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# The logistic S.D.E.

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**Abstract** We consider the logistic S.D.E which is obtained by addition of a diffusion coefficient of the type  $\beta\sqrt{x}$  to the usual and deterministic Verhulst-Volterra differential equation. We show that this S.D.E is the limit of a sequence of birth and death Markov chains. This permits to interpret the solution  $V_t$  as the size at time  $t$  of a self-controlled tumor which is submitted to a radiotherapy treatment. We mainly focus on the family of stopping times  $T_\varepsilon$ , where  $T_\varepsilon$  is the first hitting of level  $\varepsilon > 0$  by  $(V_t)$ . We calculate their Laplace transforms and also the first moment of  $T_\varepsilon$ . Finally we determine the asymptotic behavior of  $T_\varepsilon$ , as  $\varepsilon \rightarrow 0$ .

**Keywords** Logistic equation, tumor, radiotherapy treatment, Laplace transforms, birth and death process, diffusion processes, first hitting time,

**AMS Classification** 60F15, 60F17, 60G17, 60G35, 60G40, 60H10, 60J27, 60J60, 60J65.

## 1 Introduction

1) The (deterministic) logistic equation:

$$(1.1) \quad \begin{cases} \frac{dV(t)}{dt} = r V(t) \left(1 - \frac{V(t)}{\kappa}\right) - c V(t) \\ V(0) = v \geq 0 \end{cases}$$

where the parameters  $r, c, v$  and  $\kappa$  are positive, was first introduced by Verhulst [32], then studied by Volterra [33] and more recently by a lot of authors, see for instance [28],[23],[33],[29],[11]. At the beginning, equation (1.1) has been introduced to model the time-evolution of the size  $V(t)$  of a population. The goal was to add a limit to the exponential growth described by Malthus (1798) introducing both external constraints of the environment and the natural self limitation of the phenomenon.

The solution of (1.1) can be easily calculated, see Proposition 3.1 below. The simple and explicit formula for  $V(t)$  permits to determine its limit behavior when time goes to infinity.

Afterwards, an extension to the Verhulst equation was considered by many authors, among them we can cite: [27],[31],[25],[19],[34],[15],[17],[14],[16]. The stochastic generalization of (1.1) is called the *stochastic logistic equation* and takes the form:

$$(1.2) \quad \begin{cases} dV_t = V_t (a - b V_t) dt + \beta V_t dB_t \\ V_0 = v > 0 \end{cases}$$

where  $(B_t)_{t \geq 0}$  a standard Brownian motion starting from zero and  $a, b, \beta$  are real parameters. Note that if  $\beta = 0$ ,  $a = r - c$  and  $b = \frac{r}{\kappa}$  then (1.2) and (1.1) coincide. In other words, (1.2) is actually an extension of (1.1). Introducing noise to the initial deterministic model is reasonable to take into account uncertainty related to either the internal growth of the population or external influences as interaction with the other species and environmental factors. See for instance [24],[22], [20] for applications in the field of dynamics of population.

**2)** The Brownian motion  $(B_t)$  in (1.2) is a random perturbation which is arbitrary added to the O.D.E. (1.1). In particular the form of the diffusion coefficient is unexplained. One of our aims is to prove that the form of (1.2) appears naturally when we model the behavior of a tumor submitted to a treatment of either radiotherapy or phototherapy. Such treatment is applied to a population of cancer cells which evolve independently from each other. This aggregate of cells constitutes the tumor and its size corresponds to the number of all the elements. We have to take into account two antagonist forces. The first one is the natural duplication of cells. The second has two parts, the first one is the effect of the treatment and the second results from the self-limitation of the tumor (for instance a maximal possible volume).

We denote by  $X_n(t)$  the size of the tumor at time  $t$ . Let  $n := X(0)$  be its initial size. We suppose that  $(X_n(t))$  is a birth and death process, i.e. a Markov chain in continuous time valued in  $\mathbb{N}$  and with jumps equal to  $\pm 1$ . More precisely we propose (see Section 2.1 for details) the following infinitesimal behavior:

$$\begin{aligned} \mathbb{P}(X_n(t + \Delta t) = i + 1 | X_n(t) = i) &= [ri + \frac{\beta^2}{2}i^2] \Delta t + o(\Delta t), \quad \forall i \in \mathbb{N} \\ \mathbb{P}(X_n(t + \Delta t) = i - 1 | X_n(t) = i) &= [ci + (\frac{\beta^2}{2}i^2 + \frac{r}{\kappa} \frac{1}{n})i^2] \Delta t + o(\Delta t), \quad \forall i \in \mathbb{N}^*. \end{aligned}$$

One interesting feature of the above model is its small number of parameters and  $r, c$  and  $\kappa$  have biological significance:  $r$  is the intrinsic growth rate of the tumor,  $c$  is its rate of decay due to cancer treatment,  $\kappa$  is the carrying capacity of the environment. It can be proved, see Proposition 2.3, that the proportion of cancer cells  $(\frac{X_n(t)}{n})$  converges in distribution as  $n \rightarrow \infty$  to the diffusion process  $V(t)$  solution of (1.2) where  $v = 1$ ,  $a = r - c$  and  $b = \frac{r}{\kappa}$ . Therefore this approximation scheme based on biological considerations allows to consider  $V(t)$  as the "limit size" at time  $t$  of a tumor submitted to a treatment. This approach is developed in detail in Section 2.

3) S.D.E. (1.2) which is valued in  $[0, \infty[$  admits a unique and explicit solution, see [15], Theorem 2.2, page 167. We also study under which conditions  $V_t$  is either recurrent or transient. We prove that in some cases either  $V_t$  admits a unique and invariant probability measure (p.m.) or goes to 0 as  $t \rightarrow \infty$ . All these results are given in Propositions 3.3 and 3.6. We also study the moments of  $V_t$ , see Proposition 3.8. Then we interpret the above results when  $V_t$  models the size of a tumor at time  $t$ . This allows to have a better understanding of the role of the parameters. We partially recover the asymptotic dynamics of the Verhulst's function, cf (3.16), however new interesting effects appear due to the presence of the stochastic perturbation. Randomness introduces more flexibility than the deterministic setting.

4) In the rest of the paper, i.e. Section 4, we study the first hitting times of fixed levels by the logistic process  $(V_t)$ . These random times have a clear biological interpretation, since they represent the first time when the size of the tumor reaches a given level  $\epsilon > 0$ . We can calculate the Laplace transform of the hitting time  $T_\epsilon := \inf\{t \geq 0, V_t = \epsilon\}$ .

**Theorem 1.1** *The Laplace transform of the hitting time  $T_\epsilon$  is given by:*

$$(1.3) \quad \mathbb{E}_v[\exp(-\lambda T_\epsilon)] = \left(\frac{v}{\epsilon}\right)^{\sqrt{\frac{2\lambda}{\beta^2} + q^2 + q}} \frac{\varphi(\lambda, v)}{\varphi(\lambda, \epsilon)}, \quad \lambda \geq 0$$

where

$$\varphi(\lambda, x) = \begin{cases} U\left(\sqrt{\frac{2\lambda}{\beta^2} + q^2 + q}, 1 + 2\sqrt{\frac{2\lambda}{\beta^2} + q^2}; \frac{2bx}{\beta^2}\right) & \text{if } \epsilon \leq v \\ M\left(\sqrt{\frac{2\lambda}{\beta^2} + q^2 + q}, 1 + 2\sqrt{\frac{2\lambda}{\beta^2} + q^2}; \frac{2bx}{\beta^2}\right) & \text{if } \epsilon \geq v \end{cases}$$

$U$  is the Tricomi hypergeometric function and  $M$  is the Kummer function.

Definitions of functions  $U$  and  $M$  are given by (4.29), resp. (4.28). Theoretically, identity (1.3) gives the law of the random variable  $T_\epsilon$ , but it is impossible to inverse this Laplace transform. However the explicit form of the Laplace transform of  $T_\epsilon$  permits to get interesting consequences. We can prove that  $T_\epsilon$  admits exponential moments, see Theorem 4.1. Taking the  $\lambda$ -derivative at 0 in (4.29) gives the value of the first order moment of  $T_\epsilon$ , see Propositions 4.6 and 4.8. In particular if  $a = \beta^2$  and  $\epsilon < V_0$ , then:

$$\mathbb{E}[T_\epsilon | V_0 = v] = \frac{1}{b} \left( \frac{1}{v} - \frac{1}{\epsilon} \right).$$

If we go back to the interpretation of  $V_t$  as the size of a tumor at time  $t$ , it is crucial to know how fast the tumor can be reduced to a very small level  $\epsilon$ . In other words, we would like to determine the asymptotic behavior of  $T_\epsilon$ , as  $\epsilon \rightarrow 0$ . The complete and explicit results are given in Theorem 4.3 and the three different regimes are governed by the values of the parameter  $q := \frac{1}{2} - \frac{a}{\beta^2}$ .

We have postponed in Section 5 all the technical proofs of results stated in Sections 2, 3 and 4.

## 2 An approximation of the logistic diffusion by continuous Markov chains

### 2.1 Biological considerations

Markov chains have been already associated with the logistic equation, see [31],[19], [25] and [30] but the convergence to the limit diffusion process has never been proven. Our setting is the one of modeling evolution of a tumor exposed to a treatment, for instance radiotherapy. Random models of tumors have been already considered, see for example [2] and [3]. We propose a very simple model in which a tumor is a collection of independent and non-interacting cells. The initial number  $n$  of cancer cells is considered as a parameter and  $X_n(t)$  stands for the number of malignant cells at time  $t$  and we assume that  $(X_n(t))_{t \geq 0}$  is a Markov chain in continuous time. We prove, see Proposition 2.3, the convergence of  $X_n(t)/n$  as  $n$  tends to infinity to the solution of the logistic equation (1.2). This shows that the logistic S.D.E. admits a biological interpretation and can be considered itself as a pertinent model of tumor size.

Since  $X_n(t)$  is a Markov chain, its behavior is determined by its infinitesimal dynamics. We suppose that it depends on two sequences  $(\alpha(i))_{i \in \mathbb{N}}$ ,  $(\gamma(i))_{i \in \mathbb{N}}$  of real numbers and a function  $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}$ . The first sequence expresses the natural (intrinsic) growth of the tumor. The second one and the function  $\delta$  will take into account the response of cancer cells to the treatment and the environment respectively. We detail how these two antagonist forces act.

- Each cell can duplicate: it dies and gives birth to two new cells. More precisely, a cell gives rise to two new cells between  $t$  and  $t + \Delta t$  with probability:  $\alpha(X_t) \Delta t$ . However, from a mathematical point of view it is more convenient to consider this phenomenon as the addition of one cell to the population. Since all the cells evolve independently from each other, then the offspring distribution is the binomial law  $\mathcal{B}(i; \alpha(i)\Delta t)$  where  $i$  stands for  $X_t$ , i.e.

$$(2.4) \quad \mathbb{P}(X_n(t + \Delta t) = i + 1 | X_n(t) = i) = i \alpha(i) \Delta t + o(\Delta t), \quad \forall i \in \mathbb{N}.$$

- The death of a cancer cell is due to two factors. The first one comes from the limited size of the underlying environment, i.e. when the tumor reaches a "fixed" size, its growing is stopped. We suppose that each cell will die with probability  $\delta(\frac{i}{n})$  between  $t$  and  $t + \Delta t$ . The second cause of death comes from the treatment applied to the tumor, each cell can die with probability  $\gamma(i)$  in the time interval  $[t, t + \Delta t]$ . Finally, we can summarize death of malignant cells as

$$(2.5) \quad \mathbb{P}(X_n(t + \Delta t) = i - 1 | X_n(t) = i) = i [\gamma(i) + \delta(\frac{i}{n})] \Delta t + o(\Delta t), \quad \forall i \in \mathbb{N}^*.$$

The size evolution of the tumor is then well captured by the process  $(X_n(t))$ . However, to deal with the case  $n \rightarrow \infty$ , the right quantity to consider is the proportion  $V_n(t)$  of cancer cells among the whole population, i.e.

$$(2.6) \quad V_n(t) := \frac{1}{n} X_n(t), \quad t \geq 0.$$

It is clear that  $(V_n(t))$  is a continuous time Markov chain which takes its values in  $\frac{\mathbb{N}}{n}$ .

## 2.2 Study of the Markov chain $V_n(t)$

Recall that  $(\alpha(i))_{i \in \mathbb{N}}$  and  $(\gamma(i))_{i \in \mathbb{N}}$  are two sequences of real numbers and  $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}$  is a real valued function.

For all  $x \in \frac{\mathbb{N}}{n}$ , we set:

$$(2.7) \quad q_n(x, y) := \begin{cases} nx\alpha(nx) & \text{if } y = x + \frac{1}{n} \\ nx(\gamma(nx) + \delta(x)) & \text{if } y = x - \frac{1}{n} \text{ and } x > 0 \\ -q_n(x, x + \frac{1}{n}) - q_n(x, x - \frac{1}{n}) & \text{if } y = x \text{ and } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$q_n(x) := -q_n(x, x) = nx\alpha(nx) + nx(\gamma(nx) + \delta(x)), \quad x \in \frac{\mathbb{N}}{n}.$$

It is well-known that a Markov chain in continuous time is characterized by its generator, i.e. its  $Q$ -matrix, see for instance in [26]. According to (2.4), (2.5) and (2.6), we adopt the following definition.

**Definition 2.1** ( $V_n(t), t \geq 0$ ) is the continuous time Markov chain which takes its values in  $\frac{\mathbb{N}}{n}$  and with  $Q$ -matrix:  $Q_n := (q_n(x, y))_{x, y \in \frac{\mathbb{N}}{n}}$ .

We briefly recall the dynamics of the Markov chain  $(V_n(t))$ . Let  $(Y_n(k))_{k \in \mathbb{N}}$  be the skeleton associated with  $V_n(t)$  :  $(Y_n(k))$  is a Markov chain with values in the set  $\{\frac{i}{n}; i \in \mathbb{N}\}$ . Its transition matrix  $\Pi_n$  verifies

$$(2.8) \quad \Pi_n(0, 0) + \Pi_n(0, \frac{1}{n}) = 1, \quad \Pi_n(\frac{i}{n}, \frac{i+1}{n}) + \Pi_n(\frac{i}{n}, \frac{i-1}{n}) = 1 \quad i \geq 1$$

and the above coefficients of matrix  $\Pi_n$  can be expressed in terms of the ones of matrix  $Q_n$  :

$$(2.9) \quad \begin{cases} \mathbb{P}(Y_n(k+1) = x + \frac{1}{n} | Y_n(k) = x) = \Pi_n(x, x + \frac{1}{n}) = \frac{q_n(x, x + \frac{1}{n})}{q_n(x)} \\ \mathbb{P}(Y_n(k+1) = x - \frac{1}{n} | Y_n(k) = x) = \Pi_n(x, x - \frac{1}{n}) = \frac{q_n(x, x - \frac{1}{n})}{q_n(x)}. \end{cases}$$

Let  $\xi_1, \xi_2, \dots$  be a collection of i.i.d. random variables with exponential distribution, independent from the Markov chain  $(Y_n(k))$ . Suppose that  $V_n(0) = Y_n(0) = v_0$  where  $v_0 \in \frac{\mathbb{N}}{n}$ . Then,  $V_n(t)$  remains at level  $v_0$  up to  $T_1 := \frac{\xi_1}{q_n(v_0)}$ . At that first jump time,  $V_n$  moves to  $Y_n(1)$  :

$$V_n(t) = v_0, \quad \forall t \in [0, T_1[, \quad V_n(T_1) = Y_n(1).$$

Set  $Y_n(1) = v_1$ . The next jump time for  $(V_n(t))$  occurs at  $T_2 := T_1 + \frac{\xi_2}{q_n(v_1)}$  and so on.

The process  $(V_n(t))$  is a Markov process and its infinitesimal generator is easy to determine in terms of the  $Q_n$ -matrix, see for instance [26], p. 94.

**Proposition 2.2** *Let  $\mathcal{L}_n$  be the infinitesimal generator of the process  $(V_n(t); t \geq 0)$ . Then, for any  $x \in \frac{\mathbb{N}}{n}$ , and  $f : \frac{\mathbb{N}}{n} \rightarrow \mathbb{R}$ , we have*

$$\mathcal{L}_n f(x) = q_n(x) \left[ \Pi_n(x, x - \frac{1}{n}) f(x - \frac{1}{n}) + \Pi_n(x, x + \frac{1}{n}) f(x + \frac{1}{n}) - f(x) \right]$$

where the coefficients  $\Pi_n(x, x - \frac{1}{n})$  and  $\Pi_n(x, x + \frac{1}{n})$  are given by (2.9).

We now investigate the asymptotic behavior of  $(V_n(t))$  as  $n \rightarrow \infty$ . Since  $(V_n(t))$  is a Markov process we study the convergence of its infinitesimal generator  $\mathcal{L}_n$ .

We introduce two conditions:

$$(2.10) \quad \lim_{i \rightarrow \infty} \frac{\alpha(i)}{i} = \lim_{i \rightarrow \infty} \frac{\gamma(i)}{i} = \frac{\beta^2}{2} > 0$$

and

$$(2.11) \quad \lim_{i \rightarrow \infty} (\alpha(i) - \gamma(i)) = c_0 \in \mathbb{R}.$$

**Proposition 2.3** *1. Suppose that (2.10) and (2.11) hold. Then, for any  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  of class  $C^2$ ,*

$$\lim_{n \rightarrow \infty} \mathcal{L}_n f(x) = \mathcal{L} f(x), \quad \forall x \geq 0$$

with

$$\mathcal{L} f(x) = \frac{\beta^2}{2} x^2 f''(x) + x(c_0 - \delta(x)) f'(x).$$

*2. In the particular case :*

$$(2.12) \quad \alpha(i) = \frac{\beta^2}{2} i + r, \quad \gamma(i) = \frac{\beta^2}{2} i + c, \quad \delta(x) = \frac{r}{\kappa} x$$

conditions (2.10) and (2.11) are satisfied with  $c_0 := r - c$  and the above operator  $\mathcal{L}$  takes the form:

$$\mathcal{L} f(x) = \frac{\beta^2}{2} x^2 f''(x) + x \left( r - c - \frac{r}{\kappa} x \right) f'(x).$$

Moreover, the sequence of Markov chains  $((V_n(t))_{t \geq 0})_{n > 0}$  converges in law to the process  $(V_t)_{t \geq 0}$ , solution of the SDE (2.13) with infinitesimal generator  $\mathcal{L}$ , i.e.

$$(2.13) \quad dV_t = V_t \left( r - c - \frac{r}{\kappa} V_t \right) dt + \beta V_t dB_t$$

with  $V_0 = 1$ .

The proof of Proposition 2.3 is postponed in Section 5.1 and we show that conditions (2.10) and (2.11) are quite natural.

**Remark 2.4** *1. The coefficient  $\beta$  of the diffusion (2.13) comes from the common rate of growth of  $\alpha(i)$  and  $\gamma(i)$  which is of the type  $(\beta^2/2)i$ .*

*2. Proposition 2.3 gives a limit procedure for approximating the law of  $(V_t)_{t \geq 0}$ .*

3. It is convenient to introduce :

$$(2.14) \quad a = r - c, \quad b = \frac{r}{\kappa}.$$

Then, it is clear that (1.2) and (2.13) are equivalent, where  $v = 1$ .

4. We can study SDE (1.2) for itself. However, we will keep here in memory that the diffusion  $V_t$  can model the size of a cancer tumor. In that case the parameters  $r, c, \kappa$  and  $v$  are biologically significant and permit to interpret the results.

Four parameters influence the evolution of the tumor size:

- the initial size  $v > 0$ ;
- the intrinsic growth rate  $r$  of the tumor;
- the rate of decrease  $c$  of the tumor due to the treatment;
- the carrying capacity of the environment  $\kappa$  which is the maximal size the environment can support without negative impact. This capacity could vary according to time and context. It is nevertheless difficult to take into account these fluctuations and we restrict ourselves to  $\kappa$  being a constant. This parameter is actually a fictive upper bound since we will see in Section 3 below that the diffusion can exceed  $\kappa$  but with a "small" probability.

### 3 Study of the logistic diffusion

We now focus on equation (1.2). Relations (2.14) imply that equations (1.1) and (3.15) are equivalent.

The case without noise corresponds to  $\beta = 0$ . Then equation (1.2) reduces to the Verhulst equation:

$$(3.15) \quad \begin{cases} dV_t = V_t (a - b V_t) dt, \\ V_0 = v > 0. \end{cases}$$

(3.15) is a classical ordinary differential equation which can be easily solved.

**Proposition 3.1** *Let  $a, b$  and  $v$  be real numbers such that  $b, v > 0$ .*

1. Equation (3.15) has a unique solution given by:

$$(3.16) \quad V(t) = \begin{cases} \frac{a v e^{at}}{a + b v (e^{at} - 1)} & \text{if } a \neq 0, \\ \frac{v}{1 + bvt} & \text{if } a = 0. \end{cases}$$

2. The function  $V$  takes its values in  $(0, \infty)$ . If  $a > bv$  (resp.  $a < bv, a = bv$ ) then  $V$  is an increasing (resp. decreasing, constant) function. Moreover  $\lim_{t \rightarrow \infty} V(t)$  equals  $a/b$  (resp. 0) when  $a > 0$  (resp.  $a \leq 0$ ).



**Remark 3.2** *It is interesting to interpret item 2 of the above proposition when  $V(t)$  represents the size of tumor at time  $t$ . This actually means that  $a$  and  $b$  are expressed in terms of  $r, c$  and  $\kappa$  via (2.14) whose interpretation is given by Remark 2.4. First, observe that Proposition 2.3 holds when  $\beta = 0$ . In that case the sequence of Markov chains  $(V_n)_{n \in \mathbb{N}}$  converges in distribution to the continuous function  $V$  given by (3.16). Moreover, relations (2.12) reduce to:*

$$\alpha(i) = r, \quad \gamma(i) = c, \quad \delta(i) = \frac{r}{\kappa}i.$$

*Note that  $a > bv$  is equivalent to  $c < r(1 - \frac{V_0}{\kappa})$ . This condition means that the level of the treatment is low and cannot compensate the natural duplication of tumor cells. This is coherent since in that case it is expected that the tumor grows.*

*Note that, in any case, the limit size of the tumor is less than  $\kappa$  since:*

$$(3.17) \quad \lim_{t \rightarrow \infty} V(t) = \frac{a}{b} = \left(1 - \frac{c}{r}\right)\kappa < \kappa.$$

We introduce a parameter  $q$  which governs the behavior of the diffusion (see Proposition 3.3 below):

$$(3.18) \quad q = \frac{1}{2} - \frac{a}{\beta^2}.$$

Note that:

$$q \leq 0 \quad \Leftrightarrow \quad a \geq \frac{\beta^2}{2}.$$

$\gamma(u, v)$  denotes the Gamma-distribution with shape parameter  $u > 0$  and scale parameter  $v > 0$ , i.e. the law with density :

$$\gamma(u, v)(x) := \frac{1}{\Gamma(u)v^u} x^{u-1} e^{-\frac{x}{v}} \mathbf{1}_{(0; \infty)}(x).$$

As usual,  $\gamma(u)$  stands for  $\gamma(u, 1)$ .

**Proposition 3.3** 1. Equation (1.2) admits a unique positive solution  $V$  and

$$(3.19) \quad V_t = \frac{\exp\left\{\beta B_t + \left(a - \frac{\beta^2}{2}\right)t\right\}}{\frac{1}{v} + b \int_0^t \exp\left\{\beta B_s + \left(a - \frac{\beta^2}{2}\right)s\right\} ds}, \quad t \geq 0.$$

2. The diffusion  $V$  is recurrent if and only if  $q \leq 0$ .
3. If  $q < 0$ , the diffusion  $V$  converges in law towards the unique stationary probability distribution  $\gamma(-2q, \frac{\beta^2}{2b})$ .
4. If  $q > 0$ , the diffusion goes a.s. to zero when time goes to infinity.

**Remark 3.4** *The comparison theorem related to one dimensional diffusions (see for instance [18] Prop 2.18 p. 293) can be applied in our context. Let us consider diffusions  $V_t$  and  $V'_t$  being the respective solutions of S.D.E. (1.2) with respective drift coefficients  $a, a'$ :*

$$dV_t = (aV_t - bV_t^2)dt + \beta V_t dB_t; \quad V_0 = v > 0$$

$$dV'_t = (a'V'_t - bV_t'^2)dt + \beta V'_t dB_t; \quad V'_0 = v > 0$$

*such that  $a \leq a'$ . Then  $\mathbb{P}(V_t \leq V'_t; \forall t \geq 0) = 1$ .*

**Remark 3.5** *Let us show that, in the setting of tumor, the behavior of  $V(t)$  is more complex and therefore reacher than the one of the solution of the Verhulst equation. In particular, we have a better understanding of the role of the carrying capacity  $\kappa$ .*

1. *The condition  $q < 0$  is equivalent to  $r > c + \frac{\beta^2}{2}$  and means that the self-replication force is greater than the joint effect of noise and the effect of the treatment. In that case, the size of the tumor evolves after a long period of time to a random state which is gamma distributed. Moreover, by (2.14), the limit size  $V_\infty$  can be written in the following form:  $V_\infty = \left(\frac{\beta^2}{2r} Z\right)\kappa$  with  $Z \sim \gamma(-2q)$  and the law of  $Z$  does not depend on  $\kappa$ . Using moreover (2.14), we get:*

$$\mathbb{E}(V_\infty) = \frac{\beta^2}{2r} (-2q)\kappa = \left(1 - \frac{c + \frac{\beta^2}{2}}{r}\right)\kappa < \kappa.$$

*Note that we recover (3.17) if  $\beta = 0$ .*

*Then, in average, the limit size of the tumor is less than  $\kappa$ . However it can exceed this threshold but the probability that the tumor size is beyond  $\kappa$  is equal to  $\mathbb{P}\left(Z > \frac{2r}{\beta^2}\right) > 0$  and does not depend on  $\kappa$ . However, this quantity is really small if the intrinsic growth rate  $r$  is large.*

2. *Otherwise, if  $q > 0$ , there is resorption of the tumor, as in the case when  $\beta = 0$ , see Proposition 3.1.*
3. *It is clear from (1.2) that  $V_0(t) := E(V(t))$  does not solve the Verhulst equation. However, we will show, see Remark 3.9 below that the solution of (1.2) converges to the deterministic logistic equation, when the intensity  $\beta$  of noise goes to 0.*

Since the coefficients of equation (1.2) are locally Lipschitz, strong uniqueness holds (see for instance page 287 of [18]).

The stochastic logistic equation can be solved explicitly in a more general context than ours. Indeed, replacing the constants  $a$ ,  $b$  and  $\beta$  in (1.2) by functions leads to:

$$(3.20) \quad \begin{cases} dV_t = V_t (a(t) - b(t) V_t) dt + \beta(t) V_t dB_t \\ V_0 = v > 0, t \in [0, T]. \end{cases}$$

Under additional assumptions an explicit solution of (3.20) has been given in [15], Theorem 2.2, page 167 which takes the form (3.19) in our setting.

The existence of the invariant probability measure has been proved in [34]. As for the asymptotic behavior when  $q < 0$ , it follows from Proposition 3.6 below, see Subsection 5.2 for details.

We complete the above proposition by giving the behavior of the diffusion near the two boundary points 0 and  $\infty$ . The classification of the boundary points of a diffusion can be found for instance either in [13] p. 108 or in [5] or in [18]. Let us introduce:

$$(3.21) \quad s(x) = \begin{cases} \int_1^x y^{2q-1} \exp\left\{\frac{2}{\beta^2} b(y-1)\right\} dy & \text{if } x > 1, \\ -\int_x^1 y^{2q-1} \exp\left\{\frac{2}{\beta^2} b(y-1)\right\} dy & \text{if } 0 < x < 1, \end{cases}$$

and

$$(3.22) \quad m(dx) = \frac{2}{\beta^2} x^{-2q-1} e^{-\frac{2}{\beta^2} b(x-1)} \mathbf{1}_{\{x>0\}} dx.$$

**Proposition 3.6** 1. The point 0 is neither an exit nor a starting point, i.e. the process  $V$  cannot start from 0 and neither visits 0 in finite time.

2. The scale function and speed measure of  $V$  are  $s$  and  $m$  respectively.

3. Let  $T_\varepsilon$  be the hitting time of level  $\varepsilon$ ,  $T_\varepsilon := \inf\{t \geq 0, V_t = \varepsilon\}$ .

(a) For any  $\varepsilon \leq v$ ,  $\mathbb{P}(T_\varepsilon < +\infty | V_0 = v) = 1$ .

(b) When  $\varepsilon \geq v$ ,

$$\mathbb{P}(T_\varepsilon < +\infty | V_0 = v) = \begin{cases} 1 & \text{if } q \leq 0 \\ \frac{s(v) - s(0^+)}{s(\varepsilon) - s(0^+)} & \text{if } q > 0. \end{cases}$$

Note that in the case  $q > 0$ ,

$$(3.23) \quad s(v) - s(0^+) = e^{-\frac{2b}{\beta^2} v} \int_0^v y^{2q-1} \exp\left\{\frac{2b}{\beta^2} y\right\} dy = \frac{e^{-\rho}}{2q} v^{2q} M(2q, 1 + 2q, \rho v)$$

where  $\rho := \frac{2b}{\beta^2}$  and  $M$  is the Tricomi function defined by (4.28).

The proof of Proposition 3.6 is based on the explicit calculation of the scale function and speed measure of  $V$ . Using standard analysis we can verify the given criterium related to the boundary points, see Section 5.2 for details.

**Remark 3.7** Suppose that  $q > 0$  and  $\varepsilon \geq v$ . Let  $(\bar{V}_t)$  be the one-sided maximum of  $(V_t)$ , i.e.  $\bar{V}_t := \max_{0 \leq u \leq t} V_u$ . Taking the limit  $t \rightarrow \infty$  in the identity  $\mathbb{P}(T_\varepsilon > t | V_0 = v) = \mathbb{P}(\bar{V}_t < \varepsilon | V_0 = v)$  and using item 3 (b) of Proposition 3.6 lead to:

$$\mathbb{P}(\bar{V}_\infty < \varepsilon | V_0 = v) = \mathbb{P}(T_\varepsilon = \infty | V_0 = v) = \frac{s(\varepsilon) - s(v)}{s(\varepsilon) - s(0^+)}.$$

Therefore relation (3.21) implies that the distribution function of  $\bar{V}_\infty$  is determined.

We now study the integrability of  $V_t$ .

**Proposition 3.8** Let  $V$  be the solution of equation (1.2).

1. For any  $t > 0$  and  $k \geq 2$ , the random variable  $V_t^k$  is integrable.

2. Set  $m_k(t) := \mathbb{E}(V_t^k)$ . Then, the sequence of functions  $(m_k(t))_{k \geq 1}$  satisfies the following recursive equations:

$$(3.24) \quad m_k(t) = v^k + k\left(a + \frac{k-1}{2}\beta^2\right) \int_0^t m_k(s) ds - kb \int_0^t m_{k+1}(s) ds, \quad k \geq 1, t \geq 0.$$

3. For any  $k \geq 1$  and  $t \geq 0$ ,

$$(3.25) \quad m_k(t) \leq \max \left\{ v^k, \left( \frac{2a + (k-1)\beta^2}{2b} \right)^k \right\}.$$

4. Let  $V^0$  be the solution of the Verhulst equation (3.15) with initial condition  $V^0(0) = V_0$ . Then,

$$(3.26) \quad \mathbb{E}(V_t) < V^0(t), \quad \forall t > 0.$$

The proof is a consequence of the Itô formula, see Section 5.3.

**Remark 3.9** 1. Let  $(V_t^\beta)_{t \geq 0}$  be the process defined by (3.19). It is clear that  $V_t^\beta$  converges a.s. and uniformly on compact sets as  $\beta \rightarrow 0$  to the Verhulst solution  $V_t$  given by (3.16). Consequently any solution of (1.2) converges as  $\beta$  goes to 0 to the deterministic logistic equation.

2. Suppose that  $q < 0$ . Thanks to the explicit form of the solution (3.19), we can recover the fact that  $V_t$  converges in distribution towards  $\gamma\left(-2q, \frac{\beta^2}{2b}\right)$ , as  $t \rightarrow \infty$ . Indeed, we modify (3.19) as:

$$\begin{aligned} V_t &= \left[ \frac{e^{-(a-\frac{\beta^2}{2})t-\beta B_t}}{v} + b \int_0^t e^{-(a-\frac{\beta^2}{2})(t-s)-\beta(B_t-B_s)} ds \right]^{-1} \\ &= \left[ \frac{e^{-(a-\frac{\beta^2}{2})t-\beta B_t}}{v} + b \int_0^t e^{-(a-\frac{\beta^2}{2})u-\beta(B_t-B_{t-u})} du \right]^{-1}. \end{aligned}$$

For any fixed  $t > 0$ , the process  $(B_t, (B_t - B_{t-u})_{0 \leq u \leq t})$  is distributed as the process  $(B_t, (B_u)_{0 \leq u \leq t})$ . Consequently:

$$(3.27) \quad V_t \stackrel{(d)}{=} \left[ \frac{e^{-(a-\frac{\beta^2}{2})t-\beta B_t}}{v} + b \int_0^t e^{-(a-\frac{\beta^2}{2})u-\beta B_u} du \right]^{-1}.$$

Note that  $q < 0$  implies that  $a > \frac{\beta^2}{2}$ . Then, we deduce :

$$\lim_{t \rightarrow \infty} e^{-(a-\frac{\beta^2}{2})t-\beta B_t} = 0, \quad a.s.$$

From [7], we know that we have the following identity in law:

$$\int_0^\infty e^{\mu_1 B_s - \mu_2 s} ds \stackrel{(d)}{=} \frac{2}{\mu_1^2 Z}, \quad \mu_2 > 0$$

where  $Z$  is a  $\gamma\left(\frac{2\mu_2}{\mu_1^2}\right)$  distributed random variable.

Finally, taking the limit  $t \rightarrow \infty$  in (3.27), we deduce that  $V_t$  converges in distribution to  $V_\infty \sim \gamma\left(-2q, \frac{\beta^2}{2b}\right)$ .

## 4 Hitting times

Recall  $(V_t)_{t \geq 0}$  is the process solving (1.2). We adopt the classical notation: under the p.m.  $\mathbb{P}_v$  the logistic process starts at level  $v$  and  $\mathbb{E}_v$  stands for the related expectation. For any  $\varepsilon > 0$ , we denote by  $T_\varepsilon$  the first passage time at level  $\varepsilon$  of  $(V_t)_{t \geq 0}$ :

$$T_\varepsilon := \inf\{t \geq 0, V_t = \varepsilon\}$$

with the convention  $\inf \emptyset = +\infty$ .

### 4.1 Calculation of the Laplace transform of $T_\varepsilon$ and asymptotic distributions

We introduce the hypergeometric confluent functions of the first and second type  $M$  and  $U$  which will play a central role in our study. We begin with

$$(4.28) \quad M(a, b, t) = {}_1F_1(a; b; t) = \sum_{n=0}^{\infty} \frac{(a)_n t^n}{(b)_n n!}, \quad a, t \in \mathbb{C}, b \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$$

where  $(a)_0 = 1$  and  $(a)_n = a(a+1)(a+2) \cdots (a+n-1)$  when  $n \geq 1$ .

${}_1F_1(a; b; t)$  is known as the hypergeometric confluent function (see [1] page 504). The second function of interest is the Tricomi function which is given as

$$(4.29) \quad U(a, b, t) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a, b, t) + \frac{\Gamma(b-1)}{\Gamma(a)} t^{1-b} M(a-b+1, 2-b, t), \quad a, t \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}.$$

Function  $U$  can be extended by continuity when  $b$  is an integer, see Step 1 of Section 5.4. Functions  $M$  and  $U$  are also denoted respectively  $\Phi$  and  $\Psi$  in the literature, see Sections 9.9 and 9.10 in [21].

Recall that, from (3.18), we have  $q = \frac{1}{2} - \frac{a}{\beta^2}$ . It is convenient to introduce:

$$(4.30) \quad \rho = \frac{2b}{\beta^2}$$

and

$$(4.31) \quad z_\lambda = \sqrt{\frac{2\lambda}{\beta^2} + q^2} \text{ for all } \lambda \geq -\frac{\beta^2 q^2}{2}.$$

In the calculus of the Laplace transform of  $T_\varepsilon$ , we discriminate  $\varepsilon \leq v$  and  $\varepsilon \geq v$ .

**Theorem 4.1** 1. Let  $v, \varepsilon > 0$ . The Laplace transform of  $T_\varepsilon$  is given by:

$$(4.32) \quad \mathbb{E}_v [\exp(-\lambda T_\varepsilon) 1_{\{T_\varepsilon < \infty\}}] = \left(\frac{v}{\varepsilon}\right)^{\sqrt{\frac{2\lambda}{\beta^2} + q^2} + q} \frac{F\left(\sqrt{\frac{2\lambda}{\beta^2} + q^2} + q, 1 + 2\sqrt{\frac{2\lambda}{\beta^2} + q^2}, \rho v\right)}{F\left(\sqrt{\frac{2\lambda}{\beta^2} + q^2} + q, 1 + 2\sqrt{\frac{2\lambda}{\beta^2} + q^2}, \rho \varepsilon\right)}$$

where  $\lambda \geq 0$  and

$$F := \begin{cases} U & \text{if } v \geq \varepsilon \\ M & \text{if } v \leq \varepsilon. \end{cases}$$

2. There exists a negative real number  $\sigma_c(\varepsilon, v)$  such that:

$$\mathbb{E}_v [\exp(\lambda T_\varepsilon) 1_{\{T_\varepsilon < \infty\}}] < \infty, \quad \forall \lambda < -\sigma_c(\varepsilon, v).$$

**Remark 4.2** 1. Recall that according to Proposition 3.6,  $\mathbb{P}_v(T_\varepsilon < \infty) = 1$  (resp.  $\mathbb{P}_v(T_\varepsilon < \infty) < 1$ ) if  $\varepsilon \leq v$  (resp.  $\varepsilon > v$ ). Taking the limit  $\lambda \rightarrow 0$  in (4.32) we recover the results given in Proposition 3.6.

2. We discuss in Remarks 5.8 and 5.9 the maximal value of  $\lambda > 0$  so that

$$\mathbb{E}_v [\exp(\lambda T_\varepsilon) 1_{\{T_\varepsilon < \infty\}}] \text{ is finite.}$$

3. By the strong Markov property:

$$u \leq v \leq w, \quad \mathbb{E}_w [e^{-\lambda T_u}] = \mathbb{E}_w [e^{-\lambda T_v}] \mathbb{E}_v [e^{-\lambda T_u}].$$

As a consequence, if the left-hand side is finite for  $\lambda < 0$ , the two factors of the right-hand side are finite for this same  $\lambda$ . It implies that the abscissa of convergence  $\sigma_c(\varepsilon, v)$  (introduced in [35]) of the Laplace transform  $\lambda \mapsto \mathbb{E}_v [\exp(-\lambda T_\varepsilon)]$  verifies:

$$0 < v \leq w < \varepsilon \Rightarrow \sigma_c(\varepsilon, v) \geq \sigma_c(\varepsilon, w), \quad 0 < \varepsilon < v \leq w \Rightarrow \sigma_c(\varepsilon, v) \leq \sigma_c(\varepsilon, w).$$

Theoretically, the law of  $T_\varepsilon$  is completely determined the knowledge of its Laplace transform given in Theorem 4.1. However we cannot go further, i.e. calculate the density function of this random variable. This leads naturally to investigate the asymptotic behavior of  $T_\varepsilon$  as  $\varepsilon \rightarrow 0$ . From Proposition 3.6 the random variable  $T_\varepsilon$  is finite if  $\varepsilon \leq v$  and it is clear that  $\varepsilon \mapsto T_\varepsilon$  is decreasing and goes to infinity as  $\varepsilon \rightarrow 0$ . We can actually determine the rate of convergence.

**Theorem 4.3** Recall that the parameters  $\rho$  and  $q$  have been defined by (4.30) and (3.18) respectively.

1. When  $q > 0$ , then  $\sqrt{-\ln \varepsilon} \left( \frac{T_\varepsilon}{-\ln \varepsilon} - \frac{1}{\beta^2 q} \right)$  converges in law to  $\frac{Z}{\beta^2 \sqrt{q^3}}$  where  $Z$  is a standard normal r.v., as  $\varepsilon \rightarrow 0$ .
2. When  $q < 0$ ,  $\varepsilon^{2|q|} T_\varepsilon$  converges in law, when  $\varepsilon$  goes to 0, to  $\frac{\rho^{2q} \Gamma(2|q|)}{\beta^2 |q|} \xi$  where  $\xi$  is an exponential random variable.
3. When  $q = 0$ , then  $\frac{T_\varepsilon}{(\ln \varepsilon)^2}$  converges in law, when  $\varepsilon$  goes to 0, to the hitting time of  $\frac{1}{\sqrt{2a}}$  by a reflected Brownian motion starting at 0.

**Remark 4.4** *The asymptotic behavior of  $T_\varepsilon$  as  $\varepsilon \rightarrow 0$  also holds in mean. Indeed, we deduce directly from Proposition 4.8 below:*

$$\mathbb{E}_v(T_\varepsilon) \sim \begin{cases} \frac{1}{\beta^2 q} \ln(1/\varepsilon) & \text{when } q > 0 \\ \frac{\rho^{2q} \Gamma(2|q|)}{\beta^2 |q|} \varepsilon^{2q} & \text{when } q < 0 \\ \frac{1}{2a} (\ln \varepsilon)^2 & \text{when } q = 0. \end{cases}$$

**Remark 4.5** 1. *According to Proposition 3.3, we know that in the case where  $q > 0$ , then  $V_t$  goes to 0 as  $t \rightarrow 0$ . One way to measure how fast  $V_t$  goes to 0 is to determine the rate of convergence of  $T_\varepsilon$  as  $\varepsilon \rightarrow 0$ . Indeed, we can easily prove that item 1 of Theorem 4.3 is equivalent to  $\lim_{t \rightarrow \infty} \frac{\ln \underline{V}_t}{t} = -\beta^2 q$  in the probability sense, where  $\underline{V}_t := \min_{0 \leq u \leq t} V_u$ .*

2. *When  $q < 0$ , it is easy to deduce from item 2 of Theorem 4.3 that  $t(\underline{V}_t^{-2q})$  converges in distribution to an exponential random variable with rate parameter  $\frac{\beta^2 |q|}{\Gamma(2|q|)} (\frac{\beta^2}{2b})^{2q}$ , as  $t \rightarrow \infty$ .*

3. *When  $q = 0$ , then item 3 of Theorem 4.3 implies that  $\frac{1}{\sqrt{t}} \ln(1/\underline{V}_t)$  converges in distribution to  $\frac{1}{\sqrt{T^*}}$ , where  $T^* := \inf \{t \geq 0, |B_t| = \frac{1}{\sqrt{2a}}\}$ . Recall that its Laplace transform equals  $\frac{1}{\cosh(\sqrt{\frac{\lambda}{a}})}$  and its expectation is  $\frac{1}{2a}$ .*

## 4.2 First moment of the passage time

We distinguish two cases according to the respective positions of the starting point  $v$  and the level  $\varepsilon$ . We begin with  $v < \varepsilon$ . Recall that according to (4.30),  $\rho = \frac{2b}{\beta^2}$ . We introduce two functions involved in the following development of function  $M$ ,

$$(4.33) \quad M(\alpha + x, 1 + \alpha + 2x; y) = \hat{b}_\alpha(y) + b_\alpha(y)x + \mathcal{O}(x^2) \text{ where } \alpha \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$$

$$\hat{b}_\alpha(x) = \sum_{n=0}^{\infty} \frac{\alpha}{\alpha + n} \frac{x^n}{n!} = M(\alpha, \alpha + 1, x)$$

$$b_\alpha(x) = \sum_{n=0}^{\infty} \frac{\alpha}{\alpha + n} \left( \psi(1 + \alpha) - \psi(1 + \alpha + n) + \frac{1}{\alpha} - \frac{1}{\alpha + n} \right) \frac{x^n}{n!}$$

where  $x \in \mathbb{R}$ ,  $\alpha \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$  and  $\psi = \frac{\Gamma'}{\Gamma}$  is the digamma function.

**Proposition 4.6** *Suppose  $v < \varepsilon$ .*

1. If  $q > 0$ ,

$$(4.34) \quad \mathbb{P}_v(T_\varepsilon < \infty) = \left(\frac{v}{\varepsilon}\right)^{2q} \frac{\hat{b}_{2q}(\rho v)}{\hat{b}_{2q}(\rho \varepsilon)}.$$

2. As for the expectation of  $T_\varepsilon$ , we have in the case  $q \geq 0$ :

$$(4.35) \quad \mathbb{E}_v[T_\varepsilon \mathbf{1}_{\{T_\varepsilon < \infty\}}] = \frac{1}{\beta^2 q} \left( \ln\left(\frac{\varepsilon}{v}\right) + \frac{b_{2q}(\rho \varepsilon)}{\hat{b}_{2q}(\rho \varepsilon)} - \frac{b_{2q}(\rho v)}{\hat{b}_{2q}(\rho v)} \right)$$

and otherwise ( $q < 0$ ):

$$(4.36) \quad \mathbb{E}_v[T_\varepsilon] = \frac{1}{\beta^2 |q|} \left[ \ln\left(\frac{\varepsilon}{v}\right) + \sum_{n=1}^{\infty} \frac{1}{(1-2q)_n} \frac{(\rho \varepsilon)^n}{n} - \sum_{n=1}^{\infty} \frac{1}{(1-2q)_n} \frac{(\rho v)^n}{n} \right].$$

**Remark 4.7** 1. When  $v < \varepsilon$  and  $q > 0$ , we have already given, in item 3 b) of Proposition 3.6, the value of  $\mathbb{P}_v(T_\varepsilon < \infty)$ . Using identity (3.23), we can easily prove that probability equals the right hand-side of (4.34).

2. Recall that according to Proposition 3.6,  $\mathbb{P}_v(T_\varepsilon < \infty) = 1$  for any  $q \leq 0$ .

**Proposition 4.8** When  $v > \varepsilon$ , we obtain the first moment of the hitting time to  $\varepsilon$  starting from  $v$ , according to the value of the parameter  $q$ :

$$(4.37) \quad \mathbb{E}_v[T_\varepsilon] = f_q(v) - f_q(\varepsilon)$$

where

(i) If  $2q \in ]0, \infty[ \setminus \{1, 2, \dots\}$  then

$$(4.38) \quad f_q(x) = \frac{1}{\beta^2 q} \left[ \ln x + \sum_{n=1}^{\infty} \frac{1}{n(1-2q)_n} (\rho x)^n + \Gamma(-2q) \sum_{n=0}^{\infty} \frac{2q}{n!(2q+n)} (\rho x)^{2q+n} \right].$$

(ii) If  $2q = m \in \{1, 2, \dots\}$  then

$$(4.39) \quad \begin{aligned} f_q(x) &= \frac{2}{\beta^2 m} \left[ \ln x + \frac{1}{(m-1)!} \sum_{n=1}^{m-1} \frac{(-1)^n (m-n-1)!}{n} (\rho x)^n \right. \\ &\quad \left. + \frac{(-1)^m}{(m-1)!} \sum_{n \geq 0} \frac{1}{(n+m)n!} \left( \psi(n+1) - \ln(\rho x) + \frac{1}{n+m} \right) (\rho x)^{n+m} \right]. \end{aligned}$$

with  $\psi$  the digamma function.

(iii) If  $2q = 0$  then

$$(4.40) \quad \begin{aligned} f_q(x) &= \frac{2}{\beta^2} \left[ -(\gamma + \ln \rho) \ln x - \frac{1}{2} (\ln x)^2 + \sum_{n=1}^{\infty} \left( -H_n - \frac{1}{n} + \gamma + \ln \rho \right) \frac{(\rho x)^n}{n! n} \right] \\ &\quad - \frac{2}{\beta^2} \ln x \sum_{n=1}^{\infty} \frac{(\rho x)^n}{n! n}. \end{aligned}$$



with  $\gamma$  is the Euler constant and  $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$  is the harmonic series.

(iv) If  $2q \in ]-\infty, 0[ \setminus \{-1, -2, \dots\}$  then

$$(4.41) \quad f_q(x) = \frac{1}{\beta^2 |q|} \left[ \ln x + \sum_{n=1}^{\infty} \frac{1}{n(1-2q)_n} (\rho x)^n + \Gamma(-2q) (\rho x)^{2q} \hat{b}_{2q}(\rho x) \right].$$

(v) If  $2q = -m \in \{-1, -2, \dots\}$  then

$$(4.42) \quad f_q(x) = \frac{2}{\beta^2} (m-1)! \sum_{l=1}^m \frac{1}{l} \frac{(\rho x)^{-l}}{(m-l)!}.$$

**Remark 4.9** 1. It is actually possible to prove that for any  $x > 0$ , the map  $q \rightarrow f_q(x)$  is continuous, see Section 5.7, step (ii).

2. The function  $f_q$  takes a simple form when  $2q \in \{1, 2, \dots\}$ ; more specifically when  $q = 1/2$ , i.e.  $m = 1$ , we have

$$f_q(x) = \frac{2}{\beta^2} \left[ \ln x - \sum_{n=1}^{\infty} \frac{(\rho x)^n}{n!} \left( \psi(n) - \ln(\rho x) + \frac{1}{n} \right) \right].$$

Using  $\psi(n) + \frac{1}{n} = H_n - \gamma$  in the above identity and (4.37) leads to

$$\begin{aligned} \mathbb{E}_v[T_\varepsilon] &= \frac{2}{\beta^2} \left[ \ln \left( \frac{v}{\varepsilon} \right) \right. \\ &\quad - \sum_{n=1}^{\infty} \frac{(\rho v)^n}{n!} (H_n - \gamma - \ln(\rho v)) \\ &\quad \left. + \sum_{n=1}^{\infty} \frac{(\rho \varepsilon)^n}{n!} (H_n - \gamma - \ln(\rho \varepsilon)) \right]. \end{aligned}$$

But  $H_n = \sum_{k=1}^n \frac{(-1)^{k-1}}{k} \binom{n}{k}$  implies  $\sum_{n=1}^{\infty} \frac{z^n}{n!} H_n = \exp(z) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k! k} z^k$ , and thus

$$\begin{aligned} \mathbb{E}_v[T_\varepsilon] &= \frac{2 \exp(\rho v)}{\beta^2} \left[ \gamma + \ln(v) - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k! k} (\rho v)^k \right] \\ &\quad + \frac{2 \exp(\rho \varepsilon)}{\beta^2} \left[ \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k! k} (\rho \varepsilon)^k - \gamma - \ln(\varepsilon) \right], \end{aligned}$$

$$\begin{aligned} \mathbb{E}_v[T_\varepsilon] &= \frac{2 \exp(\rho v)}{\beta^2} \left[ \gamma + \ln(v) + \int_0^{\rho v} \frac{e^{-t} - 1}{t} dt \right] \\ &\quad + \frac{2 \exp(\rho \varepsilon)}{\beta^2} \left[ \int_0^{\rho \varepsilon} \frac{1 - e^{-t}}{t} dt - \gamma - \ln(\varepsilon) \right]. \end{aligned}$$

## 5 Proofs

### 5.1 Proof of Proposition 2.3

The infinitesimal generator given in Proposition 2.2 can be decomposed in:

$$\mathcal{L}_n(f)(x) = \mathcal{L}_n^+ f(x) + \mathcal{L}_n^- f(x)$$

with, for every  $x \in \frac{\mathbb{N}}{n}$

$$\mathcal{L}_n^+ f(x) = q_n(x) \Pi_n \left( x, x + \frac{1}{n} \right) \left( f \left( x + \frac{1}{n} \right) - f(x) \right),$$

$$\mathcal{L}_n^- f(x) = q_n(x) \Pi_n \left( x, x - \frac{1}{n} \right) \left( f \left( x - \frac{1}{n} \right) - f(x) \right).$$

From the definition of  $\Pi_n$  (see (2.9)), we have:

$$\mathcal{L}_n^\pm f(x) = q \left( x, x \pm \frac{1}{n} \right) \left( f \left( x \pm \frac{1}{n} \right) - f(x) \right).$$

Using the Taylor-Young formula we get:

$$f \left( x \pm \frac{1}{n} \right) - f(x) = \pm \frac{1}{n} f'(x) + \frac{1}{2n^2} f''(x) + o \left( \frac{1}{n^2} \right)$$

which leads to:

$$\mathcal{L}_n^\pm f(x) = \pm \frac{q \left( x, x \pm \frac{1}{n} \right)}{n} f'(x) + \frac{q \left( x, x \pm \frac{1}{n} \right)}{2n^2} f''(x) + o \left( \frac{1}{n^2} \right).$$

Therefore

$$\mathcal{L}_n(f)(x) = \tau_n^1(x) f'(x) + \tau_n^2(x) \frac{f''(x)}{2}$$

with

$$\tau_n^1(x) = \frac{q \left( x, x + \frac{1}{n} \right) - q \left( x, x - \frac{1}{n} \right)}{n}, \quad \tau_n^2(x) = \frac{q \left( x, x + \frac{1}{n} \right) + q \left( x, x - \frac{1}{n} \right)}{n^2}.$$

From (2.7) we get:

$$(5.43) \quad \tau_1(x) = x [\alpha(nx) - \gamma(nx) - \delta(x)],$$

$$(5.44) \quad \tau_2(x) = x \left[ x \frac{\alpha(nx)}{nx} + x \frac{\gamma(nx)}{nx} + \frac{\delta(x)}{n} \right].$$

To assure the convergence of  $\mathcal{L}_n f(x)$  when  $n$  goes to infinity, it seems natural to ask that both  $\tau_n^1(x)$  and  $\tau_n^2(x)$  converge when  $n$  goes to infinity. According to (5.44), the convergence of  $\tau_n^2(x)$  is established if

$$\lim_{i \rightarrow \infty} \frac{\alpha(i)}{i} = \frac{\beta_1^2}{2} \quad \text{and} \quad \lim_{i \rightarrow \infty} \frac{\gamma(i)}{i} = \frac{\beta_2^2}{2}, \quad \text{with } \beta_1, \beta_2 \geq 0.$$

However, if  $\beta_1 \neq \beta_2$ , then  $\alpha(nx) - \gamma(nx) \sim \frac{\beta_1^2 - \beta_2^2}{2} nx$  and consequently  $\tau_n^1(x)$  diverges as  $n \rightarrow \infty$ . Therefore we impose that  $\beta_1 = \beta_2 = \beta$ . Note that  $\tau_n^1(x)$  converges if and only if (2.11) holds.

Finally, under (2.10) and (2.11) we have proven item 1 of Proposition 2.3.

## 5.2 Proof of Propositions 3.3 and 3.6

For a real valued diffusion with generator :

$$Lf(x) = \frac{1}{2}\sigma^2(x)f''(x) + b(x)f'(x),$$

the associated scale function  $s$  and speed measure  $m$  can be calculated easily. Recall, see for instance [5], Chap. II that we have:

$$s'(x) = \exp \left\{ - \int^x \frac{2b(y)}{\sigma^2(y)} dy \right\}, \quad m(dx) = m'(x)dx = \frac{2}{\sigma^2(x)s'(x)}dx.$$

Since  $\sigma(x) = \beta x$  and  $b(x) = x(a - bx)$  then (3.21) and (3.22) follow directly.

1) We claim that 0 is not an exit time. It is clear that (3.21) and (3.22) imply:

$$(5.45) \quad s'(x) \sim kx^{2q-1}, \quad m'(x) \sim k'x^{-2q-1} \text{ when } x \rightarrow 0.$$

Since  $m(x, 1) = \int_x^1 m'(y) dy$ , we deduce that  $m(0, 1) < \infty$  iff  $q < 0$  and in that case:

$$\int_0^1 m(x, 1) s'(x) dx \sim m(0, 1) \int_0^1 s'(x) dx = +\infty.$$

When  $q > 0$ , then  $m(x, 1) \sim kx^{-2q}$  as  $x \rightarrow 0+$  and consequently

$$\int_0^1 m(x, 1) s'(x) dx = +\infty.$$

The case  $q = 0$  can be treated similarly.

2) We prove that 0 is not a starting point for the diffusion  $(V_t)$ .

• If  $q > 0$ , then

$$\int_0^1 (s(1) - s(x)) m'(x) dx \sim (s(1) - s(0_+)) \int_0^1 m'(x) dx = +\infty.$$

• If  $q = 0$ , then  $s(1) - s(x) \sim -k \ln x$  when  $x \rightarrow 0$ . Therefore:

$$\int_x^1 (s(1) - s(y)) m'(y) dy \sim k \int_x^1 \frac{\ln y}{y} dy \rightarrow \infty, \quad x \rightarrow 0.$$

• The case  $q < 0$  can be studied similarly.

3) Since 0 is neither an exit nor a starting boundary, then 0 is natural. Consequently, it is attractive iff  $s(0^+) > -\infty$ . Relation (5.45) obviously implies that  $s(0^+) > -\infty \Leftrightarrow q > 0$ .

4) We now prove that  $V$  is recurrent iff  $q \leq 0$ . Recall that if  $0 < u < v < w$ , then

$$\mathbb{P}_v(T_w < T_u) = \frac{s(v) - s(u)}{s(w) - s(u)}$$

where  $\mathbb{P}_v$  is the conditional probability  $\mathbb{P}(\cdot | V_0 = v)$ ,  $s$  is the scale function and

$$T_x := \inf\{t \geq 0, V_t = x\}$$

with the convention:  $\inf \emptyset = \infty$ .

a) We consider  $\varepsilon < v < z$ . Then,

$$\mathbb{P}_v(T_\varepsilon < +\infty) = \lim_{z \rightarrow +\infty} \mathbb{P}_v(T_\varepsilon < T_z) = \lim_{z \rightarrow +\infty} \frac{s(z) - s(v)}{s(z) - s(\varepsilon)} = 1$$

since identity (3.21) implies  $s(+\infty) = \infty$ .

b) Let us now consider that  $\varepsilon > v > z$ . Since 0 is not an exit boundary for the diffusion:

$$\mathbb{P}_v(T_\varepsilon < +\infty) = \lim_{z \rightarrow 0} \mathbb{P}_v(T_\varepsilon < T_z) = \lim_{z \rightarrow 0} \frac{s(v) - s(z)}{s(\varepsilon) - s(z)} = \begin{cases} 1 & \text{if } q \leq 0 \\ \frac{s(v) - s(0^+)}{s(\varepsilon) - s(0^+)} < 1 & \text{if } q > 0. \end{cases}$$

As a result, the process  $(V_t)$  is recurrent iff  $q \leq 0$ .

In the case  $q > 0$  the numerator and the denominator can be simplified. Indeed, from (3.21), we have

$$s(x) - s(0^+) = e^{-\frac{2b}{\beta^2}x} \int_0^x y^{2q-1} \exp\left\{\frac{2b}{\beta^2}y\right\} dy.$$

Expanding the exponential gives:

$$s(x) - s(0^+) = x^{2q} e^{-\rho} \sum_{k \geq 0} \frac{\rho^k}{(k+2q)k!} x^k = \frac{e^{-\rho}}{2q} M(2q, 1+2q, \rho x)$$

where  $\rho := \frac{2b}{\beta^2}$  and  $M$  is the Tricomi function defined by (4.28).

5) Suppose that  $q > 0$ . We have to prove that  $\lim_{t \rightarrow \infty} V_t = 0$  almost surely. Note that 0 being an attractive boundary we know that  $V_t$  converges to 0 at infinity with positive probability. Let  $0 < \varepsilon < v/2$  and let us introduce the following sequence of stopping times:  $\sigma_0 := 0$ ,  $\sigma_1 := \inf\{t \geq 0, V_t = \varepsilon\}$  and inductively

$$\sigma_{2k} := \inf\{t \geq \sigma_{2k-1}, V_t = 2\varepsilon\}, \quad \sigma_{2k+1} := \inf\{t \geq \sigma_{2k}, V_t = \varepsilon\}, \quad k \geq 1.$$

According to the above steps 4) a) and b), for any  $k \geq 1$  we have:

$$\mathbb{P}(\sigma_{2k-1} - \sigma_{2k-2} < \infty | \sigma_{2k-2} < \infty) = 1, \quad \rho := \mathbb{P}(\sigma_{2k} - \sigma_{2k-1} < \infty | \sigma_{2k-1} < \infty) < 1.$$

Since  $\rho$  does not depend on  $k$ , these relations imply the existence of a finite random number  $K$  such that  $\sigma_{2K-1} < \infty$  and  $\sigma_{2K} = \infty$  a.s. This means that  $V_t \leq 2\varepsilon$  for any  $t \geq \sigma_{2K-1}$ . In other words  $\lim_{t \rightarrow \infty} V_t = 0$  a.s.

### 5.3 Proof of Proposition 3.8

1) Let  $t > 0$  and  $k \geq 2$ . Using (1.2) and Itô formula we get:

$$(5.46) \quad V_t^k = v^k + k \int_0^t V_s^{k-1} [V_s(a - bV_s) ds + \beta V_s dB_s] + \frac{k(k-1)}{2} \beta^2 \int_0^t V_s^k ds.$$

Define :  $T_n := \inf\{t \geq 0, V_t = n\}$  and

$$\varphi_n(t) := \mathbb{E}(V_{t \wedge T_n}^k).$$

Replacing  $t$  by  $t \wedge T_n$  in (5.46), using the facts that  $V_t \geq 0$  and  $b > 0$  and taking the expectation we get :

$$\varphi_n(t) \leq v^k + \left(ka + \frac{k(k-1)}{2}\beta^2\right) \int_0^t \varphi_n(s) ds. \quad t \geq 0.$$

Gronwall's lemma implies that:

$$\varphi_n(t) = \mathbb{E}(V_{t \wedge T_n}^k) \leq v^k \exp \left\{ t \left(ka + \frac{k(k-1)}{2}\beta^2\right) \right\}.$$

Taking  $n \rightarrow \infty$  in the above inequality and using Fatou lemma give:

$$\mathbb{E}(V_t^k) \leq v^k \exp \left\{ t \left(ka + \frac{k(k-1)}{2}\beta^2\right) \right\}.$$

Consequently,  $\max_{0 \leq u \leq t} E(V_u^k) < \infty$  for any  $k \geq 1$  and  $t \geq 0$ . Therefore we can take the expectation in (5.46). This gives item 2.

**2)** Using Hölder inequality and the fact that  $V_t$  is a non-constant random variable for  $t > 0$ , we have:

$$m_k(t) = \mathbb{E}(V_t^k) < (\mathbb{E}(V_t^{k+1}))^{\frac{k}{k+1}} = m_{k+1}(t)^{\frac{k}{k+1}}.$$

Consequently, relation (3.24) implies

$$(5.47) \quad m'_k(t) < km_k(t) \left( a + \frac{k-1}{2}\beta^2 - bm_k(t)^{1/k} \right), \quad t > 0.$$

Since  $m'_k(t) < 0$  if  $m_k(t) > \left( \frac{2a + (k-1)\beta^2}{2b} \right)^k$  we deduce (3.25).

Inequality (3.26) can be proved similarly using identity (5.47) with  $k = 1$ , i.e.  $m'_1(t) < am_1(t) - bm_1(t)^2$  and  $V'_0(t) = aV_0(t) - bV_0(t)^2$ .

## 5.4 Proof of Theorem 4.1

We begin with two preliminary sections devoted to notations. The proof of Theorem 4.1 actually starts at step 3.

### Step 1: definition of function $U$

When  $b \neq 0, \pm 1, \pm 2, \dots$ , then  $U(a, b; t)$  has been defined by (4.29). Function  $U$  can be extended to any  $a, b \in \mathbb{R}$  as follows:

1. When  $-a \notin \mathbb{N}$ ,  $b = n + 1$  where  $n \in \mathbb{N}$  (cf formula (9.10.6) page 264 in [21]):

$$U(a, n+1; t) = \frac{(-1)^{n+1}}{\Gamma(a-n)} \sum_{k=0}^{\infty} \frac{(a)_k}{(n+k)!k!} [\psi(a+k) - \psi(1+k) - \psi(n+1+k) + \ln t] t^k \\ + \frac{1}{\Gamma(a)} \sum_{k=0}^{n-1} \frac{(-1)^k (n-k-1)! (a-n)_k}{k!} t^{k-n}, \quad t \in \mathbb{R}, a \neq n.$$

where  $\psi(z) = \Gamma'(z)/\Gamma(z)$  is the the digamma function.

We get the particular case  $a = n, b = n + 1$  using the recurrence relation (9.10.16) page 266 in [21]:

$$U(n, n+1; t) = \frac{t}{t+n} U(n, n+2; t).$$

2. When  $a = -m$  ( $m \in \mathbb{N}$ ) and  $b = n + 1$ , according to formula (9.10.7) page 264 in [21]:

$$U(-m, n + 1; t) = (-1)^m \frac{(m + n)!}{n!} M(-m, n + 1; t), \quad n \in \mathbb{N}.$$

3. Using identity formula (9.10.8) page 265 in [21], we get:

$$U(a, -n; t) = t^{1+n} U(1 + a + n, 1 + n; t), \quad a, t \in \mathbb{R}, n \in \mathbb{N}.$$

This identity permits to come down to the two above cases.

Then  $U(a, b; t)$  is meaningful for arbitrary values of the parameters  $a$  and  $b$ . Moreover  $U(a, b; t)$  is an analytic function of  $t > 0$  and an entire function of  $a$  and  $b$ .

### Step 2: notations

It is convenient to recall the values of  $q$ ,  $\rho$  and  $z_\lambda$  given by (3.18), (4.30) and (4.31) resp.:

$$\rho = \frac{2b}{\beta^2}, \quad q = \frac{1}{2} - \frac{a}{\beta^2}, \quad z_\lambda = \sqrt{\frac{2\lambda}{\beta^2} + q^2} \text{ for all } \lambda \geq -\frac{\beta^2 q^2}{2}.$$

Let  $z \mapsto \sqrt{z}$  be the analytic continuation of the square root function to the set  $\mathbb{C} \setminus ]-\infty, 0[$  and define:

$$(5.48) \quad r_1(z) := \sqrt{\frac{2z}{\beta^2} + q}, \quad z \in \mathbb{C} \setminus ]-\infty, -\frac{q^2 \beta^2}{2}],$$

$$(5.49) \quad r_2(\lambda) := i \sqrt{-\frac{2\lambda}{\beta^2} - q^2}, \quad \lambda \in ]-\infty, -\frac{q^2 \beta^2}{2}].$$

### Step 3: proof of item 1

We recall the classical method which permits to calculate its Laplace transform. Let  $\mathcal{L}$  be the generator of process  $(V_t)$ , i.e.

$$(5.50) \quad \mathcal{L}f(x) := \frac{\beta^2}{2} x^2 f''(x) + x(a - bx)f'(x), \quad x > 0.$$

Consider an eigenfunction  $f_\lambda$  associated with the eigenvalue  $\lambda > 0$  of  $\mathcal{L}$ , i.e. a solution of

$$(5.51) \quad \mathcal{L}f = \lambda f.$$

**Lemma 5.1** *Let  $v, \varepsilon > 0$  and  $f_\lambda$  be a solution of (5.51) and such that it is bounded on any interval of the type  $[\varepsilon, \infty[$  (resp.  $[0, \varepsilon]$ ) when  $\varepsilon \leq v$  (resp.  $\varepsilon \geq v$ ), then*

$$(5.52) \quad f_\lambda(\varepsilon) \mathbb{E}_v(e^{-\lambda T_\varepsilon} 1_{\{T_\varepsilon < \infty\}}) = f_\lambda(v).$$

If moreover  $f_\lambda(\varepsilon) \neq 0$ , we have:

$$(5.53) \quad \mathbb{E}_v(e^{-\lambda T_\varepsilon} 1_{\{T_\varepsilon < \infty\}}) = \frac{f_\lambda(v)}{f_\lambda(\varepsilon)}.$$

*Proof:* Since  $f_\lambda$  solves (5.51), the process  $(f_\lambda(V_t)e^{-\lambda t}, t \geq 0)$  is a local martingale. If we choose  $f_\lambda$  such that it is bounded on any interval of the type  $[\varepsilon, \infty[$  (resp.  $[0, \varepsilon]$ ) when  $\varepsilon \leq v$  (resp.  $\varepsilon \geq v$ ) then  $(f_\lambda(V_{t \wedge T_\varepsilon})e^{-\lambda t \wedge T_\varepsilon}, t \geq 0)$  is a bounded process and is therefore a martingale. The stopping theorem gives (5.52)

□

This approach leads us to first determine the eigenfunctions of  $\mathcal{L}$  and their behaviors in the vicinity of 0 and  $+\infty$  (see Lemma 5.2). Secondly we prove, cf Lemma 5.3, that these functions do not vanish.

**Lemma 5.2** *Let  $\lambda \geq -\frac{\beta^2 q^2}{2}$ . We introduce the two following functions:*

$$\begin{aligned} h_\lambda(x) &= x^{z_\lambda+q} U(z_\lambda + q, 1 + 2z_\lambda; \rho x), \quad x > 0 \\ \tilde{h}_\lambda(x) &= x^{z_\lambda+q} M(z_\lambda + q, 1 + 2z_\lambda; \rho x), \quad x > 0 \end{aligned}$$

where  $M$  and  $U$  have been defined by (4.28) and in the above step 1(4.29).

1. For any solution  $f_\lambda$  of (5.51) there exists two constants  $C_1$  and  $C_2$  such that:

$$(5.54) \quad f_\lambda(x) = C_1 h_\lambda(x) + C_2 \tilde{h}_\lambda(x), \quad x > 0.$$

2. The function  $h_\lambda$  (resp.  $\tilde{h}_\lambda$ ) is the unique solution, up to a multiplicative constant, to the equation (5.51) being bounded on every interval  $[\varepsilon, +\infty[$  with  $\varepsilon > 0$  (resp. on  $]0, v]$ , with  $v > 0$ ).

*Proof:* a) According to (5.50), the function  $f_\lambda$  is a solution of the following equation:

$$(5.55) \quad \frac{\beta^2}{2} x^2 f''(x) + (ax - bx^2) f'(x) = \lambda f(x), \quad x > 0.$$

We consider the function  $u$  associated with the function  $f$  by the relation :

$$(5.56) \quad f(x) = x^{-\frac{a}{\beta^2}} e^{\frac{b}{\beta^2}x} u(x).$$

By consecutive derivations we get:

$$(5.57) \quad \frac{f'(x)}{f(x)} + \frac{a - bx}{\beta^2 x} = \frac{u'(x)}{u(x)}$$

and

$$(5.58) \quad \frac{f''(x)}{f(x)} - \left( \frac{f'(x)}{f(x)} \right)^2 - \frac{a}{\beta^2 x^2} = \frac{u''(x)}{u(x)} - \left( \frac{u'(x)}{u(x)} \right)^2.$$

Squaring the two sides of (5.57) gives:

$$(5.59) \quad \left( \frac{u'(x)}{u(x)} \right)^2 - 2 \left( \frac{a - bx}{\beta^2 x} \right) \left( \frac{f'(x)}{f(x)} \right) - \left( \frac{f'(x)}{f(x)} \right)^2 = \left( \frac{a - bx}{\beta^2 x} \right)^2.$$

Substituting (5.59) to (5.58) leads to:

$$\frac{f''(x)}{f(x)} = \frac{u''(x)}{u(x)} - \left(\frac{a-bx}{\beta^2 x}\right)^2 - 2\left(\frac{a-bx}{\beta^2 x}\right) \left(\frac{f'(x)}{f(x)}\right) + \frac{a}{\beta^2 x^2}.$$

Using (5.57), we obtain:

$$\frac{f''(x)}{f(x)} = \frac{u''(x)}{u(x)} + \left(\frac{a-bx}{\beta^2 x}\right)^2 - 2\left(\frac{a-bx}{\beta^2 x}\right) \left(\frac{u'(x)}{u(x)}\right) + \frac{a}{\beta^2 x^2}.$$

We divide (5.55) by  $f$  and then use (5.57) to get:

$$u''(x) = \left(\frac{2\lambda - a}{\beta^2 x^2} + \left(\frac{a - bx}{\beta^2 x}\right)^2\right) u(x).$$

The change of variables  $x = \frac{\beta^2 t}{2b}$  (i.e.  $t = \frac{2b}{\beta^2} x$ ) leads to a Whittaker's equation (page 505 of [1] or [12]), with  $w(t) = u(x)$ .

$$w''(t) + \left[-\frac{1}{4} + \frac{a}{\beta^2} \frac{1}{t} + \left(\frac{1}{4} - z_\lambda^2\right) \frac{1}{t^2}\right] w(t) = 0.$$

This Whittaker's equation admits two linearly independent solutions (see [4], Section 6.9) given by:

$$M_{a/\beta^2, z_\lambda}(t) = \exp\left(-\frac{t}{2}\right) t^{z_\lambda + \frac{1}{2}} M(z_\lambda + q, 1 + 2z_\lambda; t)$$

and

$$W_{a/\beta^2, z_\lambda}(t) = \exp\left(-\frac{t}{2}\right) t^{z_\lambda + \frac{1}{2}} U(z_\lambda + q, 1 + 2z_\lambda; t)$$

where  $M$  and  $U$  are the confluent hypergeometric functions of the first and second kind, defined by (4.28) and (4.29). Then  $u(x) = w(t)$  is a linear combination of these functions  $M$  and  $U$ . Using (5.56) we get (5.54).

b) We now determine the asymptotic behaviors of  $h_\lambda(x)$  and  $\tilde{h}_\lambda(x)$  as  $x \rightarrow 0$  and  $x \rightarrow \infty$ . Since  $\lim_{x \rightarrow 0} M(a, b; x) = 1$  we deduce

$$\lim_{x \rightarrow 0} \tilde{h}_\lambda(x) = \lim_{x \rightarrow 0} \left(x^{z_\lambda + q} M(z_\lambda + q, 1 + 2z_\lambda; \rho x)\right) = 0.$$

Using (4.29) we have:  $h_\lambda(x) \sim Cte x^{q-z_\lambda}$  when  $x \rightarrow 0$ . Since  $z_\lambda > q$ ,  $\lim_{x \rightarrow 0} h_\lambda(x) = \infty$ .

Recall that from [1] p. 504 (13.1.8) and (13.1.4) respectively, we have:

$$U(a, b; t) \sim t^{-a}, \quad M(a, b; t) \sim \frac{\Gamma(b)}{\Gamma(a)} e^t t^{a-b}, \quad t \rightarrow \infty.$$

Consequently:

$$(5.60) \quad \lim_{x \rightarrow \infty} \tilde{h}_\lambda(x) = \infty, \quad \lim_{x \rightarrow \infty} h_\lambda(x) = 1.$$

□

**Lemma 5.3** For any  $x > 0$ ,  $h_\lambda(x)$  and  $\tilde{h}_\lambda(x)$  are positive.



*Proof* The definition (4.28) of function  $M$  implies that  $\tilde{h}_\lambda(x) > 0$  for any  $x > 0$ . According to (5.52) and Lemma 5.2, we have

$$h_\lambda(\varepsilon)\mathbb{E}_v(e^{-\lambda T_\varepsilon}) = h_\lambda(v), \quad \varepsilon \leq v.$$

Therefore, if  $h_\lambda(\varepsilon) = 0$ , then  $h_\lambda(v) = 0$  for any  $v \in [\varepsilon, \infty[$ . This generates a contradiction since  $h_\lambda$  is analytic on  $]0, \infty[$ . Consequently,  $h_\lambda(x) \neq 0$  for any  $x > 0$ . The second identity in (5.60) implies that  $h_\lambda(x)$  is positive. □

It is clear that Lemmas 5.1-5.3 and (5.53) imply item 1 of Theorem 4.1.

#### Step 4: the method to prove item 2

Our strategy is based on the holomorphic extension of the Laplace transform of a non-negative random variable  $X$ . Recall (see [35], chapter 2, page 37) that the convergence abscissa of its Laplace transform:

$$L_X(\lambda) := \mathbb{E}(e^{-\lambda X}), \quad \lambda \geq 0.$$

is the unique real number, possibly infinite,  $\sigma_c \in [-\infty, 0]$  such that:

$$\mathbb{E}(e^{-sX}) < \infty \quad \text{for all } s > \sigma_c \quad \text{and} \quad \mathbb{E}(e^{-sX}) = \infty \quad \text{for all } s < \sigma_c.$$

Then,  $\widehat{L}_X(z) := \mathbb{E}(e^{-zX})$  is the unique holomorphic extension of  $L$  to the domain  $\{z \in \mathbb{C}, \operatorname{Re}(z) > \sigma_c\}$ .

Our approach is based on the following general result.

#### Lemma 5.4 (Landau Theorem)

Let  $F$  be an holomorphic function in a domain  $\mathcal{D}_F$  which contains the real half-line  $(\sigma_F; +\infty)$  where  $\sigma_F \leq 0$  and such that

$$F(\lambda) = L_X(\lambda), \quad \forall \lambda \geq 0.$$

1. Then,  $\sigma_c \leq \sigma_F$ .
2. If  $F$  is not bounded in the neighborhood of  $\sigma_F$ , then  $\sigma_c = \sigma_F$ .

*Proof* The proof is inspired by the one of Theorem 5b, page 58 in [35]. By analytic continuation,

$$(5.61) \quad F(\lambda) = \widehat{L}_X(\lambda), \quad \lambda > \max\{\sigma_c, \sigma_F\}.$$

Suppose that  $\sigma_c > \sigma_F$  and let us show that this inequality generates a contradiction, i.e.

$$(5.62) \quad \mathbb{E}(e^{(-\sigma_c + \varepsilon)X}) < \infty, \quad \text{for some } \varepsilon > 0.$$

Since  $F$  is analytic at  $\sigma_c$ , there exists  $0 < \varepsilon < \sigma_c - \sigma_F$  such that:

$$F(\sigma_c - \varepsilon) = \sum_{k \geq 0} \frac{(-2\varepsilon)^k}{k!} F^{(k)}(\sigma_c + \varepsilon).$$

Relation (5.61) implies

$$F^{(k)}(\sigma_c + \varepsilon) = \widehat{L}_X^{(k)}(\sigma_c + \varepsilon) = (-1)^k \mathbb{E}(X^k e^{-(\sigma_c + \varepsilon)X})$$

and therefore

$$\begin{aligned} F(\sigma_c - \varepsilon) &= \sum_{k \geq 0} \frac{(2\varepsilon)^k}{k!} \mathbb{E}(X^k e^{-(\sigma_c + \varepsilon)X}) \\ &= \mathbb{E}\left(\sum_{k \geq 0} \frac{(2\varepsilon X)^k}{k!} e^{-(\sigma_c + \varepsilon)X}\right) \\ &= \mathbb{E}(e^{(-\sigma_c + \varepsilon)X}). \end{aligned}$$

This proves (5.62). Item 2 is direct consequence of the fact that  $\widehat{L}_X$  is holomorphic in  $\{z \in \mathbb{C}, \operatorname{Re}(z) > \sigma_c\}$ .

□

**Step 5: proof of item 2, when  $v \geq \varepsilon$**

According to Lemmas 5.1 and 5.2, the function

$$g_1(z) = \left(\frac{v}{\varepsilon}\right)^{r_1(z)+q} \frac{U(r_1(z) + q, 1 + 2r_1(z); \rho v)}{U(r_1(z) + q, 1 + 2r_1(z); \rho \varepsilon)}$$

seems to be the good candidate to give an holomorphic extension of the Laplace transform of  $T_\varepsilon$  under  $\mathbb{P}_v$ . We prove a technical result (see Lemma 5.5 below) which permits to prove that  $g_1$  is holomorphic on a suitable set which contains any real number  $\lambda$  such that  $\lambda > -\frac{q^2 \beta^2}{2}$ , see Lemma 5.5. Then, the proof of item 2 of Theorem 4.1 follows immediately. We begin with notations. We introduce

$$\mathcal{U}_* = \{z \in \mathbb{C}; U(z + q, 1 + 2z; \rho \varepsilon) \neq 0\}$$

and function  $g$  as:

$$(5.63) \quad g(z) = \left(\frac{v}{\varepsilon}\right)^{z+q} \frac{U(z + q, 1 + 2z; \rho v)}{U(z + q, 1 + 2z; \rho \varepsilon)}, \quad z \in \mathcal{U}_*.$$

We introduce two new sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$ :

$$\begin{aligned} \mathcal{U}_1 &= \{z \in \mathbb{C} \setminus ]-\infty, -\frac{q^2 \beta^2}{2}]; r_1(z) = \sqrt{\frac{2z}{\beta^2} + q^2} \in \mathcal{U}_*\}, \\ \mathcal{U}_2 &= \{\lambda \in ]-\infty, -\frac{q^2 \beta^2}{2}]; r_2(\lambda) = i\sqrt{-\frac{2\lambda}{\beta^2} - q^2} \in \mathcal{U}_*\}. \end{aligned}$$

The following result is a preliminary step to prove that function  $g_1$  is holomorphic on  $\mathcal{U}_1 \cup \mathcal{U}_2$  (cf Lemma 5.5 below).

**Lemma 5.5** 1. Set  $\mathcal{U}_*$  and function  $g$  are symmetric, i.e.  $z \in \mathcal{U}_* \Rightarrow -z \in \mathcal{U}_*$  and  $g(-z) = g(z)$ .

2.  $\mathcal{U}_*$  is an open set.

3.  $g$  is a meromorphic function over  $\mathbb{C}$  and its restriction to  $\mathcal{U}_*$  is holomorphic.

4. The set  $(\mathcal{U}_1 \cup \mathcal{U}_2)^c$  is discrete and therefore the set  $\mathcal{U}_1 \cup \mathcal{U}_2$  is open in  $\mathbb{C}$ .

*Proof:* a) It is easy to prove that relation (4.29) implies:

$$U(-z + q, 1 - 2z; x) = x^{2z} U(z + q, 1 + 2z; x), \quad x \in \mathbb{R}.$$

We deduce that  $\mathcal{U}_*$  is symmetric and

$$g(-z) = \left(\frac{v}{\varepsilon}\right)^{-z+q} \frac{(\rho v)^{2z} U(z + q, 1 + 2z; \rho v)}{(\rho \varepsilon)^{2z} U(z + q, 1 + 2z; \rho \varepsilon)} = g(z), \quad z \in \mathcal{U}_*.$$

From [21] page 261 section 9.9, we have:

$$(5.64) \quad \frac{M(a, b; z)}{\Gamma(b)} = \sum_{k \geq 0} \frac{(a)_k}{\Gamma(b+k)} \frac{z^k}{k!}.$$

Recall Gamma function admits an holomorphic extension to  $\{z \in \mathbb{C}, -z \notin \mathbb{N}\}$ . It follows that, for any real  $t$ , the maps  $z \mapsto M(z + q, 1 + 2z; t)$  and  $z \mapsto M(q - z, 1 - 2z; t)$  are holomorphic when  $z \notin \{0, \pm \frac{1}{2}, \pm \frac{2}{2}, \pm \frac{3}{2}, \dots\}$ . It is easy to deduce items 2 and 3 from (4.29).

c) Since function  $r_1$  (resp.  $r_2$ ) is continuous, the set  $\mathcal{U}_1 = r_1^{-1}(\mathcal{U}_*)$  is open both in  $\mathbb{C} \setminus ] - \infty, -\frac{q^2 \beta^2}{2}]$  and  $\mathbb{C}$ . However  $\mathcal{U}_2 = r_2^{-1}(\mathcal{U}_*)$  is only open in  $] - \infty, -\frac{q^2 \beta^2}{2}]$ .

d) Item 3 implies that

$$\begin{aligned} \mathcal{U}_{1,c} &:= \{z \in \mathbb{C} \setminus ] - \infty, -\frac{q^2 \beta^2}{2}]; r_1(z) \in \mathcal{U}_*^c\} \\ &= \{z \in \mathbb{C} \setminus ] - \infty, -\frac{q^2 \beta^2}{2}]; U(r_1(z) + q, 1 + 2r_1(z); \rho \varepsilon) = 0\} \end{aligned}$$

is discrete.

Similarly, by analyticity,

$$\begin{aligned} \mathcal{U}_{2,c} &:= \{\lambda \in ] - \infty, -\frac{q^2 \beta^2}{2}]; r_2(\lambda) \in \mathcal{U}_*^c\} \\ &= \{\lambda \in ] - \infty, -\frac{q^2 \beta^2}{2}]; U(r_2(\lambda) + q, 1 + 2r_2(\lambda); \rho \varepsilon) = 0\} \end{aligned}$$

is also discrete.

Note that  $\mathcal{U}_{1,c}$  (resp.  $\mathcal{U}_{2,c}$ ) is the complement of  $\mathcal{U}_1$  (resp.  $\mathcal{U}_2$ ) in  $\mathbb{C} \setminus ] - \infty, -\frac{q^2 \beta^2}{2}]$  (resp.  $] - \infty, -\frac{q^2 \beta^2}{2}]$ ) and  $(\mathcal{U}_1 \cup \mathcal{U}_2)^c = \mathcal{U}_{1,c} \cup \mathcal{U}_{2,c}$ .

In order to prove that the union of  $\mathcal{U}_{1,c}$  and  $\mathcal{U}_{2,c}$  is discrete, it is sufficient to prove that every sequence of complex numbers in  $\mathcal{U}_{1,c}$  cannot have an accumulation point in  $\mathcal{U}_{2,c}$ . Suppose, contrary to our claim, there exists a sequence  $(\lambda_n)_{n \geq 1}$  of points in  $\mathcal{U}_{1,c}$  converging to  $\lambda \in \mathcal{U}_{2,c}$ . Using the definition of  $\mathcal{U}_{1,c}$  and  $\mathcal{U}_{2,c}$ , we have:

$$U(r_1(\lambda_n) + q, 1 + 2r_1(\lambda_n); \rho \varepsilon) = 0, \quad \forall n \geq 1.$$

and

$$U(r_2(\lambda) + q, 1 + 2r_2(\lambda); \rho \varepsilon) = U(-r_2(\lambda) + q, 1 - 2r_2(\lambda); \rho \varepsilon) = 0.$$

Since  $\lambda$  is a negative real number, the sequence

$$(r_1(\lambda_n))_{n \geq 1} = (\sqrt{2\lambda_n/\beta^2 + q^2})_{n \geq 1}$$

presents at least one of the two accumulation points

$$i\sqrt{-2\lambda/\beta^2 - q^2} = r_2(\lambda) \text{ and } -i\sqrt{-2\lambda/\beta^2 - q^2} = -r_2(\lambda).$$

As a consequence, one of these two points is an accumulation point of zeros of the meromorphic function  $z \mapsto U(z + q, 1 + 2z, \frac{2b\varepsilon}{\beta^2})$ . This generates a contradiction, since a meromorphic function has isolated zeros.

□

**Lemma 5.6** *The function  $g_1$  defined on the open set  $\mathcal{U}_1 \cup \mathcal{U}_2$  by*

$$g_1(z) = \begin{cases} g(r_1(z)) & \text{if } z \in \mathcal{U}_1 \\ g(r_2(z)) & \text{if } z \in \mathcal{U}_2 \end{cases}$$

*is holomorphic on  $\mathcal{U}_1 \cup \mathcal{U}_2$ , meromorphic on  $\mathbb{C}$  and the set of poles of  $g_1$  is included in the discrete set  $(\mathcal{U}_1 \cup \mathcal{U}_2)^c$ .*

*Proof:* 1) The restriction of function  $g_1$  to  $\mathcal{U}_1$  could be written as the composition of two holomorphic functions:  $g_1|_{\mathcal{U}_1} = g \circ r_1|_{\mathcal{U}_1}$ ; function  $g_1$  is therefore holomorphic on  $\mathcal{U}_1$ .

If  $g_1$  is continuous in  $\mathcal{U}_1 \cup \mathcal{U}_2$ , according to ([10] exercise 11, pages 100 and 463)  $g_1$  is holomorphic on  $\mathcal{U}_1 \cup \mathcal{U}_2$ . Function  $g_1$  is clearly continuous on  $\mathcal{U}_1$ . We now proceed to show the continuity at any point  $\lambda \in \mathcal{U}_2$ . Let  $(\lambda_n)_{n \geq 1}$  be a sequence of points in  $\mathcal{U}_1 \cup \mathcal{U}_2$ , converging to  $\lambda$ . The two accumulation points of  $(\lambda_n)_{n \geq 1}$  are  $\pm r_2(\lambda)$ . However, since  $g$  is an even function, the sequence  $(g_1(\lambda_n))_{n \geq 1}$  converges to  $g(r_2(\lambda)) = g_1(\lambda)$ .

2) It is clear that the points of  $(\mathcal{U}_1 \cup \mathcal{U}_2)^c$  are the zero of the denominator of (5.63), they are poles or removable singularities of the function  $g_1$ . The function  $g_1$  is meromorphic and the set of its non-essential singularities is the discrete set  $(\mathcal{U}_1 \cup \mathcal{U}_2)^c$ .

□

**Lemma 5.7** *Function  $g_1$  is analytic at any real point  $\lambda \geq -\frac{\beta^2 q^2}{2}$  (resp.  $\lambda \geq 0$ ) if  $q \geq 0$  (resp.  $q < 0$ ).*

*Proof:* As function  $g_1$  is analytic on the set  $\mathcal{U}_1 \cup \mathcal{U}_2$ , we will prove that  $\mathcal{U}_1 \cup \mathcal{U}_2$  contains the interval  $[-\frac{\beta^2 q^2}{2}; +\infty[$  if  $q \geq 0$  and the interval  $[0; +\infty[$  if  $q < 0$ .

$$\begin{aligned} \mathbb{R} \cap \mathcal{U}_1 &= \{\lambda \in ]-\frac{\beta^2 q^2}{2}; +\infty[; r_1(\lambda) \in \mathcal{U}_*\} \\ &= \{\lambda \in ]-\frac{\beta^2 q^2}{2}; +\infty[; U(r_1(\lambda) + q, 1 + 2r_1(\lambda); \rho\varepsilon) \neq 0\}. \end{aligned}$$

If  $q \geq 0$  then the three numbers  $r_1(\lambda) + q$ ,  $1 + 2r_1(\lambda)$  and  $\rho\varepsilon$  are positive and so does the number  $U(r_1(\lambda) + q, 1 + 2r_1(\lambda); \rho\varepsilon)$ , according to [9] page 290 section 6.16.

Therefore, if  $q \geq 0$  then  $\mathbb{R} \cap \mathcal{U}_1 = ]-\frac{\beta^2 q^2}{2}; +\infty[$ .

Similarly, since  $U(q + r_2(-\frac{\beta^2 q^2}{2}), 1 + 2r_2(-\frac{\beta^2 q^2}{2}); \rho\varepsilon) = U(q, 1; \frac{2b\varepsilon}{\beta^2}) \neq 0$  if  $q \geq 0$ , the number  $-\frac{\beta^2 q^2}{2}$  is in  $\mathcal{U}_2$ .

If  $q < 0$  then  $U(q + r_1(0), 1 + 2r_1(0); \rho\varepsilon) = U(0, 1 + 2|q|; \rho\varepsilon) = 1 \neq 0$ ; as a consequence,  $0 \in \mathcal{U}_1$  and  $[0; +\infty[ \subset \mathcal{U}_1$ .

□

**Remark 5.8** *Set*

$$\sigma_1(q) := \max \mathcal{U}_{2,c} = \max \left\{ \lambda \in ]-\infty, -\frac{q^2\beta^2}{2}[ ; U(q + r_2(\lambda), 1 + 2r_2(\lambda); \rho\varepsilon) = 0 \right\},$$

where function  $r_2$  has been defined by (5.49) and by convention,  $\max \emptyset = -\infty$ .

1. Suppose  $q \geq 0$ . We deduce from above that  $\sigma_c(\varepsilon, v) \leq \sigma_1(q)$ , the real number  $\sigma_c(\varepsilon, v)$  equals  $\sigma_1(q)$  if it is a singularity of  $g_1$ , i.e. either  $U(q + r_2(\sigma_1(q)), 1 + 2r_2(\sigma_1(q)); \rho v) \neq 0$  or  $\sigma_1(q)$  is a zero of  $U(q + r_2(\cdot), 1 + 2r_2(\cdot); \rho v)$  with order less than the order of  $U(q + r_2(\cdot), 1 + 2r_2(\cdot); \rho\varepsilon)$ .

2. Similarly, when  $q < 0$ ,  $\sigma_c(\varepsilon, v)$  is lower than  $\max\{\sigma_1(q), \sigma_2(q)\}$ , where

$$\sigma_2(q) := \max \left\{ \lambda \in [-\frac{q^2\beta^2}{2}, 0[, U(q + r_1(\lambda), 1 + 2r_1(\lambda); \rho\varepsilon) = 0 \right\}$$

and function  $r_1$  has been defined by (5.48).

**Step 6: proof of item 2, when  $v < \varepsilon$**

We proceed similarly to the above step 5. We introduce

$$\tilde{\mathcal{U}}_* = \{z \in \mathbb{C}; M(z + q, 1 + 2z; \rho\varepsilon) \neq 0\}$$

where  $M$  is the hypergeometric confluent function defined by (4.28) and  $\tilde{g}$  is the function:

$$\tilde{g}(z) := \left(\frac{v}{\varepsilon}\right)^{z+q} \frac{M(z + q, 1 + 2z; \rho v)}{M(z + q, 1 + 2z; \rho\varepsilon)}, \quad z \in \tilde{\mathcal{U}}_*.$$

We define

$$\tilde{\mathcal{U}}_1 := \{z \in \mathbb{C} \setminus ]-\infty, -\frac{q^2\beta^2}{2}]; r_1(z) = \sqrt{\frac{2z}{\beta^2} + q^2} \in \tilde{\mathcal{U}}_*\}$$

and

$$\tilde{g}_1(\lambda) := \tilde{g}(r_1(z)), \quad z \in \tilde{\mathcal{U}}_1.$$

There is a main difference with the case  $v > \varepsilon$ : the function  $\tilde{g}$  is not even. This implies that  $\tilde{g}_1$  cannot be continuously extended to  $\{\lambda \in ]-\infty, -\frac{q^2\beta^2}{2}]; r_2(\lambda) = i\sqrt{-\frac{2\lambda}{\beta^2} - q^2} \in \tilde{\mathcal{U}}_*\}$ .

However, we can prove similarly to Lemmas 5.5-5.7:

1. The set  $\tilde{\mathcal{U}}_*$  is open and the function  $\tilde{g}$  is holomorphic on  $\tilde{\mathcal{U}}_*$ .
2. The function  $\tilde{g}_1$  is holomorphic on  $\tilde{\mathcal{U}}_1$  and meromorphic on  $\mathbb{C} \setminus ]-\infty, -\frac{q^2\beta^2}{2}]$ .
3. The function  $\tilde{g}_1$  is analytic at any real point  $\lambda > -\frac{\beta^2 q^2}{2}$  (resp.  $\lambda \geq 0$ ) if  $q \geq 0$  (resp.  $q < 0$ ).

**Remark 5.9** *Set*

$$\tilde{\sigma}_2(q) := \max \left\{ \lambda \in [-\frac{q^2\beta^2}{2}, 0[, M(q + r_1(\lambda), 1 + 2r_1(\lambda); \rho\varepsilon) = 0 \right\}.$$

1. Suppose  $q \geq 0$ . We deduce from above that  $\sigma_c(\varepsilon, v) = -\frac{q^2\beta^2}{2}$ .

2. When  $q < 0$ ,  $\sigma_c(\varepsilon, v)$  is in the interval  $[-\frac{q^2\beta^2}{2}; \tilde{\sigma}_2(q)] \subset ]-\infty; 0[$ .

## 5.5 Proof of Theorem 4.3

Since we study the asymptotic behavior of  $T_\varepsilon$  as  $\varepsilon \rightarrow 0$ , we can suppose that  $\varepsilon < v$ .

1) In the case  $q > 0$ , we will prove that the r.v.  $\sqrt{-\ln \varepsilon} \left( \frac{T_\varepsilon}{-\ln \varepsilon} - \frac{1}{\beta^2 q} \right)$  converges in law to  $\frac{1}{\beta^2 \sqrt{q^3}} Z$  where  $Z$  is a standard normal r.v.

By (4.32) and (4.31), the Laplace transform of this r.v. satisfies

$$(5.65) \quad \mathbb{E}_v \left[ \exp \left( -\lambda \sqrt{-\ln \varepsilon} \left( \frac{T_\varepsilon}{-\ln \varepsilon} - \frac{1}{\beta^2 q} \right) \right) \right] = \exp \left( \lambda \frac{\sqrt{-\ln \varepsilon}}{\beta^2 q} \right) \times \frac{v^{q+z_{\lambda(\varepsilon)}} U(q+z_{\lambda(\varepsilon)}, 1+2z_{\lambda(\varepsilon)}; \rho v)}{\varepsilon^{q+z_{\lambda(\varepsilon)}} U(q+z_{\lambda(\varepsilon)}, 1+2z_{\lambda(\varepsilon)}; \rho \varepsilon)}$$

where  $\lambda(\varepsilon) := \frac{\lambda}{\sqrt{-\ln \varepsilon}}$  and

$$z_{\lambda(\varepsilon)} = \sqrt{\frac{2\lambda(\varepsilon)}{\beta^2} + q^2} = q + \frac{\lambda}{q\beta^2\sqrt{-\ln \varepsilon}} - \frac{\lambda^2}{2q^3\beta^4(-\ln \varepsilon)} + \mathcal{O} \left( \frac{1}{(-\ln \varepsilon)^{\frac{3}{2}}} \right).$$

Using this expression or direct consequences of it in (5.65), we obtain

$$(5.66) \quad \mathbb{E}_v \left[ \exp \left( -\lambda \sqrt{-\ln \varepsilon} \left( \frac{T_\varepsilon}{-\ln \varepsilon} - \frac{1}{\beta^2 q} \right) \right) \right] = A_1(\varepsilon) \exp \left( \frac{\lambda \sqrt{-\ln \varepsilon}}{\beta^2 q} \right)$$

where,

$$A_1(\varepsilon) := v^{2q+\mathcal{O}(\frac{1}{\sqrt{-\ln \varepsilon}})} \exp \left( \frac{\lambda \sqrt{-\ln \varepsilon}}{\beta^2 q} - \frac{\lambda^2}{2q^3\beta^4} + \mathcal{O} \left( \frac{1}{\sqrt{-\ln \varepsilon}} \right) \right) A_2(\varepsilon),$$

$$A_2(\varepsilon) := \frac{U \left( 2q + \mathcal{O} \left( \frac{1}{\sqrt{-\ln \varepsilon}} \right), 1 + 2q + \mathcal{O} \left( \frac{1}{\sqrt{-\ln \varepsilon}} \right); \rho v \right)}{\varepsilon^{2q} U \left( 2q + \frac{\lambda}{q\beta^2\sqrt{-\ln \varepsilon}} + \mathcal{O} \left( \frac{1}{\ln \varepsilon} \right), 1 + 2q + \frac{2\lambda}{q\beta^2\sqrt{-\ln \varepsilon}} + \mathcal{O} \left( \frac{1}{\ln \varepsilon} \right); \rho \varepsilon \right)}.$$

Using the following identity ([21] page 505, formula (13.1.29))

$$(5.67) \quad U(a, b, t) = t^{1-b} U(1+a-b, 2-b, t), \quad t \in \mathbb{R}, b \notin \mathbb{N},$$

we obtain the convergence of the function  $U$  of the numerator  $A_2(\varepsilon)$ :

$$(5.68) \quad U \left( 2q + \mathcal{O} \left( \frac{1}{\sqrt{-\ln \varepsilon}} \right), 1 + 2q + \mathcal{O} \left( \frac{1}{\sqrt{-\ln \varepsilon}} \right); \rho v \right) = (\rho v)^{-2q} + o(1).$$

The function  $U$  at the denominator requires a more precise study; Definition (4.29) leads to the following:

$$U \left( 2q + \frac{\lambda}{q\beta^2\sqrt{-\ln \varepsilon}} + \mathcal{O} \left( \frac{1}{\ln \varepsilon} \right), 1 + 2q + \frac{2\lambda}{q\beta^2\sqrt{-\ln \varepsilon}} + \mathcal{O} \left( \frac{1}{\ln \varepsilon} \right); \rho \varepsilon \right)$$

$$\begin{aligned}
&= \frac{\Gamma(-2q - \frac{2\lambda}{q\beta^2\sqrt{-\ln\varepsilon}} + \mathcal{O}\left(\frac{1}{\ln\varepsilon}\right))}{\Gamma(-\frac{\lambda}{q\beta^2\sqrt{-\ln\varepsilon}} + \mathcal{O}\left(\frac{1}{\ln\varepsilon}\right))} M(2q + o(1), 1 + 2q + o(1); \rho\varepsilon) \\
&+ \frac{\Gamma(2q + o(1))}{\Gamma(2q + o(1))} \rho^{-2q+o(1)} \varepsilon^{-2q} \exp\left(\frac{2\lambda\sqrt{-\ln\varepsilon}}{\beta^2q} - \frac{\lambda^2}{q^3\beta^4} + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)\right) \\
&M(o(1), 1 - 2q + o(1); \rho\varepsilon).
\end{aligned}$$

When  $2q$  is not an integer, the first quotient of Gamma functions converges to 0 since  $\frac{\Gamma(-2q+2x)}{\Gamma(x)} \sim \Gamma(-2q)x$ , when  $x \rightarrow 0$ , whereas when  $2q$  is an integer, according to

$$(5.69) \quad \frac{1}{\Gamma(-n+x)} = (-1)^n n! (x - \psi(n+1)x^2 + o(x^2)), \quad x \rightarrow 0, \quad n \in \{1, 2, \dots\}$$

this ratio goes to  $\frac{(-1)^{2q}}{2(2q)!}$  as  $\varepsilon$  goes to 0. In these two cases, the quotient remains bounded when  $\varepsilon$  goes to 0.

By these considerations, the definition (4.28) of  $M$  and properties of Gamma function imply:

$$\begin{aligned}
&U\left(2q + \frac{\lambda}{q\beta^2\sqrt{-\ln\varepsilon}} + \mathcal{O}\left(\frac{1}{\ln\varepsilon}\right), 1 + 2q + \frac{2\lambda}{q\beta^2\sqrt{-\ln\varepsilon}} + \mathcal{O}\left(\frac{1}{\ln\varepsilon}\right); \rho\varepsilon\right) \\
(5.70) \quad &= \mathcal{O}(1) + (1 + o(1))\rho^{-2q+o(1)}\varepsilon^{-2q} \exp\left(\frac{2\lambda\sqrt{-\ln\varepsilon}}{\beta^2q} - \frac{\lambda^2}{q^3\beta^4} + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)\right).
\end{aligned}$$

By using (5.68) and (5.70) in (5.66), we obtain successively:

$$\begin{aligned}
&\mathbb{E}_v \left[ \exp\left(-\lambda\sqrt{-\ln\varepsilon} \left(\frac{T_\varepsilon}{-\ln\varepsilon} - \frac{1}{\beta^2q}\right)\right) \right] = \exp\left(\frac{2\lambda\sqrt{-\ln\varepsilon}}{\beta^2q}\right) \\
&\quad \times \frac{v^{2q+\mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)} \exp\left(-\frac{\lambda^2}{2q^3\beta^4} + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)\right) (\rho v)^{-2q}}{\varepsilon^{2q} \left[ (1 + o(1))\rho^{-2q+o(1)}\varepsilon^{-2q} \exp\left(\frac{2\lambda\sqrt{-\ln\varepsilon}}{\beta^2q} - \frac{\lambda^2}{q^3\beta^4} + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)\right) + \mathcal{O}(1) \right]}, \\
&\mathbb{E}_v \left[ \exp\left(-\lambda\sqrt{-\ln\varepsilon} \left(\frac{T_\varepsilon}{-\ln\varepsilon} - \frac{1}{\beta^2q}\right)\right) \right] = \exp\left(\frac{2\lambda\sqrt{-\ln\varepsilon}}{\beta^2q}\right) \\
&\quad \times \frac{\exp\left(-\frac{\lambda^2}{2q^3\beta^4} + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)\right) (1 + o(1))}{\exp\left(\frac{2\lambda\sqrt{-\ln\varepsilon}}{\beta^2q} - \frac{\lambda^2}{q^3\beta^4} + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)\right) + \mathcal{O}(1)}, \\
&\mathbb{E}_v \left[ \exp\left(-\lambda\sqrt{-\ln\varepsilon} \left(\frac{T_\varepsilon}{-\ln\varepsilon} - \frac{1}{\beta^2q}\right)\right) \right] = \exp\left(\frac{\lambda^2}{2q^3\beta^4} + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)\right).
\end{aligned}$$

The Laplace transform of  $T_\varepsilon$  being analytic in a neighbourhood of zero, we can conclude that the characteristic function of  $\sqrt{-\ln\varepsilon} \left(\frac{T_\varepsilon}{-\ln\varepsilon} - \frac{1}{\beta^2q}\right)$  satisfies:

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}_v \left[ \exp\left(i\xi\sqrt{-\ln\varepsilon} \left(\frac{T_\varepsilon}{-\ln\varepsilon} - \frac{1}{\beta^2q}\right)\right) \right] = \exp\left(\frac{-\xi^2}{2q^3\beta^4}\right).$$

The Lévy continuity theorem establishes the result.

2) We now consider the case  $q < 0$ .

For every  $\varepsilon \in ]0, v]$ , set  $\lambda(\varepsilon) := \lambda\varepsilon^{2|q|}$ . Using standard analysis, we get:

$$z_{\lambda(\varepsilon)} = |q| + \frac{\varepsilon^{2|q|}}{\beta^{2|q|}} + \mathcal{O}(\varepsilon^{4|q|}), \quad \varepsilon \rightarrow 0.$$

a) Recall that the Laplace transform of  $T_\varepsilon$  is given by (4.32). We study the limits of the numerator and the denominator. The first one is straightforward since

$$v^{z_{\lambda(\varepsilon)}+q} = v^{\frac{\lambda\varepsilon^{2|q|}}{\beta^{2|q|}} + \mathcal{O}(\varepsilon^{4|q|})} \xrightarrow{\varepsilon \rightarrow 0} 1$$

implies that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{z_{\lambda(\varepsilon)}+q} U(z_{\lambda(\varepsilon)} + q, 1 + 2z_{\lambda(\varepsilon)}; \rho v) = 1.$$

b) Next, we determine the limit of the denominator of (4.32). According to the definition (4.29) of the function  $U$ , we have:

$$\begin{aligned} U(z_{\lambda(\varepsilon)} + q, 1 + 2z_{\lambda(\varepsilon)}; \rho\varepsilon) &= \frac{\Gamma(-2z_{\lambda(\varepsilon)})}{\Gamma(q - z_{\lambda(\varepsilon)})} M(q + z_{\lambda(\varepsilon)}, 1 + 2z_{\lambda(\varepsilon)}; \rho\varepsilon) \\ &+ \frac{\Gamma(2z_{\lambda(\varepsilon)})}{\Gamma(q + z_{\lambda(\varepsilon)})} (\rho\varepsilon)^{-2z_{\lambda(\varepsilon)}} M(q - z_{\lambda(\varepsilon)}, 1 - 2z_{\lambda(\varepsilon)}; \rho\varepsilon). \end{aligned}$$

Note that:

$$\lim_{\varepsilon \rightarrow 0} \Gamma(2z_{\lambda(\varepsilon)}) = \Gamma(2|q|), \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{z_{\lambda(\varepsilon)}+q} = 1, \quad \lim_{\varepsilon \rightarrow 0} M(q + z_{\lambda(\varepsilon)}, 1 + 2z_{\lambda(\varepsilon)}; \rho\varepsilon) = 1.$$

Using moreover

$$(5.71) \quad \Gamma(x) = \frac{1}{x} - \gamma + o(1), \quad x \rightarrow 0,$$

we have:

$$\frac{1}{\Gamma(q + z_{\lambda(\varepsilon)})} (\rho\varepsilon)^{-2z_{\lambda(\varepsilon)}} = \rho^{2q + \mathcal{O}(\varepsilon^{2|q|})} \varepsilon^{2q + \mathcal{O}(\varepsilon^{2|q|})} \frac{1}{\Gamma} \left( \frac{\lambda\varepsilon^{2|q|}}{\beta^{2|q|}} + \mathcal{O}(\varepsilon^{4|q|}) \right) \xrightarrow{\varepsilon \rightarrow 0} \rho^{2q} \frac{\lambda}{\beta^{2|q|}}.$$

• We begin with studying the case where  $2q$  is not an integer, i.e.  $2|q| \notin \mathbb{N}$ . We have successively:

$$\lim_{\varepsilon \rightarrow 0} \frac{\Gamma(-2z_{\lambda(\varepsilon)})}{\Gamma(q - z_{\lambda(\varepsilon)})} = \frac{\Gamma(2q)}{\Gamma(2q)} = 1, \quad \lim_{\varepsilon \rightarrow 0} M(q - z_{\lambda(\varepsilon)}, 1 - 2z_{\lambda(\varepsilon)}; \rho\varepsilon) = 1.$$

Finally,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{z_{\lambda(\varepsilon)}+q} U(z_{\lambda(\varepsilon)} + q, 1 + 2z_{\lambda(\varepsilon)}; \rho\varepsilon) = 1 + \rho^{2q} \Gamma(2|q|) \frac{\lambda}{\beta^{2|q|}},$$

and



$$(5.72) \quad \lim_{\varepsilon \rightarrow 0} \mathbb{E}_v \left[ e^{-\lambda \varepsilon^{2|q|} T_\varepsilon} \right] = \frac{1}{1 + \rho^{2q} \Gamma(2|q|) \frac{\lambda}{\beta^{2|q|}}}.$$

• The case  $-2q \in \mathbb{N}$  is more complicated.

According to the asymptotic expansion (5.69) of the function  $\frac{1}{\Gamma}$  in the vicinity of  $2q = -2|q|$ , we deduce:

$$\frac{\Gamma(-2z_{\lambda(\varepsilon)})}{\Gamma(q - z_{\lambda(\varepsilon)})} = \frac{\frac{1}{\Gamma}(q - z_{\lambda(\varepsilon)})}{\frac{1}{\Gamma}(-2z_{\lambda(\varepsilon)})} = \frac{\frac{1}{\Gamma}(-2|q| - \frac{\lambda}{\beta^{2|q|}} \varepsilon^{2|q|} + \mathcal{O}(\varepsilon^{4|q|}))}{\frac{1}{\Gamma}(-2|q| - 2\frac{\lambda}{\beta^{2|q|}} \varepsilon^{2|q|} + \mathcal{O}(\varepsilon^{4|q|}))} \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{2}, \quad 2|q| \in \mathbb{N},$$

$$\frac{\Gamma(2z_{\lambda(\varepsilon)})}{\Gamma(q + z_{\lambda(\varepsilon)})} (\rho\varepsilon)^{-2z_{\lambda(\varepsilon)}} \xrightarrow{\varepsilon \rightarrow 0} \Gamma(2|q|) \rho^{2q} \frac{\lambda}{\beta^{2|q|}}.$$

It remains to study  $M(q - z_{\lambda(\varepsilon)}, 1 - 2z_{\lambda(\varepsilon)}; \rho\varepsilon)$ . From (5.64), it can be written as:

$$M(q - z_{\lambda(\varepsilon)}, 1 - 2z_{\lambda(\varepsilon)}; \rho\varepsilon) = 1 + \sum_{k=1}^{\infty} \frac{\Gamma(1 - 2z_{\lambda(\varepsilon)})}{\Gamma(1 - 2z_{\lambda(\varepsilon)} + k)} (q - z_{\lambda(\varepsilon)})_k \rho^k \frac{\varepsilon^k}{k!}.$$

▷ The terms of the series corresponding to  $k \in \{1, \dots, 2|q| - 1\}$  converge to 0 since relation (5.69) implies that the ratio  $\frac{\Gamma(1 - 2z_{\lambda(\varepsilon)})}{\Gamma(1 - 2z_{\lambda(\varepsilon)} + k)}$  is bounded.

▷ The terms of the series corresponding to  $k \in \{2|q| + 1, \dots\}$  converge to 0 when  $\varepsilon$  goes to 0, uniformly in  $k$ : the product  $(q - z_{\lambda(\varepsilon)})_k$  contains  $k - 1$  bounded factors and one factor containing  $\varepsilon^{2|q|}$ ; more precisely,

$$\begin{aligned} (q - z_{\lambda(\varepsilon)})_k &= \left( -2|q| - \frac{\lambda}{\beta^{2|q|}} \varepsilon^{2|q|} + \mathcal{O}(\varepsilon^{4|q|}) \right)_k \\ &\sim (-1)^{2|q|+1} (2|q|)! (-2|q| + k - 1)! \frac{\lambda}{\beta^{2|q|}} \varepsilon^{2|q|}, \quad \varepsilon \rightarrow 0. \end{aligned}$$

$$\frac{\Gamma(1 - 2z_{\lambda(\varepsilon)})}{\Gamma(1 - 2z_{\lambda(\varepsilon)} + k)} \sim \frac{1}{(-1)^{2|q|-1} (2|q| - 1)! \left( -2\frac{\lambda}{\beta^{2|q|}} \varepsilon^{2|q|} \right) ((-2|q| + k)!)}.$$

▷ The only remaining term corresponds to  $k = 2|q|$ :

$$(q - z_{\lambda(\varepsilon)})_{2|q|} \sim (-1)^{2|q|} (2|q|)!$$

and

$$\frac{\Gamma(1 - 2z_{\lambda(\varepsilon)})}{\Gamma(1 - 2z_{\lambda(\varepsilon)} + 2|q|)} \sim \frac{1}{(-1)^{2|q|-1} (2|q| - 1)! \left( -2\frac{\lambda}{\beta^{2|q|}} \varepsilon^{2|q|} \right)}.$$

Consequently

$$\lim_{\varepsilon \rightarrow 0} (q - z_{\lambda(\varepsilon)})_{2|q|} \frac{\Gamma(1 - 2z_{\lambda(\varepsilon)})}{\Gamma(1 - 2z_{\lambda(\varepsilon)} + 2|q|)} \rho^{2|q|} \frac{\varepsilon^{2|q|}}{(2|q|)!} = \frac{\beta^{2|q|}}{2\lambda} \frac{\rho^{2|q|}}{\Gamma(2|q|)},$$

$$\lim_{\varepsilon \rightarrow 0} M \left( q - z_{\lambda(\varepsilon)}, 1 - 2z_{\lambda(\varepsilon)}; \rho\varepsilon \right) = 1 + \frac{\beta^2 |q|}{2\lambda} \frac{\rho^{2|q|}}{\Gamma(2|q|)}.$$

Finally we get (5.72).

**3)** Let us now consider the case  $q = 0$ , then  $\frac{2}{\beta^2} = \frac{1}{a}$  and  $\rho = \frac{b}{a}$ .

For every  $\varepsilon \in ]0, v]$ , let  $\lambda(\varepsilon) = \frac{\lambda}{(\ln \varepsilon)^2}$ . Then, according to (4.31) we have:

$$z_{\lambda(\varepsilon)} = \sqrt{\frac{\lambda}{a}} \frac{1}{\ln(1/\varepsilon)}.$$

As previously, we determine the limits of the numerator and the denominator of (4.32). It is clear that we have:

$$\lim_{\varepsilon \rightarrow 0} v^{z_{\lambda(\varepsilon)}} = 1, \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{z_{\lambda(\varepsilon)}} = e^{-\sqrt{\frac{\lambda}{a}}}.$$

We deduce the limit of the numerator of (4.32):

$$\lim_{\varepsilon \rightarrow 0} v^{z_{\lambda(\varepsilon)}} U \left( z_{\lambda(\varepsilon)}, 1 + 2z_{\lambda(\varepsilon)}; \frac{bv}{a} \right) = U \left( 0, 1; \frac{bv}{a} \right) = 1.$$

We now determine the limit of the denominator of (4.32) using (4.29), (5.71) and

$$\lim_{\varepsilon \rightarrow 0} \frac{\Gamma(-2z_{\lambda(\varepsilon)})}{\Gamma(-z_{\lambda(\varepsilon)})} = \lim_{\varepsilon \rightarrow 0} \frac{\Gamma(2z_{\lambda(\varepsilon)})}{\Gamma(z_{\lambda(\varepsilon)})} = \frac{1}{2},$$

$$\lim_{\varepsilon \rightarrow 0} M \left( z_{\lambda(\varepsilon)}, 1 + 2z_{\lambda(\varepsilon)}; \frac{b\varepsilon}{a} \right) = \lim_{\varepsilon \rightarrow 0} M \left( -z_{\lambda(\varepsilon)}, 1 - 2z_{\lambda(\varepsilon)}; \frac{b\varepsilon}{a} \right) = 1.$$

Combining the previous limits, we get:

$$\lim_{\varepsilon \rightarrow 0} \left\{ \varepsilon^{z_{\lambda(\varepsilon)}} U \left( z_{\lambda(\varepsilon)}, 1 + 2z_{\lambda(\varepsilon)}; \frac{b\varepsilon}{a} \right) \right\} = \frac{1}{2} (1 + e^{2\sqrt{\frac{\lambda}{a}}}) e^{-\sqrt{\frac{\lambda}{a}}} = \cosh \left( \sqrt{\frac{\lambda}{a}} \right)$$

and finally,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}_v [e^{-\lambda(\varepsilon)T_\varepsilon}] = \frac{1}{\cosh \left( \sqrt{\frac{\lambda}{a}} \right)}.$$

This is the Laplace transform of the hitting time of the level  $1/\sqrt{2a}$  by a reflected Brownian motion starting at 0, see for instance ([18] p. 100).

□

## 5.6 Proof of Proposition 4.6

Let  $v < \varepsilon$ .

(i) We first prove the identities (4.34) and (4.35) concerning the case  $q > 0$ . Recall that from Theorem 4.1 we have

$$(5.73) \quad \mathbb{E}_v [\exp(-\lambda T_\varepsilon) 1_{\{T_\varepsilon < \infty\}}] = \frac{v^{z_\lambda+q} M(z_\lambda + q, 1 + 2z_\lambda, \rho v)}{\varepsilon^{z_\lambda+q} M(z_\lambda + q, 1 + 2z_\lambda, \rho \varepsilon)} = \frac{N_1(v)}{N_1(\varepsilon)}$$

with  $\rho = 2b/\beta^2$  and

$$(5.74) \quad z_\lambda = \sqrt{\frac{2\lambda}{\beta^2} + q^2} = q + \frac{\lambda}{\beta^2 q} + o(\lambda).$$

By the identity (4.33) with  $\alpha = 2q$  and  $x := \frac{\lambda}{\beta^2 q} + o(\lambda) = o(1)$ , we obtain

$$\begin{aligned} N_1(v) &= v^{2q} \left( 1 + \ln v \frac{\lambda}{\beta^2 q} + o(\lambda) \right) \left( \hat{b}_{2q}(\rho v) + b_{2q}(\rho v) \frac{\lambda}{\beta^2 q} + o(\lambda) \right) \\ &= v^{2q} \left[ \hat{b}_{2q}(\rho v) + \left( \hat{b}_{2q}(\rho v) \ln v + b_{2q}(\rho v) \right) \frac{\lambda}{\beta^2 q} + o(\lambda) \right]. \end{aligned}$$

Then, we deduce (4.34) and (4.35).

(ii) In the case  $q < 0$ ,  $z_\lambda = -q - \frac{\lambda}{\beta^2 q} + o(\lambda)$  and therefore

$$x := z_\lambda + q = -\frac{\lambda}{\beta^2 q} + o(\lambda) = o(1), \quad \lambda \rightarrow 0.$$

According to the definition of  $M$ , i.e. formula (4.28), we have:

$$(5.75) \quad M(x, 1 - 2q + 2x, \rho v) = 1 + \left( \sum_{k \geq 1} \frac{1}{(1 - 2q)_k} \frac{(\rho v)^k}{k} \right) x + o(x).$$

Consequently,

$$\begin{aligned} N_1(v) &= \left( 1 - \ln v \frac{\lambda}{\beta^2 q} + o(\lambda) \right) \left( 1 - \left[ \sum_{k \geq 1} \frac{1}{(1 - 2q)_k} \frac{(\rho v)^k}{k} \right] \frac{\lambda}{\beta^2 q} + o(\lambda) \right) \\ &= 1 - \left( \ln v + \left[ \sum_{k \geq 1} \frac{1}{(1 - 2q)_k} \frac{(\rho v)^k}{k} \right] \right) \frac{\lambda}{\beta^2 q} + o(\lambda). \end{aligned}$$

This gives (4.36). □

## 5.7 Proof of Proposition 4.8

We suppose here that  $v > \varepsilon$ . We proceed similarly to the proof of Proposition 4.6. We begin with the case  $q > 0$ . Recall that from Theorem 4.1 we have

$$(5.76) \quad \mathbb{E}_v [\exp(-\lambda T_\varepsilon)] = \frac{v^{z_\lambda+q} U(z_\lambda + q, 1 + 2z_\lambda, \rho v)}{\varepsilon^{z_\lambda+q} U(z_\lambda + q, 1 + 2z_\lambda, \rho \varepsilon)} = \frac{N_2(v)}{N_2(\varepsilon)}$$

with  $\rho = 2b/\beta^2$ ,  $z_\lambda$  which has been defined by (5.74) and:

$$N_2(v) := v^{z_\lambda+q}U(z_\lambda + q, 1 + 2z_\lambda, \rho v).$$

(i) Case  $2q \in [0, \infty \setminus \{1, 2, \dots\}]$

We write  $z_\lambda = q + x$  where  $x := \frac{\lambda}{\beta^2 q} + o(\lambda) = o(1)$ . We set:

$$N_2(x, v) := N_2(v) = v^{2q+x}U(2q + x, 1 + 2q + 2x, \rho v).$$

By the definition (4.29) of the function  $U$ , we have:

$$(5.77) \quad U(2q + x, 1 + 2q + 2x, \rho v) = \frac{\Gamma(-2q - 2x)}{\Gamma(-x)} M(2q + x, 1 + 2q + 2x, \rho v) \\ + \frac{\Gamma(2q + 2x)}{\Gamma(2q + x)} (\rho v)^{-2q-2x} M(-x, 1 - 2q - 2x; \rho v).$$

Using (5.69), (5.75) and

$$(5.78) \quad \Gamma(x_0 + x) = \Gamma(x_0)(1 + \psi(x_0)x + o(x)), \quad x \rightarrow 0, \quad x_0 \in \mathbb{R} \setminus \{0, -1, \dots\},$$

we get

$$U(x + 2q, 1 + 2x + 2q, \rho v) = (\rho v)^{-2q} \left\{ 1 + \left[ \psi(2q) - 2 \ln(\rho v) \right. \right. \\ \left. \left. - \sum_{n \geq 1} \frac{1}{(1 - 2q)_n} \frac{(\rho v)^n}{n} - \Gamma(-2q) \hat{b}_{2q}(\rho v) (\rho v)^{2q} \right] x + o(x) \right\}$$

and

$$N_2(x, v) = \rho^{-2q} \left\{ 1 + \left[ \psi(2q) - 2 \ln \rho - \ln v \right. \right. \\ \left. \left. - \sum_{n \geq 1} \frac{1}{(1 - 2q)_n} \frac{(\rho v)^n}{n} - \Gamma(-2q) \hat{b}_{2q}(\rho v) (\rho v)^{2q} \right] x + o(x) \right\}.$$

We easily deduce (4.38).

(ii) In the case where  $2q = m$ , where  $m$  is a positive integer, we can proceed as in the above step. However to perform this method needs tedious calculations. We propose another approach which is based on the fact this case can be obtained by a limit procedure. Indeed let  $f_q$  be the function defined by (4.38) where  $q > 0$  and  $2q$  does not belong to  $\{1, 2, \dots\}$ . We set:

$$2q = m + y, \quad -1 < y < 1.$$

We will take later  $y \rightarrow 0$ , i.e.  $q \rightarrow m/2$ .

We modify  $f_q$  as follows:

$$f_q(x) = A_1(q) + A_2(q)$$

where  $x > 0$  is fixed and:

$$\begin{aligned} A_1(q) &:= \frac{1}{\beta^2 q} \left( \ln x + \sum_{n=1}^{m-1} \frac{1}{n(1-2q)_n} (\rho x)^n \right), \\ A_2(q) &:= \frac{1}{\beta^2 q} \left( \sum_{n \geq 0} \lambda_n(y) (\rho x)^{n+m} \right), \\ \lambda_n(y) &:= \frac{(m+y)\Gamma(-m-y)}{(n+m)\Gamma(1+n-y)} \left( \frac{(\rho x)^y \Gamma(1+n-y)}{\left(1 + \frac{y}{n+m}\right) n!} - 1 \right). \end{aligned}$$

By (5.78) and (5.69), we have:

$$\lim_{y \rightarrow 0} \lambda_n(y) = \frac{(-1)^m}{(n+m)n! (m-1)!} \left( \psi(n+1) - \ln(\rho x) + \frac{1}{n+m} \right).$$

and

$$(5.79) \quad f_{\frac{m}{2}}(x) = \lim_{q \rightarrow \frac{m}{2}} f_q(x) \text{ exists}$$

where

$$\begin{aligned} f_{\frac{m}{2}}(x) &= \frac{2}{\beta^2 m} \left[ \ln x + \frac{1}{(m-1)!} \sum_{n=1}^{m-1} \frac{(-1)^n (m-n-1)!}{n} (\rho x)^n \right. \\ &\quad \left. + \frac{(-1)^m}{(m-1)!} \sum_{n \geq 0} \frac{1}{(n+m)n!} \left( \psi(n+1) - \ln(\rho x) + \frac{1}{n+m} \right) (\rho x)^{n+m} \right]. \end{aligned}$$

We claim that this result permits to prove (4.39). Indeed suppose that  $2q = m$  where  $m \in \{1, 2, \dots\}$ . Recall that  $2q = 1 - \frac{2a}{\beta^2}$ . Let  $(a_n)_{n \geq 1}$  and  $(a'_n)_{n \geq 1}$  such that:

$$a_n < a < a'_n, \quad 2q_n := 1 - \frac{2a_n}{\beta^2} \notin \{1, 2, \dots\}, \quad 2q'_n := 1 - \frac{2a'_n}{\beta^2} \notin \{1, 2, \dots\}$$

and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a'_n = a$ .

Denote  $(V_t^{(a)})_{t \geq 0}$  the diffusion solution of (1.2) with initial value  $v$ , parameters  $a, b$  and  $\beta$ , where  $a, v$  and  $\beta$  are fixed. Then, Remark 3.4 tells us

$$V_t^{(a_n)} \leq V_t^{(a)} \leq V_t^{(a'_n)}, \quad \forall t \geq 0.$$

Since  $\varepsilon < v$ , that relation implies:  $T_\varepsilon^{(a_n)} \leq T_\varepsilon^{(a)} \leq T_\varepsilon^{(a'_n)}$ . Taking the expectation and using the previous step we obtain:

$$f_{q_n}(v) - f_{q_n}(\varepsilon) = \mathbb{E}_v(T_\varepsilon^{(a_n)}) \leq \mathbb{E}_v(T_\varepsilon^{(a)}) \leq f_{q'_n}(v) - f_{q'_n}(\varepsilon) = \mathbb{E}_v(T_\varepsilon^{(a'_n)}).$$

Finally (4.39) follows taking the limit  $n \rightarrow \infty$  in the above inequality and using (5.79).

(iii) Case  $q = 0$ . We have:

$$x := z_\lambda = \sqrt{\frac{2\lambda}{\beta^2}} = o(1), \quad \lambda \rightarrow 0.$$

and in (5.76), the numerator  $N_2(v)$  equals  $N_2(x, v) = (\rho v)^x U(x, 1 + 2x, \rho v)$ . Note that we add an extra term  $\rho^x$  which cancels when we consider the ratio  $N_2(x, v)/N_2(x, \varepsilon)$ . With this choice,  $x \mapsto N_2(x, v)$  is symmetric, see below.

According to the definition (4.29) of the function  $U$  and (5.77), we get:

$$N_2(x, v) = \frac{\Gamma(-2x)}{\Gamma(-x)} (\rho v)^x M(x, 1 + 2x; \rho v) + \frac{\Gamma(2x)}{\Gamma(x)} (\rho v)^{-x} M(-x, 1 - 2x; \rho v).$$

By

$$(5.80) \quad \frac{1}{\Gamma}(x) = x + \gamma x^2 + o(x^2), \quad x \rightarrow 0$$

and

$$M(x, 1 + 2x; y) = 1 + x \left( \sum_{n=1}^{\infty} \frac{y^n}{n! n} \right) + x^2 \left( \sum_{n=1}^{\infty} \left( -H_n - \frac{1}{n} \right) \frac{y^n}{n! n} \right) + o(x^2), \quad x \rightarrow 0,$$

we obtain

$$\begin{aligned} \frac{\Gamma(-2x)}{\Gamma(-x)} (\rho v)^x M(x, 1 + 2x; \rho v) &= \frac{1}{2} + x \left[ \frac{\gamma}{2} + \frac{\ln(\rho v)}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(\rho v)^n}{n! n} \right] \\ + x^2 \left[ \frac{\pi^2}{8} + \frac{\gamma^2}{4} + \frac{\ln^2(\rho v)}{4} - \frac{1}{2} \sum_{n=1}^{\infty} \left( H_n + \frac{1}{n} \right) \frac{(\rho v)^n}{n! n} + \frac{\gamma}{2} \ln(\rho v) + \frac{\gamma}{2} \sum_{n=1}^{\infty} \frac{(\rho v)^n}{n! n} \right. \\ &\quad \left. + \frac{1}{2} \ln(\rho v) \sum_{n=1}^{\infty} \frac{(\rho v)^n}{n! n} \right] + o(x^2). \end{aligned}$$

By adding the same function in  $-x$ , we get:

$$\begin{aligned} N_2(x, v) &= 1 + x^2 \left[ \frac{\pi^2}{4} + \frac{\gamma^2}{2} + \frac{\ln^2(\rho v)}{2} - \sum_{n=1}^{\infty} \left( H_n + \frac{1}{n} \right) \frac{(\rho v)^n}{n! n} + \gamma \ln(\rho v) \right. \\ &\quad \left. + \gamma \sum_{n=1}^{\infty} \frac{(\rho v)^n}{n! n} + \ln(\rho v) \sum_{n=1}^{\infty} \frac{(\rho v)^n}{n! n} + o(x^2) \right]. \end{aligned}$$

Making the quotient  $\frac{N_2(x, v)}{N_2(x, \varepsilon)}$  and replacing  $x^2$  by  $\frac{2\lambda}{\beta^2}$  give (4.40).

(iv) and (v) We now consider  $q < 0$ . Then,

$$x := z_\lambda + q = -\frac{\lambda}{\beta^2 q} + o(\lambda) = o(1), \quad \lambda \rightarrow 0.$$

and the denominator  $N_2(v)$  in (5.76) is equal to  $N_2(x, v) = v^x U(x, 1 - 2q + 2x, \rho v)$ . By (4.29), we have:

$$U(x, 1 - 2q + 2x; \rho v) = \frac{\Gamma(2q - 2x)}{\Gamma(2q - x)} M(x, 1 - 2q + 2x, \rho v)$$

$$+\frac{\Gamma(-2q+2x)}{\Gamma(x)}(\rho v)^{2q-2x}M(2q-x,1+2q-2x,\rho v).$$

a) We first deal with the case  $2q \in ]-\infty, 0[\setminus\{-1, -2, \dots\}$ . By formula (4.29) defining the function  $U$ , (5.69), (5.78), (5.80) and (5.75) we get:

$$U(x, 1+2x-2q; \rho v) = 1 + \left[ \sum_{n=1}^{\infty} \frac{(\rho v)^n}{n(1-2q)_n} - \psi(2q) + \Gamma(-2q)(\rho v)^{2q} \hat{b}_{2q}(\rho v) \right] x + o(x).$$

and we deduce (4.41).

b) The case  $-2q = m \in \mathbb{N}^*$  can be treated either as in the previous step or a limit procedure, see the above item (ii). The second approach is based on the following modification of the function  $f_q$  defined by (4.41) (when  $q < 0$  and  $q \notin \{-1, -2, \dots\}$ ):

$$\begin{aligned} f_q(x) = & \frac{1}{\beta^2 |q|} \left[ \ln x + \sum_{n=1}^{\infty} \frac{\Gamma(1+2|q|)}{\Gamma(1+2|q|+n)} \frac{(\rho x)^n}{n} + \Gamma(2|q|) \sum_{n=0}^{m-1} \frac{2|q|}{n-2|q|} \frac{(\rho x)^{n-2|q|}}{n!} \right. \\ & \left. + \Gamma(2|q|) \frac{2|q|}{m-2|q|} \frac{(\rho x)^{m-2|q|}}{m!} + \Gamma(2|q|) \sum_{n=m+1}^{\infty} \frac{2|q|}{n-2|q|} \frac{(\rho x)^{n-2|q|}}{n!} \right]. \end{aligned}$$

Taking the limit  $q \rightarrow -m/2$  in the above identity implies (4.42).

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