi_{\mu}(g)v_{\mu}, v_{\mu})$ is a $K_{\mathbb{C}}$ -bi-invariant holomorphic function of $g \in G_{\mathbb{C}}$. Considered as function on Y , it is the spherical function on Y of spectral parameter μ . Considered as a function on X , it agrees with the spherical function $\varphi_{\mu+\rho}$ defined by (5).

1.4. Eigenspace representations. Let $S(\mathfrak{a}_{\mathbb{C}})^W$ be the commutative algebra of W -invariant polynomial functions on $\mathfrak{a}_{\mathbb{C}}^{*}$. Recall that $\mathbb{D}(X)$ denotes the commutative algebra of G -invariant differential operators on X . The Harish-Chandra isomorphism is a surjective isomorphism $\gamma : \mathbb{D}(X) \rightarrow S(\mathfrak{a}_{\mathbb{C}})^W$ such that $\gamma(\Delta)(\lambda) = \langle \rho, \rho \rangle - \langle \lambda, \lambda \rangle$. See e.g. [6, Ch. II, Theorems 4.3 and 5.18, and p. 299].

Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^{*}$. The joint eigenspace $\mathcal{E}_{\lambda}(X)$ for the algebra $\mathbb{D}(X)$ is

$$\mathcal{E}_{\lambda}(X) = \{f \in C^{\infty}(X) : Df = \gamma(D)(\lambda)f \text{ for all } D \in \mathbb{D}(X)\}. \quad (15)$$

See [7, Ch. II, §2, no. 3 and Ch. III, §6]. The group G acts on $\mathcal{E}_{\lambda}(X)$ by the left regular representation:

$$[T_{\lambda}(g)f](x) = f(g^{-1}x) \quad (g \in G, x \in X). \quad (16)$$

Notice that $\mathcal{E}_{\lambda}(X) = \mathcal{E}_{w\lambda}(X)$ (and hence $T_{\lambda} = T_{w\lambda}$) for all $w \in W$. The subspace of K -fixed elements in $\mathcal{E}_{\lambda}(X)$ is 1-dimensional and spanned by Harish-Chandra's spherical function φ_{λ} . The closed subspace $\mathcal{E}_{(\lambda)}(X)$ generated by the translates $T_{\lambda}(g)\varphi_{\lambda}$ of φ_{λ} , with $g \in G$, is the

unique closed irreducible subspace of $\mathcal{E}_\lambda(\mathbf{X})$. The restriction of T_λ to $\mathcal{E}_{(\lambda)}(\mathbf{X})$ is quasisimple and admissible. See e.g. [6, Ch. IV, Theorem 4.5]. Furthermore, the representation T_λ is irreducible if and only if $1/\Gamma_{\mathbf{X}}(\lambda) \neq 0$ where $\Gamma_{\mathbf{X}}$ is the Gamma function attached to \mathbf{X} , as in [7, Ch. III, §7, Theorem 6.2]. For $\mathbf{X} = \mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ we have

$$\Gamma_{\mathbf{X}}(\lambda) = \prod_{\alpha \in \Sigma} \Gamma\left(\frac{3}{4} + \frac{\lambda_\alpha}{2}\right) \Gamma\left(\frac{1}{4} + \frac{\lambda_\alpha}{2}\right) = \prod_{\alpha \in \Sigma^+} \frac{2\pi^2}{\cos(\pi\lambda_\alpha)}. \quad (17)$$

Thus T_λ is reducible if and only if there is $\alpha \in \Sigma$ so that $i\lambda_\alpha \in i(\mathbb{Z} + \frac{1}{2})$. In the present case, this is equivalent to λ being a singularity of the Plancherel density $[c_{\mathrm{HC}}(\mathbf{i}\lambda)c_{\mathrm{HC}}(-\mathbf{i}\lambda)]^{-1}$.

The space $\mathcal{E}_{\lambda, \mathrm{G}}(\mathbf{X})$ of G -finite elements in $\mathcal{E}_\lambda(\mathbf{X})$ is a (possibly zero) invariant subspace of $\mathcal{E}_\lambda(\mathbf{X})$. Recall that it consists of all functions $f \in \mathcal{E}_\lambda(\mathbf{X})$ such that the vector space spanned by the left translates $T_\lambda(g)f$ of f , with $g \in \mathrm{G}$, is finite dimensional.

The following proposition holds for arbitrary Riemannian symmetric spaces of the non-compact type $\mathbf{X} = \mathrm{G}/\mathrm{K}$, where G is a noncompact connected semisimple Lie group with finite center and K a maximal compact subgroup of G . It characterizes the $\mathcal{E}_{\lambda, \mathrm{G}}(\mathbf{X})$, when non-zero, as finite-dimensional spherical representations of G .

Proposition 1. $\mathcal{E}_{\lambda, \mathrm{G}}(\mathbf{X}) \neq \{0\}$ if and only if there is $w \in \mathrm{W}$ so that

$$-(w\lambda)_\alpha - \rho_\alpha \in \mathbb{Z}^+ \quad \text{for all } \alpha \in \Sigma^+. \quad (18)$$

In this case, $\mathcal{E}_{\lambda, \mathrm{G}}(\mathbf{X})$ is finite dimensional and irreducible under G . It is the finite-dimensional spherical representation of highest restricted weight $-w\lambda - \rho$. All finite dimensional spherical representations of G arise in this fashion.

Proof. This is [7, Ch. II, Proposition 4.16]. □

Remark 1. Notice that (18) implies that $\lambda \in \mathfrak{a}^*$. Recall that a finite-dimensional representation π is spherical if and only if its contragredient $\tilde{\pi}$ is spherical. Moreover, if π has highest restricted weight $-w\lambda - \rho$, then $\tilde{\pi}$ has highest restricted weight $-w_0(-w\lambda - \rho) = w_0w\lambda - \rho$, where w_0 denotes the longest Weyl group element. See e.g. [7, Corollary 4.13 and Theorem 4.12]. Thus $\mathcal{E}_{\lambda, \mathrm{G}} \neq \{0\}$ if and only if $\mathcal{E}_{-w_0\lambda, \mathrm{G}} \neq \{0\}$, which is equivalent to the existence of $w \in \mathrm{W}$ so that $(w\lambda)_\alpha - \rho_\alpha \in \mathbb{N}$ for all $\alpha \in \Sigma^+$. For a dominant $\lambda \in \mathfrak{a}^*$, this condition is therefore equivalent to $\lambda_\alpha - \rho_\alpha \in \mathbb{N}$ for all $\alpha \in \Sigma^+$. In turn, for $\mathrm{G} = \mathrm{SL}(3, \mathbb{R})$, the latter condition is equivalent to $\lambda_\alpha \in \mathbb{N} + \frac{1}{2}$ for $\alpha \in \{\alpha_{1,2}, \alpha_{2,3}\}$ when $\lambda \in \mathfrak{a}^*$ is dominant.

1.5. The resolvent of the Laplacian. As in subsection 1.2, let Δ be the nonnegative Laplacian on \mathbf{X} for the G -invariant Riemannian structure associated with the Cartan-Killing form of $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$. The spectrum of Δ is the half-line $[\rho_{\mathbf{X}}^2, +\infty)$ where $\rho_{\mathbf{X}}^2 = \langle \rho, \rho \rangle = 12$. The resolvent $R_\Delta(u) = (\Delta - u)^{-1}$ is therefore a holomorphic function of $u \in \mathbb{C} \setminus [\rho_{\mathbf{X}}^2, +\infty)$. It is actually convenient to consider the change of variable $z^2 = u - \rho_{\mathbf{X}}^2$, and reduce the study of R_Δ to that of

$$R(z) = (\Delta - \rho_{\mathbf{X}}^2 - z^2)^{-1} = R_\Delta(\rho_{\mathbf{X}}^2 + z^2). \quad (19)$$

Here we are choosing the single-valued holomorphic branch of the square root function on $\mathbb{C} \setminus [0, +\infty)$ mapping -1 to i . (Later the notation $\sqrt{\cdot}$ will be reserved to a different choice of holomorphic branch; see (55).) Hence R is a holomorphic function of $z \in \mathbb{C}^+$, where $\mathbb{C}^+ := \{w \in \mathbb{C} : \mathrm{Im} w > 0\}$ is the upper half-space, with values in the space of bounded linear operators on $L^2(\mathbf{X})$.

An explicit formula for $R(z)$ can be obtained by means of the Plancherel Theorem for the Helgason-Fourier transform; see e.g. [8, Section 1.4]. It follows, in particular, that for every $z \in \mathbb{C}^+$ and $f \in C_c^\infty(\mathbf{X})$, the distribution $R(z)f$ is in fact a C^∞ function on \mathbf{X} , given by the formula

$$[R(z)f](y) = \frac{1}{|\mathbb{W}|} \int_{\mathfrak{a}^*} \frac{1}{\langle \lambda, \lambda \rangle - z^2} (f \times \varphi_{i\lambda})(y) \frac{d\lambda}{c_{\text{HC}}(\mathbf{i}\lambda)c_{\text{HC}}(-\mathbf{i}\lambda)}, \quad y \in \mathbf{X}. \quad (20)$$

The symbol \times in (20) denotes the convolution on \mathbf{X} . Recall that for sufficiently regular functions $f_1, f_2 : \mathbf{X} \rightarrow \mathbb{C}$, the convolution $f_1 \times f_2$ is the function on \mathbf{X} defined by $(f_1 \times f_2) \circ \pi = (f_1 \circ \pi) * (f_2 \circ \pi)$. Here $\pi : \mathbf{G} \rightarrow \mathbf{X} = \mathbf{G}/\mathbf{K}$ is the natural projection and $*$ denotes the convolution product of functions on \mathbf{G} .

The convolution $(f \times \varphi_{i\lambda})(y)$ can be described in terms of the Helgason-Fourier transform of f . In fact, for $f \in C_c^\infty(\mathbf{X})$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $y = g \cdot o \in \mathbf{X}$, we have by [7, Ch. III, Lemma 1.2 and proof of Theorem 1.3] that

$$(f \times \varphi_\lambda)(y) = \int_{\mathbf{B}} \mathcal{F}f(\lambda, b) e_{\lambda, b}(y) db \quad (21)$$

is the spherical Fourier transform of the \mathbf{K} -invariant function $f_y \in C_c^\infty(\mathbf{X})$ given by

$$f_y(g_1) = \int_{\mathbf{K}} f(gkg_1 \cdot o) dk.$$

It follows by the Paley-Wiener Theorem that for every fixed $y \in \mathbf{X}$ the function $(f \times \varphi_{i\lambda})(y)$ is a Weyl-group-invariant entire function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and there exists a constant $R \geq 0$ (depending on y and on the size of the support of f) so that for each $N \in \mathbb{N}$

$$\sup_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*} e^{-R|\text{Im}\lambda|} (1 + |\lambda|)^N |(f \times \varphi_{i\lambda})(y)| < \infty. \quad (22)$$

2. AN INITIAL HOLOMORPHIC EXTENSION OF THE RESOLVENT

We keep the notation introduced in the previous section. Recall in particular that the bottom of the spectrum of the Laplacian is $\rho_{\mathbf{X}}^2 = \langle \rho, \rho \rangle = \langle \alpha, \alpha \rangle$ for all $\alpha \in \Sigma$. In the following we denote by $\rho_{\mathbf{X}}$ the positive square root of $\rho_{\mathbf{X}}^2$.

Let $\{e_1, e_2\}$ be the standard basis in \mathbb{R}^2 . We identify \mathfrak{a}^* with \mathbb{R}^2 by

$$\alpha_{1,2} = e_1 \quad \text{and} \quad \alpha_{2,3} = -\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2. \quad (23)$$

In this way the inner product $\langle \cdot, \cdot \rangle$ is $\rho_{\mathbf{X}}^2$ -times the usual Euclidean inner product on \mathbb{R}^2 . One deduces from (13) that

$$w_{1,2} = e_1 + \frac{\sqrt{3}}{3}e_2 \quad \text{and} \quad w_{2,3} = 2\frac{\sqrt{3}}{3}e_2. \quad (24)$$

The above identification yields for

$$\lambda = \lambda_{1,2}w_{1,2} + \lambda_{2,3}w_{2,3} = x_1e_1 + x_2e_2 = x_1 + ix_2 \in \mathfrak{a}^* \equiv \mathbb{R}^2 \equiv \mathbb{C} \quad (25)$$

the relations:

$$\begin{aligned} \lambda_{1,2} &= x_1, \\ \lambda_{2,3} &= -\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2, \\ \lambda_{1,3} &= \lambda_{1,2} + \lambda_{2,3} = \frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2. \end{aligned} \quad (26)$$

Set

$$\boldsymbol{\xi} = e^{i\frac{\pi}{3}}. \quad (27)$$

In the above coordinates, one can rewrite (10) as

$$\begin{aligned} [c_{\text{HC}}(\mathbf{i}\lambda)c_{\text{HC}}(-\mathbf{i}\lambda)]^{-1} &= c_0 x_1 \operatorname{th}(\pi x_1) (x_1 \frac{1}{2} + x_2 \frac{\sqrt{3}}{2}) \operatorname{th}(\pi(x_1 \frac{1}{2} + x_2 \frac{\sqrt{3}}{2})) \\ &\quad \times (-x_1 \frac{1}{2} + x_2 \frac{\sqrt{3}}{2}) \operatorname{th}(\pi(-x_1 \frac{1}{2} + x_2 \frac{\sqrt{3}}{2})) \\ &= c_0 \prod_{u \in \{1, \boldsymbol{\xi}, \boldsymbol{\xi}^2\}} \operatorname{Re}(\lambda u) \operatorname{th}(\pi \operatorname{Re}(\lambda u)). \end{aligned} \quad (28)$$

Let $S^1 = \{w \in \mathbb{C}; |w| = 1\}$ be the unit circle and let σ denote the rotation-invariant probability measure on S^1 :

$$\int_{S^1} f(w) d\sigma(w) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta = \frac{1}{2\pi} \int_{S^1} f(w) \frac{dw}{iw}.$$

Introduce polar coordinates in $\mathfrak{a}^* \equiv \mathbb{R}^2$: for λ as in (25), set

$$\lambda = \lambda(r, w) = rw \quad \text{where} \quad \begin{cases} r^2 = x_1^2 + x_2^2, \\ w = \frac{\lambda}{|\lambda|} \in S^1. \end{cases} \quad (29)$$

Hence $\langle \lambda(r, w), \lambda(r, w) \rangle = \rho_X^2 r^2$. Moreover, (20) becomes

$$[R(z)f](y) = \frac{1}{|W|} \int_0^\infty \frac{1}{\rho_X^2 r^2 - z^2} F(f, r, y) r dr \quad (30)$$

where

$$\begin{aligned} F(f, y, r) &= \int_{S^1} (f \times \varphi_{\mathbf{i}\lambda(r, w)})(y) \frac{d\sigma(w)}{c_{\text{HC}}(\mathbf{i}\lambda(r, w))c_{\text{HC}}(-\mathbf{i}\lambda(r, w))} \\ &= \frac{1}{2\pi} \int_{S^1} (f \times \varphi_{\mathbf{i}\lambda(r, w)})(y) \frac{1}{c_{\text{HC}}(\mathbf{i}\lambda(r, w))c_{\text{HC}}(-\mathbf{i}\lambda(r, w))} \frac{dw}{iw}. \end{aligned} \quad (31)$$

In the following, we will omit the dependence on $f \in C_c^\infty(\mathbf{X})$ and $y \in \mathbf{X}$ from the notation, and write $F(r)$ instead of $F(f, r, y)$, and $R(z)$ instead of $[R(z)f](y)$.

Set

$$c(z) = \frac{z + z^{-1}}{2} \quad (z \in \mathbb{C}^\times), \quad (32)$$

$$s(z) = \frac{z - z^{-1}}{2} \quad (z \in \mathbb{C}^\times). \quad (33)$$

If $z \in S^1$, then $c(z) = \operatorname{Re} z$ and $s(z) = i \operatorname{Im} z$. Hence (28) gives for $r > 0$ and $w \in S^1$:

$$\begin{aligned} [c_{\text{HC}}(\mathbf{i}\lambda(r, w))c_{\text{HC}}(-\mathbf{i}\lambda(r, w))]^{-1} &= c_0 \prod_{u \in \{1, \boldsymbol{\xi}, \boldsymbol{\xi}^2\}} \operatorname{Re}(ruw) \operatorname{th}(\pi \operatorname{Re}(ruw)) \\ &= c_0 r^3 \prod_{u \in \{1, \boldsymbol{\xi}, \boldsymbol{\xi}^2\}} c(uw) \operatorname{th}(\pi r c(uw)). \end{aligned} \quad (34)$$

We extend the function $(r, w) \in [0, +\infty) \times S^1 \rightarrow \lambda(r, w) = rw \in \mathfrak{a}^* = \mathbb{R}^2$ by \mathbb{C} linearity to a map from $\mathbb{C} \times S^1$ to $\mathfrak{a}_\mathbb{C}^*$ by setting

$$\lambda(z, w) = \lambda(x, w) + \mathbf{i}\lambda(x, w) \quad (z = x + iy \in \mathbb{C}, w \in S^1), \quad (35)$$

where $\lambda(x, w) = xw = \frac{x}{|x|}\lambda(|x|, w)$ for $(x, w) \in (-\infty, 0) \times S^1$.

Proposition 2. *Let $f \in C_c^\infty(\mathbf{X})$ and $y \in \mathbf{X}$ be fixed.*

- (a) *The function $F(z)$, defined by (31) for $z = r > 0$, extends to an even holomorphic function of $z \in \mathbb{C} \setminus i((-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty))$. Moreover, $F(z)/z^6$ is bounded near $z = 0$.*
- (b) *Fix $x_0 > 0$ and $y_0 > 0$. Let*

$$\begin{aligned} Q &= \{z \in \mathbb{C}; \operatorname{Re} z > x_0, y_0 > \operatorname{Im} z \geq 0\} \\ U &= Q \cup \{z \in \mathbb{C}; \operatorname{Im} z < 0\} \end{aligned}$$

Then there is a holomorphic function $H : U \rightarrow \mathbb{C}$ such that

$$R(\rho_{\mathbf{X}}z) = H(z) + \frac{\pi i}{|W|\rho_{\mathbf{X}}^2} F(z) \quad (z \in Q). \quad (36)$$

In particular, the function $R(z) = [R(z)f](y)$ extends holomorphically from \mathbb{C}^+ to $\mathbb{C} \setminus ((-\infty, 0] \cup i(-\infty, -\frac{1}{2}\rho_{\mathbf{X}}])$.

Proof. According to (34), the integrand in the definition (31) of F is

$$\frac{1}{2\pi i w} (f \times \varphi_{i\lambda(z, w)})(y) z^3 \prod_{u \in \{1, \xi, \xi^2\}} c(uw) \operatorname{th}(\pi z c(uw)), \quad (37)$$

which is a meromorphic function of $z \in \mathbb{C}$. Since $\operatorname{th}(\pi v)$ is holomorphic on $v \in \mathbb{C} \setminus i(\mathbb{Z} + \frac{1}{2})$ and $|c(w)| \leq 1$ for all $w \in S^1$, the function F extends to a holomorphic function of $z \in \mathbb{C} \setminus i((-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, +\infty))$. The change of variable given by the antipodal map $w \mapsto -w$ of S^1 shows that F is even. The last part of (a) follows from (37) and the Dominated Convergence Theorem.

Since,

$$\frac{2r}{r^2 - z^2} = \frac{1}{r - z} + \frac{1}{r + z},$$

we have

$$\begin{aligned} 2|W|\rho_{\mathbf{X}}^2 R(\rho_{\mathbf{X}}z) &= 2 \int_0^\infty \frac{F(r)}{r^2 - z^2} r dr \\ &= \int_0^\infty \frac{F(r)}{r - z} dr + \int_0^\infty \frac{F(r)}{r + z} dr. \end{aligned} \quad (38)$$

The functions

$$\mathbb{C} \setminus [0, +\infty) \ni z \mapsto \int_0^\infty \frac{F(r)}{r - z} dr \in \mathbb{C} \quad (39)$$

$$\mathbb{C} \setminus (-\infty, 0] \ni z \mapsto \int_0^\infty \frac{F(r)}{r + z} dr \in \mathbb{C} \quad (40)$$

are holomorphic.

Let $z_0 = x_0 + iy_0$. Fix in the first quadrant a curve γ_+ that starts at 0, goes to the right and up above z_0 and then becomes parallel to the positive real line and goes to infinity. We suppose that Q is interior to the region bounded by γ_+ and the positive real line.

Let M, m be fixed positive numbers. Notice that, by (22), the function $(f \times \varphi_{\mathbf{i}\lambda(z,w)})(y)$ is rapidly decreasing in the strip $\{z \in \mathbb{C}; |\operatorname{Im} z| \leq M\}$. Moreover $\operatorname{th}(\pi v)$ is bounded in the domain $\{v \in \mathbb{C}; |v| \leq 1/3\} \cup \{v \in \mathbb{C}; \operatorname{Re} v \geq m\}$. Let $z \in Q$. Then, by Cauchy's theorem,

$$\int_0^\infty \frac{F(r)}{r-z} dr = \int_{\gamma_+} \frac{F(\zeta)}{\zeta-z} d\zeta + 2\pi i F(z). \quad (41)$$

The term

$$\int_{\gamma_+} \frac{F(\zeta)}{\zeta-z} d\zeta$$

has a holomorphic extension to the forth and third quadrant through the positive reals and the negative imaginary numbers. Moreover,

$$\int_0^\infty \frac{F(r)}{r^2-z^2} r dr = \frac{1}{2} \left(\int_0^\infty \frac{F(r)}{r+z} dr + \int_{\gamma_+} \frac{F(\zeta)}{\zeta-z} d\zeta \right) + \pi i F(z).$$

□

Remark 2. Write the elements of S^1 as $w = e^{i\theta}$. So $c(w) = \cos \theta$. Since

$$\cos \theta \operatorname{th}\left(\frac{\pi i}{2} \cos \theta\right) = -\pi \cos \theta \frac{\sin\left(\frac{\pi}{2} \cos \theta\right)}{\cos\left(\frac{\pi}{2} \cos \theta\right)}$$

and since for θ near 0 the above expression is approximately equal to

$$-\pi \frac{1}{\frac{\pi}{2}(1-\cos \theta)} = -\frac{2}{(1-\cos \theta)} \approx -\frac{2}{\theta^2},$$

the function (37) is not absolutely integrable if $z = \frac{i}{2}$. Therefore it is not clear if F extends from the right or left half plane to a continuous function on any set containing parts of the imaginary axis larger than $(-\frac{i}{2}, \frac{i}{2})$.

The extension of $R(z)$ across $(-\infty, 0]$ can be deduced from the results of Mazzeo and Vasy [13] and of Strohmaier [22]. They require an additional change of variables.

Let \log denote the holomorphic branch of the logarithm defined on $\mathbb{C} \setminus]-\infty, 0]$ by $\log 1 = 0$. Set $\tau = \log z$. It gives a biholomorphism between \mathbb{C}^+ and the strip $S_{0,\pi} := \{\tau \in \mathbb{C} : 0 < \operatorname{Im} \tau < \pi\}$. Let $f \in C_c^\infty(X)$ and $y \in X$ be fixed. Define

$$[R_{\log}(\tau)f](y) = [R(\rho_X e^\tau)f](y) = \frac{1}{|W|} \int_{\mathfrak{a}^*} \frac{1}{\langle \lambda, \lambda \rangle - \rho_X^2 e^{2\tau}} (f \times \varphi_{\mathbf{i}\lambda})(y) \frac{d\lambda}{c_{\text{HC}}(\mathbf{i}\lambda)c_{\text{HC}}(-\mathbf{i}\lambda)}. \quad (42)$$

Polar coordinates in \mathfrak{a}^* give now

$$[R_{\log}(\tau)f](y) = \frac{1}{\rho_X^2 |W|} \int_{-\infty}^{+\infty} \frac{1}{e^{2t} - e^{2\tau}} F(e^t) e^{2t} dt, \quad (43)$$

where $F(r) = F(f, y, r)$ is as in (31). The evenness of the function F becomes $i\pi$ -periodicity of the function $F(e^t)$.

As for the functions F and R , we will omit the dependence of f and y in the notation and write $R_{\log}(\tau)$ instead of $[R_{\log}(\tau)f](y)$.

Proposition 3. *Let $f \in C_c^\infty(\mathbf{X})$ and $y \in \mathbf{X}$ be fixed. The function $R_{\log}(\tau) = [R_{\log}(\tau)f](y)$ extends holomorphically from $S_{0,\pi}$ to the open set*

$$U_{\log} = \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z} \setminus \{0\}} \left(i\pi \left(n + \frac{1}{2} \right) + [\log(\frac{1}{2}), +\infty) \right)$$

and satisfies the identity:

$$R_{\log}(\tau + i\pi) = R_{\log}(\tau) + \frac{i\pi}{\rho_{\mathbf{X}}^2 |\mathbb{W}|} F(e^\tau) \quad (\tau \in U_{\log}). \quad (44)$$

Consequently,

$$[R(\rho_{\mathbf{X}} z e^{i\pi})f](y) = [R(\rho_{\mathbf{X}} z)f](y) + \frac{i\pi}{\rho_{\mathbf{X}}^2 |\mathbb{W}|} F(z) \\ (z \in \mathbb{C} \setminus (i(-\infty, -\frac{1}{2}] \cup i[\frac{1}{2}, +\infty) \cup (-\infty, 0])). \quad (45)$$

Proof. This is [22, Proposition 4.3] with $f(x) = F(x)x$ for $x \in [0, +\infty)$; see also [13, Theorem 1.3]. \square

Since $F(0) = 0$, we obtain the following corollary.

Corollary 4. *Let $f \in C_c^\infty(\mathbf{X})$ and $y \in \mathbf{X}$ be fixed. Then the function $R(z) = [R(z)f](y)$ satisfies*

$$R(ze^{2i\pi}) = R(z) + \frac{2i\pi}{\rho_{\mathbf{X}}^2 |\mathbb{W}|} F(\rho_{\mathbf{X}}^{-1} z) \\ (z \in \mathbb{C} \setminus ((-\infty, 0] \cup i(-\infty, -\frac{1}{2}\rho_{\mathbf{X}}] \cup i[\frac{1}{2}\rho_{\mathbf{X}}, +\infty))). \quad (46)$$

and extends holomorphically from $\mathbb{C} \setminus ((-\infty, 0] \cup i(-\infty, -\frac{1}{2}\rho_{\mathbf{X}}])$ to a logarithmic Riemann surface branched along $(-\infty, 0]$, with the preimages of $i((-\infty, -\frac{1}{2}\rho_{\mathbf{X}}] \cup [\frac{1}{2}\rho_{\mathbf{X}}, +\infty))$ removed.

Proposition 2 and Corollary 4 show that all possible resonances of the resolvent $R(z)$ of the Laplacian of \mathbf{X} are located along the half-line $i(-\infty, -\frac{1}{2}\rho_{\mathbf{X}}]$. Because of (36), the study of $R(z)$ along $i(-\infty, -\frac{1}{2}\rho_{\mathbf{X}}]$ is reduced to that of $F(z)$ along $i(-\infty, -\frac{1}{2}]$. This will be done in the following section.

3. MEROMORPHIC EXTENSION

In this section we complete the meromorphic extension of the resolvent of the Laplacian. We first need a different expression for the function F from (31).

Recall that, for fixed $f \in C_c^\infty(\mathbf{X})$ and $y \in \mathbf{X}$,

$$F(z) = F(f, y, z) = \frac{1}{2\pi} \int_{S^1} (f \times \varphi_{i\lambda(z,w)})(y) z^3 \prod_{u \in \{1, \xi, \xi^2\}} c(uw) \operatorname{th}(\pi z c(uw)) \frac{dw}{iw},$$

where $\xi = e^{i\pi/3}$ and $c(z)$ is as in (32). Observe that

$$F(z) = \frac{1}{2\pi} \int_{S^1} \left(f \times \frac{\varphi_{i\lambda(z,w)} + \varphi_{-i\lambda(z,w)}}{2} \right) (y) z^3 \prod_{u \in \{1, \xi, \xi^2\}} c(uw) \operatorname{th}(\pi z c(uw)) \frac{dw}{iw},$$

where now $(f \times \frac{\varphi_{i\lambda(z,w)} + \varphi_{-i\lambda(z,w)}}{2})(y)$ is an even function of z and w .

Let

$$\phi_{z,u}(w) = \frac{zc(uw)}{iw} \operatorname{th}(\pi zc(uw)) \quad (zc(uw) \notin i(\mathbb{Z} + \frac{1}{2}), z \in \mathbb{C}, w \in \mathbb{C}^\times), \quad (47)$$

$$\psi_z(w) = \frac{1}{2\pi} \left(f \times \frac{\varphi_{i\lambda(z,w)} + \varphi_{-i\lambda(z,w)}}{2} \right) (y)(iw)^2 \quad (z \in \mathbb{C}, w \in \mathbb{C}^\times). \quad (48)$$

Hence

$$F(z) = \int_{S^1} \psi_w(z) \prod_{u \in \{1, \xi, \xi^2\}} \phi_{z,u}(w) dw. \quad (49)$$

Note that

$$\phi_{z,u}(w) = \phi_{\pm z, \pm u}(w) \quad \text{and} \quad \psi_z(w) = \psi_{-z}(w). \quad (50)$$

Since $\lambda \mapsto \varphi_{i\lambda} + \varphi_{-i\lambda}$ is Weyl group invariant and even, it is also invariant under rotations by multiples of $\pi/3$. It follows that for $u = e^{i\theta}$ with $\theta \in \{ \pm \frac{\pi}{3}, \pm \frac{2\pi}{3} \}$ we have

$$\psi_z(uw) = \psi_z(w) u^2. \quad (51)$$

Moreover

$$\phi_{z,u}(-w) = -\phi_{z,u}(w) \quad \text{and} \quad \psi_z(-w) = \psi_z(w). \quad (52)$$

3.1. Analytic properties of the function $c(z)$. To proceed further, we need some analytic properties of the functions $c(z)$ and $s(z)$. Recall that for $z \in \mathbb{C}^\times$ these functions are defined by

$$c(z) = \frac{z + z^{-1}}{2} \quad \text{and} \quad s(z) = \frac{z - z^{-1}}{2}.$$

For $r > 0$ and $a, b \in \mathbb{R} \setminus \{0\}$ let

$$\begin{aligned} D_r &= \{z \in \mathbb{C}; |z| < r\}, \\ E_{a,b} &= \left\{ \xi + i\eta \in \mathbb{C}; \left(\frac{\xi}{a} \right)^2 + \left(\frac{\eta}{b} \right)^2 < 1 \right\}. \end{aligned}$$

Their boundaries ∂D_r and $\partial E_{a,b}$ are respectively the circle of radius r and the ellipse of semi-axes $|a|, |b|$, both centered at 0. In particular, $\partial D_1 = S^1$. Furthermore, $\overline{D_r}$ is the closed disc of center 0 and radius r .

Observe that $c : D_1 \setminus \{0\} \rightarrow \mathbb{C} \setminus [-1, 1]$ is a biholomorphic function. We denote by c^{-1} its inverse. More precisely, for $0 < r < 1$, it restricts to a biholomorphic function

$$c : D_1 \setminus \overline{D_r} \rightarrow E_{c(r), s(r)} \setminus [-1, 1]. \quad (53)$$

Also,

$$c : \partial D_r \rightarrow \partial E_{c(r), s(r)}. \quad (54)$$

is a bijection. In particular,

$$Re^{i\Theta} c(\partial D_r) = Re^{i\Theta} \partial E_{c(r), s(r)}$$

is the ellipse $\partial E_{c(r), s(r)}$ rotated by Θ and dilated by R .

Let $\sqrt{\cdot}$ denote the single-valued holomorphic branch of the square root function defined on $\mathbb{C} \setminus (-\infty, 0]$ by

$$\sqrt{Re^{i\Theta}} = \sqrt{R} e^{\frac{i\Theta}{2}} \quad (R > 0, -\pi < \Theta < \pi). \quad (55)$$

A straightforward computation shows that

$$\sqrt{x+iy} = \sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} + i \operatorname{sign}(y) \sqrt{\frac{\sqrt{x^2+y^2}-x}{2}} \quad (x+iy \in \mathbb{C} \setminus \mathbb{R}^-). \quad (56)$$

Lemma 5. *The function $\sqrt{z+1}\sqrt{z-1}$ originally defined on $\mathbb{C} \setminus (-\infty, 1]$ extends to a holomorphic function on $\mathbb{C} \setminus [-1, 1]$. Also, $\sqrt{(-z)+1}\sqrt{(-z)-1} = -\sqrt{z+1}\sqrt{z-1}$.*

Proof. For $z = x + iy$ with $y \neq 0$, write

$$\begin{aligned} A &= (x+1)^2 + y^2, & a &= x+1, \\ B &= (x-1)^2 + y^2, & b &= x-1. \end{aligned}$$

Then

$$\begin{aligned} 2\sqrt{z+1}\sqrt{z-1} &= \left(\sqrt{A+a} + i \operatorname{sign}(y) \sqrt{A-a}\right) \left(\sqrt{B+b} + i \operatorname{sign}(y) \sqrt{B-b}\right) \\ &= \left(\sqrt{A+a}\sqrt{B+b} - \sqrt{A-a}\sqrt{B-b}\right) \\ &\quad + i \operatorname{sign}(y) \left(\sqrt{A+a}\sqrt{B-b} + \sqrt{A-a}\sqrt{B+b}\right). \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{y \rightarrow 0^\pm} 2\sqrt{z+1}\sqrt{z-1} &= \left(\sqrt{|a|+a}\sqrt{|b|+b} - \sqrt{|a|-a}\sqrt{|b|-b}\right) \\ &\quad \pm i \left(\sqrt{|a|+a}\sqrt{|b|-b} + \sqrt{|a|-a}\sqrt{|b|+b}\right). \end{aligned}$$

If $x+1 < 0$, then $b < a < 0$. So

$$\lim_{y \rightarrow 0^\pm} \sqrt{z+1}\sqrt{z-1} = -\sqrt{|a||b|} = -\sqrt{x^2-1} \quad (57)$$

is real. Thus $\sqrt{z+1}\sqrt{z-1}$ extends holomorphically across $(-\infty, -1)$ by Schwarz's reflection principle.

Checking the last formula is straightforward. \square

Lemma 6. *For $z \in \mathbb{C} \setminus [-1, 1]$, the following formulas hold:*

$$\begin{aligned} (z - \sqrt{z+1}\sqrt{z-1})(z + \sqrt{z+1}\sqrt{z-1}) &= 1, \\ c^{-1}(z) &= z - \sqrt{z+1}\sqrt{z-1}, \\ s \circ c^{-1}(z) &= -\sqrt{z+1}\sqrt{z-1}. \end{aligned}$$

Also

$$\begin{aligned} \sqrt{x+i0+1}\sqrt{x+i0-1} &= i\sqrt{1-x^2} \quad (-1 < x < 1), \\ \sqrt{x-i0+1}\sqrt{x-i0-1} &= -i\sqrt{1-x^2} \quad (-1 < x < 1), \\ c^{-1}(x \pm i0) &= x \mp i\sqrt{1-x^2} \quad (-1 < x < 1). \end{aligned}$$

In particular, the “upper part” of the interval $[-1, 1]$ is mapped via c^{-1} onto the lower half of the unit circle and the “lower part” of the interval $[-1, 1]$ is mapped onto the upper half of the unit circle. Furthermore

$$\begin{aligned} s \circ c^{-1} : \partial E_{c(r),s(r)} &\rightarrow \partial E_{s(r),s(r)}, \\ s \circ c^{-1} : E_{c(r),s(r)} \setminus [-1, 1] &\rightarrow E_{s(r),c(r)} \setminus i[-1, 1], \end{aligned}$$

are bijections.

Proof. The first equation is obvious. The two sides of the second equation are holomorphic and equal for $z > 1$, hence equal in the whole domain of definition. Also,

$$\begin{aligned} s \circ c^{-1}(z) &= \frac{z - \sqrt{z+1}\sqrt{z-1} - (z - \sqrt{z+1}\sqrt{z-1})^{-1}}{2} \\ &= \frac{z - \sqrt{z+1}\sqrt{z-1} - (z + \sqrt{z+1}\sqrt{z-1})}{2} = -\sqrt{z+1}\sqrt{z-1} \quad (0 < r < 1). \end{aligned}$$

The two limits are easy to compute. The last statement follows from the fact that

$$s : \partial D_r \rightarrow \partial E_{s(r),c(r)}$$

and

$$s : D_1 \setminus \overline{D_r} \rightarrow E_{s(r),c(r)} \setminus i[-1, 1]$$

are bijections. \square

Lemma 7. *Let $|\zeta| = |\zeta_0| = 1$. Then*

$$(\zeta c^{-1}(\zeta_0 \mathbb{R}^+ \setminus [-1, 1])) \cap (c^{-1}(\zeta_0 \mathbb{R}^+ \setminus [-1, 1])) \neq \emptyset \text{ if and only if } \zeta = 1. \quad (58)$$

Also

$$(\zeta c^{-1}(\zeta_0 \mathbb{R} \setminus [-1, 1])) \cap (c^{-1}(\zeta_0 \mathbb{R} \setminus [-1, 1])) \neq \emptyset \text{ if and only if } \zeta = \pm 1. \quad (59)$$

Proof. Recall that c^{-1} is a bijection of $\mathbb{C} \setminus [-1, 1]$ onto $D_1 \setminus \{0\}$. If the intersection (58) is not empty then there is $0 < r < 1$ and z with $|z| = r$ such that the two points z and ζz are mapped by c to a point on the ray $\zeta_0 \mathbb{R}^+$. But the image of the circle ∂D_r is the ellipse $\partial E_{c(r),s(r)}$, which has only one point of intersection with the ray. So $c(z) = c(\zeta z)$. Hence $z = \zeta z$, which means that $\zeta = 1$. This verifies (58).

Since $c(-z) = -c(z)$ a similar argument proves (59). \square

3.2. Deformation of S^1 and residues. We keep the notation introduced above.

Lemma 8. *Suppose $z \in \mathbb{C} \setminus i((-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty))$ and $0 < r < 1$ are such that*

$$i(\mathbb{Z} + \frac{1}{2}) \cap z \partial E_{c(r),s(r)} = \emptyset. \quad (60)$$

Then

$$F(z) = F_r(z) + 2\pi i G_r(z), \quad (61)$$

where

$$F_r(z) = \int_{\partial D_r} \psi_z(w) \prod_{u \in \{1, \xi, \xi^2\}} \phi_{z,u}(w) dw, \quad (62)$$

$$G_r(z) = \sum'_{w_0} \psi_z(w_0) \operatorname{Res}_{w=w_0} \prod_{u' \in \{1, \xi, \xi^2\}} \phi_{z,u'}(w), \quad (63)$$

where \sum'_{w_0} denotes the sum over the w_0 's for which there is a $u \in \{1, \xi, \xi^2\}$ so that

$$zc(uw_0) \in i(\mathbb{Z} + \frac{1}{2}) \cap z(E_{c(r),s(r)} \setminus [-1, 1]).$$

For a fixed r , both F_r and G_r are holomorphic on the open subset of $\mathbb{C} \setminus i((-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty))$ where the condition (60) holds. Furthermore, F_r extends to a holomorphic function on the open subset of \mathbb{C} where the condition (60) holds. Also, $F_r(0) = G_r(0) = 0$.

Proof. Observe that, by (53), the condition that $w_0 \in D_1 \setminus \overline{D_r}$ and there is $u \in \{1, \xi, \xi^2\}$ so that $zc(uw_0) \in i(\mathbb{Z} + \frac{1}{2})$ is equivalent to that there is $u \in \{1, \xi, \xi^2\}$ for which $zc(uw_0) \in i(\mathbb{Z} + \frac{1}{2}) \cap z(\mathbf{E}_{c(r),s(r)} \setminus [-1, 1])$. Formula (61) follows then from (49) and the Residue Theorem.

Since $\phi_{z,u}(w)$ and $\psi_z(w)$ are holomorphic functions of the two variables z and w , we see from (54) that F_r is a holomorphic function of z in the region where (60) holds. Also F is holomorphic on $\mathbb{C} \setminus i((-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty))$, hence G_r is also holomorphic there.

Since $\phi_{0,u} = 0$ we obtain that $F_r(0) = G_r(0) = 0$. \square

We now derive an explicit formula for the function G_r in (63).

Lemma 9. *Let z and r be as in (60). Fix $u \in \{1, \xi, \xi^2\}$. Let $n_0 \in \mathbb{Z}$ be such that $i(n_0 + \frac{1}{2}) \in z(\mathbf{E}_{c(r),s(r)} \setminus [-1, 1])$ and let $w_0 \in D_1 \setminus \overline{D_r}$ be such that $zc(uw_0) = i(n_0 + \frac{1}{2})$. Then $z \neq 0$,*

$$w_0 = \frac{1}{u} c^{-1} \left(\frac{i}{z} (n_0 + \frac{1}{2}) \right) \quad (64)$$

and

$$\operatorname{Res}_{w=w_0} \prod_{u' \in \{1, \xi, \xi^2\}} \phi_{z,u'}(w) = \left(\prod_{u' \neq u} \phi_{z,u'}(w_0) \right) \cdot \frac{\frac{i}{z} (n_0 + \frac{1}{2})}{i\pi(s \circ c^{-1})(\frac{i}{z} (n_0 + \frac{1}{2}))}. \quad (65)$$

Conversely, suppose that $n_0 \in \mathbb{Z}$ is so that $i(n_0 + \frac{1}{2}) \in z(\mathbf{E}_{c(r),s(r)} \setminus [-1, 1])$, and define w_0 by (65). Then $w_0 \in D_1 \setminus \overline{D_r}$ and $zc(uw_0) = i(n_0 + \frac{1}{2})$.

Proof. Since $0 \notin i(\mathbb{Z} + \frac{1}{2})$, we see that $z \neq 0$. If $u' \neq u$, then Lemma 7 shows that the set of the singularities of the function $\phi_{z,u'}$ within the punctured open unit disc $D_1 \setminus \{0\}$ is disjoint from the set of the singularities of the function $\phi_{z,u}$. Therefore

$$\operatorname{Res}_{w=w_0} \prod_{u'} \phi_{z,u'}(w) = \left(\prod_{u' \neq u} \phi_{z,u'}(w_0) \right) \cdot \operatorname{Res}_{w=w_0} \phi_{z,u}(w).$$

Furthermore,

$$\begin{aligned} \operatorname{Res}_{w=w_0} \phi_{z,u}(w) &= \frac{zc(uw_0)}{iw_0} \operatorname{sh}(\pi zc(uw_0)) \frac{1}{\frac{d}{dw} \operatorname{ch}(\pi wc(uz))|_{w=w_0}} \\ &= \frac{zc(uw_0)}{iw_0} \operatorname{sh}(\pi zc(uw_0)) \frac{1}{\operatorname{sh}(\pi zc(uw_0)) \pi z u s(uw_0) (uw_0)^{-1}} \\ &= \frac{zc(uw_0)}{i\pi s(uw_0)} = \frac{\frac{i}{z} (n_0 + \frac{1}{2})}{i\pi(s \circ c^{-1})(\frac{i}{z} (n_0 + \frac{1}{2}))}. \end{aligned}$$

The last statement follows immediately from (53). \square

Lemma 10. *For all $z \in \mathbb{C}$ and $w \in \mathbb{C}^\times$ we have*

$$\sum_{u \in \{1, \xi, \xi^2\}} \psi_z(\frac{1}{u}w) \left(\prod_{u' \neq u} \phi_{z,u'}(\frac{1}{u}w) \right) = -3\psi_z(w) \phi_{z,1}(\xi w) \phi_{z,1}(\xi^2 w). \quad (66)$$

Proof. Notice first that

$$\begin{aligned}\phi_{z,u'}\left(\frac{1}{u}w\right) &= \frac{zc\left(\frac{u'}{u}w\right)}{i\frac{1}{u}w} \operatorname{th}\left(\pi zc\left(\frac{u'}{u}w\right)\right) \\ &= u' \frac{zc\left(\frac{u'}{u}w\right)}{i\frac{u'}{u}w} \operatorname{th}\left(\pi zc\left(\frac{u'}{u}w\right)\right) \\ &= u' \phi_{z,1}\left(\frac{u'}{u}w\right).\end{aligned}$$

By (51), we have

$$\psi_z\left(\frac{1}{u}w\right) = \psi_z(w) \frac{1}{u^2}.$$

Hence

$$\begin{aligned}\sum_u \psi_z\left(\frac{1}{u}w\right) \left(\prod_{u' \neq u} \phi_{z,u'}\left(\frac{1}{u}w\right)\right) &= \sum_u \psi_z(w) \frac{1}{u^2} \left(\prod_{u' \neq u} u' \phi_{z,1}\left(\frac{u'}{u}w\right)\right) \\ &= \sum_u \psi_z(w) \left(\prod_{u' \neq u} \frac{u'}{u} \phi_{z,1}\left(\frac{u'}{u}w\right)\right).\end{aligned}\tag{67}$$

Since

$$-\frac{u'}{u} \phi_{z,1}\left(-\frac{u'}{u}w\right) = \frac{u'}{u} \phi_{z,1}\left(\frac{u'}{u}w\right),$$

we may assume that in the above product, for each fixed u ,

$$\frac{u'}{u} = \pm \frac{1}{2} + i \frac{\sqrt{3}}{2} \in \{\boldsymbol{\xi}, \boldsymbol{\xi}^2\}.$$

Then

$$\prod_{u' \neq u} \frac{u'}{u} \phi_{z,1}\left(\frac{u'}{u}w\right) = \boldsymbol{\xi} \phi_{z,1}(\boldsymbol{\xi}w) \boldsymbol{\xi}^2 \phi_{z,1}(\boldsymbol{\xi}^2w).$$

Since $\boldsymbol{\xi}^3 = -1$, we conclude that (67) is equal to the right-hand side of (66). \square

Corollary 11. *Under the assumptions of Lemma 8,*

$$G_r(z) = \sum_{n \in S_{r,z}} G_{(n)}(z),\tag{68}$$

where

$$S_{r,z} = \left\{n \in \mathbb{Z}; i\left(n + \frac{1}{2}\right) \in z(\mathbb{E}_{c(r),s(r)} \setminus [-1, 1])\right\}\tag{69}$$

and

$$\begin{aligned}G_{(n)}(z) &= -3\psi_z\left(c^{-1}\left(\frac{i}{z}\left(n + \frac{1}{2}\right)\right)\right) \phi_{z,1}\left(\boldsymbol{\xi}c^{-1}\left(\frac{i}{z}\left(n + \frac{1}{2}\right)\right)\right) \phi_{z,1}\left(\boldsymbol{\xi}^2c^{-1}\left(\frac{i}{z}\left(n + \frac{1}{2}\right)\right)\right) \\ &\quad \times \frac{\frac{i}{z}\left(n + \frac{1}{2}\right)}{i\pi(s \circ c^{-1})\left(\frac{i}{z}\left(n + \frac{1}{2}\right)\right)}.\end{aligned}\tag{70}$$

The function (70) is holomorphic and even on $\mathbb{C} \setminus i\mathbb{R}$. It satisfies

$$G_{(n)}(z) = G_{(-n-1)}(z).\tag{71}$$

Moreover, $n \in S_{r,-z}$ if and only if $-n - 1 \in S_{r,z}$.

Proof. The equality (68) is immediate from Lemmas 9 and 10. If $z \in \mathbb{C} \setminus i\mathbb{R}$, then $\frac{i}{z}(n + \frac{1}{2}) \in \mathbb{C} \setminus i\mathbb{R}$. Hence $G_{(n)}(z)$ is holomorphic on $\mathbb{C} \setminus i\mathbb{R}$. It is even, because of (50) and (52) and since $c^{-1}(-w) = -c^{-1}(w)$ for $w \in \mathbb{C} \setminus [-1, 1]$. The equality $G_{(n)}(-z) = G_{(-n-1)}(z)$ follows from Lemmas 5 and 6 together with (50) and (52). This proves (71) as $G_{(n)}(z)$ is even. \square

Corollary 12. *Let $W \subseteq \mathbb{C}$ be a connected open set and let $0 < r < 1$ be such that*

$$i(\mathbb{Z} + \frac{1}{2}) \cap W \partial E_{c(r), s(r)} = \emptyset. \quad (72)$$

Then the set $S_r := S_{r,z}$ defined in (69) does not depend on $z \in W \setminus i\mathbb{R}$ and

$$G_r(z) = \sum_{n \in S_r} G_{(n)}(z) \quad (z \in W \setminus i\mathbb{R}). \quad (73)$$

Also,

$$n \in S_r \text{ if and only if } -n - 1 \in S_r. \quad (74)$$

Proof. The condition (72) implies that for any fixed $n \in \mathbb{Z}$ the set $\{z \in W : i(n + \frac{1}{2}) \in wE_{c(r), s(r)}\}$ is open and closed in W . Since W is connected, this set is either \emptyset or W . Hence

$$i(\mathbb{Z} + \frac{1}{2}) \cap zE_{c(r), s(r)} = i(\mathbb{Z} + \frac{1}{2}) \cap WE_{c(r), s(r)} \quad (z \in W). \quad (75)$$

Notice that if $z \in \mathbb{C} \setminus i\mathbb{R}$, then

$$i(\mathbb{Z} + \frac{1}{2}) \cap zE_{c(r), s(r)} = i(\mathbb{Z} + \frac{1}{2}) \cap (z(E_{c(r), s(r)} \setminus [-1, 1])).$$

Hence, for each $z \in W \setminus i\mathbb{R}$

$$S_{r,z} = i(\mathbb{Z} + \frac{1}{2}) \cap WE_{c(r), s(r)}. \quad (76)$$

Therefore (73) follow from Corollary 11.

Since our ellipses are invariant under the multiplication by -1 , the formula (74) follows from (69). \square

Lemma 13. *For any $iv \in i\mathbb{R}$ there is a connected neighborhood $W \subseteq \mathbb{C}$ of iv and $0 < r < 1$ such that the condition (72) holds.*

Proof. Choose $0 < r < 1$ so that

$$i(\mathbb{Z} + \frac{1}{2}) \cap iv \partial E_{c(r), s(r)} = \emptyset.$$

Then enlarge iv to W . \square

Lemma 13 shows that for any point $iv \in i\mathbb{R}$ there is a neighborhood W of that point and a radius $0 < r < 1$ so that condition (72) is satisfied. Lemma 8 and Corollary 12 show that

$$F(z) = F_r(z) + 2\pi i \sum_{n \in S_r} G_{(n)}(z) \quad (z \in W \setminus i\mathbb{R}), \quad (77)$$

where F_r extends to a holomorphic function on W . Let $N \geq 0$ be the largest element in S_r . Then (69) and (74) show that $S_r = \{-N - 1, -N, \dots, N - 1, N\}$. Moreover, $G_{(n)} = G_{(-n-1)}$

by (71). Hence, (77) may be rewritten as

$$\begin{aligned} F(z) &= F_r(z) + 2\pi i \sum_{n=-N-1}^N G_{(n)}(z) \\ &= F_r(z) + 4\pi i \sum_{n=0}^N G_{(n)}(z) \quad (z \in W \setminus i\mathbb{R}). \end{aligned} \quad (78)$$

Notice that, since $c(r) > 1$, if $iv = i(m + \frac{1}{2})$ with $m \in \mathbb{N}$, then $m \in S_r$ for all $0 < r < 1$.

Remark 3. By Lemma 8, for a fixed $0 < r < 1$, the function $F_r(z)$ is holomorphic on the set of $z \in \mathbb{C}$ for which (60) holds. The intersection of this set with $i\mathbb{R}$ consists of the points iv for which $vc(r) \notin \mathbb{Z} + \frac{1}{2}$. The function $F_r(z)$ will be therefore holomorphic in an open set containing $i(\mathbb{Z} + \frac{1}{2})$ provided $0 < r < 1$ is chosen so that $(\mathbb{Z} + \frac{1}{2}) \cap c(r)(\mathbb{Z} + \frac{1}{2}) \neq \emptyset$.

The functions $G_{(n)}(z)$ are defined and even on $\mathbb{C} \setminus i\mathbb{R}$. In the following section we fix $n \in \mathbb{Z}$ and determine a 2-sheeted Riemann surface M_n covering $\mathbb{C} \setminus \{0\}$ and branched at $z = \pm i(n + \frac{1}{2})$. Then we prove that $G_{(n)}(z)$ extends as a meromorphic function to M_n , with simple poles at the branching points above $z = \pm i(n + \frac{1}{2})$.

3.3. Meromorphic extension of the functions $G_{(n)}(z)$. The derivative of the map

$$\mathbb{C}^2 \ni (w, \zeta) \mapsto \zeta^2 - w^2 + 1 \in \mathbb{C}$$

is equal to $(-2w, 2\zeta)$. This is zero if and only if $w = \zeta = 0$. Hence the map

$$(\mathbb{C}^2 \setminus \{0, 0\}) \ni (w, \zeta) \mapsto \zeta^2 - w^2 + 1 \in \mathbb{C}$$

is a submersion. Therefore the preimage of zero

$$M = \{(w, \zeta); \zeta^2 = w^2 - 1\} \subseteq \mathbb{C}^2 \quad (79)$$

is a complex submanifold. The fibers of the surjective holomorphic map

$$\pi : M \ni (w, \zeta) \mapsto w \in \mathbb{C} \quad (80)$$

consist of two points (w, ζ) and $(w, -\zeta)$, if and only if $w \neq \pm 1$. If $w = \pm 1$, then the fibers consist of one point $(w, 0)$.

Let $S \subseteq \mathbb{C}$ and let $\tilde{S} \subseteq M$ be the preimage of S in M under the map π given by (80). We say that a function $\tilde{f} : \tilde{S} \rightarrow \mathbb{C}$ is a lift of a $f : S \rightarrow \mathbb{C}$ if there is a holomorphic section $\sigma : S \rightarrow \tilde{S}$ of the restriction of π to \tilde{S} so that

$$\tilde{f}(w, \zeta) = f(w) \quad ((w, \zeta) \in \sigma(S)).$$

We see from Lemma 6 that

$$(s \circ c^{-1})^\sim : M \ni (w, \zeta) \mapsto -\zeta \in \mathbb{C} \quad (81)$$

is a lift of

$$s \circ c^{-1} : \mathbb{C} \setminus [-1, 1] \ni w \mapsto -\sqrt{w+1}\sqrt{w-1} \in \mathbb{C}$$

and

$$(c^{-1})^\sim : M \ni (w, \zeta) \mapsto w - \zeta \in \mathbb{C} \quad (82)$$

is a lift of

$$c^{-1} : \mathbb{C} \setminus [-1, 1] \ni w \mapsto w - \sqrt{w+1}\sqrt{w-1} \in \mathbb{C}.$$

Both maps (81) and (82) are holomorphic.

Let

$$\begin{aligned} M_{(1,0)} &= \{(w, \zeta); \zeta^2 = w^2 - 1, \operatorname{Re} w > 0\}, \\ M_{(-1,0)} &= \{(w, \zeta); \zeta^2 = w^2 - 1, \operatorname{Re} w < 0\}. \end{aligned}$$

Then $M_{(1,0)}$ is an open neighborhood of $(1, 0) \in M$ and $M_{(-1,0)}$ is an open neighborhood of $(-1, 0) \in M$. Furthermore the following maps are local charts:

$$\begin{aligned} M_{(1,0)} \ni (w, \zeta) &\mapsto \zeta \in \mathbb{C} \setminus i((-\infty, -1] \cup [1, +\infty)), \quad w = \sqrt{\zeta^2 + 1}, \\ M_{(-1,0)} \ni (w, \zeta) &\mapsto \zeta \in \mathbb{C} \setminus i((-\infty, -1] \cup [1, +\infty)), \quad w = -\sqrt{\zeta^2 + 1}. \end{aligned}$$

Lemma 14. Fix $n \in \mathbb{Z}$ and let

$$M_n = \left\{ (z, \zeta) \in \mathbb{C}^\times \times (\mathbb{C} \setminus \{\pm i\}); \zeta^2 = \left(\frac{i}{z} \left(n + \frac{1}{2} \right) \right)^2 - 1 \right\}.$$

The fibers of the map

$$\pi_n : M_n \ni (z, \zeta) \mapsto z \in \mathbb{C}^\times \quad (83)$$

are $\{(z, \zeta), (z, -\zeta)\}$. In particular

$$M_n \setminus \left\{ \pm \left(i \left(n + \frac{1}{2} \right), 0 \right) \right\} \ni (z, \zeta) \mapsto z \in \mathbb{C}^\times \setminus \left\{ \pm i \left(n + \frac{1}{2} \right) \right\}$$

is a double cover.

Let $M_{n,0} = \{(z, \zeta) \in M_n; z \notin i\mathbb{R}\}$. The function $G_{(n)}$, (70), lifts to a holomorphic function $\tilde{G}_{(n)} : M_{n,0} \rightarrow \mathbb{C}$ and then extends to a meromorphic function $\tilde{G}_{(n)} : M_n \rightarrow \mathbb{C}$ by the formula

$$\tilde{G}_{(n)}(z, \zeta) = -3 \psi_z \left(\frac{i}{z} \left(n + \frac{1}{2} \right) - \zeta \right) \phi_{z,1} \left(\xi \left(\frac{i}{z} \left(n + \frac{1}{2} \right) - \zeta \right) \right) \phi_{z,1} \left(\xi^2 \left(\frac{i}{z} \left(n + \frac{1}{2} \right) - \zeta \right) \right) \cdot \frac{\frac{i}{z} \left(n + \frac{1}{2} \right)}{-i\pi\zeta}. \quad (84)$$

The function $\tilde{G}_{(n)}$ is holomorphic on the set $M_n \setminus \{(i(n + \frac{1}{2}), 0), (-i(n + \frac{1}{2}), 0)\}$. It satisfies

$$\tilde{G}_{(n)}(-z, -\zeta) = \tilde{G}_{(n)}(z, \zeta) \quad ((z, \zeta) \in M_n). \quad (85)$$

The following sets are open neighborhoods of $(i(n + \frac{1}{2}), 0)$ and $(-i(n + \frac{1}{2}), 0)$, respectively:

$$\begin{aligned} U_{i(n+\frac{1}{2})} &= \{(z, \zeta) \in M_n; (n + \frac{1}{2}) \operatorname{Im} z > 0\}, \\ U_{-i(n+\frac{1}{2})} &= \{(z, \zeta) \in M_n; (n + \frac{1}{2}) \operatorname{Im} z < 0\}. \end{aligned}$$

Furthermore the following maps are local charts:

$$\begin{aligned} \kappa_+ : U_{i(n+\frac{1}{2})} \ni (z, \zeta) &\mapsto \zeta \in \mathbb{C} \setminus i((-\infty, -1] \cup [1, +\infty)), \quad z = \frac{i(n + \frac{1}{2})}{\sqrt{\zeta^2 + 1}}, \\ \kappa_- : U_{-i(n+\frac{1}{2})} \ni (z, \zeta) &\mapsto \zeta \in \mathbb{C} \setminus i((-\infty, -1] \cup [1, +\infty)), \quad z = -\frac{i(n + \frac{1}{2})}{\sqrt{\zeta^2 + 1}}. \end{aligned} \quad (86)$$

The points $(\pm i(n + \frac{1}{2}), 0)$ are simple poles of $\tilde{G}_{(n)}$. The local expressions for $\tilde{G}_{(n)}$ in terms of the charts (86) are

$$\begin{aligned} \tilde{G}_{(n)} \circ \kappa_+^{-1}(\zeta) &= 3\psi_{\frac{i(n+\frac{1}{2})}{\sqrt{\zeta^2+1}}}(\sqrt{\zeta^2+1}-\zeta) \\ &\times \phi_{\frac{i(n+\frac{1}{2})}{\sqrt{\zeta^2+1}},1}(\xi(\sqrt{\zeta^2+1}-\zeta))\phi_{\frac{i(n+\frac{1}{2})}{\sqrt{\zeta^2+1}},1}(\xi^2(\sqrt{\zeta^2+1}-\zeta))\frac{\sqrt{\zeta^2+1}}{i\pi\zeta} \end{aligned} \quad (87)$$

and

$$\begin{aligned} \tilde{G}_{(n)} \circ \kappa_-^{-1}(\zeta) &= -3\psi_{\frac{i(n+\frac{1}{2})}{\sqrt{\zeta^2+1}}}(\sqrt{\zeta^2+1}+\zeta) \\ &\times \phi_{\frac{i(n+\frac{1}{2})}{\sqrt{\zeta^2+1}},1}(\xi(\sqrt{\zeta^2+1}+\zeta))\phi_{\frac{i(n+\frac{1}{2})}{\sqrt{\zeta^2+1}},1}(\xi^2(\sqrt{\zeta^2+1}+\zeta))\frac{\sqrt{\zeta^2+1}}{i\pi\zeta}. \end{aligned} \quad (88)$$

Furthermore,

$$2\pi i \operatorname{Res}_{\zeta=0} \tilde{G}_{(n)} \circ \kappa_+^{-1}(\zeta) = \frac{3}{2\pi} \left(n + \frac{1}{2}\right)^2 \psi_{i(n+\frac{1}{2})}(1) \quad (89)$$

and

$$2\pi i \operatorname{Res}_{\zeta=0} \tilde{G}_{(n)} \circ \kappa_-^{-1}(\zeta) = -\frac{3}{2\pi} \left(n + \frac{1}{2}\right)^2 \psi_{i(n+\frac{1}{2})}(1). \quad (90)$$

Proof. We replace w by $\frac{i}{z}(n + \frac{1}{2})$ in the discussion preceding the lemma. This explains the structure of the covering (83). Formula (84) follows from that discussion and (70). The function (84) might be singular at some point (z, ζ) with $\zeta \neq 0$ if and only if

$$\phi_{z,1}(\xi(\frac{i}{z}(n + \frac{1}{2}) - \zeta))\phi_{z,1}(\xi^2(\frac{i}{z}(n + \frac{1}{2}) - \zeta)) \quad (91)$$

is singular at (z, ζ) , $\zeta \neq 0$. Fix $k \in \{1, 2\}$ and suppose one term

$$\phi_{z,1}(\xi^k(\frac{i}{z}(n + \frac{1}{2}) - \zeta)) \quad (92)$$

is singular at (z, ζ) , $\zeta \neq 0$. Then

$$zc(\xi^k(\frac{i}{z}(n + \frac{1}{2}) - \zeta)) \in i(\mathbb{Z} + \frac{1}{2}),$$

which means

$$c(\xi^k(\frac{i}{z}(n + \frac{1}{2}) - \zeta)) \in \frac{i}{z}(\mathbb{Z} + \frac{1}{2}). \quad (93)$$

Assume first that

$$z \notin i(-\infty, -|n + \frac{1}{2}|] \cup i[|n + \frac{1}{2}|, \infty) \quad (94)$$

or equivalently

$$\frac{i}{z}(n + \frac{1}{2}) \notin [-1, 1]. \quad (95)$$

Then $c^{-1}(\frac{i}{z}(n + \frac{1}{2}))$ is in the open unit disc D_1 with the zero removed. It follows that

$$\xi^k c^{-1}(\frac{i}{z}(n + \frac{1}{2})) \in D_1 \setminus \{0\} = c^{-1}(\mathbb{C} \setminus [-1, 1]). \quad (96)$$

Because of (95), Lemma 6 implies that

$$\frac{i}{z}(n + \frac{1}{2}) - \zeta = c^{-1}(\frac{i}{z}(n + \frac{1}{2})) \quad \text{or} \quad \frac{i}{z}(n + \frac{1}{2}) - \zeta = (c^{-1}(\frac{i}{z}(n + \frac{1}{2})))^{-1}. \quad (97)$$

In the first case, (93) and (96) yield

$$\xi^k c^{-1}(\frac{i}{z}(n + \frac{1}{2})) \in c^{-1}(\frac{i}{z}(\mathbb{Z} + \frac{1}{2}) \setminus [-1, 1]).$$

Since $\xi^k \neq \pm 1$, Lemma 7 shows that this is impossible. In the second case,

$$\begin{aligned} c(\xi^k(\frac{i}{z}(n + \frac{1}{2}) - \zeta)) &= c(\xi^k(c^{-1}(\frac{i}{z}(n + \frac{1}{2})))^{-1}) = c((\xi^{-k}c^{-1}(\frac{i}{z}(n + \frac{1}{2})))^{-1}) \\ &= c(\xi^{-k}c^{-1}(\frac{i}{z}(n + \frac{1}{2}))). \end{aligned}$$

Hence, (93) and (96) yield

$$\xi^{-k}c^{-1}(\frac{i}{z}(n + \frac{1}{2})) \in c^{-1}(\frac{i}{z}(\mathbb{Z} + \frac{1}{2}) \setminus [-1, 1]).$$

Since $\xi^{-k} \neq \pm 1$, Lemma 7 shows that this is impossible.

Assume from now on that

$$z \in i(-\infty, -|n + \frac{1}{2}|] \cup i[|n + \frac{1}{2}|, \infty) \quad (98)$$

or equivalently that

$$\frac{i}{z}(n + \frac{1}{2}) \in [-1, 1]. \quad (99)$$

Let us find all the points (z, ζ) , $\zeta \neq 0$, where the term

$$\phi_{z,1}(\xi^k(\frac{i}{z}(n + \frac{1}{2}) - \zeta)) \quad (100)$$

is singular. These are the points where (93) holds.

For an integer m let $x_m = \frac{i}{z}(m + \frac{1}{2})$. Then there is a unique $\epsilon = \pm 1$ such that

$$\frac{i}{z}(n + \frac{1}{2}) = x_n \quad \text{and} \quad \zeta = -\epsilon i \sqrt{1 - x_n^2}.$$

Hence (93) is equivalent to the statement that there is $m \in \mathbb{Z}$ such that

$$c(\xi^k(x_n + \epsilon i \sqrt{1 - x_n^2})) = x_m. \quad (101)$$

By (99), the left-hand side of this equation is equal to cosine of some real angle. Hence,

$$x_m \in [-1, 1]. \quad (102)$$

Notice that $\xi^k = \delta \frac{1}{2} + i \frac{\sqrt{3}}{2}$, where $\delta = 1$ if $k = 1$ and $\delta = -1$ if $k = 2$. Therefore (101) is equivalent to

$$\delta \frac{1}{2} x_n - \epsilon \frac{\sqrt{3}}{2} \sqrt{1 - x_n^2} = x_m, \quad (103)$$

which implies

$$x_n^2 - \delta x_n x_m + x_m^2 = \frac{3}{4}. \quad (104)$$

Let $x = \frac{i}{z}$. Then (104) is equivalent to

$$x^2 = \frac{3/4}{(n + \frac{1}{2})^2 - \delta (n + \frac{1}{2})(m + \frac{1}{2}) + (m + \frac{1}{2})^2}$$

Hence,

$$1 - x_n^2 = \frac{((m + \frac{1}{2}) - \delta \frac{1}{2}(n + \frac{1}{2}))^2}{(n + \frac{1}{2})^2 - \delta (n + \frac{1}{2})(m + \frac{1}{2}) + (m + \frac{1}{2})^2}.$$

The numerator of this fraction cannot be zero. Hence $x_n^2 < 1$. Thus (99) is actually equivalent to

$$\frac{i}{z}(n + \frac{1}{2}) \in (-1, 1). \quad (105)$$

Write (103) as

$$x(\delta \frac{1}{2}(n + \frac{1}{2}) - (m + \frac{1}{2})) = \epsilon \frac{\sqrt{3}}{2} \sqrt{1 - x_n^2}.$$

This is equivalent to

$$\begin{aligned} z &= -i \frac{2}{\sqrt{3}} \left(2 \left(m + \frac{1}{2} \right) - \delta \left(n + \frac{1}{2} \right) \right) \frac{\sqrt{\left(n + \frac{1}{2} \right)^2 - \delta \left(n + \frac{1}{2} \right) \left(m + \frac{1}{2} \right) + \left(m + \frac{1}{2} \right)^2}}{\epsilon \left| 2 \left(m + \frac{1}{2} \right) - \delta \left(n + \frac{1}{2} \right) \right|} \\ \zeta &= \epsilon i \frac{1}{2} \frac{\left| 2 \left(m + \frac{1}{2} \right) - \delta \left(n + \frac{1}{2} \right) \right|}{\sqrt{\left(n + \frac{1}{2} \right)^2 - \delta \left(n + \frac{1}{2} \right) \left(m + \frac{1}{2} \right) + \left(m + \frac{1}{2} \right)^2}}. \end{aligned} \quad (106)$$

It is easy to check that the z given by (106) satisfies the following inequalities:

$$|z| > \left| n + \frac{1}{2} \right| \quad \text{and} \quad |z| > \left| m + \frac{1}{2} \right|. \quad (107)$$

Hence, (102) actually reads $x_m \in (-1, 1)$. Thus given any integer m there are two points $(z, \zeta), (-z, -\zeta) \in M_n$ (corresponding to $\epsilon = \pm 1$) such that (101) holds. This completes our task of finding all the possible singularities of the function (100).

We now prove that all these possible singularities are in fact removable.

Suppose (z, ζ) satisfies (106). Set

$$w = \frac{i}{z} \left(n + \frac{1}{2} \right) - \zeta. \quad (108)$$

According to (106), there is an integer m so that

$$z\zeta = -\frac{1}{\sqrt{3}} \left(2 \left(m + \frac{1}{2} \right) - \delta \left(n + \frac{1}{2} \right) \right). \quad (109)$$

where $\delta = 1$ if $\boldsymbol{\xi}w$ is a pole of $\phi_{z,1}$ and $\delta = -1$ if $\boldsymbol{\xi}^2w$ is a pole of $\phi_{z,1}$. Choose $k \in \{1, 2\}$ so that $\boldsymbol{\xi}^k w$ is a pole of $\phi_{z,1}$. Then $\boldsymbol{\xi}^k = \delta \frac{1}{2} + i \frac{\sqrt{3}}{2}$ and $\boldsymbol{\xi}^{3-k} = -\delta \frac{1}{2} + i \frac{\sqrt{3}}{2}$. Hence, by (101),

$$zc(\boldsymbol{\xi}^k w) = i \left(m + \frac{1}{2} \right). \quad (110)$$

Notice that

$$\begin{aligned} zw &= i \left(n + \frac{1}{2} \right) - z\zeta \\ &= i \left(n + \frac{1}{2} \right) + \frac{1}{\sqrt{3}} \left(2 \left(m + \frac{1}{2} \right) - \delta \left(n + \frac{1}{2} \right) \right) \\ &= \frac{2}{\sqrt{3}} \left[\left(i \frac{\sqrt{3}}{2} - \frac{\delta}{2} \right) \left(n + \frac{1}{2} \right) + \left(m + \frac{1}{2} \right) \right] \\ &= \frac{2}{\sqrt{3}} \left(\boldsymbol{\xi}^{3-k} \left(n + \frac{1}{2} \right) + \left(m + \frac{1}{2} \right) \right). \end{aligned}$$

Using the relation $s(\alpha) + c(\alpha) = \alpha$, we compute

$$\begin{aligned} zs(\boldsymbol{\xi}^k w) &= z\boldsymbol{\xi}^k w - zc(\boldsymbol{\xi}^k w) \\ &= \frac{2}{\sqrt{3}} \boldsymbol{\xi}^k \left(\boldsymbol{\xi}^{3-k} \left(n + \frac{1}{2} \right) + \left(m + \frac{1}{2} \right) \right) - i \left(m + \frac{1}{2} \right) \\ &= \frac{2}{\sqrt{3}} \left(\frac{1}{2} \delta \left(m + \frac{1}{2} \right) - \left(n + \frac{1}{2} \right) \right), \end{aligned} \quad (111)$$

because $\boldsymbol{\xi}^3 = -1$ and $\frac{2}{\sqrt{3}} \boldsymbol{\xi}^k - i = \frac{2}{\sqrt{3}} \left(\frac{1}{2} \delta + \frac{\sqrt{3}}{2} i \right) - i = \frac{1}{\sqrt{3}} \delta$.

We now prove that $zc(\boldsymbol{\xi}^{3-k} w) \in i\mathbb{Z}$. Recall that for all $\alpha, \beta \in \mathbb{C}^\times$ we have

$$\begin{aligned} c(\alpha\beta) &= c(\alpha)c(\beta) + s(\alpha)s(\beta), \\ s(\alpha\beta) &= s(\alpha)c(\beta) + c(\alpha)s(\beta). \end{aligned}$$

Notice also that

$$c(\boldsymbol{\xi}^{3-2k}) = \frac{1}{2} \quad \text{and} \quad s(\boldsymbol{\xi}^{3-2k}) = i\delta \frac{\sqrt{3}}{2}.$$

Using (110) and (111), we obtain

$$\begin{aligned}
zc(\boldsymbol{\xi}^{3-k}w) &= zc(\boldsymbol{\xi}^{3-2k}\boldsymbol{\xi}^k w) \\
&= zc(\boldsymbol{\xi}^{3-2k})c(\boldsymbol{\xi}^k w) + zs(\boldsymbol{\xi}^{3-2k})s(\boldsymbol{\xi}^k w) \\
&= i\frac{1}{2}(m + \frac{1}{2}) + i\delta(\frac{1}{2}\delta(m + \frac{1}{2}) - (n + \frac{1}{2})) \\
&= i(m + \frac{1}{2}) - i\delta(n + \frac{1}{2}) \in i\mathbb{Z}.
\end{aligned}$$

Thus $\phi_{z,1}(\boldsymbol{\xi}^{3-k}w) = \phi_{z,1}(\boldsymbol{\xi}^{3-k}(\frac{i}{z}(n + \frac{1}{2}) - \zeta)) = 0$ and the function $\tilde{G}_{(n)}$ extends to be holomorphic near $(z, \zeta) \in M_n$.

The symmetry property (85) as well as the local expressions of $\tilde{G}_{(n)}$ in terms of the charts are immediate from (84) and the parity properties of ψ_z and $\phi_{z,1}$ in (50) and (52).

The first residue at $\zeta = 0$ is equal to

$$\begin{aligned}
\operatorname{Res}_{\zeta=0} \tilde{G}_{(n)} \circ \kappa_+^{-1}(\zeta) &= 3\psi_{i(n+\frac{1}{2})}(1)\phi_{i(n+\frac{1}{2}),1}(\boldsymbol{\xi})\phi_{i(n+\frac{1}{2}),1}(\boldsymbol{\xi}^2)\frac{1}{i\pi} \\
&= -\frac{3}{4i\pi} \left(n + \frac{1}{2}\right)^2 \psi_{i(n+\frac{1}{2})}(1) \operatorname{th}^2\left(\frac{i\pi}{2}\left(n + \frac{1}{2}\right)\right),
\end{aligned}$$

which proves the result as $\operatorname{th}\left(\frac{\pi i}{2}\left(n + \frac{1}{2}\right)\right) = (-1)^n i$. One computes the second residue similarly, using (50). \square

Recall that the residues may depend on the choice of the chart. However the type of the singularity does not.

Remark 4. Let $n \in \mathbb{N}$ be fixed. By Lemma 5,

$$\sigma_n^+ : \mathbb{C} \setminus i\left((-\infty, -(n + \frac{1}{2})] \cup [n + \frac{1}{2}, +\infty)\right) \ni z \mapsto (z, \zeta_n^+(z)) \in M_n, \quad (112)$$

where

$$\zeta_n^+(z) = \sqrt{\frac{i}{z}\left(n + \frac{1}{2}\right) + 1} \sqrt{\frac{i}{z}\left(n + \frac{1}{2}\right) - 1}, \quad (113)$$

is a holomorphic section of the projection map (83) so that

$$\tilde{G}_{(n)} \circ \sigma_n^+ = G_{(n)}. \quad (114)$$

This in particular implies that $G_{(n)}$ is holomorphic on $\mathbb{C} \setminus i\left((-\infty, -(n + \frac{1}{2})] \cup [n + \frac{1}{2}, +\infty)\right)$ and that (114) extends to this domain by analyticity. The image of σ_n^+ is usually referred to as the physical sheet (or principal sheet) of M_n . The image in M_n of the map

$$\sigma_n^- : \mathbb{C} \setminus i\left((-\infty, -(n + \frac{1}{2})] \cup [(n + \frac{1}{2}), +\infty)\right) \ni z \mapsto (z, \zeta_n^-(z)) \in M_n$$

is the nonphysical sheet. In the above equation, we have set

$$\zeta_n^-(z) = -\zeta_n^+(z). \quad (115)$$

For $v \in \mathbb{R}^+$ let $\zeta_n^\pm(-iv) = \zeta_n^\pm(-iv + 0)$. Then, by Lemma 6,

$$\zeta_n^+(-iv) = \begin{cases} \sqrt{\left(\frac{n+1/2}{v}\right)^2 - 1} & \text{if } 0 < v \leq n + \frac{1}{2}, \\ i\sqrt{1 - \left(\frac{n+1/2}{v}\right)^2} & \text{if } v > n + \frac{1}{2}, \end{cases} \quad (116)$$

is the physical lift of $-i\mathbb{R}^+$ in M_n . The lift of $-i\mathbb{R}^+$ in M_n is the branched curve $\mathbb{R}^+ \ni v \mapsto \zeta_n^\pm(-iv)$. It is represented in Figure 1.

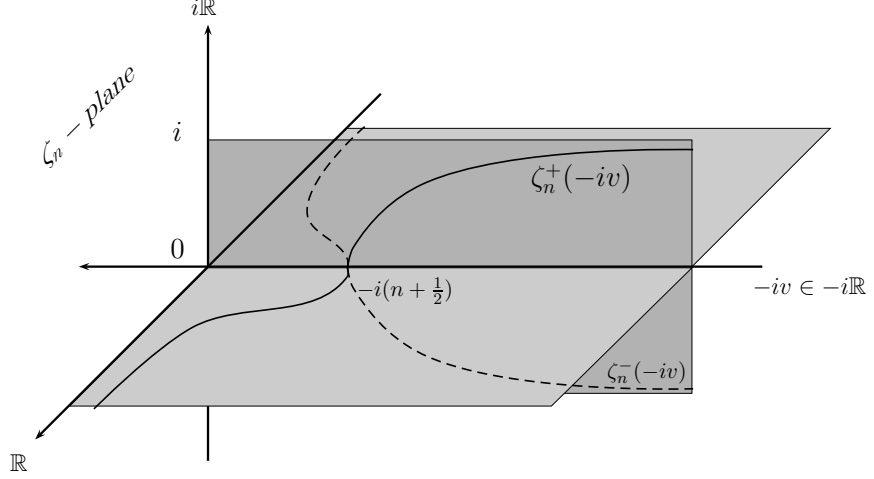


FIGURE 1. The branched curve $v \in \mathbb{R}^+ \rightarrow \pm \zeta_n^\pm(-iv)$

3.4. Meromorphic extension of F . According to (78), locally, in a neighborhood W of each point $-iv \in -i\mathbb{R}^+$, the function F can be written as

$$F(z) = F_r(z) + 4\pi i \sum_{n=0}^N G_{(n)}(z) \quad (z \in W \setminus i\mathbb{R}), \quad (117)$$

where $0 < r < 1$ and N depend v , and F_r is holomorphic in W .

In this subsection we determine a meromorphic extension of F above $-i\mathbb{R}^+$ by “putting together” the meromorphic extensions of the functions $G_{(n)}(z)$ determined in Lemma 14. To do this, we need more precise information on the parameters r and N occurring in (117).

Lemma 15. *Fix $N \in \mathbb{N}$. Suppose $0 < r < 1$ is chosen so that $c(r) < 1 + \frac{1}{2N+3}$. Then for every $0 < v < N + \frac{3}{2}$ there is an open neighborhood W_v of $-iv$ so that*

$$S_{r,z} \cap \mathbb{N} = \{n \in \mathbb{N}; -i(n + \frac{1}{2}) \in W_v E_{c(r),s(r)}\} = \{0, 1, \dots, N_v\} \quad (z \in W_v \setminus i\mathbb{R}),$$

where $N_v \leq \lfloor v \rfloor$. Here $\lfloor v \rfloor$ denotes the largest integer less than or equal to v .

Furthermore, if $\lfloor v \rfloor + \frac{1}{2} \leq v$ then $N_v = \lfloor v \rfloor$. If $v < \lfloor v \rfloor + \frac{1}{2}$ and we choose $0 < r = r_v < 1$ so that it satisfies the additional condition $c(r_v) < \frac{\lfloor v \rfloor + \frac{1}{2}}{v}$, then $N_v = \lfloor v \rfloor - 1$.

Proof. The ellipse $E_{c(r),s(r)}$ being invariant under sign change, we can work with iv instead of $-iv$.

Let $iu \in i\mathbb{R}^+$. Then $iu \in ivE_{c(r),s(r)}$ if and only if $u < c(r)v$. This shows that $i[u, +\infty) \cap ivE_{c(r),s(r)} = \emptyset$ if $iu \notin ivE_{c(r),s(r)}$. Moreover, since $c(r) > 1$, we have that $i[0, v] \subseteq ivE_{c(r),s(r)}$. In particular,

$$i(\lfloor v \rfloor + \frac{1}{2}) \in ivE_{c(r),s(r)} \quad (118)$$

if $\lfloor v \rfloor + \frac{1}{2} \leq v$. Notice also that $iu \notin iv\overline{E_{c(r),s(r)}}$ if and only if $u > c(r)v$.

Suppose $c(r) < 1 + \frac{1}{2N+3}$ and let $0 < v < N + \frac{3}{2}$. Then $1 + \frac{1}{2v} \geq 1 + \frac{1}{2N+3} > c(r)$. So $(\lfloor v \rfloor + 1) + \frac{1}{2} > v + \frac{1}{2} > vc(r)$, i.e.

$$i((\lfloor v \rfloor + 1) + \frac{1}{2}) \notin iv\overline{E_{c(r),s(r)}}. \quad (119)$$

The relations (118) and (119) still hold when we replace $ivE_{c(r),s(r)}$ with $zE_{c(r),s(r)}$ with z in a sufficiently small neighborhood W of iv . We also take W small enough so that $S_{r,z}$ is independent of $z \in W \setminus i\mathbb{R}$; see Corollary 12 and Lemma 13. The extension of (119) obtained in this way shows that $N_v \leq \lfloor v \rfloor$. Thus $N_v = \lfloor v \rfloor$ if $\lfloor v \rfloor + \frac{1}{2} \leq v$.

If $v < \lfloor v \rfloor + \frac{1}{2}$, then we can choose $0 < r = r_v < 1$ so that it also satisfies the condition $c(r_v) < \frac{\lfloor v \rfloor + \frac{1}{2}}{v}$. So $vc(r_v) < \lfloor v \rfloor + \frac{1}{2}$ yields $N_v < \lfloor v \rfloor$. Thus $N_v = \lfloor v \rfloor - 1$ in this case. \square

Corollary 16. *Let $N \in \mathbb{N}$ and $m \in \{0, 1, \dots, N\}$. Let $v \in [m + \frac{1}{2}, m + \frac{3}{2})$. Then there is an open neighborhood W_v centered at $-iv$ and $0 < r_v < 1$ so that*

$$F(z) = F_{r_v}(z) + 4\pi i \sum_{n=0}^m G_{(n)}(z) \quad (z \in W_v \setminus i\mathbb{R}), \quad (120)$$

where F_{r_v} is holomorphic in W_v and $G_{(n)}$ is the function defined in (70).

Proof. This is immediate from Lemma 15. Indeed, for $v \in [m + \frac{1}{2}, m + 1) = [\lfloor v \rfloor + \frac{1}{2}, \lfloor v \rfloor + 1)$ we have $N_v = \lfloor v \rfloor = m$, whereas for $v \in [m + 1, m + \frac{3}{2}) = [\lfloor v \rfloor, \lfloor v \rfloor + \frac{1}{2})$ we have $N_v = \lfloor v \rfloor - 1 = (m + 1) - 1 = m$. \square

In the following we suppose that we have fixed the W_v as in Corollary 16. Moreover, by possibly further shrinking them, we may also assume that every W_v is an open disk centered at $-iv$ and that

$$W_v \cap i\mathbb{R} \subseteq -i(m + \frac{1}{2}, m + \frac{3}{2}) \quad \text{if } v \in (m + \frac{1}{2}, m + \frac{3}{2}), \quad (121)$$

$$W_v \cap i\mathbb{R} \subseteq -i(m, m + 1) \quad \text{if } v = m + \frac{1}{2}. \quad (122)$$

Corollary 17. *Let $m \in \{0, 1, \dots, N\}$ and $v \in [m + \frac{1}{2}, m + \frac{3}{2})$. Suppose that $W_v \cap W_{v'} \neq \emptyset$ for some $v' \in [\frac{1}{2}, N + \frac{3}{2})$.*

If $v \in (m + \frac{1}{2}, m + \frac{3}{2})$, then $v' \in (m + \frac{1}{2}, m + \frac{3}{2})$. Moreover,

$$F_{r_{v'}}(z) = F_{r_v}(z) \quad (z \in W_v \cap W_{v'}).$$

If $v = m + \frac{1}{2}$, then $v' \in (m - \frac{1}{2}, m + \frac{3}{2})$. Moreover, if $v' \in [m + \frac{1}{2}, m + \frac{3}{2})$, then

$$F_{r_{v'}}(z) = F_{r_v}(z) \quad (z \in W_v \cap W_{v'});$$

if $v' \in (m - \frac{1}{2}, m + \frac{1}{2})$, then

$$F_{r_{v'}}(z) = F_{r_v}(z) + 4\pi i G_{(m)}(z) \quad (z \in W_v \cap W_{v'}).$$

Proof. Observe first that $W_v \cap W_{v'}$ is connected. If $v \in (m + \frac{1}{2}, m + \frac{3}{2})$ and $W_v \cap W_{v'} \neq \emptyset$, then $v' \in [m + \frac{1}{2}, m + \frac{3}{2})$ by (121). Moreover, (128) gives $F_{r_{v'}}(z) = F_{r_v}(z)$ for $z \in (W_v \cap W_{v'}) \setminus i\mathbb{R}$. This equality extends to $W_v \cap W_{v'}$ by analyticity.

Suppose now $v = m + \frac{1}{2}$. The assumption (122) ensures that if $W_v \cap W_{v'} \neq \emptyset$ then $v' \in (m - \frac{1}{2}, m + \frac{3}{2})$. When $v' \in [m + \frac{1}{2}, m + \frac{3}{2})$, the equality of the functions F_r follows as above. If $v' \in (m - \frac{1}{2}, m + \frac{1}{2})$, then

$$F(z) = F_{r_{v'}}(z) + 4\pi i \sum_{n=0}^{m-1} G_{(n)}(z) \quad (z \in W_{v'} \setminus i\mathbb{R}).$$

Therefore $F_{r_{v'}}(z) = F_{r_v}(z) + 4\pi i G_{(m)}(z)$ for $z \in (W_v \cap W_{v'}) \setminus i\mathbb{R}$. By (121), we have $W_{v'} \cap i\mathbb{R} \subset -i(m - \frac{1}{2}, m + \frac{1}{2})$. So $G_{(m)}$ is holomorphic on $W_v \cap W_{v'}$ (see Remark 4). Since $W_v \cap W_{v'}$ is connected, the previous equality holds for all $z \in W_v \cap W_{v'}$. \square

For every integer $0 \leq m \leq N$ we define

$$W_{(m)} = \bigcup_{v \in [m + \frac{1}{2}, m + \frac{3}{2})} W_v \quad (123)$$

$$F_{(m)}(z) = F_{r_v}(z) \quad (v \in [m + \frac{1}{2}, m + \frac{3}{2}), z \in W_v). \quad (124)$$

Then $W_{(m)}$ is an open neighborhood of $-i[m + \frac{1}{2}, m + \frac{3}{2})$ so that $W_{(m)} \cap i\mathbb{R} \subseteq -i(m, m + \frac{3}{2})$. Moreover, setting

$$W'_{(m)} = \bigcup_{v \in (m + \frac{1}{2}, m + \frac{3}{2})} W_v, \quad (125)$$

we have $W'_{(m)} \cap i\mathbb{R} = -i(m + \frac{1}{2}, m + \frac{3}{2})$. By Corollary 17, $F_{(m)}$ is a holomorphic extension of F_{r_v} to $W_{(m)}$ for every $v \in [m + \frac{1}{2}, m + \frac{3}{2})$.

It is convenient to introduce similar notation for a neighborhood of $(0, \frac{1}{2})$. We therefore choose for every $v \in (0, \frac{1}{2})$ and open ball W_v centered at $-iv$ so that $W_v \cap i\mathbb{R} \subseteq (0, \frac{1}{2})$ and define:

$$W_{(-1)} = W'_{(-1)} = \bigcup_{v \in (0, \frac{1}{2})} W_v \quad (126)$$

$$F_{(-1)}(z) = F(z) \quad (z \in W_{(-1)}). \quad (127)$$

Observe that $W_{(-1)} \cap i\mathbb{R} = -i(0, \frac{1}{2})$. Moreover, for every $m \in \{-1, 0, \dots, N-1\}$ the intersection $W_{(m)} \cap W_{(m+1)} = W_{(m)} \cap W_{m + \frac{3}{2}}$ is a nonempty open connected set.

Corollary 18. *For every integer $m \geq -1$ we have*

$$F(z) = F_{(m)}(z) + 4\pi i \sum_{n=0}^m G_{(n)}(z) \quad (z \in W_{(m)} \setminus i\mathbb{R}), \quad (128)$$

where $F_{(m)}$ is holomorphic in $W_{(m)}$, the $G_{(n)}$ are as in (70), and empty sums are defined to be equal to 0. Consequently, for every integer $m \geq -1$,

$$F_{(m)}(z) = F_{(m+1)}(z) + 4\pi i G_{(m+1)}(z) \quad (z \in W_{(m)} \cap W_{(m+1)} = W_{(m)} \cap W_{m + \frac{3}{2}}). \quad (129)$$

Proof. This is an immediate consequence of Corollary 16 and (123)–(127). \square

The meromorphic continuation of F across $-i[\frac{1}{2}, +\infty)$ will be done per stages, on Riemann surfaces constructed from the M_n 's. Recall that M_n denotes the Riemann surface to which $G_{(n)}$ lifts as a meromorphic function. For an integer $N \geq 0$, let

$$M_{(N)} = \{(z, \zeta) \in \mathbb{C}^\times \times \mathbb{C}^{N+1}; \zeta = (\zeta_0, \zeta_1, \dots, \zeta_N); (z, \zeta_n) \in M_n, 0 \leq n \leq N\}. \quad (130)$$

Then $M_{(N)}$ is a Riemann surface, and the map

$$\pi_{(N)} : M_{(N)} \ni (z, \zeta) \mapsto z \in \mathbb{C}^\times \quad (131)$$

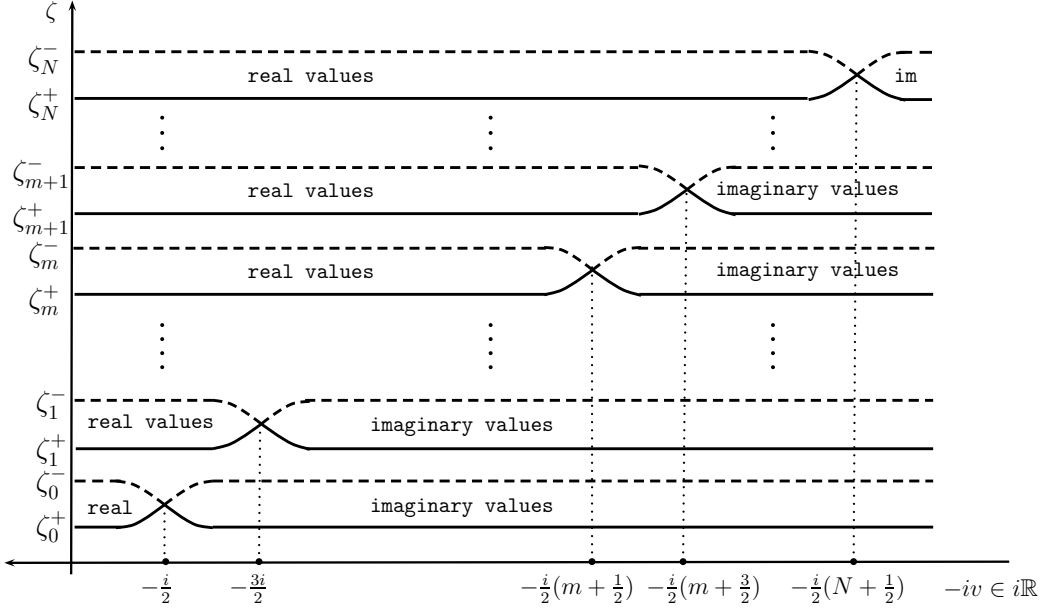


FIGURE 2. Schema of the fibre above $-i\mathbb{R}^+$ in $M_{(N)}$

is a holomorphic 2^{N+1} -to-1 cover, except when $z \in \{\pm i(n + \frac{1}{2}); n \in \mathbb{N}, 0 \leq n \leq N\}$. (This may be seen by checking, as in (79), that if a_1, a_2, \dots, a_k are non-zero complex numbers whose squares are mutually distinct, then

$$\{(w, \zeta_1, \zeta_2, \dots, \zeta_k); \zeta_j^2 = a_j^2 w^2 - 1, 1 \leq j \leq k\} \subseteq \mathbb{C}^{k+1} \quad (132)$$

is a one-dimensional complex submanifold with all the required properties.)

The ζ -coordinates ζ_n ($0 \leq n \leq N$) of the points of the fiber of $-iv \in -i\mathbb{R}^+$ in $M_{(N)}$ are uniquely determined by the condition that $(-iv, \zeta_n) \in M_n$. This means that

$$\zeta_n^2 = \left(\frac{n + \frac{1}{2}}{v}\right)^2 - 1 \quad (0 \leq n \leq N). \quad (133)$$

If $v \notin \{m + \frac{1}{2}; m \in \mathbb{N}, 0 \leq m \leq N\}$, we get exactly 2^{N+1} points, corresponding to the two sign choices for each ζ_n . Hence the fiber of $-iv \in -i\mathbb{R}^+ \setminus \{-i(m + \frac{1}{2}); m \in \mathbb{N}, 0 \leq m \leq N\}$ consists of the points

$$(-iv, \zeta_0^\pm(-iv), \dots, \zeta_N^\pm(-iv)) \quad (134)$$

where $\zeta_n^\pm(-iv)$ is as in (116). If $v = m + \frac{1}{2}$ for some $m \in \{0, \dots, N\}$, then the ζ_m -coordinate of the points of the fiber of $-i(m + \frac{1}{2})$ is zero, whereas (133) has precisely two solutions, equal to $\pm \zeta_n^+(-i(m + \frac{1}{2}))$, for each $n \in \{0, \dots, N\}$ which is different from m . Hence the fiber of $-i(m + \frac{1}{2})$ contains exactly 2^N points. These points have a real coordinate ζ_n when $N \geq n > m$, a purely imaginary coordinate ζ_n when $0 \leq n < m$, and $\zeta_m = 0$. They are the branching points of $M_{(N)}$. A schematic picture of the coordinates of the points of the fiber above $-i\mathbb{R}^+$ is drawn in Figure 2.

We want to construct a meromorphic lift of F along the branched curve

$$\gamma_N : v \in (0, N + \frac{3}{2}) \mapsto (-iv, \pm \zeta_0^+(-iv), \dots, \pm \zeta_N^+(-iv)) \in M_{(N)}, \quad (135)$$

which is the lift of $-i(0, N + \frac{3}{2})$ in $M_{(N)}$. More precisely, let $W_{(m)}$ be the open set in \mathbb{C} defined in (123) and (126). Then we meromorphically lift F to the open neighborhood

$$M_{(\gamma_N)} = \pi_{(N)}^{-1} \left(\bigcup_{m=-1}^N W_{(m)} \right) \quad (136)$$

of γ_N in $M_{(N)}$.

Notice that by (122) the radius R_m of the open disk $W_{m+\frac{1}{2}}$ satisfies $R_m < 1/2$. Moreover, using also (121), we have for $m \in \{0, 1, \dots, N-1\}$

$$\begin{aligned} W_{(m)} \cap W_{(m+1)} &= W_{(m)} \cap W_{m+\frac{3}{2}} = (W_{(m)} \setminus W_{m+\frac{1}{2}}) \cap W_{m+\frac{3}{2}}, \\ W_{(m)} \cap W_{(m+1)} \cap i\mathbb{R} &= -i]m + \frac{3}{2} - R_{m+1}, m + \frac{3}{2}[. \end{aligned}$$

Hence, for $m \in \{-1, 0, \dots, N\}$,

$$W_{(m)} \cap W_{(m+1)} \cap i(\mathbb{Z} + \frac{1}{2}) = \emptyset \quad (137)$$

and the branching point $-i(m + \frac{3}{2})$ is a boundary point of $W_{(m)} \cap W_{(m+1)}$.

For every $\varepsilon = (\varepsilon_0, \dots, \varepsilon_N) \in \{\pm 1\}^{N+1}$ we define a section

$$\sigma_\varepsilon : \mathbb{C} \setminus i((-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, +\infty)) \rightarrow M_{(N)}$$

of $\pi_{(N)}$ by setting

$$\sigma_\varepsilon(z) = (z, \varepsilon_0 \zeta_0^+(z), \dots, \varepsilon_N \zeta_N^+(z)). \quad (138)$$

This is well defined as $\zeta_n^+(z)$ is well defined at $z \in \mathbb{C} \setminus i((-\infty, -(n + \frac{1}{2})]) \cup [n + \frac{1}{2}, +\infty))$ for all $n \in \{0, 1, \dots, N\}$.

Because of (137), by possibly shrinking the open disks W_v , we can also assume that $\pi_{(N)}^{-1}(W'_{(m)})$ is the disjoint union of 2^{N+1} homeomorphic copies of $W'_{(m)}$. In particular, each of these copies is a connected set. We denote by $U_{m,\varepsilon}$ the copy containing $\sigma_\varepsilon(W'_{(m)} \setminus i\mathbb{R})$.

Similarly, by possibly shrinking the open disks $W_{m+\frac{1}{2}}$, we can furthermore assume that their preimage $\pi_{(N)}^{-1}(W_{m+\frac{1}{2}})$ in $M_{(N)}$ is the disjoint union of 2^N homeomorphic copies of $\pi_m^{-1}(W_{m+\frac{1}{2}})$. They can be parameterized by the elements

$$(\varepsilon_0, \dots, \varepsilon_{m-1}, \varepsilon_{m+1}, \dots, \varepsilon_N) \quad (\varepsilon_n \in \{\pm 1\}, 0 \leq n \leq N, n \neq m). \quad (139)$$

We denote by $\varepsilon(m^\vee)$ the element (139) obtained from $\varepsilon \in \{\pm 1\}^{N+1}$ by removing its m -th component. This means that $\varepsilon(m^\vee) = \varepsilon'(m^\vee)$ if and only if ε and ε' are equal but for their m -th component, which can be ± 1 . Modulo this identification, we can indicate the connected components of $\pi_{(N)}^{-1}(W_{m+\frac{1}{2}})$ as $U_{\varepsilon(m^\vee)}$ with $\varepsilon \in \{\pm 1\}^{N+1}$. To unify notation, we define $U_{\varepsilon(m^\vee)} = \emptyset$ for $m = -1$.

Observe that $\{U_{\varepsilon(m^\vee)} \cup U_{m,\varepsilon}; \varepsilon \in \{\pm 1\}^{N+1}, m \in \mathbb{Z}, -1 \leq m \leq N\}$ is a covering of $M_{(\gamma_N)}$ consisting of open connected sets.

Theorem 19. For $m \in \{-1, 0, \dots, N\}$, $\varepsilon \in \{\pm 1\}^{N+1}$ and $(z, \zeta) \in U_{\varepsilon(m^\vee)} \cup U_{m,\varepsilon}$ define

$$\tilde{F}(z, \zeta) = F_{(m)}(z) + 4\pi i \sum_{n=0}^m \tilde{G}_{(n)}(z, \zeta_n) + 4\pi i \sum_{\substack{m < n \leq N \\ \text{with } \varepsilon_n = -1}} [\tilde{G}_{(n)}(z, \zeta_n) - \tilde{G}_{(n)}(z, -\zeta_n)], \quad (140)$$

where the first sum is equal to 0 if $m = -1$ and the second sum is 0 if $\varepsilon_n = 1$ for all $n > m$.

Then \tilde{F} is a meromorphic lift of F to the open neighborhood $M_{(\gamma_N)}$ of the branched curve γ_N lifting $-i(0, N + \frac{3}{2})$ in $M_{(N)}$. The singularities of \tilde{F} on $M_{(\gamma_N)}$ are simple poles at the points $(-i(m + \frac{1}{2}), \zeta) \in M_{(N)}$ with $m \in \{0, 1, \dots, N\}$.

Proof. To simplify the notation, we write the coordinates of the points of $M_{(N)}$ above some fixed $z \in \mathbb{C}$ as ζ_n^\pm instead of $\zeta_n^\pm(z)$.

Let $\tilde{F}_{(m,\varepsilon)}(z, \zeta)$ denote the right-hand side of (140). Since for every $m \in \{-1, 0, \dots, N-1\}$ the function $\tilde{F}_{(m,\varepsilon)}(z, \zeta)$ is meromorphic on $U_{\varepsilon(m^\vee)} \cup U_{m,\varepsilon}$, then $\tilde{F}(z, \zeta)$ will be meromorphic on $M_{(\gamma_N)}$ provided $\tilde{F}_{(m,\varepsilon)}(z, \zeta) = \tilde{F}_{(m',\varepsilon')}(z, \zeta)$ on all nonempty intersections $(U_{\varepsilon(m^\vee)} \cup U_{m,\varepsilon}) \cap (U_{\varepsilon'((m')^\vee)} \cup U_{m',\varepsilon'})$ with $m, m' \in \{-1, 0, \dots, N\}$, $\varepsilon, \varepsilon' \in \{\pm 1\}^{N+1}$ and $(m, \varepsilon) \neq (m', \varepsilon')$.

If $(m, \varepsilon) \neq (m', \varepsilon')$, then $U_{m,\varepsilon} \cap U_{m',\varepsilon'} = \emptyset$. Hence

$$(U_{\varepsilon(m^\vee)} \cup U_{m,\varepsilon}) \cap (U_{\varepsilon'((m')^\vee)} \cup U_{m',\varepsilon'}) = (U_{\varepsilon(m^\vee)} \cap U_{\varepsilon'((m')^\vee)}) \cup (U_{\varepsilon(m^\vee)} \cap U_{m',\varepsilon'}) \cup (U_{\varepsilon'((m')^\vee)} \cap U_{m,\varepsilon}).$$

Since different $U_{\varepsilon(m^\vee)}$'s are disjoint, $U_{\varepsilon(m^\vee)} \cap U_{\varepsilon'((m')^\vee)} \neq \emptyset$ means that $U_{\varepsilon(m^\vee)} = U_{\varepsilon'((m')^\vee)}$. Since $(m, \varepsilon) \neq (m', \varepsilon')$, this means that $m = m'$, $\varepsilon_n = \varepsilon'_n$ for all $n \in \{0, 1, \dots, N\}$ with $n \neq m$ and $\varepsilon_m = -\varepsilon'_m$. The definition of $\tilde{F}_{(m,\varepsilon)}(z, \zeta)$ on the right-hand side of (140) depends on ε only for the ε_n 's with $n > m = m'$. Therefore, in this case we have $\tilde{F}_{(m,\varepsilon)}(z, \zeta) = \tilde{F}_{(m',\varepsilon')}(z, \zeta)$ for $(z, \zeta) \in U_{\varepsilon(m^\vee)} = U_{\varepsilon'((m')^\vee)}$.

Suppose now that $U_{\varepsilon'((m')^\vee)} \cap U_{m,\varepsilon} \neq \emptyset$. If $m = m'$, this means that $\varepsilon(m^\vee) = \varepsilon'((m')^\vee)$. We are therefore in the situation just considered, in which $\tilde{F}_{(m,\varepsilon)}(z, \zeta) = \tilde{F}_{(m',\varepsilon')}(z, \zeta)$ for $(z, \zeta) \in U_{\varepsilon(m^\vee)} = U_{\varepsilon'((m')^\vee)}$. If $m \neq m'$, then two cases may occur:

- (1) $m' = m + 1$ and $\varepsilon'((m')^\vee) = \varepsilon'((m+1)^\vee) = \varepsilon((m+1)^\vee)$.
- (2) $m' = m - 1$ and $\varepsilon'((m')^\vee) = \varepsilon'((m-1)^\vee) = \varepsilon((m-1)^\vee)$.

In Case (1) we need to check that $\tilde{F}_{(m,\varepsilon)}(z, \zeta) = \tilde{F}_{(m+1,\varepsilon)}(z, \zeta)$ for $(z, \zeta) \in U_{m,\varepsilon} \cap U_{\varepsilon((m+1)^\vee)}$. Since $U_{m,\varepsilon} \cap U_{\varepsilon((m+1)^\vee)}$ is connected (as homeomorphic to $W_{(m)} \cap W_{m+\frac{3}{2}}$), it suffices to check this equality on its subset $\sigma_\varepsilon((W_{(m)} \cap W_{m+\frac{3}{2}}) \setminus i\mathbb{R})$. On this subset, we have $\zeta_{m+1} = \varepsilon_{m+1} \zeta_{m+1}^+$. Hence, by (114),

$$\tilde{G}_{(m+1)}(z, \varepsilon_{m+1} \zeta_{m+1}) = \tilde{G}_{(m+1)}(z, \zeta_{m+1}^+) = G_{(m+1)}(z) \quad ((z, \zeta) \in \sigma_\varepsilon((W_{(m)} \cap W_{m+\frac{3}{2}}) \setminus i\mathbb{R})).$$

Recall that, by (129),

$$F_{(m)}(z) = F_{(m+1)}(z) + 4\pi i G_{(m+1)}(z) \quad (z \in W_{(m)} \cap W_{m+\frac{3}{2}}).$$

These equalities will be subsequently used in the computations below.

We now need to distinguish two further cases inside Case (1):

- (1.a) $\varepsilon_n = 1$ for all $n > m$,
- (1.b) there exists $n > m$ such that $\varepsilon_n = -1$.

In Case (1.a), we have $\varepsilon_{m+1} = 1$. In Case (1.b), we also have to distinguish whether $\varepsilon_{m+1} = 1$ or $\varepsilon_{m+1} = -1$. In Case (1.a) or in Case (1.b) with $\varepsilon_{m+1} = 1$, for $(z, \zeta) \in$

$\sigma_\varepsilon((W_{(m)} \cap W_{m+\frac{3}{2}}) \setminus i\mathbb{R})$ (and with empty sums equal to 0), we have:

$$\begin{aligned}
\tilde{F}_{(m,\varepsilon)}(z, \zeta) &= F_{(m)}(z) + 4\pi i \sum_{n=0}^m \tilde{G}_{(n)}(z, \zeta_n) + 4\pi i \sum_{\substack{m+1 < n \leq N \\ \text{with } \varepsilon_n = -1}} [\tilde{G}_{(n)}(z, \zeta_n) - \tilde{G}_{(n)}(z, -\zeta_n)] \\
&= F_{(m+1)}(z) + 4\pi i G_{(m+1)}(z) + 4\pi i \sum_{n=0}^m \tilde{G}_{(n)}(z, \zeta_n) \\
&\quad + 4\pi i \sum_{\substack{m+1 < n \leq N \\ \text{with } \varepsilon_n = -1}} [\tilde{G}_{(n)}(z, \zeta_n) - \tilde{G}_{(n)}(z, -\zeta_n)] \\
&= F_{(m+1)}(z) + 4\pi i \sum_{n=0}^{m+1} \tilde{G}_{(n)}(z, \zeta_n) + 4\pi i \sum_{\substack{m+1 < n \leq N \\ \text{with } \varepsilon_n = -1}} [\tilde{G}_{(n)}(z, \zeta_n) - \tilde{G}_{(n)}(z, -\zeta_n)] \\
&= \tilde{F}_{(m+1,\varepsilon)}(z, \zeta).
\end{aligned}$$

In Case (1.b) with $\varepsilon_{m+1} = -1$, then for $(z, \zeta) \in \sigma_\varepsilon((W_{(m)} \cap W_{m+\frac{3}{2}}) \setminus i\mathbb{R})$ (and with empty sums equal to 0), we have:

$$\begin{aligned}
\tilde{F}_{(m,\varepsilon)}(z, \zeta) &= F_{(m)}(z) + 4\pi i \sum_{n=0}^m \tilde{G}_{(n)}(z, \zeta_n) + 4\pi i [\tilde{G}_{(m+1)}(z, \zeta_{m+1}) - \tilde{G}_{(m+1)}(z, -\zeta_{m+1})] \\
&\quad + 4\pi i \sum_{\substack{m+1 < n \leq N \\ \text{with } \varepsilon_n = -1}} [\tilde{G}_{(n)}(z, \zeta_n) - \tilde{G}_{(n)}(z, -\zeta_n)] \\
&= F_{(m+1)}(z) + 4\pi i G_{(m+1)}(z) + 4\pi i \sum_{n=0}^m \tilde{G}_{(n)}(z, \zeta_n) \\
&\quad + 4\pi i [\tilde{G}_{(m+1)}(z, \zeta_{m+1}) - \tilde{G}_{(m+1)}(z, -\zeta_{m+1})] \\
&\quad + 4\pi i \sum_{\substack{m+1 < n \leq N \\ \text{with } \varepsilon_n = -1}} [\tilde{G}_{(n)}(z, \zeta_n) - \tilde{G}_{(n)}(z, -\zeta_n)] \\
&= F_{(m+1)}(z) + 4\pi i \sum_{n=0}^{m+1} \tilde{G}_{(n)}(z, \zeta_n) + 4\pi i \sum_{\substack{m+1 < n \leq N \\ \text{with } \varepsilon_n = -1}} [\tilde{G}_{(n)}(z, \zeta_n) - \tilde{G}_{(n)}(z, -\zeta_n)] \\
&= \tilde{F}_{(m+1,\varepsilon)}(z, \zeta).
\end{aligned}$$

This concludes Case (1).

For Case (2), we need to check that $\tilde{F}_{(m-1,\varepsilon)}(z, \zeta) = \tilde{F}_{(m,\varepsilon)}(z, \zeta)$ for $(z, \zeta) \in U_{m,\varepsilon} \cap U_{\varepsilon((m-1)^\vee)}$. As in Case (1), since $U_{m,\varepsilon} \cap U_{\varepsilon((m-1)^\vee)}$ is connected (as homeomorphic to $W_{(m)} \cap W_{m+\frac{1}{2}}$), it suffices to check this equality on $\sigma_\varepsilon((W_{(m)} \cap W_{m+\frac{1}{2}}) \setminus i\mathbb{R})$, where $\zeta_m = \varepsilon_m \zeta_m^+$. The proof parallels that of Case (1), by considering if $\varepsilon_n = 1$ for all $n > m - 1$ (Case (2.a)) or if there exists $n > m - 1$ such that $\varepsilon_n = -1$ (Case (2.b)). In Case (2.b), one moreover has to distinguish whether $\varepsilon_m = 1$ or $\varepsilon_m = -1$. The details are omitted.

This concludes the proof that $\tilde{F}(z, \zeta)$ is a meromorphic function on $M_{(\gamma_N)}$. Its singularities on $W_{(m)}$ are those of $\tilde{G}_{(m)}(z, \zeta_m)$, i.e. simple poles at the branching points above $z = -i(m + \frac{1}{2})$, with $m \in \{0, 1, \dots, N\}$.

The fact that $\tilde{F}(z, \zeta)$ is a lift of F is an immediate consequence of (114), Corollary 18 and the definition of $\tilde{F}_{(m, \varepsilon)}(z, \zeta)$ for $\varepsilon = (1, \dots, 1)$. \square

3.5. Meromorphic extension of the resolvent. Let $\mathbb{C}^- = \{z \in \mathbb{C}; \text{Im } z < 0\}$ be the lower half plane. In this subsection we meromorphically extend the resolvent $z \mapsto [R(z)f](y)$ (where $f \in C_c^\infty(\mathbf{X})$ and $y \in \mathbf{X}$ are arbitrarily fixed) from $\mathbb{C}^- \setminus i(-\infty, -\frac{1}{2}\rho_{\mathbf{X}}]$ across the half-line $i(-\infty, -\frac{1}{2}\rho_{\mathbf{X}}]$. As before, we shall omit the dependence on f and y from the notation and write $R(z)$ instead of $[R(z)f](y)$. This simplification of notation will be employed wherever it is appropriate.

Recall from Proposition 2 that on \mathbb{C}^- (and indeed on a larger domain) we can write

$$R(z) = H(\rho_{\mathbf{X}}^{-1}z) + \frac{\pi i}{|W|\rho_{\mathbf{X}}^2} F(\rho_{\mathbf{X}}^{-1}z) \quad (z \in \mathbb{C}^-), \quad (141)$$

where H is a holomorphic function. The required meromorphic extension will be therefore deduced from that of F in Section 3.4.

For any fixed integer $N \geq 0$ consider the Riemann surface above

$$\mathbb{C}_N^- = \{z \in \mathbb{C}^-; \text{Im } z > -(N + \frac{3}{2})\rho_{\mathbf{X}}\}$$

defined by

$$M_{(\mathbf{X}, N)} = \{(z, \zeta); z \in \mathbb{C}_{-, N}, \zeta = (\zeta_0, \zeta_1, \dots, \zeta_N) \in \mathbb{C}^{N+1}, \\ \zeta_n^2 = \left(\frac{i}{z}(n + \frac{1}{2})\right)^2 - \frac{1}{\rho_{\mathbf{X}}^2}, 0 \leq n \leq N\}.$$

Hence $(z, \zeta) \in M_{(\mathbf{X}, N)}$ if and only if $(\rho_{\mathbf{X}}^{-1}z, \rho_{\mathbf{X}}\zeta) \in M_{(N)}$. Then the curve

$$v \in (0, N + \frac{3}{2}) \mapsto -i\rho_{\mathbf{X}}v \in i\mathbb{R}$$

on the negative imaginary axis lifts to the curve $\gamma_{\mathbf{X}, N}$ on the Riemann surface $M_{(\mathbf{X}, N)}$ given by

$$\gamma_{\mathbf{X}, N}(v) = (-i\rho_{\mathbf{X}}v, \rho_{\mathbf{X}}^{-1}\zeta_0^\pm(-iv), \dots, \rho_{\mathbf{X}}^{-1}\zeta_N^\pm(-iv)) \quad (0 < v < N + \frac{3}{2}), \quad (142)$$

where $\zeta_n^+(-iv) = -\zeta_n^-(-iv)$ is as in (116).

From each object constructed for $M_{(N)}$ we obtain a corresponding object for $M_{(\mathbf{X}, N)}$ by replacing (z, ζ) with $(\rho_{\mathbf{X}}^{-1}z, \rho_{\mathbf{X}}\zeta)$. It will be denoted by adding “ \mathbf{X} ” to the symbol used in Section 3.4 for the corresponding object in $M_{(N)}$. For instance, for $\varepsilon \in \{\pm 1\}^{N+1}$ and $n \in \{-1, 0, 1, \dots, N\}$, the sets $U_{\mathbf{X}, n, \varepsilon}$ and $U_{\mathbf{X}, \varepsilon(n^\vee)}$ consist of the $(z, \zeta) \in M_{(\mathbf{X}, N)}$ so that the points $(\rho_{\mathbf{X}}^{-1}z, \rho_{\mathbf{X}}\zeta)$ belong to $U_{n, \varepsilon}$ and $U_{\varepsilon(n^\vee)}$, respectively. Similarly,

$$M_{(\gamma_{\mathbf{X}, N})} = \{(z, \zeta) \in M_{(\mathbf{X}, N)}; (\rho_{\mathbf{X}}^{-1}z, \rho_{\mathbf{X}}\zeta) \in M_{(\gamma_N)}\}.$$

Also, for $n \in \{-1, 0, 1, \dots, N\}$,

$$W_{(\mathbf{X}, n)} = \rho_{\mathbf{X}}^{-1}W_{(n)}, \quad W'_{(\mathbf{X}, n)} = \rho_{\mathbf{X}}^{-1}W'_{(n)}, \quad W_{\mathbf{X}, n+\frac{1}{2}} = \rho_{\mathbf{X}}^{-1}W_{n+\frac{1}{2}}.$$

Hence $W_{(\mathbf{X}, n)} = W_{\mathbf{X}, n+\frac{1}{2}} \cup W'_{(\mathbf{X}, n)}$ is an open neighborhood in \mathbb{C}_N^- of the interval $-i\rho_{\mathbf{X}}[n + \frac{1}{2}, n + \frac{3}{2}]$ if $n \neq -1$, and of $-i(0, \frac{\rho_{\mathbf{X}}}{2})$ if $n = -1$.

Let $\pi_{(\mathbf{X},N)} : M_{(\mathbf{X},N)} \rightarrow \mathbb{C}_N^-$ denote the projection $\pi_{(\mathbf{X},N)}((z, \zeta)) = z$. The Riemann surface $M_{(\mathbf{X},N)}$ admits branching points at the 2^N points above $z^{(n)} = -i(n + \frac{1}{2})\rho_{\mathbf{X}}$, where $n \in \{0, 1, \dots, N\}$. The open disk $W_{\mathbf{X},n+\frac{1}{2}}$ is centered at $z^{(n)}$. Set

$$\mathcal{E}_n = \{\varepsilon \in \{\pm 1\}^{N+1}; \varepsilon_n = 1\}. \quad (143)$$

Then the sets $U_{\mathbf{X},\varepsilon(n^\vee)}$ with $\varepsilon \in \mathcal{E}_n$ are pairwise disjoint and form an open cover of $\pi_{(\mathbf{X},N)}^{-1}(W_{\mathbf{X},n+\frac{1}{2}})$. For $\varepsilon \in \mathcal{E}_n$, we denote by $(z^{(n)}, \zeta^{(n,\varepsilon)})$ the point of the fiber $\pi_{(\mathbf{X},N)}^{-1}(z^{(n)})$ which belongs to $U_{\mathbf{X},\varepsilon(n^\vee)}$. Moreover, the map

$$\kappa_{n,\varepsilon} : U_{\mathbf{X},\varepsilon(n^\vee)} \ni (z, \zeta) \rightarrow \zeta_n \in \mathbb{C} \setminus i((-\infty, -\rho_{\mathbf{X}}^{-1}] \cup [\rho_{\mathbf{X}}^{-1}, +\infty)), \quad z = -\frac{i\rho_{\mathbf{X}}(n + \frac{1}{2})}{\sqrt{\rho_{\mathbf{X}}^2 \zeta_n^2 + 1}} \quad (144)$$

is a local chart around $(z^{(n)}, \zeta^{(n,\varepsilon)})$.

Observe also that, as in (136),

$$M_{(\gamma_{\mathbf{X},N})} = \pi_{(\mathbf{X},N)}^{-1} \left(\bigcup_{n=-1}^N W_{(\mathbf{X},n)} \right) \quad (145)$$

is an open neighborhood of the curve $\gamma_{\mathbf{X},N}$ in $M_{(\mathbf{X},N)}$. Moreover, for every $n \in \{-1, 0, \dots, N\}$, we have

$$\pi_{(\mathbf{X},N)}^{-1}(W_{(\mathbf{X},n)}) = \bigcup_{\varepsilon \in \{\pm 1\}^{N+1}} (U_{\mathbf{X},n,\varepsilon} \cup U_{\mathbf{X},\varepsilon(n^\vee)}).$$

The resolvent R can be lifted along the curve $\gamma_{\mathbf{X},N}$ to a meromorphic function on $M_{(\gamma_{\mathbf{X},N})}$. The meromorphically lifted function admits singularities at the $(N+1) \times 2^N$ branching points $(z^{(n)}, \zeta^{(n,\varepsilon)})$ with $(n, \varepsilon) \in \{0, 1, \dots, N\} \times \mathcal{E}_n$. They are simple poles. The precise situation is given by the following theorem. Recall the notation $\boldsymbol{\xi} = e^{i\pi/3}$ and the functions ψ_z and $\phi_{z,u}$ from (47) and (48).

Theorem 20. *Let $f \in C_c^\infty(\mathbf{X})$ and $y \in \mathbf{X}$ be fixed. Let $N \in \mathbb{N}$ and let $\gamma_{\mathbf{X},N}$ be the curve in $M_{(\mathbf{X},N)}$ given by (142). Then the resolvent $R(z) = [R(z)f](y)$ lifts as a meromorphic function to the neighborhood $M_{(\gamma_{\mathbf{X},N})}$ of the curve $\gamma_{\mathbf{X},N}$ in $M_{(\mathbf{X},N)}$. We denote the lifted meromorphic function by $\tilde{R}_{(N)}(z, \zeta) = [\tilde{R}_{(N)}(z, \zeta)f](y)$.*

The singularities of $\tilde{R}_{(N)}$ are simple poles at the points $(z^{(n)}, \zeta^{(n,\varepsilon)}) \in M_{(\mathbf{X},N)}$ with $z^{(n)} = -i(n + \frac{1}{2})\rho_{\mathbf{X}}$ and $(n, \varepsilon) \in \{0, 1, \dots, N\} \times \mathcal{E}_n$. Explicitly, for $(n, \varepsilon) \in \{0, 1, \dots, N\} \times \mathcal{E}_n$,

$$\tilde{R}_{(N)}(z, \zeta) = \tilde{H}_{(N,n,\varepsilon)}(z, \zeta) + C \tilde{G}_{(\mathbf{X},n)}(z, \zeta) \quad ((z, \zeta) \in U_{\mathbf{X},n,\varepsilon} \cup U_{\mathbf{X},\varepsilon(n^\vee)}), \quad (146)$$

where $\tilde{H}_{(N,n,\varepsilon)}$ is holomorphic,

$$C = \frac{12\pi^2}{\rho_{\mathbf{X}}^2 |W|} = \frac{\pi^2}{|W|}$$

is a constant (independent of N , n and ε), and

$$\begin{aligned} \tilde{G}_{(\mathbf{X},n)}(z, \zeta) &= \psi_{z\rho_{\mathbf{X}}^{-1}} \left(\rho_{\mathbf{X}} \left(\frac{i}{z} \left(n + \frac{1}{2} \right) - \zeta_n \right) \right) \phi_{z\rho_{\mathbf{X}}^{-1},1} \left(\boldsymbol{\xi} \rho_{\mathbf{X}} \left(\frac{i}{z} \left(n + \frac{1}{2} \right) - \zeta_n \right) \right) \\ &\quad \cdot \phi_{z\rho_{\mathbf{X}}^{-1},1} \left(\boldsymbol{\xi}^2 \rho_{\mathbf{X}} \left(\frac{i}{z} \left(n + \frac{1}{2} \right) - \zeta_n \right) \right) \frac{\frac{i}{z} \left(n + \frac{1}{2} \right)}{-i\pi\zeta_n} \end{aligned} \quad (147)$$

is independent of N and ε (but dependent on f and y , which are omitted from the notation). The function $\tilde{G}_{(\mathbf{X},n)}$ has a simple pole at $(z^{(n)}, \zeta^{(n,\varepsilon)})$ for all $\varepsilon \in \mathcal{E}_n$.

The local expression for $\tilde{G}_{(\mathbf{X},n)}$ in terms of the chart (144) is:

$$\begin{aligned} (\tilde{G}_{(\mathbf{X},n)} \circ \kappa_{n,\varepsilon}^{-1})(\zeta_n) &= \psi_{\frac{i(n+\frac{1}{2})}{\sqrt{\rho_{\mathbf{X}}^2 \zeta_n^2 + 1}}} \left(\sqrt{\rho_{\mathbf{X}}^2 \zeta_n^2 + 1} + \rho_{\mathbf{X}} \zeta_n \right) \\ &\quad \cdot \phi_{\frac{i(n+\frac{1}{2})}{\sqrt{\rho_{\mathbf{X}}^2 \zeta_n^2 + 1}}, 1} \left(\xi \left(\sqrt{\rho_{\mathbf{X}}^2 \zeta_n^2 + 1} + \rho_{\mathbf{X}} \zeta_n \right) \right) \\ &\quad \cdot \phi_{\frac{i(n+\frac{1}{2})}{\sqrt{\rho_{\mathbf{X}}^2 \zeta_n^2 + 1}}, 1} \left(\xi^2 \left(\sqrt{\rho_{\mathbf{X}}^2 \zeta_n^2 + 1} + \rho_{\mathbf{X}} \zeta_n \right) \right) \cdot \frac{\sqrt{\rho_{\mathbf{X}}^2 \zeta_n^2 + 1}}{i\pi \rho_{\mathbf{X}} \zeta_n}. \end{aligned} \quad (148)$$

Furthermore,

$$\operatorname{Res}_{\zeta_n=0} [(\tilde{R}_{(n)} \circ \kappa_{n,\varepsilon}^{-1})(\zeta_n) f](y) = -\frac{1}{4|W|} \left(n + \frac{1}{2} \right)^2 (f \times \varphi_{(n+\frac{1}{2})\rho})(y), \quad (149)$$

where $\varphi_{(n+\frac{1}{2})\rho}$ is the spherical function on \mathbf{X} of spectral parameter $(n + \frac{1}{2})\rho$, see (5). In particular, $\operatorname{Res}_{\zeta_n=0} [(\tilde{R}_{(n)} \circ \kappa_{n,\varepsilon}^{-1})(\zeta_n) f](y)$ is independent of ε .

Proof. According to Corollary 4 and Proposition 2, it suffices to meromorphically extend the function F given by (31), as done in the Section 3.4. The lift $\tilde{R}_{(N)}$ of the resolvent to $M_{(\gamma_{\mathbf{X}}, N)}$ as well as the expression of $\tilde{G}_{(\mathbf{X},n)}$ are then obtained from Theorem 19, with (z, ζ) replaced by $(\rho_{\mathbf{X}}^{-1}z, \rho_{\mathbf{X}}\zeta)$. In fact, the function $\tilde{H}_{(N,n,\varepsilon)}(z, \zeta)$ in (146) is the sum of the holomorphic function $H(\rho_{\mathbf{X}}^{-1}z)$ from Proposition 2 together with $\frac{\pi i}{|W|\rho_{\mathbf{X}}^2}$ times the sum of all terms in the definition of $\tilde{F}(\rho_{\mathbf{X}}^{-1}z, \rho_{\mathbf{X}}\zeta)$ on $U_{\mathbf{X},\varepsilon(n^\vee)} \cup U_{\mathbf{X},n,\varepsilon}$, as on the right-hand side of (140) with n instead of m , except for $4\pi i \tilde{G}_{(n)}(\rho_{\mathbf{X}}^{-1}z, \rho_{\mathbf{X}}\zeta)$. Thus $\tilde{H}_{(N,n,\varepsilon)}$ is holomorphic on $U_{\mathbf{X},\varepsilon(n^\vee)} \cup U_{\mathbf{X},n,\varepsilon}$. Finally,

$$\tilde{G}_{(\mathbf{X},n)}(z, \zeta) = -\frac{1}{3} \tilde{G}_{(n)}(\rho_{\mathbf{X}}^{-1}z, \rho_{\mathbf{X}}\zeta_n). \quad (150)$$

Notice that for (z, ζ) in the neighborhood $U_{\mathbf{X},\varepsilon(n^\vee)}$ of $(z^{(n)}, \zeta^{(n,\varepsilon)})$ we have

$$i \frac{\rho_{\mathbf{X}}}{z} \left(n + \frac{1}{2} \right) = -\sqrt{\rho_{\mathbf{X}}^2 \zeta_n^2 + 1}.$$

The expression (148) for $\tilde{G}_{(\mathbf{X},n)}$ in the chart $(U_{\mathbf{X},\varepsilon(n^\vee)}, \kappa_{n,\varepsilon})$ comes then from (150) together with the parity properties (50) and (52) of the functions ψ_z and $\phi_{z,1}$, as for (88).

To compute the residue (149), we have, by (146),

$$\operatorname{Res}_{\zeta_n=0} (\tilde{R}_{(N)} \circ \kappa_{n,\varepsilon}^{-1})(\zeta_n) = C \operatorname{Res}_{\zeta_n=0} (\tilde{G}_{(\mathbf{X},n)} \circ \kappa_{n,\varepsilon}^{-1})(\zeta_n). \quad (151)$$

By (90) and (150),

$$\operatorname{Res}_{\zeta_n=0} (\tilde{G}_{(\mathbf{X},n)} \circ \kappa_{n,\varepsilon}^{-1})(\zeta_n) = \frac{1}{2\pi} \left(n + \frac{1}{2} \right)^2 \psi_{i(n+\frac{1}{2})}(1). \quad (152)$$

Moreover, (48) yields

$$\psi_{i(n+\frac{1}{2})}(1) = \frac{1}{2\pi} \left(f \times \frac{\varphi_{i\lambda(i(n+\frac{1}{2}),1)} + \varphi_{-i\lambda(i(n+\frac{1}{2}),1)}}{2} \right)(y).$$

By definition, see (35), we have $\lambda(i(n + \frac{1}{2}), 1) = \mathbf{i}\lambda(n + \frac{1}{2}, 1)$. Furthermore, by (29) and (23),

$$\lambda(n + \frac{1}{2}, 1) = (n + \frac{1}{2})e_1 = (n + \frac{1}{2})\alpha_{1,2}.$$

Recall that the spherical functions φ_λ are Weyl group invariant in the spectral parameter $\lambda \in \mathfrak{a}_\mathbb{C}^*$. Recall also that the Weyl group W permutes all roots, and more precisely it acts on $\alpha_{i,j}$ by permuting the indices $i, j \in \{1, 2, 3\}$. Since $\rho = \alpha_{1,3}$ is a root, we conclude that

$$\varphi_{\pm \mathbf{i}\lambda(i(n+\frac{1}{2}),1)} = \varphi_{\pm \mathbf{i}^2\lambda((n+\frac{1}{2}),1)} = \varphi_{\mp(n+\frac{1}{2})\alpha_{1,2}} = \varphi_{(n+\frac{1}{2})\rho}.$$

Therefore

$$\psi_{i(n+\frac{1}{2})}(1) = -\frac{1}{2\pi}(f \times \varphi_{(n+\frac{1}{2})\rho})(y). \quad (153)$$

The residue (149) follows then from (151), (152) and (153). \square

4. THE RESIDUE OPERATORS

Theorem 20 gives the meromorphic lift of the resolvent of the (shifted) positive Laplace-Beltrami operator $\Delta - \langle \rho, \rho \rangle$ of \mathbf{X} to the Riemann surface $M_{(\mathbf{X},N)}$ covering $\mathbb{C}_N^- = \{z \in \mathbb{C}^- : \text{Im } z > -(N + \frac{3}{2})\rho_{\mathbf{X}}\}$, where N be any fixed nonnegative integer. For fixed $f \in C_c^\infty(\mathbf{X})$ and $y \in \mathbf{X}$, the lifted resolvent $\tilde{R}_{(N)}$ admits simple poles at the branching points of $M_{(\mathbf{X},N)}$, that is at the points $(z^{(n)}, \zeta^{(n,\varepsilon)})$ with $(n, \varepsilon) \in \{0, 1, \dots, N\} \times \{\pm 1\}^{N+1}$. The singular part of the function $(z, \zeta) \rightarrow [\tilde{R}_{(N)}(z, \zeta)f](y)$ at $(z^{(n)}, \zeta^{(n,\varepsilon)})$ is a constant multiple the function $\tilde{G}_{(\mathbf{X},n)}(z, \zeta)$ defined in (150). It is independent of the choice of $N \geq n$ and of the singular point $(z^{(n)}, \zeta^{(n,\varepsilon)})$ in the fiber above $z^{(n)} = -i\rho_{\mathbf{X}}(n + \frac{1}{2})$ in $M_{(\mathbf{X},N)}$. In terms of the canonical chart $(U_{\mathbf{X},\varepsilon(n^\vee)}, \kappa_{n,\varepsilon})$ around $(z^{(n)}, \zeta^{(n,\varepsilon)})$, the residue of $\tilde{R}_{(N)}(z, \zeta)$ at $(z^{(n)}, \zeta^{(n,\varepsilon)})$, computed in (149), is a smooth function of $y \in X$ depending linearly on $f \in C_c^\infty(X)$. We use the value of this residue to define a residue operator at $z^{(n)} = -i\rho_{\mathbf{X}}(n + \frac{1}{2})$. More precisely, we define the residue operator

$$\text{Res}_n \tilde{R} : C_c^\infty(\mathbf{X}) \rightarrow C^\infty(\mathbf{X}) \quad (154)$$

by

$$\text{Res}_n \tilde{R}f = \left(n + \frac{1}{2}\right)^2 (f \times \varphi_{(n+\frac{1}{2})\rho}) \quad (f \in C_c^\infty(\mathbf{X})). \quad (155)$$

In this section we study the range of the linear operator $\text{Res}_n \tilde{R}$ from a representation theoretic point of view.

4.1. Residue operators and eigenspace representations. Let $\lambda \in \mathfrak{a}_\mathbb{C}^*$. We consider the convolution operator

$$\mathcal{R}_\lambda : C_c^\infty(\mathbf{X}) \rightarrow C^\infty(\mathbf{X}) \quad (156)$$

defined by

$$\mathcal{R}_\lambda(f) = f \times \varphi_\lambda \quad (f \in C_c^\infty(\mathbf{X})), \quad (157)$$

where, as before, φ_λ denotes Harish-Chandra's spherical function of spectral parameter λ . We keep the notation on eigenspace representations introduced in Section 1.4.

The next proposition holds for arbitrary Riemannian symmetric spaces of the noncompact type $\mathbf{X} = \mathbf{G}/\mathbf{K}$, where \mathbf{G} is a noncompact connected semisimple Lie group with finite center and \mathbf{K} a maximal compact subgroup of \mathbf{G} . It characterizes the closure of $\mathcal{R}_\lambda(C_c^\infty(\mathbf{X}))$ inside the eigenspace representation space $\mathcal{E}_\lambda(\mathbf{X})$ and gives a necessary and sufficient condition for $\mathcal{R}_\lambda(C_c^\infty(\mathbf{X}))$ to be finite dimensional.

Proposition 21. *The space $\mathcal{R}_\lambda(C_c^\infty(\mathbf{X})) = \{f \times \varphi_\lambda : f \in C_c^\infty(\mathbf{X})\}$ is a non-zero T_λ -invariant subspace of $\mathcal{E}_\lambda(\mathbf{X})$. Its closure is the unique closed irreducible subspace $\mathcal{E}_{(\lambda)}(\mathbf{X})$ of $\mathcal{E}_\lambda(\mathbf{X})$, which is generated by the translates of the spherical function φ_λ . The space $\mathcal{R}_\lambda(C_c^\infty(\mathbf{X}))$ is finite dimensional if and only if $\mathcal{E}_{\lambda, \mathbf{G}}(\mathbf{X}) \neq \{0\}$ is the finite dimensional spherical representation of highest restricted weight $-w\lambda - \rho$ (for some $w \in \mathbf{W}$, the Weyl group). In the latter case, $\mathcal{R}_\lambda(C_c^\infty(\mathbf{X})) = \mathcal{E}_{\lambda, \mathbf{G}}(\mathbf{X})$.*

Proof. Observe first that $\mathcal{R}_\lambda(C_c^\infty(\mathbf{X})) \neq \{0\}$ as φ_λ is nonzero and continuous.

For all $D \in \mathbb{D}(\mathbf{X})$ we have $D(f \times \varphi_\lambda) = f \times D\varphi_\lambda = \gamma(D)(\lambda)(f \times \varphi_\lambda)$. See [6, Ch. II, Theorem 5.5]. So $\mathcal{R}_\lambda(C_c^\infty(\mathbf{X})) \subseteq \mathcal{E}_\lambda(\mathbf{X})$.

Let $g \in G$, and let $F^{\tau(g)}$ denote the left translate by g of the function $F : \mathbf{X} \rightarrow \mathbb{C}$. Hence, if $o = eK$ is the base point of G/K , then $F^{\tau(g)}(h \cdot o) = F(g^{-1}h \cdot o)$ for all $h \in G$. We have

$$T_\lambda(g)(f \times \varphi_\lambda) = (f \times \varphi_\lambda)^{\tau(g)} = f^{\tau(g)} \times \varphi_\lambda.$$

As $f^{\tau(g)} \in C_c^\infty(\mathbf{X})$, the subspace $\mathcal{R}_\lambda(C_c^\infty(\mathbf{X}))$ of $\mathcal{E}_\lambda(\mathbf{X})$ is T_λ -invariant.

By definition, for $f \in C_c^\infty(\mathbf{X})$,

$$(f \times \varphi_\lambda)(x) = \int_G f(g \cdot o) \varphi_\lambda(g^{-1} \cdot x) dg = \int_G f(g \cdot o) \varphi_\lambda^{\tau(g)}(x) dg \quad (x \in \mathbf{X})$$

belongs to $\mathcal{E}_{(\lambda)}(\mathbf{X})$, the closure of the subspace of $\mathcal{E}_\lambda(\mathbf{X})$ spanned by the left translates of φ_λ . So $\overline{\mathcal{R}_\lambda(C_c^\infty(\mathbf{X}))} \subseteq \mathcal{E}_{(\lambda)}(\mathbf{X})$ is non-zero, closed and T_λ -invariant. Since $\mathcal{E}_{(\lambda)}(\mathbf{X})$ is irreducible (see e.g. [6, Ch. IV, Theorem 4.5]), they must agree.

If $\mathcal{R}_\lambda(C_c^\infty(\mathbf{X}))$ is finite dimensional, then its nonzero elements are G -finite, so $\mathcal{E}_{\lambda, \mathbf{G}}(\mathbf{X}) \neq \{0\}$. By Proposition 1 we conclude that $w\lambda - \rho$ is the restricted highest weight of a finite-dimensional spherical representation for some $w \in \mathbf{W}$. Moreover $\mathcal{R}_\lambda(C_c^\infty(\mathbf{X})) = \mathcal{E}_{\lambda, \mathbf{G}}(\mathbf{X})$ by irreducibility.

Conversely, suppose $\mathcal{E}_{\lambda, \mathbf{G}}(\mathbf{X}) \neq \{0\}$ is the finite dimensional spherical representation of highest restricted weight $w\lambda - \rho$ for some $w \in \mathbf{W}$. Then $\mathcal{R}_\lambda(C_c^\infty(\mathbf{X})) = \mathcal{E}_{\lambda, \mathbf{G}}(\mathbf{X})$ by [8, Theorem 3.2]. In particular $\mathcal{R}_\lambda(C_c^\infty(\mathbf{X}))$ is finite dimensional. \square

Corollary 22. *For all $n \in \mathbb{N}$ the eigenspace representation $T_{(n+\frac{1}{2})\rho}$ of $G = \mathrm{SL}(3, \mathbb{R})$ on $\mathcal{E}_{(n+\frac{1}{2})\rho}(\mathbf{X})$ is reducible. The closure of the image $\mathrm{Res}_n \widetilde{R}(C_c^\infty(\mathbf{X}))$ of the residue operator is the infinite dimensional irreducible subspace $\mathcal{E}_{((n+\frac{1}{2})\rho)}(\mathbf{X})$ of $\mathcal{E}_{(n+\frac{1}{2})\rho}(\mathbf{X})$ spanned by the translates of the spherical function $\varphi_{(n+\frac{1}{2})\rho}$. In particular, the residue operator $\mathrm{Res}_n \widetilde{R}$ has infinite rank for all $n \in \mathbb{N}$.*

Proof. Since $\rho = \alpha_{1,3}$, we have for all $n \in \mathbb{N}$ that

$$(n + \frac{1}{2})\rho_{1,3} = n + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}.$$

(Recall from Section 1.3 that $\rho_{i,j} = \rho_{\alpha_{i,j}}$.) Thus $T_{(n+\frac{1}{2})\rho}$ is reducible by (17).

Because of Proposition 21, it remains to prove that $\mathrm{Res}_n \widetilde{R}(C_c^\infty(\mathbf{X}))$ is infinite dimensional. For this, it is enough to check that for every Weyl group element w there is a root $\alpha \in \Sigma^+$ so that $-(n + \frac{1}{2})(w\rho)_\alpha \notin \mathbb{N} + \frac{1}{2}$. In turn, by Remark 1, it suffices to check that $(n + \frac{1}{2})\rho_{i,j} \notin \mathbb{Z}$ when $n \in \mathbb{N}$ and $(i, j) = (1, 2)$ or $(i, j) = (2, 3)$. This is immediate, since $\rho_{1,2} = \rho_{2,3} = \frac{1}{2}$. \square

Remark 5. As before, let $B = K/M$. The Poisson transform of $h \in C(B)$ is the function $\mathcal{P}_\lambda h : G/K \rightarrow \mathbb{C}$ defined by

$$(\mathcal{P}_\lambda h)(y) = \int_B h(b) e_{\lambda,b}(y) db \quad (y \in G/K),$$

see e.g. [7, Ch. II, §3, no. 4, and §5, no. 4]. According to (21), the range of the residue operator $\text{Res}_n \tilde{R}$ is the image under the Poisson transform of the elements of the Paley-Wiener space $\mathcal{H}(\mathfrak{a}_\mathbb{C}^* \times B)_W$, see Section 1.2, evaluated at $\lambda = (n + \frac{1}{2})\rho$ and considered as a function of $b \in B$.

4.2. Residue operators and Langlands' classification. In this section we give a description of the $\text{SL}(3, \mathbb{R})$ -action on the range of the residue operator in terms of Langlands' classification. We will identify all infinitesimally equivalent representations of the group $G = \text{SL}(3, \mathbb{R})$.

Recall the Iwasawa decomposition $G = KAN$,

$$x = \kappa(x)a(x)n(x) \quad (x \in G),$$

from Section 1.1. Let $P = MAN \subseteq G$ be the minimal parabolic subgroup consisting of matrices with zeros below the diagonal. For a fixed $\lambda \in \mathfrak{a}_\mathbb{C}^*$ the spherical non-unitary principal series representation, $\text{Ind}_P^G(1 \otimes e^\lambda)$, is a representation of G defined on the Hilbert space $\mathcal{H}_\Pi \subseteq L^2(K)$ consisting of the right M -invariant functions, with the group action given by

$$[\Pi(g)v](k) = a(g^{-1}k)^{-\lambda-\rho} v(\kappa(g^{-1}k)) \quad (g \in G, k \in K).$$

This representation has precisely one irreducible subquotient, $\bar{\pi}(1 \otimes (\lambda))(1)$, which contains a trivial K -type, see [23, Def. 4.4.6]. If $\text{Re } \lambda$ is negative, then $\bar{\pi}(1 \otimes (\lambda))(1)$ is a subrepresentation of $\text{Ind}_P^G(1 \otimes e^\lambda)$.

Also, recall that the group G has only three nilpotent orbits in \mathfrak{g} (see e.g. [2]). They are indexed by the partitions of 3: the maximal orbit $\mathcal{O}_{1,1,1}$, the minimal orbit $\mathcal{O}_{2,1}$ and the zero orbit $\mathcal{O}_{1^3} = \{0\}$.

Proposition 23. *As a representation of G , the range of the residue operator $\text{Res}_n \tilde{R}$ is infinitesimally equivalent to $\bar{\pi}(1 \otimes (n + \frac{1}{2}))(1)$. This is a proper infinite dimensional subrepresentation of the non-unitary principal series. This representation is unitarizable if and only if $n = 0$. The wave front set of each of these representations, see [19] for the definition, is equal to $\mathcal{O}_{2,1} \cup \mathcal{O}_{1^3}$.*

The proof of Proposition 23 is based on some well know facts. Since their proofs are short we include them in our argument.

Let (Π, \mathcal{H}_Π) be an admissible representation of G realized on a Hilbert space \mathcal{H}_Π with inner product $(\cdot, \cdot)_\Pi$. The hermitian dual $(\Pi^h, \mathcal{H}_\Pi^h)$ is defined by $\mathcal{H}_\Pi^h = \mathcal{H}_\Pi$ and $\Pi^h(x) = \Pi(x^{-1})^*$, $x \in G$.

Lemma 24. *Abusing the notation in an obvious way we have*

$$(\text{Ind}_P^G(1 \otimes e^\lambda))^h = \text{Ind}_P^G(1 \otimes e^{-\bar{\lambda}}).$$

Proof. In the argument below we'll find the following "change of variables" formula useful

$$\int_K a(gk)^{-2\rho} f(\kappa(gk)) dk = \int_K f(k) dk \quad (g \in G). \quad (158)$$

It may be found for instance in [11, (7.4)]. Let (\cdot, \cdot) denote the L^2 inner product on K . For $u, v \in \mathcal{H}_\Pi$ and $g \in G$ we have

$$\begin{aligned} (\Pi^h(g)u, v) &= (u, \Pi(g^{-1})v) = \int_K u(k) \overline{a(gk)^{-\lambda-\rho} v(\kappa(gk))} dk \\ &= \int_K a(gk)^{-2\rho} \left(u(k) a(gk)^{-\bar{\lambda}+\rho} \overline{v(\kappa(gk))} \right) dk. \end{aligned} \quad (159)$$

Let $l = \kappa(gk)$. Then $gk = \kappa(gk)a(gk)n(gk)$. So,

$$\begin{aligned} k &= g^{-1}l a(gk) n(gk) \\ &= \kappa(g^{-1}l) a(g^{-1}l) n(g^{-1}l) a(gk) n(gk) \\ &= \kappa(g^{-1}l) a(g^{-1}l) a(gk) n'. \end{aligned}$$

Hence

$$k = \kappa(g^{-1}l) \quad \text{and} \quad a(g^{-1}l) a(gk) = 1.$$

Let $\psi_g(l) = k$. (This is the inverse of the map $k \rightarrow \kappa(gk)$.) The above formulas show that

$$\psi_g(l) = \kappa(g^{-1}l) \quad \text{and} \quad a(g\psi_g(l)) = a(g^{-1}l)^{-1}.$$

Since the function

$$f(l) = u(\psi_g(l)) a(g\psi_g(l))^{-\bar{\lambda}+\rho} \overline{v(l)}$$

is right M -invariant, formula (158) implies that (159) is equal to

$$\int_K u(\kappa(g^{-1}l)) a(g^{-1}l)^{\bar{\lambda}-\rho} \overline{v(l)} dl.$$

Thus

$$[\Pi^h(g)u](l) = a(g^{-1}l)^{\bar{\lambda}-\rho} u(\kappa(g^{-1}l)).$$

Since this is the action of the induced representation $\text{Ind}_P^G(1 \otimes e^{-\bar{\lambda}})$, we are done. \square

Next we show that the range of the residue operator $\text{Res}_n \tilde{R}$ is infinitesimally equivalent to $\bar{\pi}(1 \otimes \lambda)(1)$, where $\lambda = (n + \frac{1}{2})\rho$.

If 1_K denotes the constant function equal to 1 on K , then the Harish-Chandra spherical function φ_λ (considered as a K -bi-invariant function on G) is given by

$$\varphi_\lambda(x) = (\Pi(x)1_K, 1_K) \quad (x \in G),$$

where (Π, \mathcal{H}_Π) stands for the induced representation $\text{Ind}_P^G(1 \otimes \lambda)$. As our λ is real, we have

$$\varphi_\lambda(x) = (1_K, \Pi(x)1_K) \quad (x \in G).$$

Since the convolution of two functions is given by

$$(f * g)(x) = \int_G f(y) g(y^{-1}x) dy \quad (x \in G, f \in C_c(G), g \in C(G)),$$

we see that

$$(f * \varphi_\lambda)(x) = \int_G f(y) (1_K, \Pi(y^{-1}x)1_K) dy = (\Pi^h(f)1_K, \Pi(x)1_K). \quad (160)$$

Here Π^h denotes the lift of $(\Pi^h, \mathcal{H}_\Pi^h)$ to $L^1(G)$. The map

$$C_c^\infty(G) \ni f \mapsto \Pi^h(f)1_K \in \mathcal{H}_\Pi^h \quad (161)$$

intertwines the left regular representation on $C_c^\infty(G)$ with Π^h . By definition, the range of this map is generated by the action of $L^1(G)$ on the vector 1_K . Hence it is generated by the action of the group on the vector 1_K . But the representation $(\Pi^h, \mathcal{H}_\Pi^h)$ is infinitesimally equivalent to $\text{Ind}_P^G(1 \otimes (-\lambda))$. As $-\lambda$ is negative, the induced representation contains a unique irreducible subrepresentation, $\bar{\pi}(1 \otimes (-\lambda))(1)$, containing the trivial K -type, [23, Proposition 4.2.12]. Hence the range of (161) coincides with $\bar{\pi}(1 \otimes (-\lambda))(1)$.

Furthermore we have the map

$$\mathcal{H}_\Pi^h \ni u \mapsto (u, \Pi(\cdot)1_K) \in C(G). \quad (162)$$

Since the map

$$C_c^\infty(G) \ni f \mapsto f * \varphi_\lambda \in C^\infty(G) \quad (163)$$

is the composition of (162) and (161), we see that the range of (163) coincides with $\bar{\pi}(1 \otimes (-\nu))(1)$. Since replacing $C_c^\infty(G)$ by $C_c^\infty(G/K)$ does not change the range, the range of (163) is equal to the range of the residue operator. Since $\bar{\pi}(1 \otimes (-\lambda))(1)$ is isomorphic to $\bar{\pi}(1 \otimes \lambda)(1)$, the first part of the proposition follows.

Next we study some properties of the representation $\bar{\pi}(1 \otimes \lambda)(1)$. The infinitesimal character of $\bar{\pi}(1 \otimes \lambda)(1)$ is equal to the infinitesimal character of the induced representation, and therefore is represented by λ , see [23, Lemma 4.1.8]. In particular this infinitesimal character is not of the form “a highest weight plus ρ ”. Therefore $\bar{\pi}(1 \otimes \lambda)(1)$ is infinitely dimensional.

For each positive root α we have an embedding

$$\phi_\alpha : \text{GL}(2, \mathbb{R}) \rightarrow G$$

defined by

$$\begin{aligned} \phi_{\alpha_{1,2}} \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} &= \begin{pmatrix} g_{1,1} & g_{1,2} & 0 \\ g_{2,1} & g_{2,2} & 0 \\ 0 & 0 & (g_{1,1}g_{2,2} - g_{1,2}g_{2,1})^{-1} \end{pmatrix} \\ \phi_{\alpha_{2,3}} \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} &= \begin{pmatrix} (g_{1,1}g_{2,2} - g_{1,2}g_{2,1})^{-1} & 0 & 0 \\ 0 & g_{1,1} & g_{1,2} \\ 0 & g_{2,1} & g_{2,2} \end{pmatrix} \\ \phi_{\alpha_{1,3}} \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} &= \begin{pmatrix} g_{1,1} & 0 & g_{1,2} \\ 0 & (g_{1,1}g_{2,2} - g_{1,2}g_{2,1})^{-1} & 0 \\ g_{2,1} & 0 & g_{2,2} \end{pmatrix} \end{aligned}$$

Then $\phi_\alpha(\text{GL}(2, \mathbb{R}))$ is the centralizer of the kernel of $e^\alpha : A \rightarrow \mathbb{C}$ in G , and is denoted by $M^\alpha A^\alpha$ in [23, Notation 4.2.21]. As before, let $\lambda = (n + \frac{1}{2})\rho$. We see from [23, Theorem 4.2.25] that the induced representation $\text{Ind}_P^G(1 \otimes e^\lambda)$ is reducible if and only if there is α such that the induced representation $\text{Ind}_{P \cap \phi_\alpha(\text{GL}(2, \mathbb{R}))}^{\phi_\alpha(\text{GL}(2, \mathbb{R}))}(1 \otimes e^\lambda)$ is reducible. Set

$$H_{\alpha_{1,2}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_{\alpha_{1,3}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad H_{\alpha_{2,3}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

If $\alpha \in \{\alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3}\}$ the reducibility condition for $\text{Ind}_{P \cap \phi_\alpha(\text{GL}(2, \mathbb{R}))}^{\phi_\alpha(\text{GL}(2, \mathbb{R}))}(1 \otimes e^\lambda)$ reads that

$$\lambda(H_\alpha) = 2d + 1 \quad (\text{for some } d = 1, 2, 3, \dots).$$

But $\rho = \alpha_{1,3}$, so

$$\rho(H_{\alpha_{1,2}}) = 1, \quad \rho(H_{\alpha_{1,3}}) = 2, \quad \rho(H_{\alpha_{2,3}}) = 1.$$

Therefore $\text{Ind}_{\mathbb{P}}^{\mathbb{G}}(1 \otimes e^\lambda)$ is reducible if and only if

$$\begin{aligned} & n + \frac{1}{2} \quad \text{is an odd positive integer} \\ \text{or} & \quad 2(n + \frac{1}{2}) \quad \text{is an odd positive integer} \\ \text{or} & \quad n + \frac{1}{2} \quad \text{is an odd positive integer.} \end{aligned}$$

Thus $\text{Ind}_{\mathbb{P}}^{\mathbb{G}}(1 \otimes e^\lambda)$ is reducible. According to [21] the unitary dual of \mathbb{G} consists of the trivial representation, complementary series and unitarily induced representations. Our representation $\bar{\pi}(1 \otimes \frac{1}{2}\rho)(1)$ is at the end of a complementary series, hence it is unitarizable. However, for $n = 1, 2, 3, \dots$, $\bar{\pi}(1 \otimes (n + \frac{1}{2}\rho))(1)$ is not in any complementary series. Hence it is not unitarizable.

Let $\mathbb{Q} \subseteq \mathbb{G}$ be the group generated by \mathbb{P} and $\phi_{\alpha_{1,3}}(\text{GL}(2, \mathbb{R}))$. Then \mathbb{Q} is a maximal parabolic subgroup with the Levi factor equal to $\phi_{\alpha_{1,3}}(\text{GL}(2, \mathbb{R}))$. The restriction of the character e^λ to the center of $\phi_{\alpha_{1,3}}(\text{GL}(2, \mathbb{R}))$ is trivial. Therefore the induction by stages, [23, Proposition 4.1.18], shows that

$$\text{Ind}_{\mathbb{P}}^{\mathbb{G}}(1 \otimes e^\lambda) = \text{Ind}_{\mathbb{Q}}^{\mathbb{G}}(\text{Ind}_{\phi_{\alpha_{1,3}}(\text{GL}(2, \mathbb{R})) \cap \mathbb{P}}^{\phi_{\alpha_{1,3}}(\text{GL}(2, \mathbb{R}))}(1 \otimes e^\lambda) \otimes e^0). \quad (164)$$

The representation $\text{Ind}_{\phi_{\alpha_{1,3}}(\text{GL}(2, \mathbb{R})) \cap \mathbb{P}}^{\phi_{\alpha_{1,3}}(\text{GL}(2, \mathbb{R}))}(1 \otimes e^\lambda)$ has the unique Langlands quotient F_λ , which happens to be finite dimensional. Hence our Langlands quotient $\bar{\pi}(1 \otimes \frac{1}{2}\rho)(1)$ is a subquotient of

$$\text{Ind}_{\mathbb{Q}}^{\mathbb{G}}(F_\lambda \otimes e^0). \quad (165)$$

In particular the wave front set of $\bar{\pi}(1 \otimes \frac{1}{2}\rho)(1)$ is contained in the wave front set of $\text{Ind}_{\mathbb{Q}}^{\mathbb{G}}(F_\lambda \otimes e^0)$, which is equal to the closure of the nilpotent orbit induced from the zero orbit on the Lie algebra of $\phi_{\alpha_{1,3}}(\text{GL}(2, \mathbb{R}))$, i.e. to the closure of $\mathcal{O}_{2,1}$. This completes the proof of Proposition 23.

Remark 6. By (159), the image of the map (162) is the function

$$\mathbb{G} \ni g \mapsto (u, \Pi(g)1_{\mathbb{K}}) = \int_{\mathbb{K}} u(k)a(g^{-1}k)^{-\lambda-\rho} dk.$$

Consider right M -invariant functions on \mathbb{K} as functions on $B = \mathbb{K}/M$ and right \mathbb{K} -invariant functions on \mathbb{G} as functions on \mathbb{G}/\mathbb{K} . Then $\mathcal{H}_{\Pi}^h = L^2(B)$ and the range of (162) is $C^\infty(\mathbb{G}/\mathbb{K})$. Since $a(g^{-1}k)^{-\lambda-\rho} = e_{\lambda, kM}(g\mathbb{K})$, the map (162) is the Poisson transform \mathcal{P}_λ , see Remark 5.

For generic λ , the Poisson transform maps the hyperfunction vectors of spherical non-unitary principal series representation $\text{Ind}_{\mathbb{P}}^{\mathbb{G}}(1 \otimes e^\lambda)$ onto $\mathcal{E}_\lambda(\mathbb{X})$. But Corollary 22 says the resonances $(n + \frac{1}{2})\rho$ are not generic in this sense. In this case, according to [17, Thm. 2.4], the image of the Poisson transform consists of those elements $u \in \mathcal{E}_{(n+\frac{1}{2})\rho}(\mathbb{X})$ which satisfy the following additional differential equations: $\text{sym}(h)u = 0$, where h is a \mathbb{K} -harmonic polynomial on $\mathfrak{p}_{\mathbb{C}}^*$, viewed as an element of the symmetric algebra $S(\mathfrak{p}_{\mathbb{C}})$ of $\mathfrak{p}_{\mathbb{C}}$ and $\text{sym}: S(\mathfrak{p}_{\mathbb{C}}) \rightarrow U(\mathfrak{p}_{\mathbb{C}})$ is the usual symmetrization map.

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