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The contact problem in Lagrangian systems subject to bilateral and unilateral constraints, with or without sliding Coulomb's friction: A tutorial

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Abstract

This work concerns the existence and uniqueness of the acceleration and contact forces for Lagrangian systems subject to bilateral and/or unilateral constraints with or without sliding Coulomb's friction. Sliding friction is known to yield singularities in the system, such as Painlevé's paradox. Our work aims at providing sufficient conditions on the parameters of the system so that singularities are avoided (*i.e.*, the contact problem is at least solvable). To this end, the frictional problem is treated as a perturbation of the frictionless case. We provide explicit criteria, in the form of calculable upper bounds on the friction coefficients, under which the frictional contact problem is guaranteed to remain well-posed. Complementarity problems, variational inequalities, quadratic programs and inclusions in normal cones are central tools.

1 Introduction

Lagrangian systems subject to (frictional) bilateral and unilateral constraints are considered. Such systems, mathematically described in Equation (1) below, feature a very rich dynamics, because they are nonlinear, nonsmooth, and set-valued. A large number of studies have been led on their well-posedness. Their goal is to formulate conditions under which (1) possesses a solution, that is, a trajectory $(q(\cdot), \dot{q}(\cdot))$ belonging to a certain functional space (with absolutely continuous positions and right-continuous velocities of local bounded variations), and satisfying (1) for all $t \geq 0$. It also consists in determining

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whether a solution is unique for given initial data $(q(0), \dot{q}(0))$, whether solutions depend continuously on initial data or not, whether they converge to some equilibrium state (stability and control analysis [41]), *etc.* The well-posedness of the dynamical system (1) has been pioneered in the frictionless case by Schatzman [67] and Monteiro-Marques [49], and further tackled in [59, 60, 61, 6, 46, 23, 24, 68], and in [4, 7, 50, 71] when Coulomb's friction is considered.

Besides such analysis, one may in turn be interested in properties of least mathematical relevance, but of high interest for mechanical engineers. Typically, are the contact forces calculable during persistent motion phases? Are they calculable in a unique way? If some solvability results can be obtained in the absence of friction, what happens when Coulomb friction is added at some contacts? More generally, how does the system evolve during persistent contact phases in the presence of friction, e.g., does the dynamics exhibit some singular states? In Solid Mechanics, these questions are to be examined by studying the so-called *contact problem*, which assumes the mechanical state $(q(t), \dot{q}(t))$ to be known at a given time t , and considers the acceleration $\ddot{q}(t)$ and the contact forces at time t as unknowns. It thus results in a merely algebraic system, for which well-posedness remains a difficult question. Studying the contact problem is of high interest in multibody systems where one often wants to calculate the contact forces at a certain given time. Another motivation comes from event-driven numerical integration methods, where one has to solve the contact problem at a given time where possibly the state of the system may switch to another mode [1].

In this paper our goal is to study conditions under which the contact problem is well-posed. We first focus on frictionless systems, then on systems with sliding friction (single-valued law). Indeed sliding friction is known to yield hard singularities like the Painlevé paradoxes [27, 56, 5], while it has been shown in the seminal paper [58] that sticking modes can always, under some mild assumption, be continued in another contact mode (including detachment from the constraints).

For systems subject to bilateral constraints, the necessary and sufficient conditions for the existence and uniqueness of the acceleration in the presence of a singular mass matrix and redundant constraints are given in [20, 76]. The computation of constraint reactions in the redundant case is addressed via augmented Lagrangian methods in [8, 10, 65, 22], by a constraint elimination method combined with solvability analysis techniques in [81, 79, 26] and by a pseudo-inverse method in [75]. While the mechanical and geometrical interpretation of the augmented Lagrangian method is well understood [10], only recently have its convergence and robustness properties been proved in the absence of a linearly independent constraint qualification hypothesis [37]. In the case of bilateral constraints with sliding Coulomb's friction, Matrosov and Finogenko derive an implicit criterion in [47, 48], which guarantees the uniqueness of the acceleration for small enough friction coefficients, however no explicit upper bounds are given. In [5] such an upper bound may be found but concerns only systems with a single contact point. For systems subject only to unilateral constraints and sliding Coulomb's friction an existence result based on small enough friction coefficients and complementarity theory is given in [42] and an explicit upper bound is established in [58]. The contact problem with mixed (bilateral

and unilateral) constraints has received surprisingly less attention, its analysis in the frictionless case with redundant constraints and a singular mass matrix is given in [13], while in the frictional sliding case it is established in [35] that it becomes a mixed linear complementarity problem but no explicit condition for its solvability is given.

1.1 Dynamics of Lagrangian systems subject to bilateral, unilateral and frictional constraints

In a Lagrangian formalism such systems may be written generically as follows,

$$\begin{cases} M(q)\ddot{q} + F(q, \dot{q}, t) = \nabla h_{n,b}(q)\lambda_{n,b} + \nabla h_{n,u}(q)\lambda_{n,u} + H_{t,b}(q)\lambda_{t,b} + H_{t,u}(q)\lambda_{t,u} & (1a) \\ \text{Sliding friction (Coulomb)} : \lambda_{t,i} = -\mu_i |\lambda_{n,i}| \frac{v_{t,i}}{\|v_{t,i}\|}, v_{t,i} \neq 0 \quad 1 \leq i \leq m & (1b) \\ \text{Complementarity conditions} : 0 \leq h_{n,u}(q) \perp \lambda_{n,u} \geq 0 & (1c) \\ \text{Bilateral (holonomic) constraints} : h_{n,b}(q) = 0, & (1d) \end{cases}$$

where $q \in \mathbb{R}^n$ is the vector collecting the generalized coordinates q_i , $1 \leq i \leq n$, assumed to be independent when all the constraints are removed, \dot{q} is the vector of generalized velocities, $M(q) = M(q)^T$ is the inertia matrix, always assumed to be at least positive semi-definite (it may be assumed non-singular in some cases), $F(q, \dot{q}, t)$ collects internal forces (including forces deriving from a potential, plus Coriolis and centrifugal forces), as well as external actions on the system such as disturbances or control.

We consider $m = m_u + m_b$ constraints (or contacts) consisting of m_u unilateral (inequality) constraints $h_{n,u}(q) \in \mathbb{R}^{m_u}$ and m_b bilateral (equality) constraints $h_{n,b}(q) \in \mathbb{R}^{m_b}$. The matrix $\nabla h_{n,u}(q)$ (respectively $\nabla h_{n,b}(q)$) collects on each column the gradient for each unilateral constraint (for each bilateral constraint, respectively). The vectors of Lagrange multipliers $\lambda_{n,u} \in \mathbb{R}^{m_u}$ and $\lambda_{n,b} \in \mathbb{R}^{m_b}$ are associated with the unilateral and bilateral constraints, respectively. From a mechanical point of view, $\lambda_{n,u}$ and $\lambda_{n,b}$ correspond to the normal components of the contact forces in the unilateral and bilateral case, respectively, and are obtained from the local contact kinematics [30, Chapter 10] [1, Chapter 3]. The unilateral constraints and their associated Lagrange multipliers are related through the complementarity condition (1c), which is to be understood componentwise (per contact). It models the fact that for each contact i , the normal contact force should not act at a unilateral contact point if the contact is open (i.e., $h_{n,u,i}(q) > 0$), and that $\lambda_{n,u,i} > 0$ if and only if $h_{n,u,i}(q) = 0$.

The coefficients of friction at each contact point i are $\mu_i \geq 0$. We consider in this study frictional contacts in a sliding mode only (non-zero relative tangential velocities). Also the contact problem (with unknowns the acceleration and the contact forces) is considered at a given time instant. This means that by a proper choice of the local contact kinematics frames one has $\frac{v_{t,i}}{\|v_{t,i}\|} = \text{sgn}(v_{t,i})$, both for 2-dimensional and 3-dimensional friction.

Organization: We first review in detail the frictionless case in Section 2, where results for the bilaterally and unilaterally constrained cases are recalled. In Section 3, it is shown

how the problem gets all the more complex as Coulomb's friction is considered and as mixed constraints (both bilateral and unilateral) are added.

1.2 Further notations

Mathematical notations \mathbb{R}^n is the set of n -vectors with real entries, \mathbb{R}_+^n is the set of n -vectors with non-negative entries. Let a_1, a_2, \dots, a_n some given reals, then $[a] = \text{diag}(a_i)$ is the $n \times n$ diagonal matrix with entries a_i . Let $A \in \mathbb{R}^{n \times n}$ be a real square matrix, its induced norm is $\|A\|_2 = \max_{x \in \mathbb{R}^n, \|x\|_2=1} \|Ax\|_2$, where $\|x\|_2$ is the Euclidean norm on the vector space \mathbb{R}^n . This induced matrix norm is sometimes denoted as $\|A\|_{2,2}$. One has $\|A\|_2 = \sigma_{\max}(A) = \sqrt{\lambda_{\max}(AA^T)}$ [9, Proposition 9.4.7], where σ_{\max} is the maximum singular value of A , and $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are its smallest and largest eigenvalues, respectively [9, Proposition 9.4.7]. Moreover, if A is invertible, $\sigma_{\min}(A) = \frac{1}{\sigma_{\max}(A^{-1})}$, where $\sigma_{\min}(A)$ is the smallest singular value of A [9, fact 6.3.21]. A positive definite (resp. semi definite) matrix is denoted $A \succ 0$ (resp. $A \succeq 0$), it may be non-symmetric. Let $f : \mathbb{R}^n \mapsto \mathbb{R}^p$ be a differentiable function. Its Euclidean gradient is $\nabla f(x) = (\nabla f_1(x) \ \nabla f_2(x) \ \dots \ \nabla f_p(x)) \in \mathbb{R}^{n \times p}$, and its Jacobian $\frac{\partial f}{\partial x}(x) = \nabla f(x)^T$. The cardinality of a countable set \mathcal{I} is denoted as $\text{card}(\mathcal{I})$. Let $S \subseteq \mathbb{R}^n$ be a set, its orthogonal complement $S^\perp \triangleq \{x \in \mathbb{R}^n | x^T y = 0 \text{ for all } y \in S\}$ and is a subspace. Its boundary is denoted as $\text{bd}(S)$. One has $\ker(A) = \text{Im}(A^T)^\perp$, and $\ker(A) = \text{Im}(A^T)^\perp$ for any matrix A .

Mechanical notations Since contacts may be frictionless, or in contrast may involve Coulomb friction, the following conventions shall be adopted:

- Bilateral contacts: $1 \leq i \leq m_b$, i.e., $i \in \mathcal{I}_b$.
 - Frictional bilateral contacts (sliding): $1 \leq i \leq m_b^\mu$, i.e., $i \in \mathcal{I}_b^\mu$.
 - Frictionless bilateral contacts: $m_b^\mu + 1 \leq i \leq m_b$, i.e., $i \in \mathcal{I}_b^0$.
- Unilateral contacts: $m_b + 1 \leq i \leq m$, i.e., $i \in \mathcal{I}_u$.
 - Frictional unilateral contacts (sliding): $m_b + 1 \leq i \leq m_u^\mu$, i.e., $i \in \mathcal{I}_u^\mu$.
 - Frictionless unilateral contacts: $m_u^\mu + 1 \leq i \leq m$, i.e., $i \in \mathcal{I}_u^0$.

We may therefore rewrite the first line in (1) as

$$\begin{aligned}
M(q)\ddot{q} + F(q, \dot{q}, t) &= \sum_{i \in \mathcal{I}_b^\mu} \nabla h_{n,b,i}(q) \lambda_{n,b,i} + \sum_{i \in \mathcal{I}_b^0} \nabla h_{n,b,i}^0(q) \lambda_{n,b,i}^0 \\
&+ \sum_{i \in \mathcal{I}_b^\mu} H_{t,b,i}(q) \lambda_{t,b,i} + \sum_{i \in \mathcal{I}_u^\mu} \nabla h_{n,u,i}(q) \lambda_{n,u,i} \\
&+ \sum_{i \in \mathcal{I}_u^0} \nabla h_{n,u,i}^0(q) \lambda_{n,u,i}^0 + \sum_{i \in \mathcal{I}_u^\mu} H_{t,u,i}(q) \lambda_{t,u,i}
\end{aligned} \tag{2}$$

- Matrix notations : Schur complements and Delassus' operators are ubiquitous in the study of contact forces, their mechanical interpretation is that of matrices expressing the coupling of constraints. In trying to render the equations more readable we assign these matrices names which reflect the effects they are responsible for. For example bilateral/bilateral couplings are of two types:
 - normal bilateral/normal bilateral : $A_{nb}(q) \triangleq \nabla h_{n,b}(q)^T M(q)^{-1} \nabla h_{n,b}(q)$
 - normal bilateral/tangential bilateral : $A_{tb}(q) \triangleq \nabla h_{n,b}(q)^T M(q)^{-1} H_{t,b}(q)$

Unilateral/unilateral coupling matrices are defined in the same way, replacing the letter 'b' by 'u' in the above expressions. For the mixed couplings a more explicit notation is used:

- normal bilateral/tangential unilateral : $A_{nbtu}(q) \triangleq \nabla h_{n,b}(q)^T M(q)^{-1} H_{t,u}(q)$

1.3 Mechanical systems

Throughout this paper, examples will help illustrate the concepts. The different mechanical systems that will be considered are depicted on Figure 1.

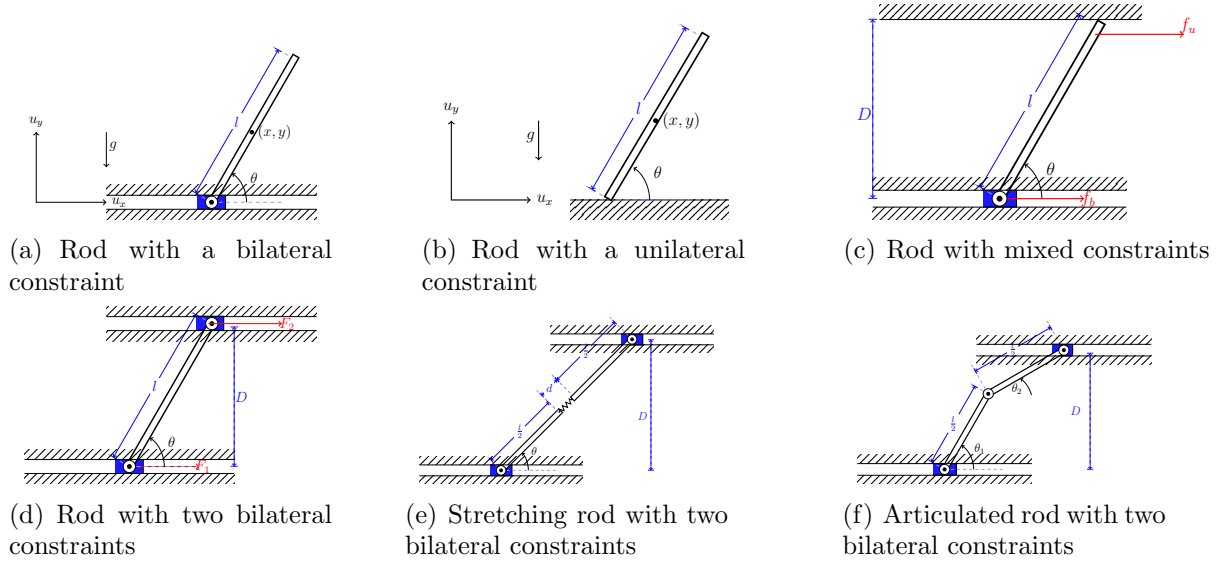


Figure 1: Mechanical systems with constraints.

2 Frictionless systems

This section starts with the classical KKT system for bilaterally constrained systems of index 1, and then elaborates on how this system is transformed when unilateral constraints are added. A thorough analysis of the ubiquitous KKT system is proposed in appendix E.

2.1 Bilaterally constrained systems

Let us assume that $m_u = 0$, i.e., $\mathcal{I}_u^0 = \emptyset$, and that $\mathcal{I}_b^0 \neq \emptyset$. The Lagrangian system (1) boils down to a differential-algebraic equation (DAE) of index 3 [70, 31],

$$\begin{cases} M(q)\ddot{q} + F(q, \dot{q}, t) = \nabla h_{n,b}(q)\lambda_{n,b} \\ h_{n,b}(q) = 0. \end{cases} \quad (3)$$

Reducing the index to 1 consists in differentiating twice the constraint $h_{n,b}(q) = 0$ to obtain $\nabla h_{n,b}(q)^T \ddot{q} + \frac{d}{dt}(\nabla h_{n,b}(q)^T) \dot{q} = 0$. This new equality combined with the dynamics yields the following linear system,

$$\underbrace{\begin{pmatrix} M(q) & -\nabla h_{n,b}(q) \\ \nabla h_{n,b}(q)^T & 0 \end{pmatrix}}_{M_b(q)} \begin{pmatrix} \ddot{q} \\ \lambda_{n,b} \end{pmatrix} = \begin{pmatrix} -F(q, \dot{q}, t) \\ -\frac{d}{dt}(\nabla h_{n,b}(q)^T) \dot{q} \end{pmatrix}. \quad (4)$$

Time-varying constraints $h_{n,b}(q, t) = 0$ may be considered in the analysis. Then $\frac{d}{dt}h_{n,b}(q, t) = \nabla h_{n,b}(q, t)^T \dot{q}(t) + \frac{\partial h_{n,b}}{\partial t}(q, t)$ and $\frac{d^2}{dt^2}h_{n,b}(q, t) = \nabla h_{n,b}(q, t)^T \ddot{q}(t) + \frac{d}{dt}(\nabla h_{n,b}(q, t)^T) \dot{q}(t) + \frac{\partial}{\partial q} \frac{\partial h_{n,b}}{\partial t}(q, t) \dot{q}(t) + \frac{\partial^2 h_{n,b}}{\partial t^2}(q, t)$. If needed it is thus sufficient to add the missing terms to step from the time-invariant to the time-varying case. Recall that $M(q)$ is assumed to be symmetric positive semi-definite (with possibly $\text{rank}(M(q)) < n$). The matrix in the left-hand side, denoted as $M_b(q)$ and often called the *DAE matrix*, is ubiquitous not only in the analysis and numerics of Lagrangian systems with holonomic constraints [70, 31, 40, 76, 73, 20, 80], but also in convex quadratic minimization. In the latter context, the $M_b(q)$ matrix is referred to as the *KKT matrix* and system (4) corresponds to the *KKT system* of the following quadratic minimization problem [66, §10.1.1],

$$\begin{aligned} \ddot{q}(t) &= \underset{z}{\text{argmin}} \frac{1}{2} z^T M(q) z + F(q, \dot{q}, t)^T z \\ &\text{subject to} \quad \nabla h_{n,b}(q)^T z + \frac{d}{dt}(\nabla h_{n,b}(q)^T) \dot{q} = 0, \end{aligned} \quad (5)$$

which can be, in the case when $M(q)$ is non-singular, interpreted mechanically as the Gauss principle of least constraints¹ applied to a Lagrangian system subject to bilateral holonomic constraints [75].

The DAE or KKT matrix $M_b(q)$ is a positive semi-definite matrix. The fact that $M_b(q)$ has a skew-symmetric part in (4) is not intrinsic to the problem. Indeed it stems from an arbitrary choice in the way the bilateral constraints are introduced in the dynamics (1), and thus of the choice of the sign of the multiplier. By changing the sign of the multiplier, the problem may be analyzed equivalently without the minus sign in $-\nabla h_{n,b}(q)$, thus setting $M_b(q)$ symmetric (meanwhile causing the loss of positiveness for $M_b(q)$). This is actually the convention adopted in most of the DAE literature. Next proposition gathers solvability results for the system (4) from Optimization [66, §10.1.1] and Mechanics [20].

¹Gauss' principle of least constraints, applied to a constrained motion, amounts to minimizing the kinetic distance between the actual generalized acceleration z of the body and the generalized acceleration z^* it would have in the absence of constraints (here, $z^* = M(q)^{-1}F(q, \dot{q}, t)$), over all compatible accelerations.

Proposition 1 Consider the KKT system in (4).

- (i) Let $m_b < n$ and $\nabla h_{n,b}(q)$ have full (column) rank m_b . Then $M_b(q)$ is non-singular (or equivalently, given (q, \dot{q}) and any arbitrary right-hand side vector of (4), there exists a unique solution $(\ddot{q}, \lambda_{n,b})$ to (4)) if and only if

$$\ker(M(q)) \cap \ker(\nabla h_{n,b}(q)^T) = \{0\}. \quad (6)$$

Moreover, if $M(q) \succ 0$, then the multiplier λ_{nb} can be computed in closed form by solving the linear equation

$$A_{nb}(q)\lambda_{n,b} + w_b(q, \dot{q}, t) = 0, \quad (7)$$

where $A_{nb}(q) = A_{nb}(q)^T \succ 0$, corresponding to the Schur complement² of $M(q)$ in the DAE matrix $M_b(q)$, and where $w_b(q, \dot{q}, t) = -\nabla h_{n,b}(q)^T M(q)^{-1} F(q, \dot{q}, t) + \frac{d}{dt}(\nabla h_{n,b}(q)^T \dot{q})$. Injecting the expression $\lambda_{n,b} = -A_{nb}(q)^{-1} w_b(q, \dot{q}, t)$ in the dynamics then yields a dynamical system which sets the submanifold $\{(q, \dot{q}) \in \mathbb{R}^n \times \mathbb{R}^n \mid h_{n,b}(q) = 0 \text{ and } \nabla h_{n,b}(q)^T \dot{q} = 0\}$ invariant.

- (ii) Let $\nabla h_{n,b}(q)$ have arbitrary rank and satisfy the compatibility of constraints, i.e., $-\frac{d}{dt}(\nabla h_{n,b}(q)^T \dot{q}) \in \text{Im}(\nabla h_{n,b}(q)^T)$. Then, given (q, \dot{q}) and an arbitrary force vector $F(q, \dot{q}, t)$,
 - A solution $(\ddot{q}, \lambda_{n,b})$ of (4) exists, and
 - The acceleration \ddot{q} and the generalized contact force $\nabla h_{n,b}(q)\lambda_{n,b}$ are unique,
if and only if condition (6) holds.

Proof It follows from the material in section E items (iii) (iii') and (iv). ■

Item (ii) obviously relies on weaker assumptions than item (i), since $\nabla h_{nb}(q)$ is not required to have full column rank m_b , but rather satisfies the compatibility of constraints, as in [20]. In particular, we consider in (ii) cases where $m_b > n$, i.e., where $M_b(q)$ is necessarily a singular matrix, and where some constraints are redundant. Note that provided $M(q) \succ 0$, the Schur complement satisfies $A_{nb}(q) = A_{nb}(q)^T \succeq 0$. Furthermore, we have $\ker(A_{nb}(q)) = \ker(\nabla h_{nb}(q))$. In particular, $A_{nb}(q)$ is non-singular if and only if $\nabla h_{nb}(q)$ has full column rank $m_b \leq n$. Singular mass matrices and redundant constraints are common features of multibody dynamical systems subject to bilateral constraints, due to rotation parameterization [69], or redundant generalized coordinates [76, 77].

²The Schur complement of the invertible matrix A_{11} in the $m \times n$ matrix $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, is the matrix $A_{22} - A_{21} A_{11}^{-1} A_{12}$. The Schur complement of the invertible matrix A_{22} in A , is the matrix $A_{11} - A_{12} A_{22}^{-1} A_{21}$.

Example: 2D bilaterally constrained rigid rod Let us study the sliding rigid rod depicted in Figure 1(a), which can be seen as a rigid pendulum with a frictionless sliding base. The only forces applied here are gravity (oriented downwards) and the net (normal) contact force λ_{nb} (taken positive by convention when oriented upwards). Let $m > 0$ be the mass of the rod and $l > 0$ its total length. Let us choose $q = (x, y, \theta)^T$, where x and y are the coordinates of the center of mass of the rod, and $\theta \in [0, \pi]$ is the angle between the horizontal line and the main axis of the rod. The dynamic equations read

$$\begin{cases} m\ddot{x} &= 0 \\ m\ddot{y} &= -mg + \lambda_{nb} \\ \frac{ml^2}{12}\ddot{\theta} &= -\frac{l}{2} \cos \theta \lambda_{nb} \\ h_{n,b}(q) &= y - \frac{l}{2} \sin \theta = 0, \end{cases} \quad (8)$$

where the last equality expresses the prismatic constraint applied onto the bottom tip of the rod. Note that the multiplier $\lambda_{nb} \in \mathbb{R}$ may take positive (upward net force) or negative (downward net force) values. Formulating (8) as the canonical system (4), one obtains $M(q) = M = \text{diag}(m, m, \frac{ml^2}{12})$, $\nabla h_{n,b}(\theta) = (0, 1, -\frac{l}{2} \cos \theta)^T$, $F(\dot{q}, q, t) = F = (0, mg, 0)^T$. Observe that both M and $\nabla h_{nb}(\theta)$ have full rank, hence this case study falls into the first (the most classical) category (i) examined in Proposition 1. M being full rank, the Schur complement of the DAE matrix is well-defined and can be computed as the scalar number $A_{nb}(\theta) = \nabla h_{nb}(\theta)^T M^{-1} \nabla h_{nb}(\theta) = \frac{1+3 \cos^2 \theta}{m}$. The reduced equation for λ_{nb} then boils down to a linear scalar equation $A_{nb}(\theta) \lambda_{nb} + w_b(\theta, \dot{\theta}) = 0$, where $w_b(\theta, \dot{\theta}) = \frac{d}{dt} (\nabla h_{nb}^T) \dot{q} - g = \frac{l}{2} \dot{\theta}^2 \sin \theta - g$. As expected, $A_{nb}(\theta)$ is non-singular and thus the solution for λ_{nb} exists and is unique. Similarly, from the dynamic equation (8) there exists a unique solution for the generalized acceleration \ddot{q} of the system. Finally we get

$$\lambda_{nb} = m \left(\frac{\frac{l}{2} \dot{\theta}^2 \sin \theta + g}{1 + 3 \cos^2 \theta} \right) \quad \ddot{q} = \left(0, -g + \frac{\frac{l}{2} \dot{\theta}^2 \sin \theta + g}{1 + 3 \cos^2 \theta}, -\frac{6}{l} \cos \theta \left(\frac{\frac{l}{2} \dot{\theta}^2 \sin \theta + g}{1 + 3 \cos^2 \theta} \right) \right)^T.$$

2.2 Unilaterally constrained systems

Now we assume that all contacts are unilateral, $m_b = 0$, i.e., $\mathcal{I}_b = \emptyset$. Let us recall the so-called *contact linear complementarity problem (LCP)*.

2.2.1 Construction of the contact LCP

Proposition 2 Let $h(\cdot)$ and $\lambda(\cdot)$ be two functions of time, and let $0 \leq h(t) \perp \lambda(t) \geq 0$ for all t . Assume that $h(\cdot)$ is continuous, $\dot{h}(\cdot)$, $\ddot{h}(\cdot)$ and $\lambda(\cdot)$ are right-continuous at some time t . (i) Let $h(t) = 0$, then $0 \leq \dot{h}(t) \perp \lambda(t) \geq 0$. (ii) Let $h(t) = 0$ and $\dot{h}(t) = 0$, then $0 \leq \ddot{h}(t) \perp \lambda(t) \geq 0$.

Proof: (i) For any $t' \geq t$ one has $h(t') - q(t) = \int_t^{t'} \dot{h}(s) ds$. Suppose that $\dot{h}(t) < 0$. Since $\dot{h}(\cdot)$ is right-continuous, there exists $\epsilon > 0$ such that $\dot{h}(s) < 0$ for all $s \in [t, t + \epsilon)$. Thus

for any $t' \in [t, t + \epsilon)$ one has $h(t') < 0$ which is impossible. Thus one has $\dot{h}(t) \geq 0$. Now let $\dot{h}(t) > 0$, by continuity there exists $\epsilon > 0$ such that for all $t' \in (t, t + \epsilon)$, one has $\dot{h}(t') > 0$. Consequently $h(t') > 0$ for all $t' \in (t, t + \epsilon)$, and $\lambda(t) = 0$. Now suppose that $\lambda(t) > 0$, thus $h(t) = 0$. Assume that $\dot{h}(t) > 0$ so that $h(t') > 0$ for all $t' \in (t, t + \epsilon)$, so that $\lambda(t') = 0$ for all $t' \in (t, t + \epsilon)$: this is a contradiction and consequently $\dot{h}(t) \leq 0$. From the non-negativeness one infers $\dot{h}(t) = 0$. Hence (i) is proved. Part (ii) is proved in a similar way. ■

One sees that Proposition 2 involves the lexicographical inequality $(h(t) \dot{h}(t) \ddot{h}(t)) \succcurlyeq 0$ ³. Lexicographical inequalities are ubiquitous in the analysis of unilaterally constrained dynamical systems [53, 78, 2, 30, 34, 72, 74]. Assume that $q(\cdot)$ and $h_{n,u}(\cdot)$ are continuous, while $\dot{q}(\cdot)$, $\ddot{q}(\cdot)$ and $\lambda_{n,u}(\cdot)$ are right-continuous. Replacing $h(t)$ with $h_{n,u,i} \circ q(t)$ and $\lambda(t)$ with $\lambda_{n,u,i}(t)$ allows one to assert that $h_{n,u,i}(q(t)) = 0$ implies that $0 \leq \nabla h_{n,u,i}(q(t))^T \dot{q}(t) \perp \lambda_{n,u,i}(t) \geq 0$, while $h_{n,u,i}(q(t)) = 0$ and $\nabla h_{n,u,i}(q(t))^T \dot{q}(t) = 0$ implies $0 \leq \nabla h_{n,u,i}(q(t))^T \ddot{q}(t) + w_{u,i}(q, \dot{q}, t) \perp \lambda_{n,u,i}(t) \geq 0$, where $w_u(q, \dot{q}, t)$ is in (11). In fact since we disregard impacts we may even assume that $\dot{q}(\cdot)$ is continuous. In view of this the following *mixed LCP (MLCP)* whose unknown is the acceleration, and which is the counterpart of (4) for unilateral constraints, holds:

$$\begin{cases} M(q)\ddot{q} + F(q, \dot{q}, t) = \nabla h_{n,u}(q)\lambda_{n,u} \\ 0 \leq \lambda_{n,u} \perp \nabla h_{n,u}(q)^T \ddot{q} + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} \geq 0. \end{cases} \quad (9)$$

2.2.2 Analysis of the contact LCP

In case $M(q) \succ 0$, the MLCP (9) is easily transformed by elimination of \ddot{q} to construct the contact LCP, that is the counterpart of (7):

Definition 1 (Frictionless Contact LCP) *Let $\mu_i = 0$ for all $1 \leq i \leq m$, $m_b = 0$, $m_u > 0$, and $M(q) \succ 0$. The frictionless contact LCP is given by:*

$$0 \leq \lambda_{n,u} \perp A_{nu}(q)\lambda_{n,u} + w_u(q, \dot{q}, t) \geq 0, \quad (10)$$

where $A_{nu}(q) = \nabla h_{n,u}(q)^T M(q)^{-1} \nabla h_{n,u}(q)$ is the Delassus' matrix, and

$$w_u(q, \dot{q}, t) = -\nabla h_{n,u}(q)^T M(q)^{-1} F(q, \dot{q}, t) + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q}. \quad (11)$$

If at a given time t the solution of (10) satisfies $\lambda_{n,u}(t) > 0$, then the contact mode exists at t . If $\lambda_{n,u}(t) = 0$ it may be that another mode has to be considered, depending on $w_u(q, \dot{q}, t)$ being > 0 or null (componentwise). In a more general situation it may occur that some components of the multiplier are positive while others are zero.

The next proposition gathers results from Moreau [51, 52], Lötstedt [43], Pang and Trinkle [57], Brogliato and Goeleven [11, 13].

³ $(x_1 \ x_2 \dots x_n) \succcurlyeq 0$ if the first non-zero entry $x_j > 0$, or all entries are zero.

Proposition 3 (Frictionless, unilateral constraints) *Let $\mu_i = 0$ for all $1 \leq i \leq m$, $m_b = 0$, $m_u > 0$. Let also $M(q) \succ 0$.*

- (i) *The LCP (10) has a unique solution for any $w_u(q, \dot{q}, t)$ if and only if the constraint functions $h_{n,u,i}(q)$ are independent ($\Rightarrow m \leq n$).*
- (ii) *If $\lambda_{n,u,1}$ and $\lambda_{n,u,2}$ are any two solutions of the LCP in (10), then $\nabla h_{n,u}^T(q)\lambda_{n,u,1} = \nabla h_{n,u}^T(q)\lambda_{n,u,2}$ and $w_u(q, \dot{q}, t)^T \lambda_{n,u,1} = w_u(q, \dot{q}, t)^T \lambda_{n,u,2}$. If the LCP in (10) has at least one solution at t , and given unique (q, \dot{q}) , then \ddot{q} is unique.*
- (iii) *The LCP in (10) is solvable if for any $z \in \mathbb{R}^m$ in the set of solutions of the homogeneous LCP: $0 \leq z \perp A_{nu}(q)z \geq 0$, one has $z^T w_u(q, \dot{q}, t) \geq 0$.*
- (iv) *The implication $\lambda_{n,u} \geq 0, \nabla h_{n,u}(q)\lambda_{n,u} = 0 \Rightarrow \lambda_{n,u}^T \frac{d}{dt}(\nabla h_{n,u}(q)^T)\dot{q} \geq 0$ holds if and only if the LCP in (10) is solvable.*
- (v) *If $\frac{d}{dt}(\nabla h_{n,u}(q)^T)\dot{q} \in \text{Im}(A_{nu}(q))$ then the LCP in (10) is solvable.*

Let now $M(q) \succeq 0$ and the gap functions $h_{n,u,i}(q)$, $1 \leq i \leq m_u$ satisfy the Mangasarian-Fromovitz constraint qualification.

- (vi) *The MLCP in (9) is solvable if $T_{\Phi_u}(q) \cap \ker(M(q)) = \{0\}$, where $T_{\Phi_u}(q) = \{v \in \mathbb{R}^n | \nabla h_{n,u}(q)^T v \geq 0\}$, $\Phi_u = \{q \in \mathbb{R}^n | h_{n,u}(q) \geq 0\}$.*

Proof: Item (i) follows from the symmetry of $A_{nu}(q)$ and the important fact that $\ker(A_{nu}(q)) = \ker(\nabla h_{n,u}(q))$, since the symmetric positive semi definite matrix $A_{nu}(q)$ is a P-matrix if and only if it is invertible. Item (ii) is a direct application of Theorem 1 in section B, and implies that the mere existence of a solution to the LCP in (10) assures that the acceleration is unique, independently of n , m and the constraints jacobian matrix rank. Item (iii) follows from Theorem 2, since $A_{nu}(q)$ is positive semi definite and hence copositive. Item (iv) is stated in [57, pp.211-212] without proof, we give the details of how to prove (iv) and (v) for completeness. For the “only if” part in item (iv) one remarks that the condition $(\lambda_{n,u} \geq 0, \nabla h_{n,u}(q)\lambda_{n,u} = 0)$ is equivalent to $(\lambda_{n,u} \geq 0, \lambda_{n,u} \in \text{Ker}(A_{nu}(q)))$, which is in turn equivalent to $\lambda_{n,u}$ being a solution of the homogeneous LCP($A_{nu}(q), 0$). Next, one has the inequality $\lambda_{n,u}^T (-\nabla h_{n,u}(q)^T M(q)^{-1} F(q, \dot{q}, t) + \frac{d}{dt}(\nabla h_{n,u}(q)^T)\dot{q}) \geq 0$, the first term being zero and the second non-negative by hypothesis. Hence the conditions for item (iii) hold and the LCP($A_{nu}(q), w_u(q, \dot{q}, t)$) is solvable. Conversely, if LCP($A_{nu}(q), w_u(q, \dot{q}, t)$) is solvable then there exists a feasible element z such that $A_{nu}(q)z + w_u(q, \dot{q}, t) \geq 0$. Let $\lambda_{n,u}$ be any element such that $\lambda_{n,u} \geq 0, \nabla h_{n,u}(q)\lambda_{n,u} = 0$. Then one has the inequality $\lambda_{n,u}^T (A_{nu}z + w_u) \geq 0$. From the structures of $w_u(q, \dot{q}, t)$ and $A_{nu}(q)$ one deduces that $\lambda_{n,u}^T \frac{d}{dt}(\nabla h_{n,u}(q)^T)\dot{q} \geq 0$. Thus the implication in (iv) holds. Item (v) gives sufficient conditions for the hypothesis of (iv) to hold. Any element in the range of $A_{nu}(q)$ is orthogonal to the nullspace of $A_{nu}(q)$ (which is the same as $\ker(\nabla h_{n,u}(q))$). Hence, if $\frac{d}{dt}(\nabla h_{n,u}(q)^T)\dot{q} \in \text{Im}(A_{nu}(q))$ then

for all non-negative $\lambda_{n,u}$ in the nullspace of $A_{nu}(q)$ one has $\lambda_{n,u}^T \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} = 0$, so the hypothesis of (iv) holds. The proof of item (vi) relies on the results in [3], see [13]. ■

It is interesting to compare Proposition 3 (vi) and (6): the null space is replaced by the tangent cone. In [11] item (iv) is stated as $\frac{d}{dt}(\nabla h_{n,u}(q)) \dot{q} = 0$, however due to the structure of $w_u(q, \dot{q}, t)$ it may be replaced by this inequality as noted in [57, pp.211-212]. It is noteworthy that in item (iv) the force $F(q, \dot{q}, t)$ plays no role. Let us note that if $w_u(q, \dot{q}, t) \geq 0$ the LCP in (10) is solvable, since $\lambda_{n,u} = 0$ is a solution (possibly not the only one). This is however a rather stringent requirement. Corollary 3.8.12 in [18] may also be used in our context. Indeed, given that $A_{nu}(q) = A_{nu}(q)^T \succeq 0$, the LCP in (10) is equivalent to the convex quadratic program

$$\lambda_{n,u} = \operatorname{argmin}_{z \geq 0} \frac{1}{2} z^T A_{nu}(q) z + w_u(q, \dot{q}, t)^T z \quad (12)$$

This QP is solvable if and only if it is bounded by below on the set $\{z \mid z \geq 0, z \in \ker(A_{nu}(q))\}$, hence retrieving the condition of item (iv). We shall see that when friction is considered the positivity may be kept, however the symmetry is usually lost.

The uniqueness of \ddot{q} may be deduced following another path that necessitates some convex analysis and variational inequalities theory (while proving Proposition 3 requires tools from Complementarity Theory [18]). The complementarity conditions in (9) are equivalently rewritten as the inclusion in a normal cone $\lambda_{n,u} \in -N_{\mathbb{R}_+^{m_u}}(\nabla h_{n,u}(q)^T \ddot{q} + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q})$. Then using the chain rule of convex analysis (see *e.g.* Proposition A.2 in [14]) it follows that:

$$\nabla h_{n,u}(q) \lambda_{n,u} \in -N_{K(q, \dot{q})}(\ddot{q}) \quad (13)$$

where

$$K(q, \dot{q}) = \{x \in \mathbb{R}^n \mid \nabla h_{n,u}(q)^T x + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} \geq 0\} \quad (14)$$

is convex polyhedral. We suppose here that $K(q, \dot{q})$ is non-empty, for otherwise the problem we are dealing with is simply meaningless. This translates into the classical constraint qualification for the chain rule [64, Theorem 23.9]

Assumption 1 $\operatorname{Im}(\nabla h_{n,u}(q)^T)$ contains a point in $\mathbb{R}_+^{m_u} - \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q}$.

Therefore the MLCP (9) is equivalently rewritten as the inclusion:

$$M(q) \ddot{q} + F(q, \dot{q}, t) \in -N_{K(q, \dot{q})}(\ddot{q}) \quad (15)$$

Proposition 4 Let $M(q) \succ 0$, m and n be arbitrary integers. The inclusion in (15) has a unique solution \ddot{q} .

The proof follows from Theorem 6 in [3] since $M(q) \succ 0$. Consequently the MLCP (9) with unknown \ddot{q} is well-posed also. This is in fact what Moreau proved in [51, 52] using another reasoning. The case when $M(q)$ has low rank $r < n$ may be tackled *via* the tools in [3], however this is outside the scope of this article, see [13]. It is nevertheless noteworthy that the formulations in (15) or (4) do not *a priori* rely on the assumption that

$M(q) \succ 0$, and this will be the case of all their extensions in the sequel. The existence and uniqueness of the acceleration for given position and velocity, is closely linked to Gauss' principle of mechanics (which is consequently extended to systems with unilateral constraints). Indeed since $M(q)$ is symmetric and at least positive semi definite, (15) is equivalent to (compare with (5)):

$$\begin{aligned} \ddot{q} &= \operatorname{argmin}_{z \in K(q, \dot{q})} \frac{1}{2} z^T M(q) z + z^T F(q, \dot{q}, t) \\ &= \operatorname{proj}_{M(q)} [K(q, \dot{q}); -M(q)^{-1} F(q, \dot{q}, t)] \quad (\text{in case } M(q) \succ 0). \end{aligned} \tag{16}$$

The existence and uniqueness of \ddot{q} in Proposition 3 is clearly sufficient only since it relies on the existence of a solution $\lambda_{n,u}$ of the contact LCP, while the acceleration existence relies on the existence of the generalized force $F_{n,u}(q) \triangleq \nabla h_{n,u}(q) \lambda_{n,u}$. Further quadratic programs obtained for the unilaterally constrained case may be found in [11, §4.4.1], using Dorn's duality. They summarize various results (including (12) and (16)), some of which were obtained by Moreau [52, 51] and Lötstedt [43]. It is noteworthy that (15) is written in a more general setting in [29, Equ. (8.11)], such that $K(q, \dot{q}) = T_{T_{\Phi_u}(q)}(\dot{q})$. Here we deal with the case when $h_{n,u}(q) = 0$ and $\nabla h_{n,u}(q)^T \dot{q} = 0$, *i.e.* $q \in \operatorname{bd}(\Phi_u)$ and $\dot{q} \in \operatorname{bd}(T_{\Phi_u}(q))$, for it is clear that if $h_{n,u}(q) = 0$ and $\nabla h_{n,u}(q)^T \dot{q} > 0$, or if $h_{n,u}(q) > 0$, then $\ddot{q} \in \mathbb{R}^n$.

Remark 1 *As announced in the introduction, the existence and uniqueness issues of solutions $(q(\cdot), \dot{q}(\cdot))$ of the dynamical system (1) over \mathbb{R}^+ is not tackled here, where only the contact complementarity problems are studied. It is however noteworthy that well-posedness results as in [60] or [6], rely on the full rank of the Delassus' matrix computed at the active constraints. This demonstrates that the contact LCP well-posedness and the dynamical system's well-posedness, have strong connections.*

Example: Unilaterally constrained rigid rod Consider the system in Figure 1(b) with a unilateral constraint. The rod is allowed to slide on the horizontal plane but it may also take off (note that θ is assumed to belong to $[0, \pi]$ so that the whole rod should remain in the upper half space). This example corresponds to the classical example of Painlevé, albeit without friction. The dynamics of the system reads

$$\begin{cases} m\ddot{x} & = 0 \\ m\ddot{y} & = -mg + \lambda_{nu} \\ I\ddot{\theta} & = -\frac{l}{2} \cos \theta \lambda_{nu} \\ 0 \leq h_{nu}(\theta) = y - \frac{l}{2} \sin(\theta) \perp \lambda_{nu} \geq 0 \end{cases}$$

where the complementarity expresses the unilateral constraint applied onto the bottom tip of the rod. In contrast with the bilateral case, the multiplier $\lambda_{nu} \in \mathbb{R}^+$ can take positive values only, meaning that the contact force should always be oriented upwards. Derivations of M , F , and ∇h_{nu} are identical to the bilateral case (replacing ∇h_{nb} with ∇h_{nu}), and the LCP in λ_{nu} reads

$$0 \leq A_{nu}(\theta) \lambda_{nu} + w_u(\theta, \dot{\theta}) \perp \lambda_{nu} \geq 0$$

with $A_{nu}(\theta) = \frac{1+3\cos^2\theta}{m}$ and $w_u(\theta, \dot{\theta}) = \frac{l}{2}\dot{\theta}^2 \sin\theta - g$. Since $A_{nu}(\theta) \succ 0$, the solution for λ_{nu} (and thus for \ddot{q}) exists and is unique. This is in accordance with Proposition 3(i), which applies here since ∇h_{nu} has full column rank.

2.3 Unilaterally/bilaterally constrained systems

We now consider the case when both bilateral and unilateral constraints are involved. Strangely enough, it is only recently that the analysis of this case has received attention [11, 13], though it may represent the most common case in practice [62]. Let us write the bilateral constraints on the acceleration level. The MLCP in (9) is augmented as follows:

$$\begin{cases} M(q)\ddot{q} + F(q, \dot{q}, t) = \nabla h_{n,b}(q)\lambda_{n,b} + \nabla h_{n,u}(q)\lambda_{n,u} \\ 0 \leq \lambda_{n,u} \perp \nabla h_{n,u}(q)^T \ddot{q} + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} \geq 0 \\ \nabla h_{n,b}(q)^T \ddot{q} + \frac{d}{dt}(\nabla h_{n,b}(q)^T) \dot{q} = 0. \end{cases} \quad (17)$$

Let $M(q) \succ 0$. The equality is used to rewrite (17) as the following MLCP, which extends both (7) and (9):

$$\begin{cases} (a) \quad M(q)\ddot{q} + F(q, \dot{q}, t) = \nabla h_{n,b}(q)\lambda_{n,b} + \nabla h_{n,u}(q)\lambda_{n,u} \\ (b) \quad \nabla h_{n,b}(q)^T M(q)^{-1} \nabla h_{n,b}(q) \lambda_{n,b} + \nabla h_{n,b}(q)^T M(q)^{-1} \nabla h_{n,u}(q) \lambda_{n,u} \\ \quad - \nabla h_{n,b}(q)^T M(q)^{-1} F(q, \dot{q}, t) + \frac{d}{dt}(\nabla h_{n,b}(q)^T) \dot{q} = 0 \\ (c) \quad 0 \leq \lambda_{n,u} \perp \nabla h_{n,u}(q)^T \ddot{q} + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} \geq 0 \end{cases} \quad (18)$$

In the following we are going to follow different paths for the analysis of this MLCP.

2.3.1 Analysis of acceleration inclusions

Existence and uniqueness of \ddot{q} via a first inclusion: Let us try to mimic the developments of section 2.2, in particular the steps that lead to the inclusion (15). The MLCP in (18) is a mixture of equality and complementarity constraints. Let us make the following assumption:

Assumption 2 $M(q) \succ 0$ and the $m_b \times m_b$ matrix $A_{nb}(q)$ is positive definite for all $q \in \mathbb{R}^n$ ($\Leftrightarrow \nabla h_{n,b}(q)$ has rank m_b).

We may use the second equality in (18) to express $\lambda_{n,b}$ as a function of $\lambda_{n,u}$ (the couplings between both set of multipliers comes from the matrix $A_{nbnu}(q) \triangleq \nabla h_{n,b}(q)^T M(q)^{-1} \nabla h_{n,u}(q)$; this shows why jumps in $\lambda_{n,u}$ may induce jumps in $\lambda_{n,b}$). Inserting this expression in the

dynamics one obtains:

$$\begin{aligned} & M(q)\ddot{q} + P(q)F(q, \dot{q}, t) + \nabla h_{n,b}(q)(\nabla h_{n,b}(q)^T M(q)^{-1} \nabla h_{n,b}(q))^{-1} \frac{d}{dt}(\nabla h_{n,b}(q)^T) \dot{q} \\ &= P(q) \nabla h_{n,u}(q) \lambda_{n,u} \end{aligned} \quad (19)$$

$$0 \leq \lambda_{n,u} \perp \nabla h_{n,u}(q)^T \ddot{q} + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} \geq 0$$

with:

$$P(q) = I - \nabla h_{n,b}(q) A_{nb}(q)^{-1} \nabla h_{n,b}(q)^T M(q)^{-1} \quad (20)$$

This is equivalent to an inclusion that extends (15):

$$M(q)\ddot{q} + P(q)F(q, \dot{q}, t) + \nabla h_{n,b}(q) A_{nb}(q)^{-1} \frac{d}{dt}(\nabla h_{n,b}(q)^T) \dot{q} \in -P(q) N_{K(q,\dot{q})}(\ddot{q}) \quad (21)$$

Use was made of (13) to obtain (21). Three facts about $P(q) \in \mathbb{R}^{n \times n}$ are hence: $\nabla h_{n,b}(q)^T M(q)^{-1} P(q) = 0$, $M(q)^{-1} P(q) \in \mathbb{R}^{n \times n}$ is symmetric positive semi definite with rank $r = n - m_b$, and $\text{rank}(P(q)) \leq n - 1$. The first fact is obvious from (20), the second fact follows from Lemmas 2 and 3 in [11], and the third fact is a consequence of [9, Fact 3.8.6]. The first fact may be used to prove that the submanifold $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n | h_{n,b}(x) = 0, \nabla h_{n,b}(q)^T y = 0\}$ is invariant under the dynamics in (21), which means that the multiplier $\lambda_{n,b}$ satisfying (18) plays the role of a contact force that maintains the system on the bilateral constraint. We deduced uniqueness of \ddot{q} from (15). However (21) is more complex because of the presence of the singular matrix $P(q)$ which is an idempotent matrix. $P(q)$ is not symmetric, hence it is not a projector according to [9, definition 3.1.1], but it may be named a projector according to [39, §5.8] (onto $\text{Im}(P(q))$ along $\ker(P(q))$). Let us rewrite (21) more compactly as:

$$\ddot{q} + D(q, \dot{q}, t) \in -M(q)^{-1} P(q) N_{K(q,\dot{q})}(\ddot{q}) \quad (22)$$

with an obvious definition of $D(q, \dot{q}, t)$. Since $M(q)^{-1} P(q) = (M(q)^{-1} P(q))^T \succeq 0$ there exists a unitary matrix $S(q)$ such that $S(q) M(q)^{-1} P(q) S(q)^T = \begin{pmatrix} P_r(q) & 0 \\ 0 & 0 \end{pmatrix}$, with $P_r(q) = P_r(q)^T \succ 0$ of dimension $(n - m_b) \times (n - m_b)$. Let $z \triangleq S(q)\ddot{q}$, then (22) is rewritten equivalently as:

$$z + S(q)D(q, \dot{q}, t) \in - \begin{pmatrix} P_r(q) & 0 \\ 0 & 0 \end{pmatrix} S(q) N_{K(q,\dot{q})}(S(q)^T z) \quad (23)$$

The chain rule of convex analysis [64, Theorem 23.9] allows us to state that $S(q) N_{K(q,\dot{q})}(S(q)^T z) = N_{\Phi(q,\dot{q})}(z)$ with $\Phi(q, \dot{q}) = \{z \in \mathbb{R}^n | S(q)^T z \in K(q, \dot{q})\} = (S(q)^T)^{-1}(K(q, \dot{q}))$. Let $x \in \mathbb{R}^n$, then we denote x_r the vector made of its first r entries and x_{n-r} the vector made of its last $n - r$ entries. From (23) one infers that:

$$z_{n-r} = -(S(q)D(q, \dot{q}))_{n-r} \Leftrightarrow S_{n-r}(q)\ddot{q} = -S_{n-r}(q)D(q, \dot{q}, t), \quad (24)$$

where $S_{n-r}(q) \in \mathbb{R}^{(n-r) \times n}$ is the matrix made of the last $n - r$ rows of $S(q)$, while $S_r(q) \in \mathbb{R}^{r \times n}$ is the matrix made of the first r rows of $S(q)$. The following assumption is made:

Assumption 3 *The set $\Phi(q, \dot{q}) = \{z \in \mathbb{R}^n, z_{n-r} = -(S(q)D(q, \dot{q}, t))_{n-r} | S(q)^T z \in K(q, \dot{q})\}$ is non-empty.*

It is noteworthy that Assumption 3 imposes some constraint on the term $D(q, \dot{q}, t)$ defined from (22). Let us denote $\Phi(q, \dot{q})$ as $\Phi_r(q, \dot{q})$ since the free variable is z_r , i.e.,

$$\begin{aligned} \Phi_r(q, \dot{q}) &= \{z_r \in \mathbb{R}^r | (\nabla h_{n,u}(q)^T S(q)^T)_r z_r + (\nabla h_{n,u}(q)^T S(q)^T)_{n-r} (-S(q)D(q, \dot{q}, t))_{n-r} \\ &\quad + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} \geq 0\} \end{aligned} \quad (25)$$

Actually Assumption 3 states that $\Phi_r(q, \dot{q})$ is non-empty. We can rewrite equivalently $N_{\Phi(q, \dot{q})}(z)$ as $\begin{pmatrix} \partial_{z_r} \Psi_{\Phi(q, \dot{q})}(z) \\ \partial_{z_{n-r}} \Psi_{\Phi(q, \dot{q})}(z) \end{pmatrix} = \begin{pmatrix} \partial_{z_r} \Psi_{\Phi_r(q, \dot{q})}(z_r) \\ \partial_{z_{n-r}} \Psi_{\Phi(q, \dot{q})}(z) \end{pmatrix}$, where $\Psi_{\Phi_r(q, \dot{q})}(z_r) = \psi_{\mathbb{R}_{m_u}^+} \circ [(\nabla h_{n,u}(q)^T S(q)^T)_r z_r + w_r(q, \dot{q})]$, with $w_r(q, \dot{q}) \triangleq (\nabla h_{n,u}(q)^T S(q)^T)_{n-r} (-S(q)D(q, \dot{q}, t))_{n-r} + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q}$. Thus from (23) and (24) we get:

$$z_r + (S(q)D(q, \dot{q}, t))_r \in -P_r(q) N_{\Phi_r(q, \dot{q})}(z_r) \quad (26)$$

Since $P_r(q) = P_r(q)^T \succ 0$, so is its inverse and we can apply again Theorem 6 in [3] to conclude about the existence and uniqueness of a solution z_r to (26). Since $\ddot{q} = S(q)^{-1}z$ we thus proved the following.

Proposition 5 *Let $F(q, \dot{q}, t)$ be given. Suppose that Assumptions 1, 2 and 3 hold. The inclusion in (22) has a unique solution z , and the MLCP in (18) has a unique solution \ddot{q} .*

Using (18) (a) one concludes that $\nabla h_{n,b}(q)\lambda_{n,b} + \nabla h_{n,u}(q)\lambda_{n,u}$ is unique as well. Uniqueness of the multipliers holds under an additional rank assumption on the gradients. Notice that using (26) one obtains:

$$\begin{aligned} &P_r(q)^{-1}z_r + P_r(q)^{-1}(S(q)D(q, \dot{q}, t))_r \in -N_{\Phi_r(q, \dot{q})}(z_r) \\ \Leftrightarrow &z_r = \text{proj}_{P_r(q)^{-1}[\Phi_r(q, \dot{q}); (S(q)D(q, \dot{q}, t))_r]} \\ \Leftrightarrow &\text{Find } z_r \in \Phi_r(q, \dot{q}) : \langle P_r(q)^{-1}z_r + P_r(q)^{-1}(S(q)D(q, \dot{q}, t))_r, v - z_r \rangle \geq 0 \text{ for all } v \in \Phi_r(q, \dot{q}) \end{aligned} \quad (27)$$

where the last formalism is a variational inequality of the first kind. Gauss' principle applies to systems subject to unilateral and bilateral constraints (as shown in [11] through various quadratic problems). It is noteworthy that a direct interpretation of the inclusion in (22) as in (27), is not possible due to the singular matrix $P(q)$.

Existence and uniqueness of \ddot{q} via a second inclusion: As we have seen under Assumption 1, one has using (18) (c): $\nabla h_{n,u}(q)\lambda_{n,u} \in -N_{K(q,\dot{q})}(\ddot{q})$. Similarly one has $\nabla h_{n,b}(q)\lambda_{n,b} \in -N_{K_b(q,\dot{q})}(\ddot{q})$ with $K_b(q,\dot{q}) = \{z \in \mathbb{R}^n \mid \nabla h_{n,b}(q)^T z + \frac{d}{dt}(\nabla h_{n,b}(q)^T)\dot{q} = 0\}$, the assumption being here that $K_b(q,\dot{q})$ is non-empty: this inclusion enforces the bilateral constraints.

Assumption 4 *The set $K(q,\dot{q}) \cap K_b(q,\dot{q})$ is non-empty.*

Then we have the following [13] (compare with (5) (16) (27)):

Proposition 6 *Let Assumptions 1 and 4 hold. The MLCP in (17) is equivalent to:*

$$\begin{aligned} M(q)\ddot{q} + F(q,\dot{q},t) &\in -N_{K(q,\dot{q}) \cap K_b(q,\dot{q})}(\ddot{q}) \\ \Leftrightarrow \text{Find } \ddot{q} \in K(q,\dot{q}) \cap K_b(q,\dot{q}) : \langle M(q)\ddot{q} + F(q,\dot{q},t), v - \ddot{q} \rangle &\geq 0 \text{ for all } v \in K(q,\dot{q}) \cap K_b(q,\dot{q}) \\ \Leftrightarrow \ddot{q} = \operatorname{argmin}_{z \in K(q,\dot{q}) \cap K_b(q,\dot{q})} \frac{1}{2}z^T M(q)z + z^T F(q,\dot{q},t) & \end{aligned} \quad (28)$$

Notice that no assumption on the rank of $M(q)$ has been made to get (28), contrary to (18). In order to get the right-hand side of the inclusion in (28) we used [64, Theorem 23.8] about the subdifferential of the sum of convex functions, and the fact that the functions we are dealing with are polyhedral.

Corollary 1 *Let $F(q,\dot{q},t)$ be given. Suppose that $M(q) \succ 0$, and that Assumptions 1 and 4 hold. Then \ddot{q} is given by:*

$$\ddot{q} = \operatorname{proj}_{M(q)}[K(q,\dot{q}) \cap K_b(q,\dot{q}); -M(q)^{-1}F(q,\dot{q},t)] \quad (29)$$

In fact Assumption 4 implies that Assumption 1 holds. Let us notice that $K(q,\dot{q}) \cap K_b(q,\dot{q})$ is convex hence the projection is indeed uniquely defined. The inclusions in (28) and (26) are not equivalent because no assumption on the rank of $M(q)$ has been made to get (28), contrary to (26) (indeed Assumption 2 holds only if $M(q) \succ 0$). Again, one may analyze (28) with $M(q) \succeq 0$ only, relying on results in [3] and [13]. This may be particularly interesting when bilateral constraints are present. Indeed singular mass matrices are often due to the use of redundant generalized coordinates together with bilateral holonomic constraints [76, 77, 33]. We may use [3, Corollary 4] to cope with (28) with a low rank $M(q)$, but this would bring us too far away from the main topic of this paper. Some results in this direction may be found in [13].

Existence and uniqueness of \ddot{q} and $\lambda_{n,b}$ via a third inclusion: Let us investigate again how one may dispense with Assumption 2 (as well as the underlying assumption that $M(q) \succ 0$). In (28) and (26) the multiplier $\lambda_{n,b}$ does not appear explicitly because it has been eliminated in the process used to obtain these two inclusions. Let us investigate a path that yields a direct extension of the KKT system (4). Using (13) with $K(q, \dot{q})$ in (14) it is possible to rewrite (17) as follows:

$$\begin{pmatrix} M(q) & -\nabla h_{n,b}(q) \\ \nabla h_{n,b}(q)^T & 0 \end{pmatrix} \begin{pmatrix} \ddot{q} \\ \lambda_{n,b} \end{pmatrix} + \begin{pmatrix} F(q, \dot{q}, t) \\ \frac{d}{dt}(\nabla h_{n,b}(q)^T \dot{q}) \end{pmatrix} \in \begin{pmatrix} -N_{K(q, \dot{q})}(\ddot{q}) \\ 0 \end{pmatrix} \quad (30)$$

The inclusion (30) represents the merging between (4) and (15). As noticed in section 2.1, the DAE matrix $M_b(q)$ in the left-hand side of (30) is positive semi definite, non-symmetric. Let us denote the second term in the left-hand side as $F_b(q, \dot{q}, t)$. We have that $\{0\} = N_{\mathbb{R}^{m_b}}(\lambda_{n,b})$. Let us define $z \triangleq \begin{pmatrix} \ddot{q} \\ \lambda_{n,b} \end{pmatrix}$, and $f_{(q, \dot{q})}(z) = \Psi_{K(q, \dot{q})}(\ddot{q}) + \Psi_{\mathbb{R}^{m_b}}(\lambda_{n,b})$. Using Assumption 1 we may therefore rewrite equivalently (compare with (15) and (28)):

$$M_b(q)z + F_b(q, \dot{q}, t) \in -\partial f_{(q, \dot{q})}(z) \Leftrightarrow$$

$$\text{Find } z \in \mathbb{R}^{n+m_b} : \langle M_b(q)z + F_b(q, \dot{q}, t), v - z \rangle + f_{(q, \dot{q})}(v) - f_{(q, \dot{q})}(z) \geq 0, \text{ for all } v \in \mathbb{R}^{n+m_b} \quad (31)$$

It is noteworthy that despite $M_b(q) \in \mathbb{R}^{(n+m_b) \times (n+m_b)}$ is positive semi definite it may have full rank due to its skew-symmetric part. Using section E item (iii') it follows that $M_b(q)$ is invertible if and only if $\ker(M(q)) \cap \ker(\nabla h_{n,b}(q)^T) = \{0\}$ and $\text{rank}(\nabla h_{n,b}(q)^T) = m_b$. One may then conclude about the well-posedness of the system $M_b(q)z + F_b(q, \dot{q}, t) = 0$.

However direct conclusions on the well-posedness of (31) cannot be drawn even if $M_b(q)$ satisfies the conditions recalled in section E, since the well-posedness of inclusions like (31) relies on positivity-like properties (P-matrices, copositive matrices, positive definiteness), and not merely on rank assumptions. This makes a strong difference between problems involving unilateral constraints, and those with bilateral constraints only.

We note that the criteria presented in [3, 13] rely on semi-complementarity problems that use recession functions and cones. Introducing such mathematical tools is not possible here for the sake of brevity. Nevertheless the crucial subspace which plays a role in the various criteria proposed in [3] is $\ker(M_b(q) + M_b(q)^T) = \{z \in \mathbb{R}^{n+m_b} | M(q)\ddot{q} = 0\}$ (see [3, Corollary 3] that applies to non-symmetric positive semi definite matrices).

Remark 2 *The other (mechanically equivalent) option in (4) or (30) is to replace $-\nabla h_{n,b}(q)$ with $\nabla h_{n,b}(q)$. As pointed out in section 2.1, $M_b(q)$ is then symmetric, however never positive semi definite (a necessary condition for its positive semi definiteness being that $-\nabla h_{n,b}(q)^T M(q)^{-1} \nabla h_{n,b}(q)$ be positive semi definite, which is possible only if $\nabla h_{n,b}(q) = 0$). The criteria based on positiveness and its variants, cannot be used.*

The solvability Theorem 2.1 in [45] applies to (30) only if $\frac{d}{dt}(\nabla h_{n,b}(q)^T \dot{q}) \geq 0$ (componentwise non-negativity), which is a rather restrictive case that holds in practice for all

velocities, only if $\frac{d}{dt}(\nabla h_{n,b}(q)^T) = 0$ (therefore it has practical interest for the static case where $\dot{q} = 0$ only). This in turn implies that the index 1 formalism (constraints written on the acceleration level) and the index 2 formalism (constraints written on the velocity level) possess the same canonical form.

Conclusions: three inclusions have been constructed which extend the KKT system (4) and the inclusion (15), when both bilateral and unilateral constraints are present: (26), (28) and (31). They rely on different assumptions which are used to transform the MLCP in (17). There is a significant discrepancy between Assumptions 1, 3 and 4 (which are necessary to secure that the problems make sense, *i.e.* one does not consider a system with an empty admissible space), and Assumption 3. Assumption 3 (with the prerequisite that $M(q) \succ 0$) allows us to calculate $\lambda_{n,b}$ and inject its expression in the dynamics in order to obtain a projected dynamics on the bilateral constraints. On the contrary in (28) the bilateral constraints are enforced directly in the normal cone. Finally (30) uses only Assumption 1. Both (28) and (30) hold with $M(q) \succeq 0$, and possibly redundant constraints.

2.3.2 Analysis of contact LCP with a bilateral distortion

Another path is chosen in [11]. Instead of augmenting the unilateral constraints set, one uses as above the second equality in (18) to get an expression of $\lambda_{n,b}$ as a function of $\lambda_{n,u}$, and inserts it in the dynamics (the first line of (18)). Let Assumption 2 hold. Then the *constrained LCP* is obtained in the same way as (10):

$$0 \leq \lambda_{n,u} \perp A_c(q)\lambda_{n,u} + w_c(q, \dot{q}, t) \geq 0 \quad (32)$$

where the new Delassus' matrix is:

$$A_c(q) \triangleq \nabla h_{n,u}(q)^T M_c(q)^{-1} \nabla h_{n,u}(q) \quad (33)$$

with $M_c(q)^{-1} \triangleq M(q)^{-1}P(q)$, $P(q)$ is in (20). One sees that $M_c(q)^{-1}$ is the symmetric positive semi definite matrix⁴ that appears in the right-hand side of (22): it is the mass matrix with a distortion due to the bilateral holonomic constraints. Hence $A_c(q) = A_c(q)^T$ is at least positive semi definite. Finally one has:

$$w_c(q, \dot{q}, t) = \nabla h_{n,u}(q)^T M_c(q)^{-1} F_M(q, \dot{q}, t) + \nabla h_{n,u}(q)^T M^{-1}(q) \frac{d}{dt}(\nabla h_{n,b}(q)^T) \dot{q} + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} \quad (34)$$

with $F_M(q, \dot{q}, t) \triangleq -M(q)F(q, \dot{q}, t)$. Clearly the LCP in (32) (33) (34), is the extension of the LCP in (10) (11), where one replaces $M(q)^{-1}$ by $M_c(q)^{-1}$ and $w_u(q, \dot{q}, t)$ by $w_c(q, \dot{q}, t)$. A proposition quite in the same spirit as Proposition 3 may be stated, which concatenates Proposition 6, Corollary 1, and Lemma 4 in [11], as well as results from [13].

⁴This is denoted as the inverse of some matrix though there is no inverse, just to mimic the case without bilateral constraints.

Proposition 7 (Frictionless, unilateral/bilateral constraints) *Let Assumption 2 hold. Then:*

- (i) *The LCP in (32) has a unique solution for any $w_c(q, \dot{q}, t)$ if and only if $A_c(q) \succ 0$, equivalently $(\nabla h_{n,u}(q) \ \nabla h_{n,b}(q))$ is full rank.*
- (ii) *Let $\lambda_{n,u,1}$ and $\lambda_{n,u,2}$ be two solutions of the contact LCP in (32). Then $\nabla h_{n,u}(q)(\lambda_{n,u,1} - \lambda_{n,u,2}) \in \ker(M_c(q)^{-1})$, and $(\lambda_{n,u,1}^T - \lambda_{n,u,2}^T)w_c(q, \dot{q}) = 0$.*
- (iii) *Let $\text{Im}(\nabla h_{n,u}(q)) \cap \text{Im}(\nabla h_{n,b}(q)) = \{0\}$ for all $q \in \mathbb{R}^n$. Let $\lambda_{n,u,1}$ and $\lambda_{n,u,2}$ be two solutions of the LCP in (32). Then $\nabla h_{n,u}(q)(\lambda_{n,u,1} - \lambda_{n,u,2}) = 0$. If the LCP in (32) is solvable then the acceleration \ddot{q} exists and is unique.*
- (iv) *The LCP in (32) is solvable if for any $z \in \mathbb{R}^n$ in the set of solutions of the homogeneous LCP $0 \leq z \perp A_c(q)z \geq 0$ one has $z^T w_c(q, \dot{q}, t) \geq 0$; if $\frac{d}{dt}(\nabla h_{n,b}(q)) = 0$ and $\frac{d}{dt}(\nabla h_{n,u}(q)) = 0$ and $\text{Im}(\nabla h_{n,u}(q)) \cap \text{Im}(\nabla h_{n,b}(q)) = \{0\}$ for all $q \in \mathbb{R}^n$, the LCP in (32) is solvable.*
- (v) *The implication $\lambda_{n,u} \geq 0, \nabla h_{n,u}(q)\lambda_{n,u} = 0 \Rightarrow \lambda_{n,u}^T \frac{d}{dt}(\nabla h_{n,u}(q)^T)\dot{q} \geq 0$ holds if and only if the LCP in (32) is solvable.*
- (vi) *If $(\nabla h_{n,u}(q)^T M^{-1}(q) \frac{d}{dt}(\nabla h_{n,b}(q)^T) + \frac{d}{dt}(\nabla h_{n,u}(q)^T))\dot{q} \in \text{Im}(A_c(q))$, then the LCP in (32) (33) (34) is solvable.*

Let now $M(q) \succeq 0$ and the constraints $h_{n,u,i}(q)$, $1 \leq i \leq m_u$ satisfy the Mangasarian-Fromovitz constraint qualification. Then:

- (vii) *The MLCP in (17) (equivalently (28)) is solvable if $\ker(\nabla h_{n,b}(q)) \cap T_{\Phi_u}(q) \cap \ker(M(q)) = \{0\}$.*

Item (v) stems from similar arguments as item (iv) of Proposition 3, using the specific form of $w_c(q, \dot{q}, t)$ in (34), and similarly for item (vi) (these facts were not noticed in [11]). Item (iv), that is Lemma 4 in [11], therefore states sufficient conditions only as a direct consequence of Theorem 2 in section C. As in Proposition 3, both positivity and symmetry of $A_c(q)$ are used. It is noteworthy that the independence of the constraints $h_{n,u,i}(q)$ is not sufficient to assure (i) as in Proposition 3, because $M_c(q)^{-1}$ is not a full rank matrix. Conditions which guarantee that $A_c(q) > 0$ are given in [11], mainly relying on the kinetic angles values. It has been noticed in another context [38, Lemma 2] that a matrix with the same structure as $A_c(q)$, is the Schur complement of $A_{nb}(q)$ in $\begin{pmatrix} \nabla h_{n,b}(q)^T \\ \nabla h_{n,u}(q)^T \end{pmatrix} M(q)^{-1} (\nabla h_{n,b}(q) \ \nabla h_{n,u}(q)) = \begin{pmatrix} A_{nb}(q) & A_{nbnu}(q) \\ A_{nbnu}(q) & A_{nu}(q) \end{pmatrix}$. Hence $A_c(q) = A_{nu}(q) - A_{nbnu}(q)^T A_{nb}(q)^{-1} A_{nbnu}(q)$ is positive definite if and only if $\begin{pmatrix} \nabla h_{n,b}(q)^T \\ \nabla h_{n,u}(q)^T \end{pmatrix}$ has full row rank $\Leftrightarrow \ker(\nabla h_{n,b}(q) \ \nabla h_{n,u}(q)) = \{0\}$ [38, Lemma 4] (a condition easier to check than the necessary and sufficient condition of [11, Proposition 10]). Uniqueness of the acceleration in item (iii) holds since the right-hand side of the dynamic equation is equal to $P(q)\nabla h_{n,u}(q)\lambda_{n,u,1}$.

Remark 3 Consider solvability conditions in Propositions 3 and 7. Using Corollary 3.8.12 in [18], one has that solvability of the LCP($w_u(q, \dot{q}, t), A_{nu}(q)$) is equivalent to the boundedness from below of the quadratic function $f(z) = z^T w_u(q, \dot{q}, t) + z^T A_{nu}(q)z$ for all $z \geq 0$ (and the same for $w_c(q, \dot{q}, t)$ and $A_c(q)$). It is known that such $f(z)$ is bounded from below for all z (not necessarily non-negative) if and only if $w_{nu}(q, \dot{q}, t) \in \text{Im}(A_{nu}(q))$ [66, §a.5.5]. Thus the condition that $w_u(q, \dot{q}, t) \in \text{Im}(A_{nu}(q))$ may be used for solvability of the contact LCP, as a sufficient condition. In view of the structures of the terms $w_u(q, \dot{q}, t)$ and $w_c(q, \dot{q}, t)$ in (11) and (34), if $\frac{d}{dt}(\nabla h_{n,u}(q)) \in \text{Im}(A_{nu}(q))$ (resp. $[\nabla h_{n,u}(q)^T M(q)^{-1} \frac{d}{dt}(\nabla h_{n,b}(q)) + \frac{d}{dt}(\nabla h_{n,u}(q))] \dot{q} \in \text{Im}(A_c(q))$), then $w_u(q, \dot{q}, t) \in \text{Im}(A_{nu}(q))$ (resp. $w_c(q, \dot{q}, t) \in \text{Im}(A_c(q))$) if and only if $F(q, \dot{q}, t)$ (resp. $F_M(q, \dot{q}, t)$) belong to $\text{Im}(\nabla h_{n,u}(q))$. This is more restrictive than the conditions in Proposition 3 (iv) and (v) and 7 (v) and (vi).

2.3.3 Analysis of the MLCP (18)

Let us analyse the contact MLCP in (18) with relaxed Assumption 2.

Proposition 8 Let us consider (18). Suppose that $\frac{d}{dt}(\nabla h_{n,b}(q)^T) \dot{q} \in \text{Im}(\nabla h_{n,b}(q)^T)$. Then solvability of the contact LCP holds if and only if $\nabla h_{n,u}(q)^T [z + y] + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} \geq 0$ for some $z \in \mathbb{R}^n$ and $y \in \ker(\nabla h_{n,b}(q)^T)$.

Proof: From (18) (b) it follows using the Proposition's assumption that there exists $z \in \mathbb{R}^n$ and $y \in \ker(\nabla h_{n,b}(q)^T)$ such that $\nabla h_{n,b}(q) \lambda_{n,b} = -\nabla h_{n,u}(q) \lambda_{n,u} + F(q, \dot{q}, t) + M(q)z + M(q)y$. Inserting this expression in (18) (c) and using (18) (a), one finds that $\lambda_{n,u}$ has to satisfy the degenerate LCP:

$$0 \leq \lambda_{n,u} \perp \nabla h_{n,u}(q)^T [z + y] + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} \geq 0 \quad (35)$$

Clearly this complementarity problem has a solution if and only if $\nabla h_{n,u}(q)^T [z + y] + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} \geq 0$. ■

Applying Theorem 2 in section C and noting that in the degenerate case with $M = 0$ one has $\mathcal{Q}_0 = \mathcal{Q}_0^* = \mathbb{R}_+^{m_u}$, provides a sufficient condition. It is noteworthy that the Proposition's assumption is implied by Assumption 2. If (18) (b) is seen as a linear equation with unknown $\nabla h_{n,b}(q) \lambda_{n,b}$, this condition is necessary and sufficient. Lötstedt analyzes in [43, Lemma 5.2] a mixed LCP that corresponds to (30) or to (18) written at the multipliers level with the assumption that $\frac{d}{dt}(\nabla h_{n,b}(q)^T) \dot{q} = 0$ and $\frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} = 0^5$ (the MLCP in (5.2a)–(5.2c) in [43] is easily obtained from (18) with this assumption). This allows one to relax Assumption 2 to $\text{rank} \nabla h_{n,b}(q) \leq \min(n, m_b)$ and $\text{rank} \nabla h_{n,u}(q) \leq \min(n, m_u)$, which guarantee existence and uniqueness of $\nabla h_{n,b}(q) \lambda_{n,b}$ and $\nabla h_{n,u}(q) \lambda_{n,u}$ ⁶. We tend to believe that the analysis of (30) using the tools on well-posedness of variational inequalities in [3] could yield similar results in a more general setting.

⁵Or, more generally, that $\frac{d}{dt}(\nabla h_{n,b}(q)^T) \dot{q} \in \text{Im}(\nabla h_{n,b}(q)^T)$ and $\frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} \in \text{Im}(\nabla h_{n,u}(q)^T)$.

⁶Theorem 1 in [17] is wrongly used in the proof of [43, Lemma 5.2], however this does not call into question the result which is right.

Example: The rod with one bilateral and one unilateral constraint Consider the system on Figure 1(c), where the upper tip of the rod is unilaterally constrained and the lower tip is bilaterally constrained. The external forces are horizontal pulling forces f_b, f_u at each tip. One obtains the matrices involved in the dynamics: $M = \text{diag}(m, m, \frac{ml^2}{12})$, $\nabla h_{n,b}(\theta) = (0, 1, -\frac{l}{2} \cos(\theta))^T$, $\nabla h_{n,u}(\theta) = (0, -1, -\frac{l}{2} \cos(\theta))^T$, $F = (-f_b - f_u, 0, -\frac{l}{2} (f_b - f_u) \sin(\theta))^T$, $w_c(\theta, \dot{\theta}, t) = -\frac{3 \sin(\theta) \cos(\theta)}{m(3 \cos^2(\theta) + 1)} (2(f_b - f_u) + lm \cos(\theta) \dot{\theta}^2)$, and $A_c(\theta) = \frac{12 \cos^2(\theta)}{m(3 \cos^2(\theta) + 1)}$. The LCP in (32) then becomes :

$$0 \leq \lambda_{nu} \perp \frac{12 \cos^2(\theta)}{m(3 \cos^2(\theta) + 1)} \lambda_{nu} - \frac{3 \sin(\theta) \cos(\theta)}{m(3 \cos^2(\theta) + 1)} (2(f_b - f_u) + lm \cos(\theta) \dot{\theta}^2) \geq 0.$$

It has a unique solution for all $\theta \neq \frac{\pi}{2}$. If $\theta = \frac{\pi}{2}$ then the LCP becomes $0 \leq \lambda_{n,u} \perp 0 \geq 0$ so any non negative $\lambda_{n,u}$ is a solution: this configuration is indeterminate.

3 Systems with sliding Coulomb's friction

We now turn our attention to systems that are subject to Coulomb's friction. As announced above we only deal with sliding friction, hence encapsulating the 3D case as well (indeed since the system is analysed at a given time t , it is always possible to consider that one of the two tangent vectors $t_{1,i}$ or $t_{2,i}$ on Figure 2 is colinear with v_i).

3.1 The Lagrange dynamics with sliding friction

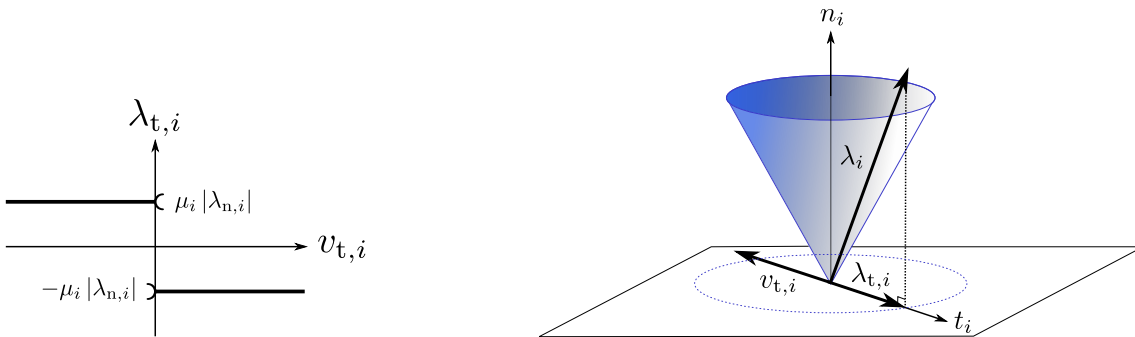


Figure 2: Sliding Coulomb friction can be expressed compactly in 2D as $\lambda_{t,i} = -\mu_i |\lambda_{n,i}| \xi_i$ where $\xi_i \in \{-1; 1\}$ is the (monovalued) sign of $v_{t,i}$. In 3D, the same law holds when expressed in the local frame (n_i, t_i) , where t_i is a tangent vector oriented along the sliding line (with arbitrary direction).

We shall assume in the sequel that any contact point lies in the sliding mode of Coulomb friction. In 2D and 3D, this allows one to build for each contact point a local kinematical frame (n_i, t_i) , where n_i is the normal at contact and t_i is oriented along the sliding line (see Figure 2 for an illustration). Using this local frame, one can define the

(scalar) tangential velocity $v_{t,i} \neq 0$ as $v_{t,i} = v_i \cdot t_i$ where v_i is the 3D velocity at contact point i , and formulate sliding Coulomb's friction as

$$\lambda_{t,i} = -\mu_i |\lambda_{n,i}| \xi_i \quad \text{where } \xi_i \in \{-1; 1\} \text{ is univoquely given by the sign of } v_{t,i}.$$

Let $v_{t,b}$ be the vector that gathers local tangential velocities for the bilaterally constrained contact points, and $v_{t,u}$ its unilateral counterpart. Then the local operators $H_{t,b}$ and $H_{t,u}$ define a local mapping between local and generalized tangential velocities, written as $v_{t,b} = H_{t,b}(q)^T \dot{q}$ and $v_{t,u} = H_{t,u}(q)^T \dot{q}$. Let $F_{t,b}$ denote the generalized force acting on the system when the normal contact force has been deleted. From the principle of virtual works it follows that $v_{t,b}^T \lambda_{t,b} = \dot{q}^T H_{t,b}(q) \lambda_{t,b} = \dot{q}^T F_{t,b}$. Hence $F_{t,b} = H_{t,b}(q) \lambda_{t,b}$, and similarly for the unilateral constraints. Then, using (1a) and (1b), one obtains

$$M(q)\ddot{q} + F(q, \dot{q}, t) = \nabla h_{n,b}(q) \lambda_{n,b} + \nabla h_{n,u}(q) \lambda_{n,u} - H_{t,b}(q) [\mu_b] [\xi] |\lambda_{n,b}| - H_{t,u}(q) [\mu_u] [\xi] \lambda_{n,u} \quad (36)$$

where $[x_b]$ denotes the diagonal matrix $\text{diag}(x_i)$ for $1 \leq i \leq m_b$ and $[x_u]$ the diagonal matrix $\text{diag}(x_i)$ for $m_b + 1 \leq i \leq m$, and where $|\lambda_{n,b}| = (|\lambda_{n,b,1}| |\lambda_{n,b,2}| \dots |\lambda_{n,b,m_b}|)^T \in \mathbb{R}^{m_b}$. Note that unilateral normal forces $\lambda_{n,u}$ do not need to be expressed in terms of absolute value since they should always remain positive due to the LCP constraint (1c). Equation (36) serves as a starting point to analyse (1) with sliding Coulomb's friction.

The Lagrange dynamics with unilateral constraints, Coulomb's friction and impacts as in (36) are very well developed in [41, Chapter 5], while their Lyapunov stability is analyzed in detail in [41, Chapter 7]. Since we are dealing with the impactless contact, (36) has to be completed with (1c) and (1d). Notice that $\lambda_{n,b}^T \nabla h_{n,b}(q)^T \dot{q} = 0$ since $\nabla h_{n,b}(q)^T \dot{q} = 0$. From Proposition 2 one has $0 \leq \lambda_{n,u,i}(t) \perp \nabla h_{n,u,i}(q(t))^T \dot{q}(t) \geq 0 \Leftrightarrow \lambda_{n,u,i}(t) \in -\partial \Psi_{\mathbb{R}_+}(\nabla h_{n,u,i}(q(t))^T \dot{q}(t))$ for all i such that $h_{n,u,i}(q(t)) = 0$. Suppose that all m_u unilateral contacts are active, then from [64, Theorem 23.9] it follows that $\nabla h_{n,u}(q) \lambda_{n,u} \in -\partial \Psi_{T_{\Phi_u}(q)}(\dot{q})$ with $T_{\Phi_u}(q) = \{z \in \mathbb{R}^n | \nabla h_{n,u}(q)^T z \geq 0\}$ ⁷. These developments may be used to prove that the contact forces (including tangential ones) dissipate energy, provided $\lambda_{n,u}$ and $\lambda_{n,b}$ exist.

It is noteworthy however that Coulomb's law cannot be written as an *associated* law⁸ [32, §4]. This means that the righ-hand side of (36) together with the complementarity conditions, cannot be written compactly as the normal cone to some convex set. De Saxcé's bipotential function [21] allows one to recover an associated form at the local kinematics level, however at the price of using a modified tangential velocity [1, §3.9.2].

3.2 Bilaterally constrained systems

In this section we analyse different cases, and we show that various results may be obtained on the contact problem's well-posedness, depending on the assumptions.

⁷The notation for this set is chosen to recall that under some constraint qualification (like Mangasarian-Fromovitz), $T_{\Phi_u}(q)$ is the tangent cone polar to the normal cone.

⁸A contact law is associated if it can be expressed as an inclusion in the subdifferential of a convex, proper function, *i.e.* it admits a convex pseudo-potential.

3.2.1 All frictional bilateral constraints

Suppose that $\mathcal{I}_b^0 = \emptyset$, i.e., $m_b^\mu = m_b$ and that all contacts are sliding: $\xi_i \triangleq \text{sgn}(v_{t,b,i}) \in \{-1, 1\}$. Using (36) the extension of the KKT system in (4) is:

$$\underbrace{\begin{pmatrix} M(q) & -\nabla h_{n,b}(q) + H_{t,b}(q)[\mu_b \xi] [\text{sgn}(\lambda_{n,b})] \\ \nabla h_{n,b}(q)^T & 0 \end{pmatrix}}_{\triangleq M_{b\mu\xi}(q)} \begin{pmatrix} \ddot{q} \\ \lambda_{n,b} \end{pmatrix} = \begin{pmatrix} -F(q, \dot{q}, t) \\ -\frac{d}{dt}(\nabla h_{n,b}(q)^T \dot{q}) \end{pmatrix} \quad (37)$$

and we may denote the matrix in the left-hand side of (37) as $M_{b\mu\xi}(q)$. Comparing with (4) one sees that friction modifies the system's matrix, not its right-hand side. Bilateral friction introduces a non-linearity or rather piecewise linearity in the variable $\lambda_{n,b}$, which prevents one from using the same linear algebra results as in section 2.1 and appendix E.

What we call a mode in item (i) of the next proposition, is a set of non-zero tangential velocities $v_{t,b,i}$ with a given sign.

Proposition 9 *Let (q, \dot{q}) , $F(q, \dot{q}, t)$ be given and suppose that all contacts are sliding ($v_{t,b,i} \neq 0$). Let $\mu_{\max} \triangleq \max_{1 \leq i \leq m_b} \mu_i$*

(a) *Consider that $M(q) \succ 0$ and $\nabla h_{n,b}(q)$ have full column rank m_b . Then if:*

$$\mu_{\max} < \mu_{\max}^b(q) \triangleq \frac{\sigma_{\min}(A_{nb}(q))}{\sigma_{\max}(A_{tb}(q))} \quad (38)$$

the matrix $M_{b\mu\xi}(q)$ in (37) is full rank $n + m_b$ ⁹. Then there exists a mode (possibly non-unique) in which \ddot{q} and $\lambda_{n,b}$ can be calculated uniquely.

(b) *Suppose that $\nabla h_{n,b}(q)$ is full column rank m_b . Then $-\nabla h_{n,b}(q) + H_{t,b}(q)[\mu_b \xi \text{sgn}(\lambda_{n,b})]$ is full rank m_b if:*

$$2\mu_{\max}\sigma_{\max}(\nabla h_{n,b}(q)^T H_{t,b}(q)) + \mu_{\max}^2 \lambda_{\max}(H_{t,b}(q)^T H_{t,b}(q)) < \sigma_{\min}(\nabla h_{n,b}(q)^T \nabla h_{n,b}(q)). \quad (39)$$

Then there exists a mode (possibly non-unique) in which \ddot{q} and $\lambda_{n,b}$ can be calculated uniquely.

Proof: (a) The proof relies on section E (vi) where \hat{A} plays the role of $M_{b\mu\xi}(q)$, F is $\nabla h_{n,b}(q)$, and P is $H_{t,b}(q)[\mu_b \xi] [\text{sgn}(\lambda_{n,b})]$. If (38) holds, the system may be solved by supposing that $[\text{sgn}(v_{t,b})] [\text{sgn}(\lambda_{n,b})] = I$, then choosing the right signums for the variables ξ_i (i.e. choosing a suitable mode). (b) The proof for (39) relies on section E (v) and on the properties of induced matrix norms [9, Corollary 9.4.9] and of singular values [9, Definition 5.6.1] to deduce the inequality in (39). Using section E (i) we know that the matrix corresponding to \tilde{A} in (71) has rank $n + m_b$, hence \ddot{q} and $-\nabla h_{n,b}(q) + H_{t,b}(q)[\mu_b \xi \text{sgn}(\lambda_{n,b})] \lambda_n$ are unique. Then (39) guarantees that λ_n is unique in a certain mode that may be found as in case (a). ■

⁹Recall that $A_{nb}(q) \triangleq \nabla h_{n,b}(q)^T M(q)^{-1} \nabla h_{n,b}(q)$ and $A_{tb}(q) \triangleq \nabla h_{n,b}(q)^T M(q)^{-1} H_{t,b}(q)$.

Notice that in case $\lambda_{n,b,i} = 0$ for some i , it is possible to assign any value to $\text{sgn}(\lambda_{n,b,i})$ since one has trivially in this case $|\lambda_{n,b,i}| = \text{sgn}(\lambda_{n,b,i})\lambda_{n,b,i}$. The possible mode at the contact point i is then free ($v_{t,b,i} > 0, < 0, = 0$): clearly uniqueness of the mode does not hold.

A direct way of analyzing the existence and uniqueness of solutions $(\ddot{q}, \lambda_{n,b})$ of (37) is to form the analog of (7),

$$A_{nb}(q)\lambda_{n,b} - A_{tb}(q)[\mu\xi] |\lambda_{n,b}| + w_b(q, \dot{q}, t) = 0, \quad (40)$$

where the absolute value is meant componentwise. By inspecting each of the 2^{m_b} possible signums of $\lambda_{n,b}$ one obtains 2^{m_b} linear systems. One may solve each such system and obtain a candidate solution $\lambda_{n,b}$. If the signums of the newly computed $\lambda_{n,b}$ coincide with those of the assumption, a solution has indeed been found, otherwise it is rejected. If no solution or several solutions are found, the system is said to be wedged or jammed. Such an exhaustive procedure would yield necessary and sufficient conditions on the parameter values for a unique solution of problem (40) to exist (see [5] in the case of a single contact). However, in practice, it becomes intractable for systems with more than a few contact points.

In the rest of this section we present two alternative ways to derive sufficient conditions for equation (37) to have a unique solution. Problem (40) can be treated as a perturbation of the frictionless problem (7) and the non-linearity introduced by bilateral friction can be treated via a fixed point argument or via complementarity theory.

Proposition 10 *Let (q, \dot{q}) , $F(q, \dot{q}, t)$ be given. Let (q, \dot{q}) , $F(q, \dot{q}, t)$ be given. Let $A_{nb}(q)$ be positive definite. Suppose that all contacts are sliding ($v_{t,b,i} \neq 0$) and that*

$$\max_{1 \leq i \leq m} \mu_i < \mu_{\max}^b(q) = \frac{\sigma_{\min}(A_{nb}(q))}{\sigma_{\max}(A_{tb}(q))}. \quad (41)$$

Then the bilateral sliding friction problem (37) has a unique solution $(\ddot{q}, \lambda_{n,b})$.

Proof: Under the rank assumptions made in the proposition, system (37) is equivalent to

$$\begin{cases} \ddot{q} = M(q)^{-1} \nabla h_{n,b}(q) \lambda_{n,b} - M(q)^{-1} H_{t,b}(q) [\mu_b] [\text{sgn}(v_{t,b})] |\lambda_{n,b}| - M(q)^{-1} F(q, \dot{q}, t) \\ A_{nb}(q) \lambda_{n,b} - A_{tb}(q) [\mu\xi] |\lambda_{n,b}| + w_b(q, \dot{q}, t) = 0 \end{cases} \quad (42)$$

Since $A_{nb}(q)$ can be inverted, $\lambda_{n,b}$ is a solution of (37) if and only if $\lambda_{n,b} = A_{nb}^{-1} A_{tb} [\mu\xi] |\lambda_{n,b}| - A_{nb}^{-1} w_b$. Let $T(x) \triangleq A_{nb}^{-1} A_{tb} [\mu\xi] |x| - A_{nb}^{-1} w_b$, we will show that the mapping $T(\cdot)$ is contracting under (41). Indeed one has:

$$\begin{aligned} \|T(x) - T(y)\|_2 &= \|A_{nb}^{-1} A_{tb} [\mu\xi] (|x| - |y|)\|_2 \leq \|A_{nb}^{-1} A_{tb} [\mu\xi]\|_2 \| |x| - |y| \|_2 \\ &\leq \|A_{nb}^{-1} A_{tb} [\mu\xi]\|_2 \|x - y\|_2 \end{aligned} \quad (43)$$

for any $x, y \in \mathbb{R}^{mb}$. By the property of induced matrix norms [9, Proposition 9.4.3], and the fact that $\|a\|_2 = \max_i |a_i|$, we have $\max_{1 \leq i \leq m} \mu_i < \frac{1}{\|A_{nb}^{-1}\|_2 \|A_{tb}\|_2}$ which implies $\|A_{nb}^{-1} A_{tb} [\mu \xi]\|_2 < 1$. Hence $T(\cdot)$ is contracting. Using the Banach fixed point theorem one concludes that a unique solution $\lambda_{n,b}$ to (42) exists. Existence and uniqueness of the acceleration in (42) follows from that of $\lambda_{n,b}$. ■

It is quite interesting to see that the same friction coefficient upperbound appears in both Proposition 10 and Proposition 9 (a), despite the fact they do not yield the same result. A result similar to Proposition 10 can be derived to guarantee existence and uniqueness of a solution to (37) by casting it into a complementarity problem.

Proposition 11 *Let (q, \dot{q}) , $F(q, \dot{q}, t)$ be given. Let $A_{nb}(q)$ be positive definite. Suppose that all contacts are sliding ($v_{t,b,i} \neq 0$) and that*

$$\max_{1 \leq i \leq m} \mu_i < \frac{1}{3} \mu_{max}^b(q)$$

where $\mu_{max}^b(q)$ is defined in (41). Then the bilateral sliding friction problem (37) has a unique solution $(\ddot{q}, \lambda_{n,b})$.

Proof: The proof consists of three steps: reformulating (40) as a horizontal LCP (hLCP¹⁰), casting the hLCP matrix as a perturbation of a positive definite matrix and applying Corollary 3 in section D. One can write $[\text{sgn}(\lambda_{n,b})] \lambda_{n,b} = |\lambda_{n,b}|$. Following [35] let us introduce $\lambda^+ \triangleq \frac{|\lambda_{n,b}| + \lambda_{n,b}}{2}$ and $\lambda^- \triangleq \frac{|\lambda_{n,b}| - \lambda_{n,b}}{2}$. Solving (40) boils down to find λ^+ and λ^- such that

$$\begin{cases} (A_{nb}(q) - A_{tb}(q)[\mu_b \xi]) \lambda^+ - (A_{nb}(q) + A_{tb}(q)[\mu_b \xi]) \lambda^- + w_b(q, \dot{q}, t) = 0 \\ 0 \leq \lambda^+ \perp \lambda^- \geq 0 \end{cases} \quad (44)$$

As pointed out in [35], a solution to the hLCP (44) exists and is unique if and only if $A_{nb} - A_{tb}[\mu_b \xi]$ is invertible and $(A_{nb} - A_{tb}[\mu_b \xi])^{-1}(A_{nb} + A_{tb}[\mu_b \xi])$ is a P-matrix. The upperbound on the friction coefficient assures $\|A_{nb}^{-1} A_{tb}[\mu_b \xi]\|_2 \leq \|A_{nb}^{-1} A_{tb}\|_2 \|\mu_b\|_2 \leq \frac{1}{3} < 1$, securing that $I - A_{nb}^{-1} A_{tb}[\mu_b \xi]$ is non-singular and that its inverse can be written using Taylor series expansion. The hLCP matrix, whose positive definiteness we want to enforce, can be rewritten as $(A_{nb} - A_{tb}[\mu_b \xi])^{-1}(A_{nb} + A_{tb}[\mu_b \xi]) = (I - A_{nb}^{-1} A_{tb}[\mu_b \xi])^{-1} A_{nb}^{-1} A_{nb} (I + A_{nb}^{-1} A_{tb}[\mu_b \xi]) = (\sum_{k=0}^{+\infty} (A_{nb}^{-1} A_{tb}[\mu_b \xi])^k) (I + A_{nb}^{-1} A_{tb}[\mu_b \xi]) = I + 2 \underbrace{\sum_{k=1}^{+\infty} (A_{nb}^{-1} A_{tb}[\mu_b \xi])^k}_{\triangleq K_\mu(q, \dot{q})}$. From

Corollary 3 in section D, it follows that if $\|K_\mu(q, \dot{q})\|_2 < 1$ then $I + K_\mu(q, \dot{q}) \succ 0$ is a P-matrix. The upperbound $\frac{1}{3} \mu_{max}^b(q)$ is chosen specifically so that this condition holds: $\|K_\mu(q, \dot{q})\| \leq 2 \sum_{k=1}^{+\infty} \|A_{nb}^{-1} A_{tb}\|_2^k \|\mu_b\|_2^k < 2 \sum_{k=1}^{+\infty} \|A_{nb}^{-1} A_{tb}\|_2^k \frac{1}{3^k \|A_{nb}^{-1} A_{tb}\|_2^k} < 1$. ■

¹⁰See [1, Definition 12.23].

3.2.2 Examples

Let us present several examples which illustrate the foregoing developments.

Bilaterally constrained rod with friction We now add Coulomb friction to the mechanical system in Figure 1(a) and point out how the equations of motion are affected. The system under consideration is a rigid pendulum with a frictional sliding base. Its dynamics is given by

$$\begin{cases} m\ddot{x} &= -\mu\xi |\lambda_{nb}| \\ m\ddot{y} &= -mg + \lambda_{nb} \\ \frac{ml^2}{12}\ddot{\theta} &= -\frac{l}{2}\cos\theta\lambda_n - l/2\sin\theta\mu\xi |\lambda_{nb}| \\ h_{n,b}(\theta) &= y - \frac{l}{2}\sin(\theta) = 0 \end{cases}$$

With respect to the frictionless case, the additional terms are $H_{t,b}(\theta) = (1, 0, \frac{l}{2}\sin(\theta))^T$ and $A_{tb}(\theta) = \nabla h_{n,b}(\theta)^T M^{-1} H_{t,b}(\theta) = \frac{-3\sin\theta\cos\theta}{m}$. Thus, equation (40) boils down to the scalar equation $A_{nb}(q)\lambda_{n,b} - A_{tb}(q)\mu\xi |\lambda_{n,b}| + w_b(q, \dot{q}) = 0$. The sufficient condition of Proposition 10 turns out to be also necessary and the critical value for the friction coefficient is given by $\mu_b^{max}(\theta) = \frac{1+3\cos^2\theta}{3\sin\theta\cos\theta}$. At a given configuration q if $\mu > \mu_b^{max}(\theta)$ then no meaningful contact force $\lambda_{n,b}$ can be computed. Note that the positive definiteness of the Delassus operator $A_{nb}(q)$ is no longer sufficient (as in the frictionless case) to guarantee existence and uniqueness of a solution. One can graphically inspect the solution set of this piecewise linear scalar equation by plotting $\ddot{h}_{n,b}$ (the normal acceleration of the contact point) versus $\lambda_{n,b}$. Solutions lie on the intersection of $\ddot{h}_{n,b} = 0$ and $\ddot{h}_{n,b} = A_{nb}\lambda_{n,b} - A_{tb}\mu\xi |\lambda_{n,b}| + w_b$. As Figure 3(a) shows, when the friction coefficient is larger than the bound, the two curves either don't intersect (when $w_b < 0$) or intersect twice (when $w_b > 0$). Whereas if friction is low enough then a unique point of intersection will occur (regardless of w_b). On Figure 3(b) the same idea is illustrated but from the fixed point equation $\lambda_{n,b} = A_{nb}(q)^{-1}A_{tb}(q)\mu\xi |\lambda_{n,b}| - A_{nb}(q)^{-1}w_b(q, \dot{q})$ point of view. One sees clearly that if the slope $|A_{nb}^{-1}A_{tb}\mu|$ is smaller than one then a unique solution exists and otherwise no solution (inconsistency) or two solutions (indeterminacy) may exist.

A bilaterally constrained system with two frictional contacts We consider now the mechanical system on Figure 1(d), with both guides frictional. We have again an all frictional bilateral problem for which we can use Proposition 10. We will show that in this case the condition for existence and uniqueness on the upper bound for friction is only sufficient and we will illustrate how conservative it is. The dynamics of the system is given by

$$\begin{cases} m\ddot{x} + F_1 + F_2 &= -\mu_1\xi_1 |\lambda_1| - \mu_2\xi_2 |\lambda_2| \\ m\ddot{y} &= \lambda_1 + \lambda_2 \\ \frac{ml^2}{12}\ddot{\theta} + \frac{l}{2}\sin\theta(F_1 - F_2) &= \frac{l}{2}\cos\theta(\lambda_2 - \lambda_1) - \frac{l}{2}\sin\theta\mu_1\xi_1 |\lambda_1| + \frac{l}{2}\sin\theta\mu_2\xi_2 |\lambda_2| \\ h_{n,b}^1(\theta) &= y - \frac{l}{2}\sin(\theta) \\ h_{n,b}^2(\theta) &= y + \frac{l}{2}\sin(\theta) - D \end{cases}$$

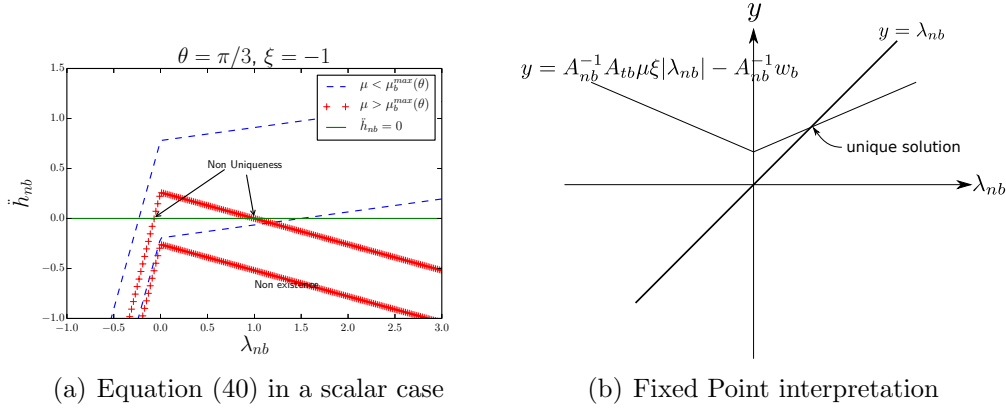


Figure 3: Bilateral sliding friction in the scalar case.

The matrices involved in the analysis of the system are: $M = \text{diag}(m, m, \frac{ml^2}{12})$, $\nabla h_{n,b}(\theta) =$

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ -\frac{l}{2} \cos(\theta) & \frac{l}{2} \cos(\theta) \end{bmatrix}, A_{nb}(\theta) = \begin{bmatrix} \frac{3}{m} \cos^2(\theta) + \frac{1}{m} & -\frac{3}{m} \cos^2(\theta) + \frac{1}{m} \\ -\frac{3}{m} \cos^2(\theta) + \frac{1}{m} & \frac{3}{m} \cos^2(\theta) + \frac{1}{m} \end{bmatrix},$$

$$H_{t,b}(\theta) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ \frac{l}{2} \sin(\theta) & -\frac{l}{2} \sin(\theta) \end{bmatrix}, A_{tb}(\theta) = \begin{bmatrix} -\frac{3}{m} \sin(\theta) \cos(\theta) & \frac{3}{m} \sin(\theta) \cos(\theta) \\ \frac{3}{m} \sin(\theta) \cos(\theta) & -\frac{3}{m} \sin(\theta) \cos(\theta) \end{bmatrix}. \text{ It}$$

has been long known [56] that the contact LCP of this system has a unique solution for the multipliers, if and only if $|\mu_1 - \mu_2| < \frac{2}{\tan \theta}$. Proposition 10 states the sufficient condition that if $\max(\mu_1, \mu_2) < \frac{\sigma_{\min}(A_{nb}(\theta))}{\sigma_{\max}(A_{tb}(\theta))}$, then the multiplier vector exists and is unique. Explicitly computing the singular values yields

$$\max(\mu_1, \mu_2) < \mu_{\max}^b(\theta) \triangleq \frac{2}{3\sqrt{-\cos(4\theta) + 1}} \sqrt{-(3\cos^2(\theta) + 1)|3\cos^2(\theta) - 1| + 9\cos^4(\theta) + 1}.$$

It can be shown that $\mu_{\max}^b(\theta) < \frac{2}{\tan \theta}$ for all $\theta \in]0; \pi/2[$. Thus there is existence and uniqueness of a solution for friction values between the two bounds. The condition stated in Proposition 10 is conservative as depicted in Figure 4(a). On Figure 4(b) $\mu_b^{\max}(\theta)$ is plotted as a function of θ . If the system is at a certain state θ and the friction at each guide is less than the value $\mu_b^{\max}(\theta)$ on the curve, then the contact force exists and is unique.

Flexible Painlevé-Klein system In this example we consider two rods of same mass and length coupled by a linear spring, constrained to be aligned and subject to the same two bilateral, frictional constraints as in the previous example, see Figure 1(e). The additional degree of freedom is d (the distance between the two rods), and $q = (x, y, d, \theta)^T$. To see how this change in the kinematics affects the friction bound of Proposition 10, $\mu_{\max}^b(q)$ is numerically computed for different configurations, see Figure 5. The matrices which are needed to compute the friction bound are:

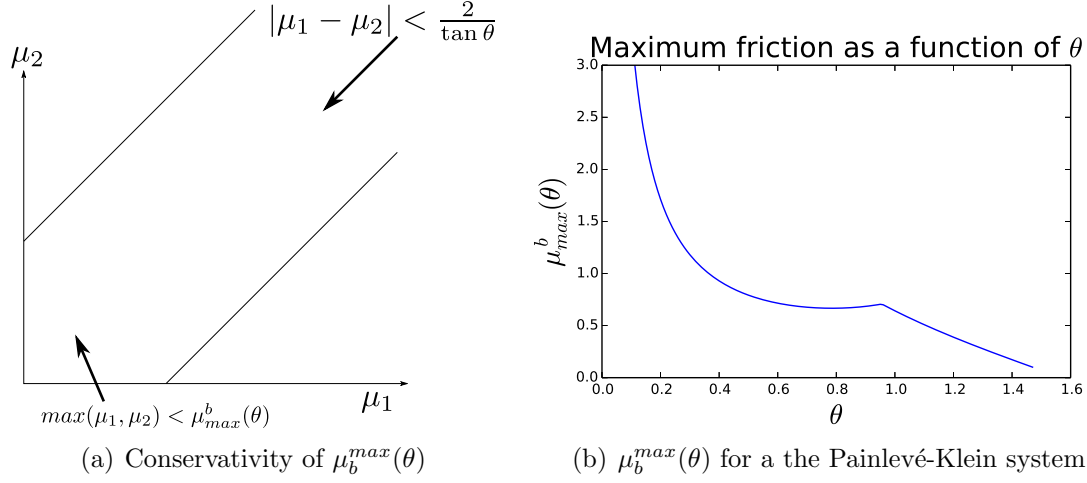


Figure 4: Friction bound for the Painlevé-Klein system

$$M(q) = \begin{bmatrix} m & 0 & \frac{m}{2} \cos(\theta) & -\frac{m}{2} (d+l) \sin(\theta) \\ 0 & m & \frac{m}{2} \sin(\theta) & \frac{m}{2} (d+l) \cos(\theta) \\ \frac{m}{2} \cos(\theta) & \frac{m}{2} \sin(\theta) & \frac{m}{2} & 0 \\ -\frac{m}{2} (d+l) \sin(\theta) & \frac{m}{2} (d+l) \cos(\theta) & 0 & \frac{m}{96} (48d^2 + 72dl + 32l^2) \end{bmatrix},$$

$$\nabla h_{n,b}(q) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & \sin(\theta) \\ 0 & (l+d) \cos(\theta) \end{bmatrix} \quad \text{and} \quad H_{t,b}(q) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & \cos(\theta) \\ 0 & -(l+d) \sin(\theta) \end{bmatrix}.$$

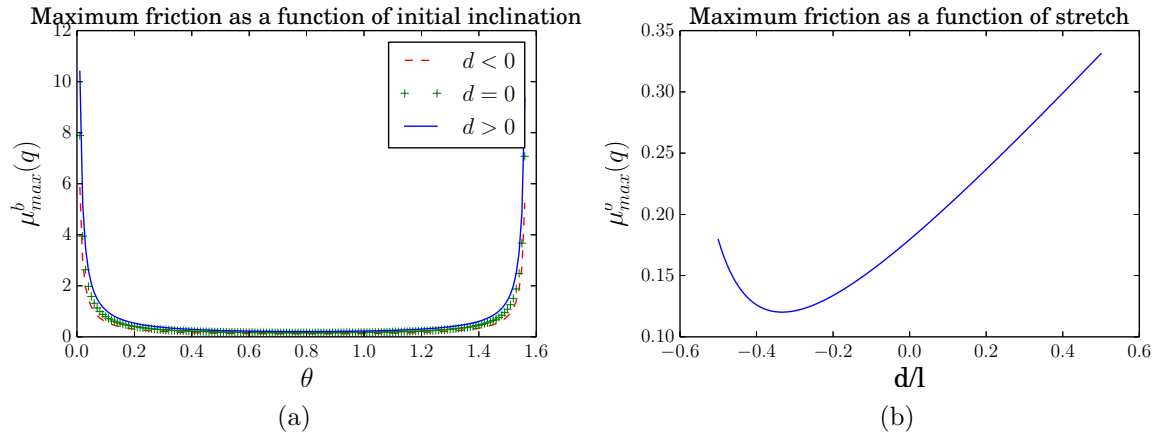


Figure 5: (a) $d = -1.2l, l, 1.2l$, (b) $\theta = \pi/3$

Figure 5 illustrates the fact that a change in θ or d influence the friction bound's conservativeness. One can also study the system in reduced coordinates and obtain a closed form expression for the necessary and sufficient conditions of existence and uniqueness of $\lambda_{n,1}$. The multiplier $\lambda_{n,1}$ exists and is unique if and only if $\mu_1 < \frac{4-3\cos^2\theta}{3\sin\theta\cos\theta}$. Therefore, just

as in the classic Painlevé example, no singularities occur when the friction coefficient is under $4/3$ [27].

Articulated Painlevé-Klein system We consider an articulated Painlevé-Klein mechanism where both ends are subject to bilateral constraints with Coulomb friction. The rigid rod is replaced by two rods of same size and mass, linked together by a rotational joint, as in Figure 1(f). The friction bound of Proposition 10 is computed to understand how the coupling of the constraints plays a role on the criterion's conservativeness. Here $q = (x_1, y_1, x_2, y_2, \theta_1, \theta_2)^T$. The matrices which are needed to compute the friction coefficient upper bound are

$$M(q) = \begin{pmatrix} m & 0 & -\frac{3l}{8}m \sin(\theta_1) & -\frac{lm}{8} \sin(\theta_2) \\ 0 & m & \frac{3l}{8}m \cos(\theta_1) & \frac{lm}{8} \cos(\theta_2) \\ -\frac{3l}{8}m \sin(\theta_1) & \frac{3l}{8}m \cos(\theta_1) & \frac{l^2m}{6} & \frac{l^2m}{16} \cos(\theta_1 - \theta_2) \\ -\frac{lm}{8} \sin(\theta_2) & \frac{lm}{8} \cos(\theta_2) & \frac{l^2m}{16} \cos(\theta_1 - \theta_2) & \frac{l^2m}{24} \end{pmatrix},$$

$$\nabla h_{n,b}(q) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & \frac{l}{2} \cos \theta_1 \\ 0 & \frac{l}{2} \cos \theta_2 \end{pmatrix} \text{ and } H_{t,b}(q) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & -\frac{l}{2} \sin \theta_1 \\ 0 & -\frac{l}{2} \sin \theta_2 \end{pmatrix}. \text{ We numerically com-}$$

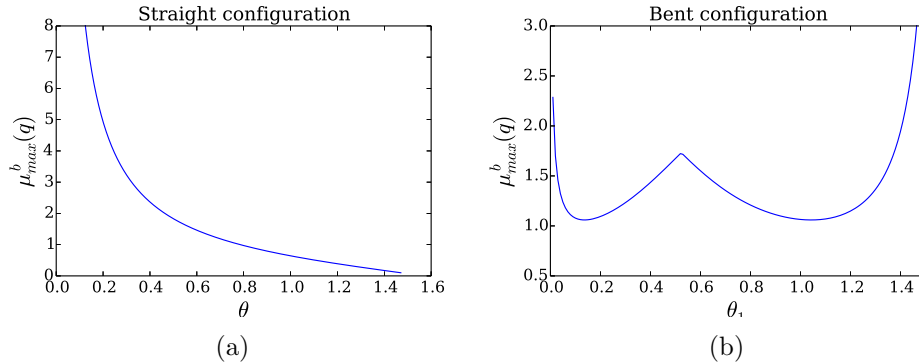


Figure 6: Evolution of $\mu_b^{max}(q)$ for (a) $q = (0, 0, \theta, \theta)$, (b) $q = (0, 0, \theta_1, \theta_2 = \arcsin(2D/l - \sin \theta_1))$

puted $\mu_{max}^b(q)$ for a variety of cases. Figure 6(a) depicts how $\mu_{max}^b(q)$ changes for a set of configurations corresponding to the case where the rod is perfectly straight, keeping every other parameter fixed. We observe that as the system approaches a vertical configuration the normal couplings become stronger and the admissible set of friction values decreases. Figure 6(b) depicts how $\mu_{max}^b(q)$ changes for initial configurations corresponding to a bent rod. The major discrepancy between this system and the previous one, compared to the other Painlevé-Klein systems with fixed θ , is that the configuration may evolve during the motion. This is also the case of the sliding rod system, and it may create additional dynamical features [27].

3.2.3 Mixed frictional/frictionless contacts

Let us now assume that $\mathcal{I}_b^0 \neq \emptyset$ and $\mathcal{I}_b^\mu \neq \emptyset$: some of the bilateral contacts are frictionless, the others are frictional. We rewrite (36) in its index 1 form:

$$\begin{cases} M(q)\ddot{q} + F(q, \dot{q}, t) = \nabla h_{n,b}^0(q)\lambda_{n,b}^0 + \nabla h_{n,b}^\mu(q)\lambda_{n,b}^\mu + H_{t,b}^\mu(q)[\mu_b][\xi][\text{sgn}(\lambda_{n,b}^\mu)] \lambda_{n,b}^\mu \\ \nabla h_{n,b}^0(q)^T \ddot{q} + \frac{d}{dt}(\nabla h_{n,b}^0(q)^T) \dot{q} = 0, \quad \nabla h_{n,b}^\mu(q)^T \ddot{q} + \frac{d}{dt}(\nabla h_{n,b}^\mu(q)^T) \dot{q} = 0 \end{cases} \quad (45)$$

In a matrix form one gets the extension of (37):

$$\begin{aligned} & \begin{pmatrix} M(q) & -\nabla h_{n,b}^0(q) & -\nabla h_{n,b}^\mu(q) + H_{t,b}^\mu(q)[\mu_b][\xi][\text{sgn}(\lambda_{n,b}^\mu)] \\ \nabla h_{n,b}^0(q)^T & 0 & 0 \\ \nabla h_{n,b}^\mu(q)^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{q} \\ \lambda_{n,b}^0 \\ \lambda_{n,b}^\mu \end{pmatrix} \\ & = \begin{pmatrix} -F(q, \dot{q}, t) \\ -\frac{d}{dt}(\nabla h_{n,b}^0(q)^T) \dot{q} \\ -\frac{d}{dt}(\nabla h_{n,b}^\mu(q)^T) \dot{q} \end{pmatrix} \end{aligned} \quad (46)$$

Due to its similarity with (37), system (46) may be analyzed along section E items (v) and (vi), and Proposition 9 may be extended accordingly. Similarly to the all frictional case, assuming $M(q) \succ 0$ and that the non-frictional constraints are independent, one obtains a reduced system where the only unknown is $\lambda_{n,b}^\mu$. Under these assumptions system (45) is equivalent to (the argument q is dropped):

$$\begin{cases} \ddot{q} = -M^{-1}F(q, \dot{q}, t) + M^{-1}\nabla h_{n,b}^0\lambda_{n,b}^0 + M^{-1}\nabla h_{n,b}^\mu\lambda_{n,b}^\mu - M^{-1}H_{t,b}^\mu[\mu_b\xi] |\lambda_{n,b}^\mu| \\ \lambda_{n,b}^0 = -(A_{nb}^{00})^{-1}A_{nb}^{0\mu}\lambda_{nb}^\mu + (A_{nb}^{00})^{-1}A_{tb}^{0\mu}[\mu_b\xi] |\lambda_{n,b}^\mu| + (A_{nb}^{00})^{-1}((\nabla h_{n,b}^0)^T M^{-1}F - \frac{d}{dt}((\nabla h_{n,b}^0)^T)\dot{q}) \\ \left(A_{nb}^{\mu\mu} - (A_{nb}^{0\mu})^T (A_{nb}^{00})^{-1} A_{nb}^{0\mu} \right) \lambda_{n,b}^\mu - \left(A_{tb}^{\mu\mu} - (A_{nb}^{0\mu})^T (A_{nb}^{00})^{-1} A_{tb}^{0\mu} \right) [\mu_b\xi] |\lambda_{n,b}^\mu| + \tilde{w}_b = 0, \end{cases} \quad (47)$$

where $A_{nb}^{00}(q) = \nabla h_{n,b}^0(q)^T M(q)^{-1} \nabla h_{n,b}^0(q)$ is the matrix of normal/normal frictionless couplings, $A_{nb}^{0\mu}(q) = \nabla h_{n,b}^0(q)^T M(q)^{-1} \nabla h_{n,b}^\mu(q)$ is the matrix of normal frictionless/normal frictional couplings, $A_{tb}^{0\mu}(q) = \nabla h_{n,b}^0(q)^T M(q)^{-1} H_{t,b}^\mu(q)$ is the matrix of normal frictionless/tangential frictional couplings, $A_{nb}^{\mu\mu}(q) = \nabla h_{n,b}^\mu(q)^T M(q)^{-1} H_{t,b}^\mu(q)$ is the matrix of normal frictional/tangential frictional couplings, and finally $\tilde{w}_b(q, \dot{q}, t) \triangleq A_{nb}^{0\mu}(A_{nb}^{00})^{-1}((\nabla h_{n,b}^0)^T M^{-1}F - \frac{d}{dt}((\nabla h_{n,b}^0)^T)\dot{q}) - (\nabla h_{n,b}^\mu)^T M^{-1}F - \frac{d}{dt}((\nabla h_{n,b}^\mu)^T)\dot{q}$ is the vector collecting all remaining terms in the last equation. Hence the problem is reduced to studying essentially the same equation as (40) in the all frictional case (which was itself the extension of (7)):

$$\tilde{A}_{nb}(q)\lambda_{n,b}^\mu - \tilde{A}_{tb}(q)[\mu_b\xi] |\lambda_{n,b}^\mu| + \tilde{w}_b(q, \dot{q}, t) = 0, \quad (48)$$

where $\tilde{A}_{nb} \triangleq A_{nb}^{\mu\mu} - (A_{nb}^{0\mu})^T (A_{nb}^{00})^{-1} A_{nb}^{0\mu}$ and $\tilde{A}_{tb} \triangleq A_{tb}^{\mu\mu} - (A_{nb}^{0\mu})^T (A_{nb}^{00})^{-1} A_{tb}^{0\mu}$. It is important to observe that $\tilde{A}_{nb}(q)$ is nothing but the Schur complement of $A_{nb}^{\mu\mu}(q)$ in the

matrix $(\nabla h_{n,b}^0 \ \nabla h_{n,b}^\mu)^T M^{-1} (\nabla h_{n,b}^0 \ \nabla h_{n,b}^\mu)$. So $\tilde{A}_{nb}(q) \succ 0$ and $A_{nb}^{00}(q) \succ 0$ if and only if $(\nabla h_{n,b}^0 \ \nabla h_{n,b}^\mu)^T M^{-1} (\nabla h_{n,b}^0 \ \nabla h_{n,b}^\mu) \succ 0$. Hence, if $(\nabla h_{n,b}^0(q) \ \nabla h_{n,b}^\mu(q))$ is full rank then $\tilde{A}_{nb}(q)$ is invertible. We deduce the following result.

Proposition 12 *Let $(q, \dot{q}), F(q, \dot{q}), t$ be given and $v_{t,b,i} \neq 0$ for all $i \in \mathcal{I}_b^\mu$. Assume that the matrix*

$$(\nabla h_{n,b}^0(q) \ \nabla h_{n,b}^\mu(q))^T M(q)^{-1} (\nabla h_{n,b}^0(q) \ \nabla h_{n,b}^\mu(q)) \quad (49)$$

is positive definite. If furthermore $\max_{1 \leq i \leq m} \mu_i < \frac{\sigma_{\min}(\tilde{A}_{nb}(q))}{\sigma_{\max}(\tilde{A}_{tb}(q))}$, then the solution $(\ddot{q}, \lambda_{n,b}^0, \lambda_{n,b}^\mu)$ of system (45) exists and is unique.

The proof is almost the same as that of Proposition 10: existence and uniqueness are obtained by a fixed point argument, and the bound on the matrix norm guarantees that an affine operator is contracting. The other hypothesis is there to ensure that $\tilde{A}_{nb}(q)$ is invertible.

Example: The Painlevé-Klein system Let us we study the classical example of Painlevé-Klein within our framework of frictional/frictionless bilateral constraints. The system consists of a rigid rod with two prismatic joints as in Figure 1(d). One of the guides is rough (it is acted on by Coulomb friction), the other is smooth. The external forces are horizontal pulling forces F_1 and F_2 . The dynamics is given by:

$$\begin{cases} m\ddot{x} + F_1 + F_2 & = -\mu_1 \xi |\lambda_{nb}^\mu| \\ m\ddot{y} & = \lambda_{n,b}^\mu + \lambda_{n,b}^0 \\ \frac{m l^2}{12} \ddot{\theta} + \frac{l}{2} \sin \theta (F_1 - F_2) & = \frac{l}{2} \cos \theta (\lambda_{n,b}^0 - \lambda_{n,b}^\mu) - \frac{l}{2} \sin \theta \mu_1 \xi |\lambda_{n,b}^\mu| \\ h_{n,b}^\mu(\theta) & = y - \frac{l}{2} \sin(\theta) \\ h_{n,b}^0(\theta) & = y + \frac{l}{2} \sin(\theta) - D \end{cases}$$

The matrices involved in the analysis of the system are: $M = \text{diag}(m, m, \frac{ml^2}{12})$, $\nabla h_{n,b}^\mu(\theta) = (0, 1, -\frac{l}{2} \cos(\theta))^T$, $\nabla h_{n,b}^0(\theta) = (0, 1, \frac{l}{2} \cos(\theta))^T$, $H_{t,b}(\theta) = (1, 0, \frac{l}{2} \sin(\theta))^T$, $\tilde{A}_{n,b}(\theta) = \frac{12 \cos^2(\theta)}{m(3 \cos^2(\theta)+1)}$, $\tilde{A}_{tb}(\theta) = \left[\frac{6 \sin(2\theta)}{m(3 \cos(2\theta)+5)} \right]$. Equation (48) boils down to a scalar equation of the form $\tilde{A}_{nb}(\theta) \lambda_{n,b}^\mu - \tilde{A}_{tb}(\theta) \mu_1 \xi |\lambda_{n,b}^\mu| + \tilde{w}_b(q, \dot{q}, t) = 0$. From Proposition 12 it follows that if $\mu_1 < \frac{2}{\tan \theta}$ then $(\ddot{x}, \ddot{y}, \ddot{\theta}, \lambda_{n,b}^\mu, \lambda_{n,b}^0)$ exists and is unique. The condition turns out to be not only sufficient but also necessary.

3.3 Unilaterally constrained systems

We now assume that $m_b = 0$ and $m_u \geq 1$. The unilateral sliding friction problem then consists in finding the accelerations and contact forces $(\ddot{q}, \lambda_{n,u})$ solutions of the mLCP:

$$\begin{cases} M(q)\ddot{q} + F(q, \dot{q}, t) = (\nabla h_{n,u}(q) - H_{t,u}(q)[\mu] [\xi])\lambda_{n,u} \\ 0 \leq \nabla h_{n,u}(q)^T \ddot{q} + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} \perp \lambda_{n,u} \geq 0. \end{cases} \quad (50)$$

3.3.1 Augmented contact LCP

If $M(q) \succ 0$ then the problem is reduced to an LCP. Let us introduce the matrix:

$$A_{u,\mu,\xi}(q) \triangleq \nabla h_{n,u}(q)^T M(q)^{-1} (\nabla h_{n,u}(q) - H_{t,u}(q)[\mu_u] [\xi]) \quad (51)$$

$$= A_{nu}(q) - A_{tu}(q)[\mu_u][\xi] \quad (52)$$

with $\xi_i = \text{sgn}(v_{t,u,i}), v_{t,u,i} \neq 0$ for $1 \leq i \leq m_u$. Then the frictional contact LCP, which extends (10), is given by:

$$0 \leq \lambda_{n,u} \perp A_{u,\mu,\xi}(q)\lambda_{n,u} + w_u(q, \dot{q}, t) \geq 0 \quad (53)$$

with $w_u(q, \dot{q}, t)$ in (11).

Proposition 13 *Let (q, \dot{q}) , $F(q, \dot{q}, t)$ be given and $v_{t,u,i} \neq 0$ for $1 \leq i \leq m_u$. Let $A_{nu}(q)$ be positive definite and suppose that:*

$$\max_{1 \leq i \leq m} \mu_i < \mu_{\max}^u(q) \triangleq \frac{\sigma_{\min}(A_{nu}(q))}{\sigma_{\max}(A_{tu}(q))} \quad (54)$$

Then the matrix $A_{u,\mu,\xi}(q)$ is positive definite, and the contact LCP (53) always has a unique solution $\lambda_{n,u}(q, \dot{q}, t, \mu)$ that is a Lipschitz continuous function of $w_u(q, \dot{q}, t)$.

The last result follows from basic complementarity theory [18]. The fact that (54) guarantees that $A_{u,\mu,\xi}(q) \succ 0$, may be proved using Corollary 3 in section D. Proposition 13 thus states that whatever the status of the tangential contacts may be (provided they are sliding) then $\lambda_{n,u}$ can always be calculated uniquely. As alluded to above, extending Propositions 3 and 7 requires some care, because even if the LCP matrix may still be positive semi definite when sliding friction is involved (because the ‘‘disturbance’’ due to friction is small) its symmetry is usually lost. Symmetry is an important property (see Theorem 1 (d) in section B). When $A_{u,\mu,\xi}(q) \succeq 0$ and is not symmetric, then using [18, Theorem 3.1.7] various properties of the solutions may be derived (however Proposition 3 (ii) cannot be stated due to the lack of symmetry): if $\lambda_{n,u}^1$ and $\lambda_{n,u}^2$ are two solutions of the LCP (53), then $\lambda_{n,u}^{1,T}(A_{u,\mu,\xi}(q) - A_{u,\mu,\xi}(q)^T)\lambda_{n,u}^2 = (\lambda_{n,u}^2 - \lambda_{n,u}^1)w_u(q, \dot{q}, t)$. Also solvability implies that the set of solutions is polyhedral. Using a characterization of copositive matrices [66, §4.41], we may state the following, which may be seen as the extension of Proposition 3 (iii) (iv).

Corollary 2 *Suppose that there exists $[\xi]$ such that the matrix $A_{tu}(q)[\mu_u][\xi]$ is elementwise non-positive. Then provided that $(0 \leq \lambda_{n,u} \perp A_{u,\mu,\xi}(q)\lambda_{n,u} \geq 0 \Rightarrow \lambda_{n,u}^T w_u(q, \dot{q}, t) \geq 0)$, the LCP in (53) is solvable.*

Indeed from [66, §4.41], the sum of a symmetric positive semi definite together with an elementwise non-negative matrices, is copositive. Hence under the stated conditions $A_{u,\mu,\xi}(q)$ is copositive. Then apply Theorem 2 in section C. Thus Corollary 2 boils down to testing whether tangential velocities signs exist such that the perturbation matrix has

all its entries non-negative. As simple examples like the classical Painlevé sliding rod show [27], this may be the case. Obviously reversing the tangential velocities signs may destroy the copositivity of $A_{u,\mu,\xi}(q)$, and may induce singularities like Painlevé paradoxes. It is noteworthy that direct extension of item (v) in Proposition 3, is generically not possible because $A_{u,\mu,\xi}(q)$ is non-symmetric. Thus the characterization of the solutions of the homogenous LCP is not straightforward.

Notice that in a sticking regime where $v_t = 0$, the LCP in (53) is not sufficient to calculate $\lambda_{n,u}$ uniquely, because the selections of the set-valued term $\text{sgn}(v_t) = \text{sgn}(0) = [-1, 1]$ are still to be determined. One has to resort to Coulomb's law at the acceleration level [30, 58] to determine which type of transition the system undergoes.

3.3.2 Inclusions and Gauss' principle

It is of interest to investigate whether or not some minimization problem for the acceleration may be obtained in the sliding frictional case. The starting point is the MLCP in (50), which is transformed using (52) into an inclusion that generalizes (15). The question is then whether or not an extension of (16) exists or not. Let $\ddot{\eta} \triangleq \nabla h_{n,u}(q)^T \ddot{q} + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q}$, and $M(q) \succ 0$. Using similar tools from convex analysis as in the foregoing sections (see (69)), one obtains:

$$\underbrace{\ddot{\eta} - \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} + \nabla h_{n,u}(q)^T M(q)^{-1} F(q, \dot{q}, t)}_{\triangleq \bar{F}_{u,\mu,\xi}(q, \dot{q}, t)} \in -A_{u,\mu,\xi}(q) \partial \Psi_{\mathbb{R}_+^{m_u}}(\ddot{\eta}) \quad (55)$$

Let the conditions of Proposition 13 be satisfied. Then one has $A_{u,\mu,\xi}(q) \partial \Psi_{\mathbb{R}_+^{m_u}}(\ddot{\eta}) = A_{u,\mu,\xi}(q) \partial \Psi_{\mathbb{R}_+^{m_u}}(A_{u,\mu,\xi}(q)^T A_{u,\mu,\xi}(q)^{-T} \ddot{\eta}) = N_{\mathcal{E}(q)}(\ddot{\zeta})$, with $\ddot{\zeta} = A_{u,\mu,\xi}(q)^{-T} \ddot{\eta}$, $\mathcal{E}(q) = \{z \in \mathbb{R}^{m_u} \mid A_{u,\mu,\xi}(q)^T z \geq 0\}$ is non-empty, closed convex (polyhedral) set, and use was made of the chain rule of convex analysis [64, Theorem 23.9]. The inclusion (55) is then equivalently rewritten as:

$$A_{u,\mu,\xi}(q)^T \ddot{\zeta} + F_{u,\mu,\xi}(q, \dot{q}, t) \in -N_{\mathcal{E}(q)}(\ddot{\zeta}) \quad (56)$$

where $F_{u,\mu,\xi}(q, \dot{q}, t) = A_{u,\mu,\xi}(q)^T \bar{F}_{u,\mu,\xi}(q, \dot{q}, t)$. The crucial fact here is that $A_{u,\mu,\xi}(q) \succ 0$ is not symmetric. However from [25, §1.1] it follows that the inclusion (56) is equivalent to the variational inequality:

$$\text{Find } \ddot{\zeta} \in \mathcal{E}(q) : \langle A_{u,\mu,\xi}(q)^T \ddot{\zeta} + F_{u,\mu,\xi}(q, \dot{q}, t), z - \ddot{\zeta} \rangle \geq 0 \text{ for all } z \in \mathcal{E}(q) \quad (57)$$

Since $A_{u,\mu,\xi}(q) \succ 0$, it follows that the mapping $x \rightarrow A_{u,\mu,\xi}(q)^T x + F_{u,\mu,\xi}(q, \dot{q}, t)$ is strongly monotone [25, p.155], so from [25, Theorem 2.3.3] the variational inequality in (57) has a unique solution $\ddot{\zeta}^*$. This solution also satisfies [25, p.158]:

$$\ddot{\zeta}^* = \text{proj}[\mathcal{E}(q); A_{u,\mu,\xi}(q)^T \ddot{\zeta}^* + F_{u,\mu,\xi}(q, \dot{q}, t)] \quad (58)$$

(56) (58) may be considered as the counterpart of (15) (16), or (22) (27), or (28) (29), for unilateral constraints with sliding Coulomb friction. Consider the LCP in (53): the extension of (12) that is the minimization associated with the LCP (10), is not clear due again to the lack of symmetry of $A_{u,\mu,\xi}(q)$.

The Classical Example of Painlevé We now add Coulomb's friction to the mechanical system in Figure 1(b) and point out how the equations of motion are affected. The system under consideration is the classical example of Painlevé. Its dynamics is given by:

$$\begin{cases} m\ddot{x} = -\mu\xi\lambda_{nu} \\ m\ddot{y} = -mg + \lambda_n \\ \frac{ml^2}{12}\ddot{\theta} = -\frac{l}{2}\cos\theta\lambda_n - l/2\sin\theta\mu\xi\lambda_{nu} \\ 0 \leq h_{nu}(\theta) = y - \frac{l}{2}\sin(\theta)\perp\lambda_{nu} \geq 0 \end{cases}$$

The additional matrices due to friction are $H_{tu}(\theta) = (1, 0, \frac{l}{2}\sin(\theta))^T$, $A_{tu}(\theta) = \frac{-3\sin\theta\cos\theta}{m}$, and $A_{u,\mu,\xi}(\theta) = \frac{1+3\cos^2(\theta)}{m} + \frac{3\sin(\theta)\cos(\theta)}{m}\mu\xi$. Hence the LCP in (53) reads:

$$0 \leq \left(\frac{1+3\cos^2(\theta)}{m} + \frac{3\sin(\theta)\cos(\theta)}{m}\mu\xi \right) \lambda_{nu} + \frac{l}{2}\dot{\theta}^2\sin\theta - g\perp\lambda_{nu} \geq 0. \quad (59)$$

The necessary and sufficient condition for a solution to exist is that the scalar $A_{u,\mu,\xi}(\theta)$ be positive. If we consider values of θ in $]0, \frac{\pi}{2}[$ then one sees that for rightward sliding ($\xi = 1$) $A_{u,\mu,\xi}(\theta)$ is always positive, regardless of the magnitude of friction (hence Corollary 2 applies and the LCP is solvable because the only solution of the homogeneous LCP is $\lambda_{n,u} = 0$, though in this scalar case one can use directly the fact that the LCP matrix is a P-matrix to conclude about existence and uniqueness). Whereas for leftward sliding ($\xi = -1$), $A_{u,\mu,\xi}(\theta)$ becomes negative for high values of friction. Indeed solving for μ in the scalar inequality $A_{u,\mu,\xi}(\theta) < 0$ one finds the critical friction value beyond which paradoxes occur, and it coincides with that of Corollary 13, namely $\mu_u^{max}(\theta) = \frac{1+3\cos^2\theta}{3\sin\theta\cos\theta}$. A graphical interpretation of these singular LCP situations is given on Figure 7, paralleling Figure 3(a) in the bilateral case.

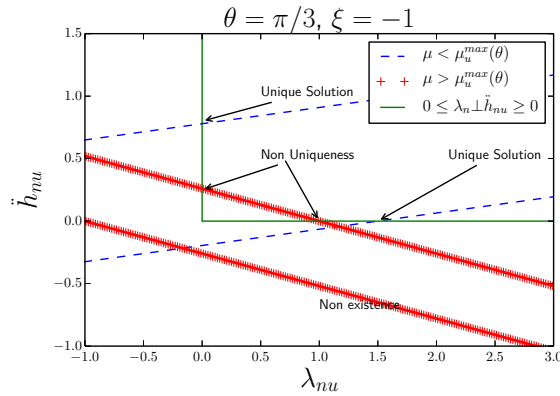


Figure 7: Singular and non-singular situations for (59).

3.4 Unilaterally/bilaterally constrained systems

Let us consider again systems as in section 2.3, but with friction at the contacts.

3.4.1 All contacts with friction

Let us assume first that all contacts are frictional, i.e., $\mathcal{I}_b^0 = \mathcal{I}_u^0 = \emptyset$. Let us consider (1) in its index 1 form, i.e., with the acceleration constraints $\nabla h_{n,b}(q)^T \ddot{q} + \frac{d}{dt}(\nabla h_{n,b}(q)^T) \dot{q} = 0$ and $0 \leq h_{n,u}(q) \perp \nabla h_{n,u}(q)^T \ddot{q} + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} \geq 0$. The dynamics in (1) becomes :

$$\begin{cases} M(q)\ddot{q} = \nabla h_{nb}(q)\lambda_{n,b} - H_{t,b}(q)[\mu_b\xi_b]|\lambda_{n,b}| + \nabla h_{nu}(q)\lambda_{n,u} - H_{t,u}(q)[\mu_u\xi_u]\lambda_{n,u} - F(q, \dot{q}, t) \\ \nabla h_{n,b}(q)^T \ddot{q} + \frac{d}{dt}(\nabla h_{n,b}(q)^T) \dot{q} = 0 \\ 0 \leq \lambda_{n,u} \perp \nabla h_{nu}(q)^T \ddot{q} + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} \geq 0. \end{cases} \quad (60)$$

By inverting the mass matrix, \ddot{q} can be expressed as a function of the multipliers. Introducing $\lambda^+ = \frac{|\lambda_{n,b}| + \lambda_{n,b}}{2}$ and $\lambda^- = \frac{|\lambda_{n,b}| - \lambda_{n,b}}{2}$ as in Proposition 11 casts the piecewise linearity induced by the absolute value into a complementarity formalism. The problem is thus transformed to determining whether for arbitrary parameter values and arbitrary vectors w_1, w_2 the following MLCP has a unique solution $(\lambda^+(q, \dot{q}), \lambda^-(q, \dot{q}), \lambda_{n,u}(q, \dot{q}))$:

$$\begin{cases} (A_{nb} - A_{tb}[\mu_b\xi_b])\lambda^+ - (A_{nb} + A_{tb}[\mu_b\xi_b])\lambda^- + (A_{nbnu} - A_{nbtu}[\mu_u\xi_u])\lambda_{n,u} + w_1 = 0 \\ 0 \leq (A_{nunb} - A_{nutb}[\mu_b\xi_b])\lambda^+ - (A_{nunb} - A_{nutb}[\mu_b\xi_b])\lambda^- + (A_{nu} - A_{tu}[\mu_u\xi_u])\lambda_{n,u} + w_2 \perp \lambda_{n,u} \geq 0 \\ 0 \leq \lambda^+ \perp \lambda^- \geq 0 \end{cases} \quad (61)$$

where

$$\begin{aligned} w_1(q, \dot{q}, t) &\triangleq -\nabla h_{n,b}(q)^T M(q)^{-1} F(q, \dot{q}, t) + \frac{d}{dt}(\nabla h_{n,b}(q)^T) \dot{q} \\ w_2(q, \dot{q}, t) &\triangleq -\nabla h_{n,u}(q)^T M(q)^{-1} F(q, \dot{q}, t) + \frac{d}{dt}(\nabla h_{n,u}(q)^T) \dot{q} \end{aligned}$$

possess the same structure as $w_u(q, \dot{q}, t)$ in (11). Assume that the conditions of Proposition 11 hold. Then $A_{nb}(q) - A_{tb}(q)[\mu_b\xi_b] \succ 0$ and the problem is to determine the existence and uniqueness of a solution $(\lambda^-(q, \dot{q}), \lambda_{n,u}(q, \dot{q}))$ (for arbitrary q, \dot{q}) of the following LCP (argument q is dropped):

$$0 \leq \begin{pmatrix} \lambda^- \\ \lambda_{n,u} \end{pmatrix} \perp \underbrace{\begin{pmatrix} A^{-1}\bar{A} & -A^{-1}B \\ CA^{-1}\bar{A} - \bar{C} & D - CA^{-1}B \end{pmatrix}}_{\triangleq M_{ub}^\mu(q)} \begin{pmatrix} \lambda^- \\ \lambda_{n,u} \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \geq 0, \quad (62)$$

where $A \triangleq A_{nb} - A_{tb}[\mu_b\xi_b]$, $\bar{A} \triangleq A_{nb} + A_{tb}[\mu_b\xi_b]$, $B \triangleq A_{nbnu} - A_{nbtu}[\mu_u\xi_u]$, $C \triangleq A_{nunb} - A_{nutb}[\mu_b\xi_b]$, $\bar{C} \triangleq A_{nunb} + A_{nutb}[\mu_b\xi_b]$, $D \triangleq A_{nu} - A_{tu}[\mu_u\xi_u]$, $z_1 \triangleq -A^{-1}w_1$ and $z_2 \triangleq w_2 + Cz_1$.

Remark 4 When all the contacts are frictionless, the LCP in (62) has the form:

$$0 \leq \begin{pmatrix} \lambda^- \\ \lambda_{n,u} \end{pmatrix} \perp \underbrace{\begin{pmatrix} I & -A_{nb}^{-1}A_{nbnu} \\ 0 & A_{nu} - A_{nbtu}^T A_{nb}^{-1} A_{nbnu} \end{pmatrix}}_{\triangleq M_{ub}^0(q)} \begin{pmatrix} \lambda^- \\ \lambda_{n,u} \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \geq 0, \quad (63)$$

The second line is an LCP involving λ_{nu} alone and is the same as the LCP in (32). Thus its solution λ_{nu} exists and is unique provided that all the constraints are independent. Under this assumption it follows that λ^- also exists and is unique: hence the matrix $M_{ub}^0(q)$ of the LCP (63) is a P-matrix.

One has the following decomposition : $M_{ub}^\mu(q) = M_{ub}^0(q) + P_\mu(q, \dot{q})$ where $\|P_\mu(q, \dot{q})\|$ tends to zero as the friction μ_i goes to zero for each contact. Once again the matrix from the frictional LCP is obtained as a perturbation of the frictionless one (63). Here sufficiently small friction coefficients will guarantee that M_{ub}^μ is a P-matrix. Let us provide some details on the matrix $P_\mu(q, \dot{q})$. We know from section 3.2 that $A^{-1}\bar{A} = I + 2K_\mu(q, \dot{q})$ with $K_\mu(q, \dot{q}) = \sum_{i=1}^{\infty} (A_{nb}^{-1}A_{tb}[\mu_b\xi_b])^i$. Using the same decomposition for the other blocks of $M_{ub}^\mu(q)$, one obtains

$$A^{-1}\bar{A} = I + 2K_\mu(q, \dot{q}) \quad (64)$$

$$-A^{-1}B = -A_{nb}^{-1}A_{nbnu} + (I + K_\mu(q, \dot{q}))A_{nb}^{-1}A_{nbtu}[\mu_u\xi_u] - K_\mu(q, \dot{q})A_{nb}^{-1}A_{nbnu} \quad (65)$$

$$CA^{-1}\bar{A} - \bar{C} = 0 - 2A_{nutb}[\mu_b\xi_b] + CK_\mu(q, \dot{q}) \quad (66)$$

$$D - CA^{-1}B = A_{nu} - A_{nbnu}^T A_{nb}^{-1} A_{nbnu} - A_{tu}[\mu_u\xi_u] + A_{nutb}[\mu_b\xi_b] A_{nb}^{-1} A_{nbnu} \\ + C((I + K_\mu(q, \dot{q}))A_{nb}^{-1}A_{nbtu}[\mu_u\xi_u] - K_\mu(q, \dot{q})A_{nb}^{-1}A_{nbnu}) \quad (67)$$

By Theorem 3 in section D, if $\|P_\mu(q, \dot{q})\|_2 < 1/\beta_2(M_{ub}^0(q))$ then $M_{ub}^\mu(q)$ is a P-matrix. One can therefore derive an explicit maximum value of all friction coefficients below which existence and uniqueness of a solution to problem (60) is guaranteed. Since each block of $P_\mu(q, \dot{q})$ is a $O(\mu)$ and $\|P_\mu(q, \dot{q})\| \leq \sqrt{\|P_{11}\|^2 + \|P_{12}\|^2 + \|P_{21}\|^2 + \|P_{22}\|^2}$ it suffices to choose μ such that $\|P_\mu(q, \dot{q})_{ij}\| < 1/2\beta_2(M_{ub}^0(q))$.

Proposition 14 *Let (q, \dot{q}) , $F(q, \dot{q}, t)$ be given and $v_{i,i} \neq 0$ for all i . Assume $M(q) \succ 0$ and $A_{nb}(q) \succ 0$. Suppose that all the constraints are independent, that the bilateral friction coefficients satisfy $\max_{1 \leq i \leq m_b} \mu_i^b < \mu_{\max}^b(q)$, and that $\|P_\mu(q, \dot{q})\|_2 < \frac{1}{\beta_2(M_{ub}^0(q))}$. Then $M_{ub}^\mu(q)$ is a P-matrix and there exists a unique solution $(\ddot{q}, \lambda_{nb}, \lambda_{nu})$ to the LCP in (62) and thus to the mixed sliding friction problem (60).*

One sees once again that there is no obvious quadratic problem that could be associated with the LCP in (62), because $M_{ub}^\mu(q)$ is guaranteed to be neither symmetric nor $\succeq 0$.

3.4.2 Frictionless bilateral contacts

When the bilateral constraints are frictionless ($\mu_{b,i} = 0$ for all $0 \leq i \leq m_b$), one can easily express λ_{nb} as a function of λ_{nu} and equation (60) becomes

$$0 \leq \lambda_{nu} \perp (D - CA^{-1}B)\lambda_{nu} + z_2 \geq 0,$$

with $D - CA^{-1}B = A_{nu} - A_{nbnu}^T A_{nb}^{-1} A_{nbnu} + (A_{nbnu}^T A_{nb}^{-1} A_{nbtu} - A_{tu})[\mu_u\xi_u]$, which is the sum of the matrix $A_c(q)$ in (33), plus a frictional perturbation term. This matrix is simpler than $M_{ub}^\mu(q)$ in (62) since it is the perturbation of a symmetric positive definite matrix, rather than that of a P-matrix. One can use the results of Section 3.3.

Proposition 15 *Let (q, \dot{q}) , $F(q, \dot{q}, t)$ be given and $v_{t,i} \neq 0$ for all i . Suppose that all constraints are independent, that the bilateral constraints are frictionless ($\mu_{b,i} = 0$ for all i), and that*

$$\max_{1 \leq i \leq m_u} \mu_{u,i} < \frac{\sigma_{\min}(A_{nu}(q) - A_{nbnu}(q)^T A_{nb}(q)^{-1} A_{nbnu}(q))}{\sigma_{\max}(A_{tu}(q) - A_{nbnu}(q)^T A_{nb}(q)^{-1} A_{nbtu}(q))} \quad (68)$$

Then there exists a unique solution $(\ddot{q}, \lambda_{nu}, \lambda_{nb})$ to problem (60).

There is no clear relationship between $\mu_{\max}^u(q)$ in (54), and the upperbound in (68).

4 Recapitulation

4.1 Contact force calculation

We now give a global view on the structures of the different contact problems we have been considering thus far. What we first wish to underline is that (for a non-singular mass matrix and constraint Jacobian) the frictionless problems are well posed. The bilateral one as a linear system in (7)

$$A_{nb}(q)\lambda_{n,b} + w_b(q, \dot{q}, t) = 0,$$

the unilateral one as an LCP in (10)

$$0 \leq A_{nu}(q)\lambda_{n,u} + w_u(q, \dot{q}, t) \perp \lambda_{n,u} \geq 0,$$

and the mixed one as the unilateral LCP with a bilateral distortion in (32)

$$0 \leq A_c(q)\lambda_{n,u} + w_c(q, \dot{q}) \perp \lambda_{n,u} \geq 0.$$

In the case of a full rank Jacobian and an invertible mass matrix all three matrices $A_{nb}(q)$, $A_{nu}(q)$, and $A_c(q)$ are positive definite. One can solve uniquely for contact forces and deduce a unique acceleration. Then in the case of small friction this well posedness is preserved. The all sliding bilateral friction problem becomes the LCP

$$0 \leq (I + 2K_\mu(q, \dot{q}))\lambda^- + w_{bf}(q, \dot{q}, t) \perp \lambda^- \geq 0,$$

for some $w_{bf}(q, \dot{q}, t)$, where $\|K_\mu(q, \dot{q}, t)\|$ tends to zero as the friction μ goes to zero. For small enough friction the positivity of the identity matrix is preserved and the LCP is well posed. The unilateral all sliding friction problem becomes (53)

$$0 \leq A_{u,\mu,\xi}(q)\lambda_{n,u} + w_u(q, \dot{q}, t) \perp \lambda_{n,u} \geq 0.$$

For small friction μ_u the well posedness of the contact problem is kept. As for the mixed all sliding friction problem one has an LCP of the form (62)

$$0 \leq (M_{ub}^0(q) + P_\mu(q, \dot{q}))\lambda + w(q, \dot{q}, t) \perp \lambda \geq 0,$$

where $\|P_\mu(q, \dot{q})\|$ tends to zero as the friction μ goes to zero. The matrix $M_{ub}^0(q)$ is a P-matrix whose terms appear in the frictionless problems and the P-matrix property is also preserved when adding the small frictional perturbation term $P_\mu(q, \dot{q})$.

4.2 Minimization problems and Gauss' principle

Although the paper it has been shown that the acceleration is the solution of a minimization problem under constraints: (5) for the frictionless bilateral case, (16) for the frictionless unilateral case, (28) for the frictionless mixed bilateral/unilateral case. All these quadratic programs may be recast into a stationary point problem

$$0 \in \partial \left(\frac{1}{2} \ddot{q}^T M(q) \ddot{q} + \dot{q}^T F(q, \dot{q}, t) + \Psi_{\tilde{K}}(\ddot{q}) \right)$$

where \tilde{K} varies from one case to the other and ∂ is the subdifferentiation from convex analysis. This was already advocated in [28, 29, 30], see also [82]. In the mixed case a different expression has been proposed in (27). The crucial property for the minimization problem to hold, is the symmetry of the mass matrix (or of $P_r(q)$ in (27)) in addition to its (semi) positive definiteness. In case of sliding Coulomb friction, symmetry is usually lost. However Gauss' principle may be formulated under the projection (58) in the unilateral case. If the contacts are sticking and Coulomb friction is formulated at the acceleration level [30, p.137] [58], then some extensions of Gauss' principle are possible [63, 36].

A Some convex analysis and complementarity theory tools

Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex proper function. Its subdifferential is denoted as $\partial f(\cdot)$ and is the set of subgradients.

Let K be a set, its indicator function is $\Psi_K(x) = 0$ if $x \in K$ and $= +\infty$ if $x \notin K$. If $K \subset \mathbb{R}^n$ is non-empty closed and convex, its normal cone at x is $N_K(x) = \{z \in \mathbb{R}^n | z^T(y - x) \leq 0 \text{ for all } y \in K\} = \partial \Psi_K(x)$. If $K = \mathbb{R}^n$ then $N_K(x) = \{0\}$ for all x . The tangent cone $T_K(x) = \{z \in \mathbb{R}^n | z^T y \leq 0 \text{ for all } y \in N_K(x)\}$. If $K = \{x \in \mathbb{R}^n | f(x) \geq 0\}$ for m continuously differentiable functions $f_i : \mathbb{R}^n \mapsto \mathbb{R}$ which satisfy the Mangasarian-Fromovitz constraint qualification [25], then $T_K(x) = \{v \in \mathbb{R}^n | v^T \nabla f_i(x) \geq 0, \text{ for all } i \in \{1, \dots, m\} \text{ such that } f_i(x) = 0\}$.

A linear complementarity problem $\text{LCP}(q, M)$ with unknown $x \in \mathbb{R}^n$ is: $x \geq 0$, $Mx + q \geq 0$, $x^T(Mx + q) = 0$. More compactly $0 \leq x \perp Mx + q \geq 0$. An LCP is said solvable if it has at least one solution. If M is a P-matrix then the LCP has a unique solution for any q [18].

Let C be a (non-necessarily convex) set of \mathbb{R}^n , then its dual set is $C^* = \{x \in \mathbb{R}^n | x^T z \geq 0 \text{ for all } z \in C\}$, which is always a closed convex cone.

Let $M \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix defining the inner product $x^T M x$. With this metric, the orthogonal projection of a vector $x \in \mathbb{R}^n$ on a convex set $K \subset \mathbb{R}^n$ is denoted as $\text{proj}_M[K; x] = \text{argmin}_{z \in K} \frac{1}{2}(z - x)^T M(z - x)$.

The following equivalences are useful. Let $x \in \mathbb{R}^n$, $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$, K is a closed

convex cone.

$$\begin{aligned}
Mx + q \in -N_K(x) &\Leftrightarrow K \ni x \perp Mx + q \in K^* \\
&\Leftrightarrow (\text{if } M = M^T \succeq 0) \ x = \operatorname{argmin}_{z \in K} \frac{1}{2} z^T Mz + q^T z \quad (69) \\
&\Leftrightarrow (\text{if } M = M^T \succ 0) \ x = \operatorname{proj}_M[K; -M^{-1}q]
\end{aligned}$$

B Theorem 3.1.7 in [18] (excerpts)

Theorem 1 *Let $M \in \mathbb{R}^{n \times n}$ be positive semi definite, and let $q \in \mathbb{R}^n$ be arbitrary. The following statements hold:*

- (a) *If z^1 and z^2 are two solutions of the LCP(M, q) then $(z^1)^T(q + Mz^2) = (z^2)^T(q + Mz^1)$.*
- (d) *If M is symmetric (as well as positive semi definite) then $Mz^1 = Mz^2$ for any two solutions z^1 and z^2 .*

C Theorem 3.8.6 in [18]

Theorem 2 *Let $M \in \mathbb{R}^{n \times n}$ be copositive and let $q \in \mathbb{R}^n$ be given. If the implication $[0 \leq v \perp Mv \geq 0] \Rightarrow [v^T q \geq 0]$ is valid, then the LCP(M, q) is solvable.*

Let \mathcal{Q}_M denote the solution set of the homogeneous LCP. This theorem can be restated equivalently as: If M is copositive and $q \in \mathcal{Q}_M^*$ then LCP(M, q) is solvable.

D Theorems 2.8 and 2.11 in [16]

Chen and Xiang [16] stated very useful criteria that guarantee that a positive definite or a P-matrix remains positive definite or P when it is subject to a small enough perturbation. We give here just an excerpt of the results in [16], and a corollaries of it.

Theorem 3 *If M is a P-matrix then all matrices A such that*

$$\beta_2(M) \|M - A\|_2 < 1$$

are P-matrices, where $\beta_2(M) := \max_{d \in [0,1]^n} \|(I - D + DM)^{-1}D\|_2$, and $D = \operatorname{diag}(d)$. When M is symmetric positive definite, $\beta_2(M) = \|M^{-1}\|_2$.

Theorem 4 *Let $M \in \mathbb{R}^{n \times n}$ be a positive definite matrix. Then every matrix*

$$A \in \left\{ A : \left\| \left(\frac{M+M^T}{2} \right)^{-1} \right\|_2 \|M - A\|_2 < 1 \right\}$$

is positive definite.

The next corollary is proved in [12].

Corollary 3 *Let $D = P + N$, where D , P and N are $n \times n$ real matrices, and $P > 0$, not necessarily symmetric. If*

$$\|N\|_2 < \frac{1}{\left\| \left(\frac{P+P^T}{2} \right)^{-1} \right\|_2} \quad (70)$$

then $D > 0$.

E KKT system: solvability and solution uniqueness

The KKT problem in (4) is ubiquitous in the study of mechanical systems with bilateral holonomic constraints. In this section it is analysed from various points of view, and proves to possess some subtleties depending on which assumptions are made on the data. Let us consider the next three problems, where $M = M^T \in \mathbb{R}^{n \times n}$ and $M \succeq 0$, $F \in \mathbb{R}^{n \times m}$, $N \in \mathbb{R}^{n \times m}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$:

$$\underbrace{\begin{pmatrix} M & I \\ F^T & 0 \end{pmatrix}}_{\triangleq \tilde{A} \in \mathbb{R}^{(n+m) \times 2n}} \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \quad (71)$$

and

$$\underbrace{\begin{pmatrix} M & -F \\ F^T & 0 \end{pmatrix}}_{\triangleq \tilde{A} \in \mathbb{R}^{(n+m) \times (n+m)}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \quad (72)$$

and

$$\underbrace{\begin{pmatrix} M & N \\ F^T & 0 \end{pmatrix}}_{\triangleq \tilde{A} \in \mathbb{R}^{(n+m) \times (n+m)}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \quad (73)$$

The three systems (71), (72) and (73) correspond to various ways to consider the system in (4), where x is for \ddot{q} , z is for $-\nabla h_{n,b}(q)\lambda_{n,b}$ and y is for $\lambda_{n,b}$.

- (i) Let us consider first (71) without any assumption on the form of z . From [9, Fact 2.10.22] one has $\text{Im}(\tilde{A}) = \text{Im} \begin{pmatrix} M \\ F^T \end{pmatrix} + \text{Im} \begin{pmatrix} I \\ 0 \end{pmatrix}$. Thus a necessary and sufficient condition for (71) to possess a solution (x, z) for any a and b , equivalently $\text{rank}(\tilde{A}) = n + m$ (which also follows from [9, Proposition 6.1.7 (iii)]), is that F^T has full rank m , i.e., F be full column rank (this implies that $n \geq m$). Uniqueness of (x, z) for any a and b holds if and only if $n = m$ and $\text{rank}(\tilde{A}) = 2n$ (this may be proved from [9, Theorem 2.6.3 ii]), in which case F is square and has full rank

n . In this case the solution is equal to $\tilde{A}^\dagger \begin{pmatrix} a \\ b \end{pmatrix}$, where \tilde{A}^\dagger is the Moore-Penrose generalized inverse of \tilde{A} [9, Proposition 6.1.7 (viii)]. One sees that M plays no role in this system.

- (ii) Let us still consider (71) assuming that $z = -Fy$ for some y , i.e., $z \in \text{Im}(F)$. One has $(x, z) \in \ker(\tilde{A}) \Leftrightarrow Mx = -z$ and $x \in \ker(F^T)$. Using $z \in \text{Im}(F)$ and $x \in \ker(F^T)$ and [9, Theorem 2.4.3] it follows that $x \perp z$. We also have $z \in \text{Im}(M)$. Using that $\text{Im}^\perp(M) = \ker(M)$, we deduce that $x \in \ker(M)$ because M is symmetric positive semi definite¹¹. Thus $x \in \ker(M) \cap \ker(F^T)$ and consequently $z = -Mx = 0$. Thus we have shown that $[(x, z) \in \ker(\tilde{A}) \text{ and } z \in \text{Im}(F)] \Rightarrow [z = 0 \text{ and } x \in \ker(M) \cap \ker(F^T)]$, and the reverse implication holds also. Let $S = \{(x, z) \in \mathbb{R}^{2n} | z \in \text{Im}(F)\}$. Then $\ker(\tilde{A}) \cap S = \{(x, z) \in \mathbb{R}^{2n} | z = 0 \text{ and } x \in \ker(M) \cap \ker(F^T)\}$. One infers that $\ker(M) \cap \ker(F^T) = \{0\} \Rightarrow \ker(\tilde{A}) \cap S = \{0\}$. From (i) existence of solutions for system (71) holds for any a and b if and only if $\text{rank}(\tilde{A}) = n + m \Leftrightarrow \text{rank}(F) = m$, hence $\dim(\ker(\tilde{A})) = 2n - n - m = n - m$. From the fact that $0 \leq \dim(\ker(\tilde{A}) \cap S) \leq \min(n - m, n + m) = n - m$ [9, Fact 2.9.14], one infers that $\ker(M) \cap \ker(F^T) = \{0\} \Rightarrow \dim(\ker(\tilde{A}) \cap S) = 0$, thus $n = m$. In this case it follows from (i) that the system has a unique solution for any a and b . Conversely the existence of solutions for arbitrary a and b and $n = m$ imply uniqueness, as well as $\ker(M) \cap \ker(F^T) = \{0\}$ since F is square full rank n .

Let us now pass to the system (72). Remark that if \bar{A} in (72) is invertible then F necessarily has full rank m . This follows from the fact that $(x, y) \in \ker(\bar{A})$ implies $x \in \ker(M) \cap \ker(F^T)$ and $y \in \ker(F)$, using similar arguments as in (ii). In particular if there are more constraints than degrees of freedom (i.e., $m > n$) then \bar{A} is not invertible, and likewise if $\text{rank}(F) = r < m$.

- (iii) Consider now the system (72). Let $\text{rank}(F) = m$ (so $m \leq n$) and M be positive semi definite. Then existence and uniqueness of both x and y for arbitrary a and b (equivalently, non-singularity of \bar{A}) holds if and only if $\ker(M) \cap \ker(F^T) = \{0\}$ (proof by direct application of [66, p.523], or using the above expression of $\ker(\bar{A})$).

The rank condition on F appears to be in fact necessary and sufficient as alluded to few lines above:

- (iii') The system (72) has a unique solution (x, y) for arbitrary a and b if and only if $\text{rank}(F) = m$ and $\ker(M) \cap \ker(F^T) = \{0\}$. This solution is equal to $\bar{A}^{-1} \begin{pmatrix} a \\ b \end{pmatrix}$.

The proof of (iii') follows from [9, Theorem 2.6.3, Proposition 6.1.7], noting that \bar{A} is square. It is sometimes wrongly stated that \bar{A} is non-singular if and only if M and F are both full rank matrices [40], which is only a sufficient condition. In fact one has from [9,

¹¹The conclusion does not hold without the positive definiteness condition.

Fact 6.4.20]:

$$\begin{aligned} \text{rank}(\bar{A}) = & \text{rank}(M) + 2\text{rank}(F) - \dim[\text{Im}(M) \cap \text{Im}(F)] \\ & - \dim\left[\text{Im}\begin{pmatrix} M \\ F^T \end{pmatrix} \cap \text{Im}\begin{pmatrix} F \\ 0 \end{pmatrix}\right] \end{aligned} \quad (74)$$

The formula in (74) shows that one may dispense with positive definiteness conditions on M , and that the non-singularity of \bar{A} results from an interplay between the matrices ranges. Consider for instance $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $F = (1 \ 0)^T$, which yields $\text{rank}(\bar{A}) = 3$.

In Contact Mechanics we wish to allow for situations where the constraints are redundant, so ($\text{rank}(F) = r < \min(m, n)$) but which are nevertheless compatible (*i.e.* $b \in \text{Im}(F^T)$), for otherwise the problem has no solution. Thus the most relevant problem is that of determining x and y such that (72) holds for arbitrary a with the additional assumption that $b \in \text{Im}(F^T)$. This problem is thus different from problems tackled in (ii) and (iii'), and corresponds to the problem tackled in [20, 19, 76].

- (iv) The necessary and sufficient condition for the existence of x and y with uniqueness of x and Fy such that (72) holds for arbitrary a is that $b \in \text{Im}(F^T)$ and $\ker(M) \cap \ker(F^T) = \{0\}$.

Proof:

\Leftarrow

Existence of x and y : By contraposition, if not ($\forall a, \exists x$ and y such that (72) holds) then there exists an $a \in \mathbb{R}^n$ such that for all x, y , $F^T x \neq b$ (so $b \notin \text{Im}(F^T)$) or $Mx - Fy \neq a$ (so $\text{Im}([M \ F]) \neq \mathbb{R}^n$, *i.e.*, $\ker(M) \cap \ker(F^T) \neq \{0\}$).

Uniqueness of x and Fy : Suppose $b \in \text{Im}(F^T)$ and $\ker(M) \cap \ker(F^T) = \{0\}$.

If (x_1, y_1) and (x_2, y_2) are two solutions of (72) then $F^T(x_1 - x_2) = 0$ and $M(x_1 - x_2) = F(y_1 - y_2)$. Hence $(x_1 - x_2) \in \ker(F^T)$ and $M(x_1 - x_2) \in \text{Im}(F) = \ker(F^T)^\perp$. Hence $(x_1 - x_2)^T M(x_1 - x_2) = 0$ and since M is symmetric positive semi definite this means $(x_1 - x_2) \in \ker(M)$. By hypothesis $\ker(M) \cap \ker(F^T) = \{0\}$ so one concludes that $x_1 - x_2 = 0$, that is, x is unique. It follows that $Fy_1 = Fy_2$ so Fy is unique as well.

\Rightarrow

Suppose that for arbitrary a , there exists x and y such that (72) holds with uniqueness of x and Fy . Then in particular $b \in \text{Im}(F^T)$. Let $x \in \ker(M) \cap \ker(F^T)$ and let (x^*, y) be the unique solution of (72). Then $(x^* + x, y)$ is also a solution of (72). Hence $x = 0$, so that $\ker(M) \cap \ker(F^T) = 0$. ■

In [20, p.319] the condition $b \in \text{Im}(F^T)$ is stated as: *acceleration constraints are compatible*. In conclusion, four types of systems are considered: system (72) with unknowns x and y in (iii), system (71) with unknowns x and z in (i), system (71) with unknowns x and z and the constraints that $z \in \text{Im}(F)$ in (ii), system (72) with unknowns x and y with uniqueness of x and Fy and arbitrary a in (iv). To complete the picture let us note that $\ker(M) \cap \ker(F^T) = \{0\} \Leftrightarrow (\text{Im}(M) + \text{Im}(F)) = \mathbb{R}^n \Leftrightarrow \text{Im}([M \ F]) = \mathbb{R}^n$ using [9, Fact

2.9.10], and we recover directly an alternative way to formulate the condition involving the kernels, sometimes used in the literature [76] [19, Equation (9)].

Problems like in (72) and (71) occur in frictionless systems. The next step is to consider systems of the form (73) for some matrix N . Such problems arise in the presence of Coulomb's friction, see (37). Using [9, Fact 6.4.20] one gets an extension of (74):

$$\begin{aligned} \text{rank}(\hat{A}) = & \text{rank}(M) + \text{rank}(F^T) + \text{rank}(N) - \dim[\text{Im}(M) \cap \text{Im}(F)] \\ & - \dim\left[\text{Im}\begin{pmatrix} M \\ F^T \end{pmatrix} \cap \text{Im}\begin{pmatrix} N \\ 0 \end{pmatrix}\right] \end{aligned} \quad (75)$$

System (73) has a unique solution (x, y) for any a and b if and only if $\text{rank}(\hat{A}) = n + m$. One has:

$$\begin{aligned} \text{Im}\begin{pmatrix} M \\ F^T \end{pmatrix} \cap \text{Im}\begin{pmatrix} N \\ 0 \end{pmatrix} = & \{z \in \mathbb{R}^{n+m} \mid \exists y_1 \in \ker(F^T), \exists y_2 \in \mathbb{R}^m, \text{ such that} \\ & z = \begin{pmatrix} My_1 \\ 0 \end{pmatrix} = \begin{pmatrix} Ny_2 \\ 0 \end{pmatrix}\} \end{aligned} \quad (76)$$

and

$$\text{Im}(M) \cap \text{Im}(F) = \{z \in \mathbb{R}^n \mid \exists y_1 \in \mathbb{R}^n, \exists y_2 \in \mathbb{R}^m, \text{ such that } z = My_1 = Fy_2\} \quad (77)$$

It is clear from (75) that the system's well-posedness depends on the interplay between M , F and N . Even if all three matrices have full rank, one may have $\text{rank}(\hat{A}) < n + m$. Suppose that $\text{rank}(M) = n$, and $\text{rank}(F^T M^{-1} N) = m$. Then $\text{Im}(M) \cap \text{Im}(F) = \text{Im}(F)$ and $\text{Im}\begin{pmatrix} M \\ F^T \end{pmatrix} \cap \text{Im}\begin{pmatrix} N \\ 0 \end{pmatrix} = \{0\}$ (since $F^T y_1 = 0 = F^T M^{-1} N y_2$). Therefore from (75) one has $\text{rank}(\hat{A}) = n + m$.

Finally we may rewrite (73) as (71) posing $z = Ny$. Then (i) applies, but (ii) usually does not except if $\text{Im}(N) \subseteq \text{Im}(F)$. Then given z , there exists a unique y if and only if N has full column rank m ($\Rightarrow m \leq n$). Let $N = -F + P$ for some matrix P .

(v) Assume that F has full column rank m (equivalently $F^T F \in \mathbb{R}^{m \times m}$ is positive definite). Let us investigate conditions that guarantee that N has full rank m . Equivalently $N^T N \in \mathbb{R}^{m \times m}$ is positive definite. One has $N^T N = F^T F - F^T P - P^T F + P^T P$. Direct application of Corollary 3 shows that a sufficient condition for $N^T N$ to be positive definite is that $\| -F^T P - P^T F + P^T P \|_2 < \frac{1}{\|(F^T F)^{-1}\|_2}$, equivalently $\sigma_{\max}(-F^T P - P^T F + P^T P) < \sigma_{\min}(F^T F)$.

Let $M = 0$, then using (75) it follows that $\text{rank}(\hat{A}) = 2m$, hence $\text{rank}(\hat{A}) = n + m$ if and only if $n = m$. This shows that depending on the interplay between the ranges of the matrices in (75), the system in (73) may be solvable with uniqueness for any a and b , for low-rank matrices M .

(vi) Let us assume that $\text{rank}(M) = n$ and study conditions such that the rank of $F^T M^{-1} N = -F^T M^{-1} F + F^T M^{-1} P$ is m . Then as shown after (77), \hat{A} has rank $n + m$ and the system (73) has a unique solution for any a and b . Using Corollary 3, $\text{rank}(F^T M^{-1} N) = m$ holds if $\text{rank}(F) = m$ and $\sigma_{\max}(F^T M^{-1} P) < \sigma_{\min}(F^T M^{-1} F)$.

It is noteworthy that the study of problem (73) may also be quite useful in the context of numerical analysis of differential algebraic equations (DAEs). Half-explicit methods involve such problems (for instance N may be the jacobian of the constraints estimated at step $i + 1$ while F is the jacobian estimated at step i) [70, §7.1] [31, §VII.6], see also [15, 44, 54, 55] for various forms of numerical KKT systems.

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