

SECOND ORDER ANALYSIS FOR STRONG SOLUTIONS IN THE OPTIMAL CONTROL OF PARABOLIC EQUATIONS*

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Abstract. In this paper we provide a second order analysis for *strong solutions* in the optimal control of parabolic equations. We consider first the case of box constraints on the control in a general setting and then, in the case of a quadratic Hamiltonian, we also impose final integral constraints on the state. For the first problem, in a rather general framework, we prove a characterization of the quadratic growth property in the *strong sense*, i.e., for admissible controls whose associated *states* are uniformly near to the state of the nominal control. Assuming a quadratic Hamiltonian in the state constrained case, we provide a sufficient second order optimality condition for the aforementioned quadratic growth property, which does not impose the uniqueness of the Lagrange multiplier set. As a consequence of our results, in the quadratic Hamiltonian case, under some continuity assumptions on the data we prove that the notions of quadratic growth in the strong sense coincide with the more standard notion of quadratic growth in the *weak sense*, i.e., with respect to controls which are uniformly near to the nominal one.

Key words. optimal control, semilinear parabolic equations, strong solutions, quadratic growth

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1. Introduction. Besides its applications in several fields such as biochemistry [18], inverse problems [2], and biology [28], the optimal control of systems governed by nonlinear parabolic equations is one of the prototypes, together with the case of optimal control of hyperbolic systems, of optimization problems of evolutive systems. We refer the reader to the monographs [22, 27] for a rather complete review of the theory, the associated numerical analysis, and some interesting applications (for the latter see in particular [22, Chapter 1]).

In this work we consider the optimal control problem of a semilinear parabolic equation where bounds constraints are imposed on the control and also, when a more particular structure is assumed for the cost and the dynamics, we consider *finitely many* constraints on the state. Thus, in the latter case, our constraints are partially polyhedral in the sense of [9]. For the sake of clarity and also because of its analogy with the corresponding study for ordinary differential equations, we suppose that we have integral constraints on the final state. In the same spirit as [3], we can consider several notions of local solutions and of local solutions satisfying a quadratic growth property. Namely, we will say that \bar{u} is a *weak local solution* of the problem if it minimizes the cost locally on the constraint set with respect to the L^∞ -norm. Moreover, if locally in the L^∞ -norm the difference between the cost of an admissible control u and the cost of \bar{u} is greater than a positive constant (independent of u) times the square of the L^2 -norm of $u - \bar{u}$, we will say that \bar{u} is a weak local solution of the problem

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satisfying the quadratic growth condition in the weak sense. A characterization of the latter property, as well as its equivalence for some specific problems with a stronger notion of solution, are studied in [25] (see also [16, 5, 15, 7] and the references therein for other studies on the topic). Another type of local solution, whose definition goes back to the beginning of the theory of calculus of variations (see [21] for detailed analysis and [4] for a short survey), is the following: the nominal control \bar{u} is a *strong local solution* of the problem if it minimizes the cost with respect to controls whose corresponding states are uniformly close to the nominal state. If the corresponding quadratic growth property holds true, then we will say that \bar{u} is a strong local solution satisfying the quadratic growth property in the strong sense. We refer the reader to section 3 for precise definitions of the notions introduced above.

Our aim in this work is to study second order optimality conditions for strong solutions satisfying the quadratic growth condition. Following the ideas in [3], the key tool is a decomposition result for the variation of the cost in terms of large and small perturbations in the L^∞ -norm of the nominal control, which is proved in Theorem 4.3. We use this decomposition result to prove our main results, which are a characterization of the quadratic growth property in the strong sense for general problems with bound constraints on the control, proved in Theorem 3.5, and, under a more particular structure, a sufficient second order optimality condition for the quadratic growth property when final state constraints are also considered (see Theorem 3.10). Using the latter second order sufficient optimality condition and the results in [25] we are able to obtain in Theorems 3.12 and 3.13 some characterizations of the quadratic growth property in the state constrained case. Loosely speaking, for the pure control constraint and the state constrained cases, our results state that if a stronger form of the classical Pontryagin's principle (see [17, 12, 14]) and the usual coercivity of the second derivative of the Lagrangian hold true, then the quadratic growth property in the strong sense is satisfied. Conversely, both aforementioned conditions are also necessary, which yields the characterization of this property.

As a consequence of the previous findings, we obtain some unexpected results. Under some continuity assumptions on the data, if the Hamiltonian associated to the problem is quadratic and strictly convex with respect to the control and, in the state constrained case, a constraint qualification condition holds true, then the notions of quadratic growth in the weak and the strong sense are *equivalent*. Of course, these arguments provide also the proof of the analogous statements, seemingly unnoticed before this work, in the case of optimal control problems of ordinary differential equations (see [10, 11]) and of semilinear elliptic equations (see [3]).

The article is organized as follows. In section 2 we fix some notation and we recall some basic results on linear parabolic equations. In section 3 we state the optimal control problem (CP_0) , where only bound constraints are imposed, and problem (CP_1) , where final state constraints are considered under the assumption that the cost is quadratic and the dynamics is affine, as functions of the control variable. Since the article is somewhat technical, we have preferred to state and comment on our main results in this section, postponing their proof until the final section. Namely, we state the characterizations of the quadratic growth property for strong solutions separately for both problems (CP_0) and (CP_1) , and also the equivalence with the corresponding property for weak solutions under some structural assumptions. At the end of section 3, some open problems regarding the state constrained case are also discussed. Section 4 is technical and devoted to the proof of the decomposition result, which is central in the proof of our results. Finally, using the results in section 4, we prove in section 5 the results stated in section 3.

2. Preliminaries. From now on we fix a nonempty bounded open set $\Omega \subseteq \mathbb{R}^d$ ($d \in \mathbb{N}$) with a *smooth* boundary. Let us also fix $T > 0$ and set $Q =]0, T[\times \Omega$ and $\Sigma =]0, T[\times \partial\Omega$. Given $s \in [1, \infty]$ and $k \in \mathbb{N}$, we denote by $\|\cdot\|_s$ and $\|\cdot\|_{W^{k,s}}$ the standard norms in $L^s(\Omega)$ and $W^{k,s}(\Omega)$, respectively. For $s_1, s_2 \in [1, \infty)$, we set $L^{s_1, s_2}(Q) := L^{s_1}([0, T]; L^{s_2}(\Omega))$ (see, e.g., [20]), which can be identified with the set of measurable functions $f : Q \rightarrow \mathbb{R}$ satisfying

$$\|f\|_{s_1, s_2} := \left(\int_0^T \left(\int_{\Omega} |f(t, x)|^{s_2} dx \right)^{\frac{s_1}{s_2}} dt \right)^{\frac{1}{s_1}} < \infty.$$

Endowed with the norm $\|\cdot\|_{s_1, s_2}$ we have that $L^{s_1, s_2}(Q)$ is a Banach space. We also denote by $\|\cdot\|_{\infty}$ the standard norm in $L^{\infty}(Q)$. For $s \in]1, \infty[$ consider the space

$$(2.1) \quad \mathcal{V}^s(Q) := L^s([0, T]; W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega)) \cap W^{1,s}([0, T]; L^s(\Omega)),$$

which endowed with the natural norm

$$\|z\|_{\mathcal{V}^s} := \|z\|_{s,s} + \|\partial_t z\|_{s,s} + \sum_{i=1}^d \|\partial_{x_i} z\|_{s,s} + \sum_{i,j=1}^d \|\partial_{x_i x_j} z\|_{s,s}$$

is a Banach space. The following properties of the spaces \mathcal{V}^s will play an important role in our analysis. For the proof we refer the reader to the monograph [6].

LEMMA 2.1. *Let $1 < s \leq s_i < \infty$ with $i = 1, 2$. Then, the following assertions hold true:*

(i) *The space $\mathcal{V}^s(Q)$ is continuously embedded in $L^{s_1, s_2}(Q)$ if*

$$(2.2) \quad \frac{1}{s} - \frac{1}{s_1} + \frac{d}{2} \left(\frac{1}{s} - \frac{1}{s_2} \right) \leq 1.$$

Moreover, if (2.2) is a strict inequality, then the embedding is compact. In particular, if

$$(2.3) \quad \left(1 + \frac{d}{2} \right) \left(\frac{1}{s} - \frac{1}{s_1} \right) \leq 1,$$

then $\mathcal{V}^s(Q)$ is continuously embedded in $L^{s_1, s_1}(Q)$ (compactly embedded if the inequality is strict).

(ii) *The space $\mathcal{V}^s(Q)$ is continuously embedded in $C^{0,1-1/s}([0, T]; L^s(\Omega))$ (the space of $(1 - 1/s)$ -Hölder mappings with values in $L^s(\Omega)$).*

(iii) *The space $\mathcal{V}^s(Q)$ is compactly embedded in $C(\overline{Q})$ (the space of continuous functions on \overline{Q}) if $s > (d + 2)/2$.*

Given $\xi_0 \in W_0^{2-\frac{2}{s}, s}(\Omega)$, with $s > (d + 2)/2$, $a \in L^{\infty}(Q)$, and $v \in L^{s,s}(Q)$, recall (see, e.g., [19, Chapter 4, Theorem 9.1]) that the *linear parabolic equation*

$$(2.4) \quad \begin{aligned} \partial_t \xi - \Delta \xi + a(t, x)\xi &= v(t, x) && \text{in } Q, \\ \xi &= 0 && \text{in } \Sigma, \\ \xi(0, \cdot) &= \xi_0(\cdot) && \text{in } \Omega \end{aligned}$$

admits a unique strong solution $\xi[\xi_0, v] \in \mathcal{V}^s(Q)$, i.e., the equation is satisfied almost everywhere and $\xi[\xi_0, v](0, \cdot)$ (which is well defined by Lemma 2.1(ii)) is equal to ξ_0 .

Moreover, there exists $c_s > 0$ such that following estimate holds true (assuming $\xi_0 = 0$, for simplicity):

$$(2.5) \quad \|\xi[0, v]\|_{\mathcal{V}^s} \leq c_s \|v\|_{s,s}.$$

Using Aubin’s theorem (see [1, 26]), it can be easily checked (see, e.g., [25]) that the linear mapping $v \in L^{2,2}(Q) \rightarrow \xi[0, v] \in L^{2,2}(Q) \cap C([0, T]; L^2(\Omega))$ is continuous when $L^{2,2}(Q)$ and $L^{2,2}(Q) \cap C([0, T]; L^2(\Omega))$ are endowed with the weak and the strong topologies, respectively. Finally, the following estimate (see [25]) will be useful in section 4:

$$(2.6) \quad \|\xi[0, v](\cdot, T)\|_1 + \|\xi[0, v]\|_{1,1} \leq c_1 \|v\|_{1,1} \quad \text{for some } c_1 > 0.$$

3. Optimal control problems and main results. Throughout the article, we consider a *controlled semilinear parabolic equation*. More precisely, given a *control* $u \in L^\infty(Q)$ and $\varphi : Q \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, we consider the equation

$$(3.1) \quad \begin{aligned} \partial_t y - \Delta y + \varphi(t, x, y, u) &= 0 && \text{in } Q, \\ y &= 0 && \text{in } \Sigma, \\ y(0, \cdot) &= y_0(\cdot) && \text{in } \Omega. \end{aligned}$$

Let us now fix some notation and standard assumptions which will guarantee the well-posedness of (3.1). For a function $\psi = \psi(t, x, y, u)$ differentiable with respect to (w.r.t.) the third and fourth coordinates we will write $\psi_{(y,u)}(t, x, y, u)$ for $D_{(y,u)}\psi(t, x, y, u)$ and $\psi_y(t, x, y, u)$, $\psi_u(t, x, y, u)$ for the partial derivatives of ψ w.r.t. y and u , respectively. Regarding second order derivatives, we will write

$$(3.2) \quad \begin{aligned} \psi_{yy}(t, x, y, u) &:= D_{yy}^2 \psi(t, x, y, u), \quad \psi_{uu}(t, x, y, u) := D_{uu}^2 \psi(t, x, y, u), \\ \psi_{yu}(t, x, y, u) &:= D_{yu}^2 \psi(t, x, y, u), \\ \psi_{(y,u)^2}(t, x, y, u)(z, v)^2 & \\ &:= \psi_{yy}(t, x, y, u)z^2 + 2\psi_{yu}(t, x, y, u)zv + \psi_{uu}(t, x, y, u)v^2 \quad \forall (z, v) \in \mathbb{R}^2. \end{aligned}$$

Our first assumption is as follows.

- (H1) (i) The initial state y_0 belongs to $W_0^{2-\frac{2}{s},s}(\Omega)$ with $s > (d + 2)/2$.
- (ii) The function φ is measurable and for all $R > 0$ there exists $c = c(R) > 0$ such that

$$-c(1 + |y|^2) \leq \varphi(t, x, y, u) \quad \forall (t, x, y) \in Q \times \mathbb{R}, \quad |u| \leq R.$$

- (iii) For almost all (a.a.) $(t, x) \in Q$ the function $\varphi(t, x, \cdot, \cdot)$ is C^2 , the mapping $(t, x) \rightarrow \varphi(t, x, \cdot, \cdot) \in C^2(\mathbb{R}^2)$ is measurable, and there exists $c = c(R)$ such that $|y| \leq R$, $|u| \leq R$ and $|y'| \leq R$, $|u'| \leq R$ imply that

$$(3.3) \quad \begin{aligned} |D_{(y,u)}\varphi(t, x, 0, 0)| + |D_{(y,u)^2}\varphi(t, x, 0, 0)| &\leq c, \\ |D_{(y,u)}\varphi(t, x, y, u) - D_{(y,u)}\varphi(t, x, y', u')| &\leq c(|y - y'| + |u - u'|), \\ |D_{(y,u)^2}\varphi(t, x, y, u) - D_{(y,u)^2}\varphi(t, x, y', u')| &\leq c(|y - y'| + |u - u'|), \end{aligned}$$

where in the above inequalities we use a matrix norm norm for the terms involving $D_{(y,u)^2}\varphi$. Denoting by $C^{\alpha,\beta}(\bar{Q})$ the space of functions defined on \bar{Q} that are α -Hölder continuous w.r.t. the time variable t and β -Hölder continuous w.r.t. the space variable x , the following result holds true (see, e.g., [12, 17, 25]).

PROPOSITION 3.1. Under assumption (H1) for any $u \in L^\infty(Q)$, (3.1) admits a unique solution $y[u] \in \mathcal{V}^s(Q) \cap C^{\beta/2, \beta}(\bar{Q})$ for all $s \in]1, \infty[$.

Remark 3.2. The result stated in the above proposition holds true under weaker assumptions than (H1) (see [12, 17, 25]). As a matter of fact, the hypothesis on the second derivatives of φ are not necessary for the well-posedness of (3.1), but they are fundamental in order to provide second order expansions of the mapping $u \mapsto y[u]$. In addition, existence of a weak solution $y \in L^2([0, T]; H_0^1(\Omega)) \cap C(\bar{Q})$ can be proved under the weaker assumptions that $y_0 \in C(\bar{\Omega})$, $y_0(x) = 0$ for all $x \in \partial\Omega$ and that Ω has a Lipschitz boundary (see, e.g., [12, 17]).

We introduce now some functions that will model the cost functional and also some constraints on the final state $y[u](T, \cdot)$. Let $\ell : Q \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $\Phi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $\Phi_E : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^{n_E}$, and $\Phi_I : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^{n_I}$ be such that the following holds.

(H2) The function ℓ is measurable and satisfies (H1)(iii) with ℓ in place of φ . Moreover, for $\psi = \Phi, \Phi_E^i, \Phi_I^j$ ($i = 1, \dots, n_E$ and $j = 1, \dots, n_I$) we have that

- (i) ψ is measurable,
- (ii) for all $x \in \Omega$, $\psi(x, \cdot)$ is C^2 , the mapping $x \mapsto \psi(x, \cdot) \in C^2(\mathbb{R})$ is measurable, and there exists a constant $c = c(R) > 0$ such that for all $y, y' \in \mathbb{R}$ with $|y| \leq R, |y'| \leq R$, we have that

$$(3.4) \quad \begin{aligned} |\psi_y(x, 0)| + |\psi_{yy}(x, 0)| &\leq c, & |\psi_y(x, y) - \psi_y(x, y')| &\leq c|y - y'|, \\ |\psi_{yy}(x, y) - \psi_{yy}(x, y')| &\leq c|y - y'|. \end{aligned}$$

Let us define the cost function $J : L^\infty(Q) \rightarrow \mathbb{R}$ as

$$(3.5) \quad J(u) := \int_Q \ell(t, x, y[u](t, x), u(t, x)) dt dx + \int_\Omega \Phi(x, y[u](T, x)) dx,$$

where we recall that $y[u]$ is defined in Proposition 3.1. Notice that under assumptions (H1)–(H2), the function J is well defined. The following assumption, requiring a quadratic (respectively, affine) dependence of the cost ℓ (respectively, the dynamics φ) with respect to u , will be required in some of our results

(H3) There exist $\underline{\ell} > 0$, functions $\ell_1, \varphi_1 \in L^\infty(Q)$ and $\ell_2, \varphi_2 : Q \times \mathbb{R} \rightarrow \mathbb{R}$ such that for a.a. $(t, x) \in Q$ we have $\ell_1(t, x) \geq \underline{\ell}$ and

$$\ell(t, x, y, u) = \ell_1(t, x)u^2 + \ell_2(t, x, y), \quad \varphi(t, x, y, u) = \varphi_1(t, x)u + \varphi_2(t, x, y).$$

Now, given $a, b \in L^\infty(Q)$ such that $\text{ess inf}\{b(t, x) - a(t, x) ; (t, x) \in Q\} > 0$, let us define the set

$$\mathcal{K}_0 := \{u \in L^\infty(Q) ; a(t, x) \leq u(t, x) \leq b(t, x) \text{ almost everywhere (a.e.) in } Q\}.$$

In this work we will consider two optimal control problems. In the first one only bound constraints are imposed on the control u , i.e., the problem is

$$(CP_0) \quad \inf_{u \in L^\infty(Q)} J(u) \quad \text{s.t.} \quad u \in \mathcal{K}_0.$$

In the second problem we consider *finitely many* state constraints on the state $y[u](T, \cdot)$. In order to introduce the problem, define $G_E : L^\infty(Q) \rightarrow \mathbb{R}^{n_E}, G_I : L^\infty(Q) \rightarrow \mathbb{R}^{n_I}$,

and $G : L^\infty(Q) \rightarrow \mathbb{R}^{n_E+n_I}$ as

$$(3.6) \quad \begin{aligned} G_E^i(u) &:= \int_\Omega \Phi_E^i(x, y[u](T, x)) dx & \text{for } i = 1, \dots, n_E, \\ G_I^j(u) &:= \int_\Omega \Phi_I^j(x, y[u](T, x)) dx & \text{for } j = 1, \dots, n_I, \\ G(u) &:= (G_E(u), G_I(u)). \end{aligned}$$

By assumption (H2) the map G is well defined. Let us set $\mathbb{R}_- := \{y \in \mathbb{R} ; y \leq 0\}$ and consider the sets

$$\mathcal{K}_1 = \{0\}^{n_E} \times \mathbb{R}_-^{n_I}, \quad \mathcal{K}_2 := \{u \in L^\infty(Q) ; G(u) \in \mathcal{K}_1\}, \quad \mathcal{K} := \mathcal{K}_0 \cap \mathcal{K}_2.$$

Our second optimal control problem is

$$(CP_1) \quad \inf_{u \in L^\infty(Q)} J(u) \quad \text{s.t. } u \in \mathcal{K}.$$

In the study of (CP_1) we will assume that (H3) is satisfied (however, we refer the reader to subsection 3.3 for some extensions and related open questions).

Now, let us fix the notions of solutions considered in this work.

DEFINITION 3.3. *We say that:*

(i) \bar{u} is a weak local solution of (CP_0) (resp., (CP_1)) if there exists $\varepsilon > 0$ such that

$$(3.7) \quad J(\bar{u}) \leq J(u) \quad \forall u \in \mathcal{K}_0 \quad (\text{resp., } \mathcal{K}) \quad \text{such that } \|u - \bar{u}\|_\infty \leq \varepsilon.$$

If, in addition, there exists $\alpha > 0$ such that

$$(3.8) \quad J(\bar{u}) + \frac{\alpha}{2} \|u - \bar{u}\|_{2,2}^2 \leq J(u) \quad \forall u \in \mathcal{K}_0 \quad (\text{resp., } \mathcal{K}) \quad \text{such that } \|u - \bar{u}\|_\infty \leq \varepsilon,$$

we will say that \bar{u} is a weak local solution satisfying the quadratic growth condition.

(ii) \bar{u} is a L^s -weak local solution ($s \in [1, \infty]$) of (CP_0) (resp., (CP_1)) if there exists $\varepsilon > 0$ such that

$$(3.9) \quad J(\bar{u}) \leq J(u) \quad \forall u \in \mathcal{K}_0 \quad (\text{resp., } \mathcal{K}) \quad \text{such that } \|u - \bar{u}\|_{s,s} \leq \varepsilon.$$

If, in addition, there exists $\alpha > 0$ such that

$$(3.10) \quad J(\bar{u}) + \frac{\alpha}{2} \|u - \bar{u}\|_{2,2}^2 \leq J(u) \quad \forall u \in \mathcal{K}_0 \quad (\text{resp., } \mathcal{K}) \quad \text{such that } \|u - \bar{u}\|_{s,s} \leq \varepsilon,$$

we will say that \bar{u} is a L^s -weak local solution satisfying the quadratic growth condition.

(iii) \bar{u} is strong local solution of (CP_0) (resp., (CP_1)) if there exists $\varepsilon > 0$ such that

$$(3.11) \quad J(\bar{u}) \leq J(u) \quad \forall u \in \mathcal{K}_0 \quad (\text{resp., } \mathcal{K}) \quad \text{such that } \|y[u] - \bar{y}\|_\infty \leq \varepsilon,$$

where $\bar{y} := y[\bar{u}]$. If, in addition, there exists $\alpha > 0$ such that

$$(3.12) \quad J(\bar{u}) + \frac{\alpha}{2} \|u - \bar{u}\|_{2,2}^2 \leq J(u) \quad \forall u \in \mathcal{K}_0 \quad (\text{resp., } \mathcal{K}) \quad \text{such that } \|y[u] - \bar{y}\|_\infty \leq \varepsilon,$$

we will say that \bar{u} is a strong local solution satisfying the quadratic growth condition.

The following remark will be very useful.

Remark 3.4. Since for $u \in \mathcal{K}_0$ and $1 \leq s_1 \leq s_2 < \infty$ we have that

$$\int_Q |u(t, x)|^{s_2} dt dx \leq M^{s_2-s_1} \int_Q |u(t, x)|^{s_1} dt dx$$

with $M := \max\{\|a\|_\infty, \|b\|_\infty\}$, it holds that $\|u\|_{s_2, s_2} \leq M^{1-\frac{s_1}{s_2}} \|u\|_{s_1, s_1}^{\frac{s_1}{s_2}}$ and thus the relative topologies of $L^{s_1, s_1}(Q) \cap \mathcal{K}_0$ and $L^{s_2, s_2}(Q) \cap \mathcal{K}_0$ are equivalent. This implies that for both problems (CP_0) and (CP_1) , for $s \in [1, \infty[$ the notions of L^s -weak solution and L^1 -weak solution (resp., L^s -weak solution and L^1 -weak solution satisfying the quadratic growth condition) are equivalent. As a consequence, using that Lemma 2.1(iii) and Lemma 4.1 in the next section imply that $\|y[u] - \bar{y}\|_\infty \leq c\|u - \bar{u}\|_{s, s}$ for a constant $c > 0$ and s large enough, we obtain that $\|u - \bar{u}\|_{1,1} \rightarrow 0$, for $u \in \mathcal{K}_0$, implies that $\|y[u] - \bar{y}\|_\infty \rightarrow 0$.

3.1. Main results for problem (CP_0) . In this section we present a characterization of property (3.12), i.e., for strong *local solutions* of problem (CP_0) , which are assumed to exist. The proof of the results in the present section will be provided in section 5.1.

Let us set some notation in order to concisely write optimality conditions for problem (CP_0) . The *Hamiltonian* $H : Q \times \mathbb{R}^3 \mapsto \mathbb{R}$ is defined as

$$(3.13) \quad H(t, x, y, p, u) := \ell(t, x, y, u) - p\varphi(t, x, y, u).$$

Given $\bar{u} \in \mathcal{K}_0$ and setting $\bar{y} := y[\bar{u}]$, the adjoint state \bar{p} associated to \bar{u} is defined as the unique weak solution of

$$(3.14) \quad \begin{aligned} -\partial_t p - \Delta p - H_y(t, x, \bar{y}, p, \bar{u}) &= 0 \text{ in } Q, \\ p &= 0 \text{ in } \Sigma, \\ p(T, \cdot) &= \Phi_y(\cdot, \bar{y}(T, \cdot)) \text{ in } \Omega. \end{aligned}$$

By (H2)(ii) and the maximum principle one has $\bar{p} \in L^\infty(Q)$. Let us define the linear form $Q_1[\bar{u}] : L^{2,2}(Q) \rightarrow \mathbb{R}$ and the quadratic form $Q_2[\bar{u}] : L^{2,2}(Q) \rightarrow \mathbb{R}$ as (recall the notation defined in (3.2))

$$(3.15) \quad \begin{aligned} Q_1[\bar{u}]v &:= \int_Q H_u(t, x, \bar{y}(t, x), \bar{p}(t, x), \bar{u}(t, x))v(t, x) dt dx, \\ Q_2[\bar{u}](v) &:= \int_Q H_{(y,u)^2}(t, x, \bar{y}(t, x), \bar{p}(t, x), \bar{u}(t, x))(\xi[v](t, x), v(t, x))^2 dt dx \\ &\quad + \int_\Omega \Phi_{yy}(x, \bar{y}(T, x))\xi[v](T, x)^2 dx, \end{aligned}$$

where $\xi[v] \in \mathcal{V}^2(Q)$ is the unique solution of

$$(3.16) \quad \begin{aligned} \partial_t \xi - \Delta \xi + \varphi_y(t, x, \bar{y}(t, x), \bar{u}(t, x))\xi &= -\varphi_u(t, x, \bar{y}(t, x), \bar{u}(t, x))v \text{ in } Q, \\ \xi &= 0 \text{ in } \Sigma, \\ \xi(0, \cdot) &= 0 \text{ in } \Omega. \end{aligned}$$

Under (H1)–(H2) we have that $Q_1[\bar{u}]$ and $Q_2[\bar{u}]$ are well defined. Now, let us recall that given a Banach space $(X, \|\cdot\|_X)$ and $K \subseteq X$, the *tangent cone* to K at u is defined as

$$(3.17) \quad T_K(u) := \{v \in X \mid \exists u(\tau) = u + \tau v + o(\tau) \in K, \tau > 0, \|o(\tau)/\tau\|_X \rightarrow 0, \text{ as } \tau \downarrow 0\}.$$

Defining

$$A_a(\bar{u}) := \{(t, x) \in Q ; \bar{u}(t, x) = a(t, x)\}, \quad A_b(\bar{u}) := \{(t, x) \in Q ; \bar{u}(t, x) = b(t, x)\},$$

it is easy to check (see, e.g., [25]) that

$$T_{\mathcal{K}_0}(\bar{u}) := \{v \in L^{2,2}(Q) ; v \geq 0 \text{ in } A_a(\bar{u}) \text{ and } v \leq 0 \text{ in } A_b(\bar{u})\}.$$

Finally, the *critical cone* to \mathcal{K}_0 at \bar{u} is defined as $C_{\mathcal{K}_0}(\bar{u}) := T_{\mathcal{K}_0}(\bar{u}) \cap \text{Ker}Q_1[\bar{u}]$, where $\text{Ker}Q_1[\bar{u}] := \{v \in L^{2,2}(Q) ; Q_1[\bar{u}]v = 0\}$. It is easily seen (see, e.g., [25]) that

$$C_{\mathcal{K}_0}(\bar{u}) = \{v \in L^{2,2}(Q) ; v \geq 0 \text{ in } A_a(\bar{u}) \text{ and } v \leq 0 \text{ in } A_b(\bar{u}), \\ H_u(t, x, \bar{y}(t, x), \bar{p}(t, x), \bar{u}(t, x))v(t, x) = 0 \text{ a.e. in } Q\}.$$

Now, we have all the elements to state our main result for problem (CP_0) which provides a *characterization* of the quadratic growth property for strong local solutions (see (3.12) in Definition 3.3(iii)).

THEOREM 3.5. *Assume (H1)–(H2). Then, the following assertions are equivalent:*

(i) *The control \bar{u} is a strong local solution of (CP_0) satisfying the quadratic growth condition (3.12).*

(ii) *There exists $\alpha > 0$ such that the following conditions hold true:*

(ii.1) *For almost all $(t, x) \in Q$ we have that*

$$(3.18) \quad H(t, x, \bar{y}(t, x), \bar{p}(t, x), \bar{u}(t, x)) + \frac{\alpha}{2}|u - \bar{u}(t, x)|^2 \leq H(t, x, \bar{y}(t, x), \bar{p}(t, x), u)$$

for all $u \in [a(t, x), b(t, x)]$.

(ii.2) *For all $v \in C_{\mathcal{K}_0}(\bar{u})$ it holds that $Q_2[\bar{u}](v) \geq \alpha\|v\|_{2,2}^2$.*

The following corollary states an interesting consequence of the previous result.

COROLLARY 3.6. *Suppose that (H1)–(H2) hold true and let $s \in [1, \infty[$. Then, \bar{u} is a strong local solution of (CP_0) satisfying the quadratic growth condition iff \bar{u} is a L^s -weak local solution of (CP_0) satisfying the quadratic growth condition.*

Thus, the notions of weak local solutions in L^s ($s \in [1, \infty[$) and strong solutions are equivalent. The next result shows that if the structural assumption (H3) holds true and the data of (CP_0) are continuous, we have the *equivalence* of the three notions of local solution in Definition 3.3 satisfying the quadratic growth condition. Note that under (H3) there exists at least a global solution \hat{u} , i.e., $J(\hat{u}) \leq J(u)$ for all $u \in \mathcal{K}_0$ (see, e.g., [27, section 5.3]).

THEOREM 3.7. *Assume (H1)–(H3) and that a, b, ℓ_1 , and φ_1 (resp., ℓ_2 and φ_2) admit extensions to \bar{Q} (resp., $\bar{Q} \times \mathbb{R}$) which are continuous. Moreover, suppose that Φ_y admits a locally Hölder continuous extension to $\bar{\Omega} \times \mathbb{R}$ satisfying $\Phi_y(x, 0) = 0$ for all $x \in \partial\Omega$. Then, for $\bar{u} \in \mathcal{K}_0$ we have the equivalence of (3.8), (3.10) (for any $s \in [1, \infty[$), and (3.12).*

We end this section with the following example.

Example 3.8. Consider the optimal control problem

$$\begin{aligned} \inf \int_Q \left[\frac{N}{2}u(t, x)^2 + \ell(t, x, y(t, x)) \right] dt dx, \\ \partial_t y - \Delta y + \varphi(y) = u + f \quad \text{in } Q, \\ \text{s.t.} \quad y = 0 \quad \text{in } \Sigma, \quad y(0, \cdot) = y_0(\cdot), \\ a \leq u \leq b \quad \text{in } Q, \end{aligned}$$

where $N > 0$, ℓ is continuous on $\bar{Q} \times \mathbb{R}$ and satisfies (H2), φ is C^2 and it is increasing, $f \in L^\infty(Q)$, $y_0 \in W_0^{2-\frac{2}{s},s}(\Omega)$, and a, b are continuous on \bar{Q} . Then, for this problem the notions of weak, L^s -weak (for any $s \in [1, \infty[$), and strong solutions satisfying the quadratic growth condition are equivalent.

On the other hand, if we consider the problem with nonconvex Hamiltonian

$$\begin{aligned} \inf \int_Q [u(t, x)^2 - u(t, x)^4 - y(t, x)^4] dt dx, \\ \partial_t y - \Delta y = u \quad \text{in } Q, \\ \text{s.t. } y = 0 \quad \text{in } \Sigma, \quad y(0, \cdot) = 0, \\ 0 \leq u \leq 2 \quad \text{in } Q, \end{aligned}$$

using [25, Theorem 5.8] we obtain that $\bar{u} \equiv 0$ is a weak local solution satisfying the quadratic growth condition. However, due to the $-u^4$ term, we have that \bar{u} is not an L^s -weak solution for any $s \in [1, \infty[$.

3.2. Main results for problem (CP₁). In contrast to the generic results presented for problem (CP₀), we will assume now that the cost function and the dynamics have the more particular form in (H3). In this case, the existence of at least one *global solution* \hat{u} , i.e., $J(\hat{u}) \leq J(u)$ for all $u \in \mathcal{K}$, follows from standard arguments (see, e.g., [27, section 5.3]). The results stated in this section will be proved in section 5.2.

Let us introduce the Lagrangian $\mathcal{L} : L^\infty(Q) \times \mathbb{R}^{n_E} \times \mathbb{R}^{n_I} \rightarrow \mathbb{R}$ defined as

$$(3.19) \quad \mathcal{L}(u, \lambda) := J(u) + \lambda^\top G(u), \quad \text{where } \lambda = (\lambda_E, \lambda_I) \in \mathbb{R}^{n_E} \times \mathbb{R}^{n_I}.$$

Now, let us fix $\bar{u} \in \mathcal{K}$ and set $\bar{y} := y[\bar{u}]$. Given $\lambda = (\lambda_E, \lambda_I) \in \mathbb{R}^{n_E} \times \mathbb{R}^{n_I}$ define

$$\Phi[\lambda](x, y) := \Phi(x, y) + \lambda_E^\top \Phi_E(x, y) + \lambda_I^\top \Phi_I(x, y) \quad \forall y \in \mathbb{R}, \quad x \in \Omega.$$

The adjoint state \bar{p}_λ associated to \bar{u} is defined as the unique weak solution of

$$\begin{aligned} (3.20) \quad -\partial_t p - \Delta p - H_y(t, x, \bar{y}(t, x), p(t, x), \bar{u}(t, x)) &= 0 \quad \text{in } Q, \\ p &= 0 \quad \text{in } \Sigma, \\ p(T, \cdot) &= \Phi_y[\lambda](\cdot, \bar{y}(T, \cdot)), \end{aligned}$$

where we recall that H is defined in (3.13). By (H2)(ii) and our assumptions on Φ_E and Φ_I , the maximum principle implies that $\bar{p}_\lambda \in L^\infty(Q)$. Similarly to the case without final state constraints, given $\lambda \in \mathbb{R}^{n_E+n_I}$, let us define the linear and quadratic forms $Q_1[\bar{u}, \lambda] : L^{2,2}(Q) \rightarrow \mathbb{R}$, $Q_2[\bar{u}, \lambda] : L^{2,2}(Q) \rightarrow \mathbb{R}$ as (recall the notation defined in (3.2))

$$\begin{aligned} (3.21) \quad Q_1[\bar{u}, \lambda]v &:= \int_Q H_u(t, x, \bar{y}(t, x), \bar{p}_\lambda(t, x), \bar{u}(t, x))v(t, x) dt dx, \\ Q_2[\bar{u}, \lambda](v) &:= \int_Q H_{(y,u)^2}(t, x, \bar{y}(t, x), \bar{p}_\lambda(t, x), \bar{u}(t, x))(\xi[v](t, x), v(t, x))^2 dt dx \\ &\quad + \int_\Omega \Phi_{yy}[\lambda](x, \bar{y}(T, x))\xi[v](T, x)^2 dx, \end{aligned}$$

where ξ is defined by (3.16). Under our assumptions (H1)–(H2) we have that $Q_1[\bar{u}, \lambda]$ and $Q_2[\bar{u}, \lambda]$ are well defined.

DEFINITION 3.9. We say that $\lambda = (\lambda_E, \lambda_I) \in \mathbb{R}^{n_E+n_I}$ is a Lagrange multiplier at $\bar{u} \in \mathcal{K}$ if

$$(3.22) \quad Q_1[\bar{u}, \lambda]v \geq 0 \quad \forall v \in T_{\mathcal{K}_0}(\bar{u}), \quad \lambda_I^j \geq 0, \quad \lambda_I^j G_I^j(\bar{u}) = 0, \quad \forall j = 1, \dots, n_I.$$

The set of Lagrange multipliers at \bar{u} is denoted as $\Lambda_L(\bar{u})$.

Note that $\lambda \in \Lambda_L(\bar{u})$ iff $\lambda_I^j \geq 0$, $\lambda_I^j G_I^j(\bar{u}) = 0$ for all $j = 1, \dots, n_I$ and a.e. in Q we have

$$H_u(t, x, \bar{y}(t, x), \bar{p}_\lambda(t, x), \bar{u}(t, x))(u - \bar{u}(t, x)) \geq 0 \quad \forall u \in [a(t, x), b(t, x)].$$

Therefore, the particular quadratic structure of the Hamiltonian assumed in (H3) implies that there exists $\alpha > 0$ such that $\lambda \in \Lambda_L(\bar{u})$ iff $\lambda_I^j \geq 0$, $\lambda_I^j G_I^j(\bar{u}) = 0$ for all $j = 1, \dots, n_I$ and a.e. in Q we have

$$(3.23) \quad H(t, x, \bar{y}(t, x), \bar{p}_\lambda(t, x), \bar{u}(t, x)) + \frac{\alpha}{2} |u - \bar{u}(t, x)|^2 \leq H(t, x, \bar{y}(t, x), \bar{p}_\lambda(t, x), u)$$

for all $u \in [a(t, x), b(t, x)]$.

Now, let us define the *critical cone* to \mathcal{K} at \bar{u} as

$$(3.24) \quad C_{\mathcal{K}}(\bar{u}) := \{v \in T_{\mathcal{K}}(\bar{u}) ; DJ(\bar{u})v = 0\},$$

where we recall that the tangent cone $T_{\mathcal{K}}(\bar{u})$ is defined in (3.17).

Let us state the main result in this section, regarding a sufficient second order optimality condition for quadratic growth in the strong sense for local solutions of (CP₁).

THEOREM 3.10. *Suppose that (H1)–(H3) hold true and that*

- (i) $\Lambda_L(\bar{u}) \neq \emptyset$,
- (ii) *there exists $\alpha > 0$ such that*

$$\max_{\lambda \in \Lambda_L(\bar{u})} Q[\bar{u}, \lambda](v) \geq \alpha \|v\|_{2,2}^2 \quad \forall v \in C_{\mathcal{K}}(\bar{u}).$$

Then, \bar{u} is a strong local solution of (CP₁) satisfying the quadratic growth condition (3.12).

In order to prove a characterization of the quadratic growth property, i.e., that conditions (i)–(iii) in the previous theorem are also *necessary*, we will need to impose some qualification conditions (see [25] for a detailed discussion on this topic).

Denoting by $B(0, \delta)$ the open ball in $\mathbb{R}^{n_E+n_I}$, centered at 0 and of radius δ , let us recall that if the *Robinson’s constraint qualification* condition (see [23])

$$(RCQ) \quad \exists \delta > 0 \text{ such that } B(0, \delta) \subseteq G(\bar{u}) + DG(\bar{u})(\mathcal{K}_0 - \bar{u}) - \mathcal{K}_2$$

holds true, then by [25, Lemma 4.2]

$$(3.25) \quad T_{\mathcal{K}}(\bar{u}) := \{v \in L^{2,2}(Q) ; v \geq 0 \text{ in } A_a(\bar{u}) \text{ and } v \leq 0 \text{ in } A_b(\bar{u}), \\ DG_E(\bar{u})v = 0, DG_I^i(\bar{u})v \leq 0 \forall i \in I(\bar{u})\},$$

where $I(\bar{u}) := \{j \in \{1, \dots, n_I\} ; G_I^j(\bar{u}) = 0\}$. Moreover, as a consequence of [14, Corollary 2.2], at any weak local solution \bar{u} of (CP₁) we have that $\Lambda_L(\bar{u})$ is a nonempty compact subset of $\mathbb{R}^{n_E+n_I}$.

We will state a characterization of the quadratic growth property by assuming that at the nominal control $\bar{u} \in \mathcal{K}$ the standard qualification condition (RCQ) is satisfied and that the critical cone is equal to the *strict critical cone* to \mathcal{K} , defined as

$$C_{\mathcal{K}}^s(\bar{u}) := \{v \in C_{\mathcal{K}}(\bar{u}) ; v = 0 \text{ a.e. in } A_a(\bar{u}) \cup A_b(\bar{u})\},$$

or that the following strengthened qualification condition holds true: there exists $\lambda \in \Lambda_L(\bar{u})$ and $\delta > 0$ such that

$$(3.26) \quad B(0, \delta) \subseteq (DG(\bar{u}) [(\mathcal{K}_0 - \bar{u}) \cap (\overline{H}_u[\lambda])^\perp] - (\mathcal{K}_1 - G(\bar{u})) \cap \lambda^\perp),$$

where $(\overline{H}_u[\lambda])^\perp$ denotes the subspace of functions $v \in L^{2,2}(Q)$ such that

$$\int_Q H_u(t, x, \bar{y}(t, x), \bar{p}_\lambda(t, x), \bar{u}(t, x))v(t, x)dt dx = 0,$$

and λ^\perp is the subspace of $\mathbb{R}^{n_E} \times \mathbb{R}^{n_I}$ which is orthogonal to $\{\lambda\}$.

Remark 3.11. Condition (3.26), introduced by Shapiro in [24] for general optimization problems and in [9] in the context of semilinear elliptic equations with finitely many state constraints, implies (RCQ) and the fact that $\Lambda_L(\bar{u})$ is a singleton (see [25, Lemma 5.4]). This qualification condition is implied by the classical surjectivity type assumption for similar problems in the context of semilinear elliptic equations (see, e.g., [13, Condition (3.1)]). We refer the reader to [25, Remark 5.3] for a detailed discussion on this matter.

For the sake of clarity, we state both characterizations as different theorems, depending on the assumed qualification condition. Using the results in [25], the proofs of the following two results are straightforward and will be given in section 5.2.

THEOREM 3.12. *Suppose that (H1)–(H3) hold true, that Robinson’s constraint qualification (RCQ) is satisfied at $\bar{u} \in \mathcal{K}$, and that $C_{\mathcal{K}}(\bar{u}) = C_{\mathcal{K}}^s(\bar{u})$. Then, the following assertions are equivalent:*

- (i) *The control \bar{u} is a strong local solution of (CP₁) satisfying the quadratic growth condition (3.12).*
- (ii) *$\Lambda_L(\bar{u}) \neq \emptyset$ and $\exists \alpha > 0$ such that for all $v \in C_{\mathcal{K}}(\bar{u})$ it holds that*

$$\max_{\lambda \in \Lambda_L(\bar{u})} Q_2[\bar{u}, \lambda](v) \geq \alpha \|v\|_{2,2}^2.$$

Condition $C_{\mathcal{K}}(\bar{u}) = C_{\mathcal{K}}^s(\bar{u})$ is satisfied if there exists $\lambda \in \Lambda_L(\bar{u})$ such that $H_u(t, x, \bar{y}(t, x), \bar{p}_\lambda(t, x), \bar{u}(t, x))$ a.e. in $A_a(\bar{u}) \cup A_b(\bar{u})$. In the absence of state constraints, this type of assumption has been proved to be useful in order to obtain asymptotic expansions for approximations of local solutions of the optimal control problem at hand (see [8] for an application in the analysis of interior point methods). In the following result, we provide the desired characterization without the assumption that $C_{\mathcal{K}}(\bar{u}) = C_{\mathcal{K}}^s(\bar{u})$, but requiring assumption (3.26), which is stronger than (RCQ).

THEOREM 3.13. *Let $\bar{u} \in \mathcal{K}$. Suppose that (H1)–(H3) hold true and that (3.26) is satisfied at some $\lambda \in \Lambda_L(\bar{u})$ (and so $\Lambda_L(\bar{u}) = \{\lambda\}$). Then, the following assertions are equivalent:*

- (i) *The control \bar{u} is a strong local solution of (CP₁) satisfying the quadratic growth condition (3.12).*
- (ii) *$\exists \alpha > 0$ such that for all $v \in C_{\mathcal{K}}(\bar{u})$ it holds that $Q_2[\bar{u}, \lambda](v) \geq \alpha \|v\|_{2,2}^2$.*

As in the case of pure control constraint (see Theorem 3.7), under some continuity assumptions on the data, we have the following equivalence of notions of local solutions satisfying the quadratic growth condition.

THEOREM 3.14. *Assume (H1)–(H3) and that a, b, ℓ_1 , and φ_1 , (resp., ℓ_2 , and φ_2) admit extensions to \bar{Q} (resp., $\bar{Q} \times \mathbb{R}$) which are continuous. Moreover, for $\psi =$*

Φ, Φ_E, Φ_I suppose that ψ_y admits a locally Hölder continuous extension to $\bar{\Omega} \times \mathbb{R}$ satisfying $\psi_y(x, 0) = 0$ for all $x \in \partial\Omega$. Then, for $\bar{u} \in \mathcal{K}$ we have the equivalence of (3.8), (3.10) (for any $s \in [1, \infty[$) and (3.12), provided that one of the two following conditions is satisfied:

- (a) The Robinson constraint qualification (RCQ) holds true and $C_{\mathcal{K}}(\bar{u}) = C_{\mathcal{K}}^s(\bar{u})$.
- (b) Condition (3.26) is satisfied at some $\lambda \in \Lambda_L(\bar{u})$ (and so $\Lambda_L(\bar{u}) = \{\lambda\}$).

3.3. Some remarks and possible extensions in the state constrained case. The proofs of the results in the previous subsection, provided in section 5, show that our statements hold true if the term $\ell_1(t, x)u^2$ in the cost is replaced by any strongly convex function of u , where the strong convexity constant is uniform w.r.t. $(t, x) \in Q$. Moreover, if (H3) holds true the reader can also verify that our results are still valid if the assumptions in (H1)–(H2) concerning the uniform Lipschitz property for the second derivatives w.r.t. to the state variable y are replaced by the weaker assumption of a uniform modulus of continuity. On the other hand, it can be interesting to relax the assumptions on the regularity of the initial data and of the boundary $\partial\Omega$. The main difficulty here is that in this case we can only expect the existence of weak solutions for (3.1) (see Remark 3.2). This fact adds some difficulties in the expansion of the cost w.r.t. perturbations of a nominal control \bar{u} proved in Theorem 4.3 in the next section.

Extensions to general cost functions and dynamics, without assuming (H3), as the ones considered for problem (CP_0) , are not straightforward and remain as interesting challenges, as we now explain.

Let us first discuss second order sufficient conditions. In analogy with [10], let us first introduce a stronger notion than the one of a Lagrange multiplier. We say that $\lambda = (\lambda_E, \lambda_I) \in \mathbb{R}^{n_E+n_I}$ is a *Pontryagin multiplier* at $\bar{u} \in \mathcal{K}$ if $\lambda \in \Lambda_L(\bar{u})$ and a.e. in Q

$$H(t, x, \bar{y}(t, x), \bar{p}_\lambda(t, x), \bar{u}(t, x)) \leq H(t, x, \bar{y}(t, x), \bar{p}_\lambda(t, x), u) \quad \forall u \in [a(t, x), b(t, x)].$$

The set of Pontryagin multipliers at \bar{u} is denoted as $\Lambda_P(\bar{u})$. The proof of Theorem 3.10 actually shows that the correct condition on the quadratic form is that there exists $\alpha > 0$ such that $\max_{\lambda \in \Lambda_P(\bar{u})} Q_2[\bar{u}, \lambda](v) \geq \alpha \|v\|_{2,2}^2$. Now, our assumption (H3) implies that the Hamiltonian H is convex w.r.t. u and so in our particular case $\Lambda_L(\bar{u}) = \Lambda_P(\bar{u})$, which allowed us to prove our results. However, in the general case $\Lambda_P(\bar{u})$ is strictly contained in $\Lambda_L(\bar{u})$. Another issue appearing in second order sufficient conditions is the fact that our assumption (H3) implies that for every $\lambda \in \mathbb{R}^{n_E+n_I}$ the quadratic form $Q_2[\bar{u}, \lambda]$ is a *Legendre form*, i.e., it satisfies that $Q_2[\bar{u}, \lambda]$ is weakly lower semicontinuous and if $v_k \in \mathcal{H}$ converges weakly to v and $Q_2[\bar{u}, \lambda](v_k)$ converges to $Q_2[\bar{u}, \lambda](v)$, we have that v_k converges strongly to v . In the general case, this assumption cannot be directly verified and has to be imposed. As a conclusion, regarding second order sufficient optimality conditions, the proof of Theorem 3.10 shows that in the general case we have the following result.

THEOREM 3.15. *Suppose that (H1)–(H2) hold true and that*

- (i) $\Lambda_P(\bar{u}) \neq \emptyset$,
- (ii) *there exists $\bar{\lambda} \in \Lambda_P(\bar{u})$ such that (3.23) is satisfied at \bar{u} ,*
- (iii) *for all $\lambda \in \Lambda_P(\bar{u})$ the quadratic form $Q[\bar{u}, \lambda]$ is a Legendre form and there exists $\alpha > 0$ such that*

$$\max_{\lambda \in \Lambda_P(\bar{u})} Q[\bar{u}, \lambda](v) \geq \alpha \|v\|_{2,2}^2 \quad \forall v \in C_{\mathcal{K}}(\bar{u}).$$

Then, \bar{u} is a strong local solution of (CP_1) satisfying the quadratic growth condition.

Now regarding second order necessary conditions, a direct application of the Pontryagin principle (see, e.g., [17, 12, 14]) and the results on second order necessary conditions in [25] show that if $\bar{u} \in \mathcal{K}$ is a strong local solution of (CP_1) and (RCQ) holds true, then there exist $\alpha > 0$ and $\lambda \in \Lambda_P(\bar{u})$ such that (3.23) holds true and for all $v \in C^s_{\mathcal{K}}(\bar{u})$ we have that $\max_{\lambda \in \Lambda_L(\bar{u})} Q_2[\bar{u}, \lambda](v) \geq \alpha \|v\|_{2,2}^2$. In accordance to (iii) in the previous theorem, the desirable second order necessary condition should imply a maximization of $Q_2[\bar{u}, \lambda](v)$ on $\Lambda_P(\bar{u})$ rather than on $\Lambda_L(\bar{u})$.

As a summary, for second order sufficient conditions the main issue is to get rid of the Legendre form assumption, while for second order necessary conditions the main problem is to obtain a maximization condition of $Q_2[\bar{u}, \lambda](v)$ on $\Lambda_P(\bar{u})$. Both problems remain as interesting future research projects, since they would allow us to establish a characterization of quadratic growth for strong local solutions for general control problems of the form (CP_1) , similar to the one established in Theorem 3.5 for problem (CP_0) .

4. Second order expansions and a decomposition result. The aim of this section is to provide a precise second order expansion for the cost J (see, e.g., (3.5)) and the Lagrangian \mathcal{L} (see (3.19)) in terms of the control, assuming (H1)–(H2). Since $J(u) = \mathcal{L}(u, 0)$ we will consider only expansions for $\mathcal{L}(\cdot, \lambda)$ which are proven to be uniform for λ in a compact set. The main result of this section is Theorem 4.3, which is the key tool to prove second order sufficient conditions for quadratic growth in the strong sense. Since this section is of a technical nature, at first reading the reader can skip the first results and pass directly to Theorem 4.3.

Throughout the section we fix $\bar{u} \in L^\infty(Q)$ and denote $\bar{y} := y[\bar{u}]$. The key point is that we consider perturbations v of \bar{u} that are not necessarily small in the $\|\cdot\|_\infty$ norm. We will first need some precise information about the effect of this type of perturbation on the mapping $u \mapsto y[u]$. We collect in the following lemma some results of this type. For a function $\psi : Q \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ we will denote $\bar{\psi}(t, x) := \psi(t, x, \bar{y}(t, x), \bar{u}(t, x))$ and given $u \in L^\infty(Q)$ we will write $\delta\psi(t, x) := \psi(t, x, \bar{y}(t, x), u(t, x)) - \bar{\psi}(t, x)$. Similarly, for $\Psi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ we set $\bar{\Psi}(x) := \Psi(x, \bar{y}(T, x))$.

LEMMA 4.1. *For $v \in L^\infty(Q)$ set $u := \bar{u} + v$ and let us define $z_1[v] \in \mathcal{V}^s(Q)$ and $z_2[v] \in \mathcal{V}^s(Q)$ as the solutions of*

$$\begin{aligned}
 \partial_t z_1 - \Delta z_1 + \bar{\varphi}_y(t, x) z_1 + \delta\varphi(t, x) &= 0 \quad \text{in } Q, \\
 z_1 &= 0 \quad \text{in } \Sigma, \\
 z_1(0, \cdot) &= 0 \quad \text{in } \Omega,
 \end{aligned}
 \tag{4.1}$$

and

$$\begin{aligned}
 \partial_t z_2 - \Delta z_2 + \bar{\varphi}_y(t, x) z_2 + \frac{1}{2} \bar{\varphi}_{yy}(t, x) z_1[v]^2 + \delta\varphi_y(t, x) z_1[v] &= 0 \quad \text{in } Q, \\
 z_2 &= 0 \quad \text{in } \Sigma, \\
 z_2(0, \cdot) &= 0 \quad \text{in } \Omega,
 \end{aligned}
 \tag{4.2}$$

respectively. Then, setting $\delta y := y[u] - \bar{y}$, $d_1[v] := \delta y - z_1[v]$, and $d_2[v] := d_1[v] - z_2[v]$, the following estimates hold true:

$$\begin{aligned}
 \|z_1[v]\|_{\mathcal{V}^s} + \|\delta y\|_{\mathcal{V}^s} &= O(\|v\|_{s,s}), \\
 \|d_1[v]\|_{2,2} + \|d_1[v](T, \cdot)\|_2 &= O(\|\delta y\|_\infty \|v\|_{2,2}), \\
 \|d_2[v]\|_{1,1} + \|d_2[v](T, \cdot)\|_1 &= O(\|\delta y\|_\infty \|v\|_{2,2}^2).
 \end{aligned}
 \tag{4.3}$$

Proof. We have that δy satisfies

$$\begin{aligned} \partial_t \delta y - \Delta \delta y &= - \left[\int_0^1 \varphi_y(t, x, \bar{y} + \tau \delta y, \bar{u} + \tau v) d\tau \right] \delta y \\ &\quad - \left[\int_0^1 \varphi_u(t, x, \bar{y} + \tau \delta y, \bar{u} + \tau v) d\tau \right] v \text{ in } Q, \\ \delta y &= 0 \text{ in } \Sigma, \\ \delta y(0, \cdot) &= 0 \text{ in } \Omega. \end{aligned}$$

Using the above equation and equation (4.1) for z_1 , the estimates for $\|\delta y\|_{\mathcal{V}^s}$ and $\|z_1[v]\|_{\mathcal{V}^s}$ follow by (H1)(iii) and (2.5). Noting that omitting the dependence on (t, x) ,

$$\begin{aligned} &\varphi(t, x, y[\bar{u} + v], \bar{u} + v) - \varphi(\bar{y}, \bar{u} + v) - \bar{\varphi}_y(t, x) \delta y \\ &= \int_0^1 [\varphi_y(t, x, \bar{y} + \tau \delta y, \bar{u} + v) - \bar{\varphi}_y(t, x)] d\tau \delta y = O(|\delta y|^2 + |v| |\delta y|), \end{aligned}$$

we easily check that d_1 satisfies

$$\begin{aligned} \partial_t d_1 - \Delta d_1 + \bar{\varphi}_y(t, x) d_1 &= O(|\delta y|^2 + |v| |\delta y|) \text{ in } Q, \\ d_1 &= 0 \text{ in } \Sigma, \\ d_1(0, \cdot) &= 0 \text{ in } \Omega. \end{aligned}$$

In particular, (2.5) implies

$$\|d_1[v]\|_{2,2} + \|d_1[v](T, \cdot)\|_2 = O\left(\left[\int_Q [|\delta y|^2 + |v| |\delta y|]^2 dt dx\right]^{\frac{1}{2}}\right) = O(\|\delta y\|_\infty \|v\|_{2,2}),$$

which gives the second estimate in (4.3). On the other hand, omitting the dependence on (t, x) and setting $y = y[\bar{u} + v]$, $u = \bar{u} + v$, we have that

$$\begin{aligned} &\varphi(y, u) - \varphi(\bar{y}, u) - \bar{\varphi}_y(t, x) z_1 - \bar{\varphi}_y(t, x) z_2 - \frac{1}{2} \bar{\varphi}_{yy}(t, x) z_1^2 - \delta \bar{\varphi}_y(t, x) z_1 \\ &= \bar{\varphi}_y(t, x) d_2 + \varphi(y, u) - \varphi(\bar{y}, u) - \bar{\varphi}_y(t, x) \delta y - \frac{1}{2} \bar{\varphi}_{yy}(t, x) z_1^2 - \delta \bar{\varphi}_y(t, x) z_1 \\ &= \bar{\varphi}_y(t, x) d_2 + \varphi_y(\bar{y}, u) \delta y + \frac{1}{2} \varphi_{yy}(\bar{y}, u) (\delta y)^2 \\ &\quad + \int_0^1 (1 - \tau) [\varphi_{yy}(\bar{y} + \tau \delta y, u) - \bar{\varphi}_{yy}(\bar{y}, u)] d\tau (\delta y)^2 - \bar{\varphi}_y(t, x) \delta y \\ &\quad - \frac{1}{2} \bar{\varphi}_{yy}(t, x) z_1^2 - \delta \bar{\varphi}_y(t, x) z_1 \\ &= \bar{\varphi}_y(t, x) d_2 + \delta \bar{\varphi}_y(t, x) d_1 + \frac{1}{2} \delta \bar{\varphi}_{yy}(t, x) (\delta y)^2 + \frac{1}{2} \bar{\varphi}_{yy}(t, x) ((\delta y)^2 - z_1^2) \\ &\quad + \int_0^1 (1 - \tau) [\varphi_{yy}(\bar{y} + \tau \delta y, u) - \bar{\varphi}_{yy}(\bar{y}, u)] d\tau (\delta y)^2 \\ &= \bar{\varphi}_y(t, x) d_2 + O(|d_1| |v| + |\delta y|^2 |v| + |d_1| (|\delta y| + |z_1|) + |\delta y|^3). \end{aligned}$$

This implies that d_2 solves

$$\begin{aligned} \partial_t d_2 - \Delta d_2 + \bar{\varphi}_y(t, x) d_2 &= O(|d_1| |v| + |\delta y|^2 |v| + |d_1| (|\delta y| + |z_1|) + |\delta y|^3) \\ &\text{in } Q, \\ d_2 &= 0 \text{ in } \Sigma, \\ d_2(0, \cdot) &= 0 \text{ in } \Omega. \end{aligned}$$

Using (2.6) we get that

$$\begin{aligned} & \|d_2[v]\|_{1,1} + \|d_2[v](T, \cdot)\|_1 \\ &= O\left(\int_Q [|d_1||v| + |\delta y|^2|v| + |d_1|(|\delta y| + |z_1|) + |\delta y|^3] dt dx \right), \end{aligned}$$

and the third estimate in (4.3) follows from the previous ones. □

Let us now fix some notation and recall the definition of the Hamiltonian H in (3.13). Given $u \in L^\infty(Q)$ and $\lambda \in \mathbb{R}^{n_E+n_I}$ we write $\delta u := u - \bar{u}$ and

$$\begin{aligned} (4.4) \quad \bar{H}[\lambda](t, x) &:= H(t, x, \bar{y}(t, x), \bar{p}_\lambda(t, x), \bar{u}(t, x)), \\ \delta \bar{H}[\lambda](t, x) &:= H(t, x, \bar{y}(t, x), \bar{p}_\lambda(t, x), u(t, x)) - \bar{H}[\lambda](t, x), \\ \bar{H}_y[\lambda](t, x) &:= H_y(t, x, \bar{y}(t, x), \bar{p}_\lambda(t, x), \bar{u}(t, x)), \\ \delta \bar{H}_y[\lambda](t, x) &:= H_y(t, x, \bar{y}(t, x), \bar{p}_\lambda(t, x), u(t, x)) - \bar{H}_y[\lambda](t, x) \end{aligned}$$

with similar notation for the second order derivatives. Using the estimates obtained in Lemma 4.1 and the definitions of the Hamiltonian and the adjoint state \bar{p}_λ (see (3.20)), we have the following result.

PROPOSITION 4.2. *Given $u \in L^\infty(Q)$ set $y := y[u]$, $\delta u := u - \bar{u}$, and $\delta y := y - \bar{y}$. The following assertions hold true:*

(i) *We have the first order expansion*

$$(4.5) \quad \mathcal{L}(u, \lambda) - \mathcal{L}(\bar{u}, \lambda) = \int_Q \delta \bar{H}[\lambda](t, x) dt dx + O(\|\delta y\|_\infty \|\delta u\|_{2,2}),$$

where the O -term is uniform for λ in a bounded set $C \subseteq \mathbb{R}^{n_E+n_I}$.

(ii) *We have the second order expansion*

$$\begin{aligned} (4.6) \quad \mathcal{L}(u, \lambda) - \mathcal{L}(\bar{u}, \lambda) &= \int_Q [\delta \bar{H}[\lambda](t, x) + \frac{1}{2} \bar{H}_{yy}[\lambda](t, x) (z_1[\delta u])^2 + \delta \bar{H}_y[\lambda](t, x) z_1[\delta u]] dt dx \\ &+ \int_\Omega \bar{\Phi}_{yy}[\lambda](z_1[\delta u](T, x))^2 dx + O(\|\delta y\|_\infty \|\delta u\|_{2,2}^2), \end{aligned}$$

where the O -term is uniform for λ in a bounded set $C \subseteq \mathbb{R}^{n_E+n_I}$.

Proof. Let us prove (i). By (3.19), omitting the dependence on (t, x) ,

$$(4.7) \quad \begin{aligned} \mathcal{L}(u, \lambda) - \mathcal{L}(\bar{u}, \lambda) &= \int_Q [\ell(y, u) - \ell(\bar{y}, \bar{u})] dt dx \\ &+ \int_\Omega [\Phi[\lambda](x, y(T, x)) - \Phi[\lambda](x, \bar{y}(T, x))] dx. \end{aligned}$$

Now, writing $\delta y_T(x) := \delta y(T, x)$, assumption (H2)(ii) implies that

$$\begin{aligned} (4.8) \quad & \int_Q [\ell(y, u) - \bar{\ell}] dt dx + \int_\Omega [\Phi[\lambda](y(T, x)) - \bar{\Phi}[\lambda]] dx \\ &= \int_Q [\ell(y, u) - \ell(\bar{y}, u) + \delta \bar{\ell}] dt dx + \int_\Omega \bar{\Phi}_y[\lambda] \delta y_T dx + O(\int_\Omega |\delta y_T|^2 dx), \\ &= \int_Q [\ell(y, u) - \ell(\bar{y}, u) + \delta \bar{\ell} + \partial_t \bar{p}_\lambda \delta y + \bar{p}_\lambda \partial_t \delta y] dt dx + O(\int_\Omega |\delta y_T|^2 dx), \end{aligned}$$

where, in the second equality, we have used that $\bar{\Phi}[\lambda](\cdot) = \bar{p}_\lambda(T, \cdot)$ and $\delta y(0, \cdot) = 0$. Using the equation satisfied by \bar{p}_λ in (3.20) and the equation satisfied by δy and

integrating by parts, we obtain

$$\begin{aligned}
 & \int_Q [\partial_t \bar{p}_\lambda \delta y + \bar{p}_\lambda \partial_t \delta y] dt dx \\
 (4.9) \quad & = \int_Q [-(\bar{\ell}_y - \bar{p}_\lambda \bar{\varphi}_y) \delta y - \bar{p}_\lambda (\varphi(y, u) - \bar{\varphi})] dt dx, \\
 & = \int_Q [-\bar{\ell}_y \delta y - \bar{p}_\lambda (\varphi(y, u) - \varphi(\bar{y}, u) - \bar{\varphi}_y \delta y + \delta \bar{\varphi})] dt dx.
 \end{aligned}$$

On the other hand, note that (H2)(ii) implies

$$\begin{aligned}
 \ell(y, u) - \ell(\bar{y}, u) - \bar{\ell}_y \delta y & = \int_0^1 (\ell_y(\bar{y} + \tau \delta y, u) - \bar{\ell}_y) \delta y dt = O(|\delta y|^2 + |\delta y| |\delta u|), \\
 \varphi(y, u) - \varphi(\bar{y}, u) - \bar{\varphi}_y \delta y & = \int_0^1 (\varphi_y(\bar{y} + \tau \delta y, u) - \bar{\varphi}_y) \delta y dt = O(|\delta y|^2 + |\delta y| |\delta u|),
 \end{aligned}$$

and so, using that the maximum principle and (H2)(ii) imply that $\|\bar{p}_\lambda\|_\infty$ is uniformly bounded for $\lambda \in C$, we get with (4.7), (4.8), and (4.9)

$$\mathcal{L}(u, \lambda) - \mathcal{L}(\bar{u}, \lambda) = \int_Q \delta \bar{H}(t, x) dt dx + O\left(\int_Q [|\delta y|^2 + |\delta y| |\delta u|] dt dx + \int_\Omega |\delta y_T|^2 dx\right),$$

and assertion (i) is obtained from Lemma 4.1. In order to prove (ii), we follow a similar strategy. Expanding up to the second order, and recalling that we denote $d_1 = \delta y - z_1$ and $d_2 = \delta y - z_1 - z_2$, we have

$$\begin{aligned}
 & \int_Q [\ell(y, u) - \bar{\ell}] dt dx \\
 & = \int_Q [\ell(y, u) - \ell(\bar{y}, u) + \delta \bar{\ell}] dt dx, \\
 & = \int_Q [\ell_y(\bar{y}, u) \delta y + \frac{1}{2} \ell_{yy}(\bar{y}, u) (\delta y)^2 + \delta \bar{\ell}] dt dx \\
 & \quad + \int_Q \int_0^1 (1 - \tau) [\ell_{yy}(\bar{y} + \tau \delta y, u) - \ell_{yy}(\bar{y}, u)] (\delta y)^2 d\tau dt dx, \\
 & = \int_Q [\delta \bar{\ell}_y \delta y + \bar{\ell}_y \delta y + \frac{1}{2} \bar{\ell}_{yy} z_1^2 + \delta \bar{\ell}] dt dx \\
 & \quad + O\left(\int_Q [|d_1| (|\delta y| + |z_1|) + |\delta u| |\delta y|^2 + |\delta y|^3] dt dx\right), \\
 & = \int_Q [\delta \bar{\ell}_y z_1 + \bar{\ell}_y (z_1 + z_2) + \frac{1}{2} \bar{\ell}_{yy} z_1^2 + \delta \bar{\ell}] dt dx \\
 & \quad + O\left(\int_Q [|d_1| |\delta u| + |d_2| + |d_1| (|\delta y| + |z_1|) + |\delta u| |\delta y|^2 + |\delta y|^3] dt dx\right),
 \end{aligned}$$

where, in the third equality, we have used that

$$\begin{aligned}
 & \ell_{yy}(\bar{y}, u) (\delta y)^2 \\
 & = \bar{\ell}_{yy} (\delta y)^2 + O(|\delta u| |\delta y|^2) = \bar{\ell}_{yy} (z_1)^2 + O(|\delta u| |\delta y|^2 + |d_1| (|z_1| + |\delta y|)) \\
 \text{and } & \int_Q \int_0^1 (1 - \tau) [\ell_{yy}(\bar{y} + \tau \delta y, u) - \ell_{yy}(\bar{y}, u)] (\delta y)^2 d\tau dt dx = O\left(\int_Q |\delta y|^3 dt dx\right)
 \end{aligned}$$

by (H2)(ii). Analogously, setting $(z_1)_T = z_1(T, \cdot)$ with a similar convention for $(d_1)_T$ and $(d_2)_T$, we have

$$\begin{aligned}
 & \int_\Omega (\Phi[\lambda](y_T) - \bar{\Phi}[\lambda]) dx \\
 & = \int_\Omega [\bar{\Phi}_y[\lambda] \delta y_T + \frac{1}{2} \bar{\Phi}_{yy}[\lambda] \delta y_T^2] dx + O\left(\int_\Omega |\delta y_T|^3 dx\right) \\
 & = \int_\Omega [\bar{\Phi}_y[\lambda] (z_1 + z_2)_T + \frac{1}{2} \bar{\Phi}_{yy}[\lambda] (z_1)_T^2] dx \\
 & \quad + O\left(\int_\Omega [|(d_2)_T| + |(d_1)_T| |(z_1)_T| + |(d_1)_T| |\delta y_T| + |\delta y_T|^3] dx\right).
 \end{aligned}$$

Therefore, by Lemma 4.1 and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
 \mathcal{L}(u, \lambda) - \mathcal{L}(\bar{u}, \lambda) &= \int_Q [\delta \bar{\ell}_y z_1 + \bar{\ell}_y(z_1 + z_2) + \frac{1}{2} \bar{\ell}_{yy} z_1^2 + \delta \bar{\ell}] dt dx \\
 (4.10) \qquad \qquad \qquad &+ \int_\Omega [\bar{\Phi}_y[\lambda](z_1 + z_2)_T + \frac{1}{2} \bar{\Phi}_{yy}[\lambda](z_1)_T^2] dx \\
 &+ O(\|\delta y\|_\infty \|\delta u\|_{2,2}^2).
 \end{aligned}$$

Finally, using (3.20), (4.1), (4.2) and integrating by parts we obtain that

$$\begin{aligned}
 &\int_\Omega \bar{\Phi}_y[\lambda](z_1 + z_2)_T dx \\
 &= \int_\Omega \bar{p}_\lambda(T, x)(z_1 + z_2)_T dx \\
 &= - \int_Q [\delta \bar{\varphi} \bar{p}_\lambda + \delta \bar{\varphi}_y z_1 \bar{p}_\lambda + \frac{1}{2} \bar{\varphi}_{yy} z_1^2 \bar{p}_\lambda + \bar{\ell}_y(z_1 + z_2)] dt dx,
 \end{aligned}$$

which combined with (4.10) yields (4.6). □

Now, let us consider a sequence $u_k \in L^\infty(Q)$ such that $\|u_k - \bar{u}\|_{2,2} \rightarrow 0$ and let A_k, B_k be a sequence of measurable subsets of $[0, T] \times \Omega$ such that $|A_k \cup B_k| = T \times |\Omega|$ and $|B_k| \downarrow 0$ as $k \uparrow \infty$ (where we set $|\cdot|$ for the Lebesgue measure of a Lebesgue measurable set). Recalling (4.1), we set

$$\begin{aligned}
 (4.11) \qquad \delta_k u &:= u_k - \bar{u}, & \delta_{A_k} u &:= \mathbb{I}_{A_k}(u_k - \bar{u}), & \delta_{B_k} u &:= \delta_k u - \delta_{A_k} u, \\
 z^k &:= z_1[\delta_k u], & z^{A_k} &:= z_1[\delta_{A_k} u], & z^{B_k} &:= z_1[\delta_{B_k} u],
 \end{aligned}$$

where \mathbb{I}_A denotes the indicator function of a measurable set A , i.e., $\mathbb{I}_A(t, x) = 1$ if $(t, x) \in A$ and it is equal to zero otherwise. Similarly to (4.4), we write $\delta_k \bar{H}[\lambda](t, x) = H(t, x, \bar{y}(t, x), \bar{p}_\lambda(t, x), u_k(t, x)) - \bar{H}[\lambda](t, x)$, with an analogous definition for $\delta_k \bar{H}_y[\lambda](t, x)$. Using Proposition 4.2, we prove now the main result of this section, which will be fundamental in section 5.

Recall that $Q_1[\bar{u}, \lambda]$ and $Q_2[\bar{u}, \lambda]$ are defined in (3.21).

THEOREM 4.3. *Suppose that $\|\delta_k u\|_{2,2} \downarrow 0$ and that $\|\delta_{A_k} u\|_\infty \downarrow 0$. Then,*

$$\begin{aligned}
 (4.12) \qquad \mathcal{L}(u_k, \lambda) - \mathcal{L}(\bar{u}, \lambda) &= \int_{B_k} \delta_k \bar{H}[\lambda](t, x) dt dx + Q_1[\bar{u}, \lambda] \delta_{A_k} u + \frac{1}{2} Q_2[\bar{u}, \lambda] (\delta_{A_k} u) + o(\|\delta_k u\|_{2,2}^2),
 \end{aligned}$$

where the o -term is uniform for λ in a compact set of $\mathbb{R}^{n_E+n_I}$.

Proof. *Step 1.* Using the expansion (4.6), we first prove that

$$\begin{aligned}
 (4.13) \qquad \mathcal{L}(u_k, \lambda) - \mathcal{L}(\bar{u}, \lambda) &= \int_{B_k} \delta_k \bar{H}[\lambda](t, x) dt dx \\
 &+ \int_Q [\bar{H}_u[\lambda](t, x) \delta_{A_k} u + \frac{1}{2} \bar{H}_{(y,u)^2}[\lambda](t, x) (z^{A_k}, \delta_{A_k} u)^2] dt dx \\
 &+ \int_\Omega \bar{\Phi}_{yy}[\lambda](z^{A_k}(T, x))^2 dx + o(\|\delta_k u\|_{2,2}^2).
 \end{aligned}$$

For notational simplicity we omit the dependence on λ for H and its derivatives. First note that since $\|\delta_{A_k} u\|_\infty \downarrow 0$, a Taylor expansion of the term $\delta_k \bar{H}(t, x)$ implies that

$$\begin{aligned}
 &\int_{A_k} \delta_k \bar{H}(t, x) dt dx \\
 &= \int_Q [\bar{H}_u(t, x) \delta_{A_k} u + \frac{1}{2} \bar{H}_{uu}(t, x) (\delta_{A_k} u)^2] dt dx + O(\|\delta_{A_k} u\|_\infty \|\delta_{A_k} u\|_{2,2}^2), \\
 &= \int_Q [\bar{H}_u(t, x) \delta_{A_k} u + \frac{1}{2} \bar{H}_{uu}(t, x) (\delta_{A_k} u)^2] dt dx + o(\|\delta_k u\|_{2,2}^2).
 \end{aligned}$$

Thus, if

$$\begin{aligned}
 & \int_Q \left[\frac{1}{2} \overline{H}_{yy}(t, x)(z^k)^2 + \delta_k \overline{H}_y z^k \right] dt dx \\
 (4.14) \quad &= \int_Q \left[\frac{1}{2} \overline{H}_{yy}(t, x)(z^{A_k})^2 + \overline{H}_{yu}(t, x) z^{A_k} \delta_{A_k} u \right] dt dx + o(\|\delta_k u\|_{2,2}^2), \\
 & \text{and } \int_\Omega \overline{\Phi}_{yy}[\lambda](z^k(T, x))^2 dx = \int_\Omega \overline{\Phi}_{yy}[\lambda](z^{A_k}(T, x))^2 dx + o(\|\delta_k u\|_{2,2}^2)
 \end{aligned}$$

hold true, relation (4.13) follows from (4.6) and (4.14) using that $\int_Q \delta_k \overline{H}$ can be decomposed as $\int_{A_k} \delta_k \overline{H} + \int_{B_k} \delta_k \overline{H}$. Let us prove (4.14). Using that $z^k = z^{A_k} + z^{B_k}$ (because $\delta_{A_k} u$ and $\delta_{B_k} u$ have disjoint supports and (4.1) has a unique solution), we have that

$$\begin{aligned}
 (4.15) \quad & \int_Q \left[\frac{1}{2} \overline{H}_{yy}(t, x)(z^k)^2 + \delta_k \overline{H}_y(t, x) z^k \right] dt dx \\
 &= \int_Q \left[\frac{1}{2} \overline{H}_{yy}(t, x)(z^{A_k})^2 + \delta_k \overline{H}_y(t, x) z^{A_k} \right] dt dx + r_k,
 \end{aligned}$$

where

$$\begin{aligned}
 |r_k| &= O \left(\int_Q \left[|(z^{A_k})^2 - (z^k)^2| + |\delta_k u| |z^{A_k} - z^k| \right] dt dx \right) \\
 &= O \left(\int_Q \left[|z^{B_k}|^2 + |z^{B_k}| |z^{A_k}| + |\delta_k u| |z^{B_k}| \right] dt dx \right).
 \end{aligned}$$

The second equality in the above relation follows from the fact that $|(z^{A_k})^2 - (z^k)^2| = |z^{B_k}| |z^{A_k} + z^k| = |z^{B_k}| |2z^{A_k} + z^{B_k}|$. Now, by (2.3), there exists $q_1 \in]1, 2[$ such that $\|z^{B_k}\|_{2,2} = O(\|\delta_{B_k} u\|_{q_1, q_1})$. By the Hölder inequality (and setting $s^* := s/(s - 1)$ for $s \in]1, \infty[$) we get

$$\begin{aligned}
 (4.16) \quad \|\delta_{B_k} u\|_{q_1, q_1} &= \left(\int_Q |\delta_{B_k} u|^{q_1} dt dx \right)^{\frac{1}{q_1}} \\
 &\leq \left(\int_Q (\mathbb{I}_{B_k})^{\left(\frac{2}{q_1}\right)^*} dt dx \right)^{\frac{1}{\left(\frac{2}{q_1}\right)^*}} \|\delta_k u\|_{2,2} = o(\|\delta_k u\|_{2,2})
 \end{aligned}$$

and so $\|z^{B_k}\|_{2,2} = o(\|\delta_k u\|_{2,2})$. Therefore, using the Cauchy–Schwarz inequality and that $\|z^{A_k}\|_{2,2} = O(\|\delta_{A_k} u\|_{2,2}) = O(\|\delta_k u\|_{2,2})$, we get $r_k = o(\|\delta_k u\|_{2,2}^2)$ and so

$$\begin{aligned}
 (4.17) \quad & \int_Q \left[\frac{1}{2} \overline{H}_{yy}(t, x)(z^k)^2 + \delta_k \overline{H}_y(t, x) z^k \right] dt dx \\
 &= \int_Q \left[\frac{1}{2} \overline{H}_{yy}(t, x)(z^{A_k})^2 + \delta_k \overline{H}_y(t, x) z^{A_k} \right] dt dx + o(\|\delta_k u\|_{2,2}^2).
 \end{aligned}$$

Now, since $\|\delta_{A_k} u\|_\infty \rightarrow 0$ we have that $\|z^{A_k}\|_\infty \rightarrow 0$ and so expanding the term $\delta_k \overline{H}_y(t, x)$ we get

$$\begin{aligned}
 (4.18) \quad \int_Q \delta_k \overline{H}_y(t, x) z^{A_k} dt dx &= \int_Q \overline{H}_{yu}(t, x) z^{A_k} \delta_k u dt dx \\
 &+ O \left(\int_Q |\delta_k u|^2 |z^{A_k}| dt dx \right) \\
 &= \int_Q \overline{H}_{yu}(t, x) z^{A_k} \delta_k u dt dx + o(\|\delta_k u\|_{2,2}^2).
 \end{aligned}$$

On the other hand, using (2.3) again, there exists $q_2 \in]2, \infty[$ such that $\|z^{A_k}\|_{q_2, q_2} = O(\|\delta_{A_k} u\|_{2,2}) = O(\|\delta_k u\|_{2,2})$. Noting that $q_2^* \in]1, 2[$, arguing as in (4.16) we get $\|\delta_{B_k} u\|_{q_2^*, q_2^*} = o(\|\delta_k u\|_{2,2})$ and so

$$\left| \int_Q \overline{H}_{yu}(t, x) z^{A_k} \delta_{B_k} u dt dx \right| = O \left(\|z^{A_k}\|_{q_2, q_2} \|\delta_{B_k} u\|_{q_2^*, q_2^*} \right) = o(\|\delta_k u\|_{2,2}^2).$$

Equation (4.18) then yields

$$(4.19) \quad \int_Q \delta_k \overline{H}_y(t, x) z^{A_k} dt dx = \int_Q \overline{H}_{yu}(t, x) z^{A_k} \delta_{A_k} u dt dx + o(\|\delta_k u\|_{2,2}^2).$$

Combining (4.17) and (4.19) gives the first identity in (4.14). In order to obtain the second one, note that Fubini's theorem implies that

$$(4.20) \quad \begin{aligned} \int_\Omega \overline{\Phi}_{yy}[\lambda](z^k(T, x))^2 dx &= 2 \int_\Omega \overline{\Phi}_{yy}[\lambda] \left[\int_0^T z^k(t, x) \partial_t z^k(t, x) dt \right] dx, \\ &= 2 \int_Q \overline{\Phi}_{yy}[\lambda] z^k(t, x) \partial_t z^k(t, x) dt dx. \end{aligned}$$

Analogously,

$$(4.21) \quad \int_\Omega \overline{\Phi}_{yy}[\lambda](z^{A_k}(T, x))^2 dx = 2 \int_Q \overline{\Phi}_{yy}[\lambda] z^{A_k}(t, x) \partial_t z^{A_k}(t, x) dt dx.$$

Therefore, using

$$z^k \partial_t z^k - z^{A_k} \partial_t z^{A_k} = (z^k - z^{A_k}) \partial_t z^k + \partial_t (z^k - z^{A_k}) z^{A_k} = z^{B_k} \partial_t z^k + \partial_t z^{B_k} z^{A_k},$$

we obtain with (4.20) and (4.21) that

$$\begin{aligned} & \left| \int_\Omega [\overline{\Phi}_{yy}[\lambda](z^k(T, x))^2 - \overline{\Phi}_{yy}[\lambda](z^{A_k}(T, x))^2] dx \right| \\ &= O \left(\int_Q [|z^{B_k}| |\partial_t z^k| + |\partial_t z^{B_k}| |z^{A_k}|] dt dx \right). \end{aligned}$$

By considering $q_1 \in]1, 2[$ and $q_2 \in]2, \infty[$ as above, the Hölder inequality implies that

$$\begin{aligned} & \left| \int_\Omega [\overline{\Phi}_{yy}[\lambda](z^k(T, x))^2 - \overline{\Phi}_{yy}[\lambda](z^{A_k}(T, x))^2] dx \right| \\ &= O \left(\|z^{B_k}\|_{2,2} \|\partial_t z^k\|_{2,2} + \|\partial_t z^{B_k}\|_{q_2^*, q_2^*} \|z^{A_k}\|_{q_2, q_2} \right) \\ &= O \left(\|\delta_{B_k} u\|_{q_1, q_1} \|\delta_k u\|_{2,2} + \|\delta_{B_k} u\|_{q_2^*, q_2^*} \|z^{A_k}\|_{q_2, q_2} \right) \\ &= o(\|\delta_k u\|_{2,2}^2), \end{aligned}$$

where we used (2.5) in order to estimate $\|\partial_t z^k\|_{2,2} = O(\|\delta_k u\|_{2,2})$ and $\|\partial_t z^{B_k}\|_{q_2^*, q_2^*} = O(\|\delta_{B_k} u\|_{q_2^*, q_2^*}) = o(\|\delta_k u\|_{2,2})$, arguing as in (4.16). Expansion (4.13) follows.

Step 2. Defining $\zeta^k := z^{A_k} - \xi^k$, where $\xi^k := \xi[\delta_{A_k} u]$ (recall (3.16)), we have that

$$(4.22) \quad \begin{aligned} \partial_t \zeta^k - \Delta \zeta^k + \overline{\varphi}_y(t, x) \zeta^k &= O(|\delta_{A_k} u|^2) \quad \text{in } Q, \\ \zeta^k &= 0 \quad \text{in } \Sigma, \\ \zeta^k(0, \cdot) &= 0 \quad \text{in } \Omega. \end{aligned}$$

By (2.5) and the fact that $\|\delta_{A_k} u\|_\infty \rightarrow 0$, we obtain that

$$\begin{aligned} & \|\zeta^k\|_{2,2} + \|\zeta^k(T, \cdot)\|_2 \\ &= O \left(\left(\int_Q |\delta_{A_k} u|^4 dt dx \right)^{\frac{1}{2}} \right) = O(\|\delta_{A_k} u\|_\infty \|\delta_{A_k} u\|_{2,2}) = o(\|\delta_{A_k} u\|_{2,2}). \end{aligned}$$

Using this estimate, it is straightforward to obtain (4.12) from (4.13). \square

Remark 4.4. When $\|\delta u\|_{2,2}$ is small, (4.12) provides a second order expansion of $\mathcal{L}(u, \lambda) - \mathcal{L}(\bar{u}, \lambda)$ that is decomposed into two principal terms. The first term $\int_{B_k} \delta_k \bar{H}[\lambda](t, x) dt dx$ takes into account large deviations of $u(t, x)$ around $\bar{u}(t, x)$, but over the set of small measure B_k . The second term

$$Q_1[\bar{u}, \lambda] \delta_{A_k} u + \frac{1}{2} Q_2[\bar{u}, \lambda](\delta_{A_k} u)$$

corresponds to a classical “weak” second order expansion (see [25]) on $\delta_{A_k} u$, which satisfies $\|\delta_{A_k} u\|_\infty$ is small.

5. Proofs of the main results. In this section we provide the proofs of the main results in section 3. We use the abbreviations for the notation introduced in the previous section.

5.1. Proofs of the main results for problem (CP₀).

Proof of Theorem 3.5. The argument follows the lines of the one in [3, Theorem 4.24]. In order to prove that (i) \Rightarrow (ii) note that Remark 3.4 implies that \bar{u} is a L^s -local solution of (CP₀) satisfying the quadratic growth condition (3.10) for every $s \in [1, \infty[$. In particular, there exists $\alpha > 0$ such that \bar{u} is a L^s -local solution of the problem

$$(CP'_0) \quad \inf \left\{ J(u) - \frac{\alpha}{2} \int_Q |u(t, x) - \bar{u}(t, x)|^2 dt dx ; u \in \mathcal{K}_0 \right\}.$$

Applying Pontryagin’s principle to problem (CP₀’) (see, e.g., [17, 14]) we directly obtain (3.18). On the other hand, since every strong solution satisfying the quadratic growth condition is a weak local solution satisfying the quadratic growth condition, we can apply second order necessary conditions for problem (CP₀’) (see, e.g., [15] and [25]) to obtain that $Q_2[\bar{u}](v) \geq \alpha \|v\|_{2,2}^2$ for all $v \in C_{\mathcal{K}_0}(\bar{u})$.

Now, suppose that (ii) holds true and that (3.12) is not satisfied. Then, there exists a sequence $u_k \in \mathcal{K}_0$ satisfying that $u_k \neq \bar{u}$, $\|y[u_k] - \bar{y}\|_\infty \rightarrow 0$ and (setting $\delta_k u := u_k - \bar{u}$)

$$(5.1) \quad J(u_k) - J(\bar{u}) \leq o(\|\delta_k u\|_{2,2}^2) \quad \text{as } k \uparrow \infty.$$

By (4.5) with $\lambda = 0$ and (3.18) we get (setting $\delta_k y := y[u_k] - \bar{y}$)

$$\frac{\alpha}{2} \|\delta_k u\|_2^2 \leq o(\|\delta_k u\|_{2,2}^2) + O(\|\delta_k y\|_\infty \|\delta_k u\|_{2,2}) = o(1),$$

which implies that $u_k \rightarrow \bar{u} \in L^{2,2}(Q)$. Now, for a.a. $(t, x) \in Q$, let us set

$$\kappa_{t,x} := \begin{cases} 1 & \text{if } \bar{H}_u(t, x) = 0, \\ 1/|\bar{H}_u(t, x)| & \text{otherwise,} \end{cases}$$

where $\bar{H}_u(t, x) := H_u(t, x, \bar{y}(t, x), \bar{p}(t, x), \bar{u}(t, x))$, and consider the set $B_k := B_k^1 \cup B_k^2$, where

$$B_k^1 := \left\{ (t, x) \in Q ; |\delta_k u(t, x)| \geq \sqrt{\|\delta_k u\|_{2,2}} \right\},$$

$$B_k^2 := \left\{ (t, x) \in Q ; \kappa_{t,x} \geq 1/\|\delta_k u\|_{2,2}^{1/4} \right\}.$$

By the Chebyshev inequality $|B_k^1| \rightarrow 0$ as $k \uparrow \infty$. By the dominated convergence theorem, we also have that $|B_k^2| = \int_Q \mathbb{I}_{\{(t,x) \in B_k^2\}} dt dx \rightarrow 0$, which implies that $|B_k| \rightarrow$

0. Defining $A_k := Q \setminus B_k$ and decomposing $\delta_k u = \delta_{A_k} u + \delta_{B_k} u$, we obtain that $\|\delta_{A_k} u\|_\infty \rightarrow 0$. Thus, letting $\lambda = 0$ in (4.12), we get

$$(5.2) \quad J(u_k) - J(\bar{u}) = \int_{B_k} \delta_k \bar{H}(t, x) dt dx + Q_1[\bar{u}] \delta_{A_k} u + \frac{1}{2} Q_2[\bar{u}] (\delta_{A_k} u) + o(\|\delta_k u\|_{2,2}^2),$$

where $\delta_k \bar{H}(t, x) := H(t, x, \bar{y}(t, x), \bar{p}(t, x), u_k(t, x)) - H(t, x, \bar{y}(t, x), \bar{p}(t, x), \bar{u}(t, x))$. Now, set $\sigma_{A_k} := \|\delta_{A_k} u\|_{2,2}$ and $\sigma_{B_k} := \|\delta_{B_k} u\|_{2,2}$, which yields $\|\delta_k u\|_{2,2}^2 = \sigma_{A_k}^2 + \sigma_{B_k}^2$. If $\sigma_{A_k} = o(\sigma_{B_k})$, then (5.2) and the continuity of the quadratic form $Q_2[\bar{u}]$ imply that

$$\int_{B_k} \delta_k \bar{H}(t, x) dt dx + Q_1[\bar{u}] \delta_{A_k} u \leq o(\sigma_{B_k}^2).$$

Since $Q_1[\bar{u}] \delta_{A_k} u \geq 0$, (3.18) would imply that $\frac{\alpha}{2} \sigma_{B_k}^2 \leq o(\sigma_{B_k}^2)$, which is impossible. Therefore, for some subsequence, we must have that $\sigma_{B_k} = O(\sigma_{A_k})$. Let us define $v_k := \delta_{A_k} u / \sigma_{A_k}$. By (3.18) we have $\int_{B_k} \delta_k \bar{H}(t, x) dt dx \geq 0$ and $Q_1[\bar{u}] \delta_{A_k} u \geq 0$. Therefore, (5.1) and (5.2) imply that

$$(5.3) \quad Q_2[\bar{u}](v_k) \leq o(1).$$

Note that if $v_k \in C_{\mathcal{K}_0}(\bar{u})$, then (5.3) would yield a contradiction with (ii.2) because $\|v_k\|_{2,2} = 1$. The problem is that a priori v_k belongs only to $T_{\mathcal{K}_0}(\bar{u})$. The idea is to decompose $v_k = v_{k,1} + v_{k,2}$ where $v_{k,1} \in C_{\mathcal{K}_0}(\bar{u})$ and prove that $\|v_{k,2}\|_{2,2} \rightarrow 0$, which together with (5.3) and (ii.2) would yield the desired contradiction. In order to construct such a decomposition, first notice that $C_{\mathcal{K}_0}(\bar{u}) = \{v \in L^{2,2}(Q) ; v(t, x) \in C_{t,x} \text{ for a.a. } (t, x) \in Q\}$, where

$$C_{t,x} := \{v \in \mathbb{R} ; \bar{H}_u(t, x)v = 0, v \geq 0 \text{ if } (t, x) \in A_a(\bar{u}) \\ \text{and } v \leq 0 \text{ if } (t, x) \in A_b(\bar{u})\}.$$

Thus, it is natural to define $v_{k,1}(t, x) := P_{C_{t,x}}(v_k(t, x))$ (where P denotes the projection operator), which is measurable in terms of (t, x) and $(t, x) \rightarrow v_{k,1}(t, x) \in C_{\mathcal{K}_0}(\bar{u})$. Using that for every $h \in \mathbb{R}$ we have, denoting $d(h, C_{t,x}) := \inf\{|z - h| ; z \in C_{t,x}\}$ and $(h)_+ := \max\{h, 0\}$,

$$(5.4) \quad d(h, C_{t,x}) \leq \kappa_{t,x} (|\bar{H}_u(t, x)h| + \mathbb{I}_{A_a}(t, x)(-h)_+ + \mathbb{I}_{A_b}(t, x)(h)_+),$$

by definition of $v_{k,1}$ and the fact that $v_k \in T_{\mathcal{K}_0}(\bar{u})$, we obtain that

$$(5.5) \quad |v_{k,2}(t, x)| = d(v_k(t, x), C_{t,x}) \\ \leq \kappa_{t,x} |\bar{H}_u(t, x)v_k(t, x)| \leq \frac{1}{\|\delta_k u\|_{2,2}^{1/4}} \bar{H}_u(t, x)v_k(t, x),$$

where we have used that $\bar{H}_u(t, x)v_k(t, x) \geq 0$ and that $v_k = \delta_{A_k} u_k / \sigma_{A_k} \equiv 0$ on B_k . Now, by the continuity of the quadratic form $Q_2[\bar{u}]$, we have that (5.1) and (5.2) imply that

$$\int_{A_k} \bar{H}_u(t, x)v_k(t, x) dt dx \leq O(\sigma_{A_k}),$$

which together with (5.5) gives $\|v_{k,2}\|_{1,1} \leq O(\sigma_{A_k} / \|\delta_k u\|_{2,2}^{1/4})$. On the other hand, by the definition of $v_{k,1}(t, x)$ and $C_{t,x}$ we have that $|v_{k,2}(t, x)| \leq |v_k(t, x)|$. And so, since

$\|v_k\|_\infty \leq \frac{\sqrt{\|\delta_k u\|_{2,2}}}{\sigma_{A_k}}$ by the definition of the set A_k , we get that

$$\|v_{k,2}\|_{2,2}^2 \leq \|v_{k,2}\|_\infty \|v_{k,2}\|_{1,1} = O\left(\frac{\sqrt{\|\delta_k u\|_{2,2}}}{\sigma_{A_k}} \frac{\sigma_{A_k}}{\|\delta_k u\|_{2,2}^{1/4}}\right) = O(\|\delta_k u\|_{2,2}^{1/4}) \rightarrow 0,$$

as $k \uparrow \infty$, which concludes the proof. \square

Proof of Corollary 3.6. If \bar{u} is a L^s -weak local solution of (CP_0) satisfying the quadratic growth condition, then arguing as in the proof of Theorem 3.5 we have that (ii.1) and (ii.2) in Theorem 3.5 are satisfied. Therefore, the converse implication in Theorem 3.5 implies that \bar{u} is a strong local solution of (CP_0) satisfying the quadratic growth condition. The other implication being clear (see Remark 3.4), the conclusion follows. \square

Proof of Theorem 3.7. By Remark 3.4 it suffices to prove that (3.8) and (3.12) are equivalent. We only need to prove, the other implication being trivial, that if \bar{u} satisfies (3.8), then it also satisfies (3.12). Note that if \bar{u} satisfies (3.8), then, by the second order necessary optimality conditions for weak minima (see, e.g., [15] and [25]), we have that (ii.2) in Theorem 3.5 is satisfied. Thus, in order to conclude we only need to show that the quadratic growth condition (3.18) for the Hamiltonian holds at \bar{u} . Since \bar{u} is a weak solution of problem (CP'_0) , by the local Pontryagin principle for weak solutions (see [14]), we have the existence of $\varepsilon > 0$ such that

$$(5.6) \quad \begin{aligned} \bar{H}(t, x) + \frac{\alpha}{2}|u - \bar{u}(t, x)|^2 &\leq H(t, x, \bar{y}(t, x), \bar{p}(t, x), u) \\ \forall u \in [a(t, x), b(t, x)], \quad |u - \bar{u}(t, x)| &\leq \varepsilon, \end{aligned}$$

where $\bar{H}(t, x) := H(t, x, \bar{y}(t, x), \bar{p}(t, x), \bar{u}(t, x))$. Now, for all $(t, x) \in \bar{Q}$ let us define

$$(5.7) \quad \hat{u}(t, x) = \underset{u \in [a(t, x), b(t, x)]}{\operatorname{argmin}} H(t, x, \bar{y}(t, x), \bar{p}(t, x), u).$$

By the strict convexity of the Hamiltonian w.r.t. u (by (H3)) and (5.6) we get that \hat{u} is well defined and $\hat{u} = \bar{u}$ a.e. in Q . Now, by the continuity assumption on the data, the assumption on Φ_y , and [17, Proposition 2.1], we have the existence of $\beta' \in]0, 1[$ such that $\bar{p} \in C^{\beta', \beta'/2}(\bar{Q}) \cap L^2([0, T]; H_0^1(\Omega))$. Thus, using the continuity of the data again, the continuity of \bar{y} and the continuity of \bar{p} , we easily check with (5.7) that \hat{u} is continuous in \bar{Q} . Therefore, for all $(t, x) \in \bar{Q}$, $u \in [a(t, x), b(t, x)]$, and $|u - \hat{u}(t, x)| \leq \varepsilon$ we have

$$(5.8) \quad H(t, x, \bar{y}(t, x), \bar{p}(t, x), \hat{u}(t, x)) + \frac{\alpha}{2}|u - \hat{u}(t, x)|^2 \leq H(t, x, \bar{y}(t, x), \bar{p}(t, x), u).$$

Therefore, letting

$$\begin{aligned} \beta &:= \inf \{ H(t, x, \bar{y}(t, x), \bar{p}(t, x), u) - H(t, x, \bar{y}(t, x), \bar{p}(t, x), \hat{u}(t, x)) \}; \\ &\quad (t, x) \in \bar{Q}, \quad u \in [a(t, x), b(t, x)], \quad |u - \hat{u}(t, x)| \geq \varepsilon \}, \end{aligned}$$

the continuity of \hat{u} implies that $\beta > 0$. Defining $\alpha' = \min\{\beta/4M^2, \frac{\alpha}{2}\}$ we obtain that (5.8) holds with α replaced by α' and for all $v \in [a(t, x), b(t, x)]$ and so (3.18) is satisfied, which implies the result. \square

5.2. Proofs of the main results for problem (CP₁).

Proof of Theorem 3.10. If (3.12) does not hold, then there exists a sequence $u_k \neq \bar{u} \in \mathcal{K}$, with associated states $y_k := y[u_k]$, such that, setting $\delta_k u := u_k - \bar{u}$ and $\delta_k y := y_k - \bar{y}$, one has

$$(5.9) \quad J(u_k) - J(\bar{u}) \leq o(\|\delta_k u\|_{2,2}^2) \quad \text{as} \quad \|\delta_k y\|_\infty \rightarrow 0.$$

By definition, for every $\lambda \in \Lambda_L(\bar{u})$ we have that $J(u_k) \geq \mathcal{L}(u_k, \lambda)$ and $J(\bar{u}) = \mathcal{L}(\bar{u}, \lambda)$; therefore by Proposition 4.2(i) we have that

$$\int_Q \delta_k \bar{H}[\lambda](t, x) dt dx + O(\|\delta_k u\|_{2,2} \|\delta_k y\|_\infty) = \mathcal{L}(u_k, \lambda) - \mathcal{L}(\bar{u}, \lambda) \leq o(\|\delta_k u\|_{2,2}^2),$$

where

$$\delta_k \bar{H}[\lambda](t, x) := H(t, x, \bar{y}(t, x), \bar{p}_\lambda(t, x), u_k(t, x)) - H(t, x, \bar{y}(t, x), \bar{p}_\lambda(t, x), \bar{u}(t, x)).$$

Setting $\lambda = \bar{\lambda}$, assumption (i) and (3.23) imply that $\|\delta_k u\|_{2,2} \rightarrow 0$. Now, let us define

$$A_k := \left\{ (t, x) \in Q \ ; \ |u(t, x) - \bar{u}(t, x)| \leq \sqrt{\|\delta_k u\|_{1,1}} \right\}, \quad B_k := Q \setminus A_k,$$

and $\delta_{A_k} u$ and $\delta_{B_k} u$ as in (4.11). Since $\|\delta_k u\|_{2,2} \rightarrow 0$, we obtain that $\|\delta_{A_k} u\|_\infty \rightarrow 0$ and using the Chebyshev inequality we get that $|B_k| \rightarrow 0$. Therefore, by (4.12) and (5.9), for every $\lambda \in \Lambda_L(\bar{u})$ we have that

$$(5.10) \quad \int_{B_k} \delta_k \bar{H}[\lambda](t, x) dt dx + Q_1[\bar{u}, \lambda](\delta_{A_k} u) + \frac{1}{2} Q_2[\bar{u}, \lambda](\delta_{A_k} u) \leq o(\|\delta_k u\|_{2,2}^2).$$

Suppose that $\|\delta_{A_k} u\|_{2,2} = o(\|\delta_{B_k} u\|_{2,2})$. Then, $\|\delta_k u\|_{2,2} = O(\|\delta_{B_k} u\|_{2,2})$. Since $\delta_{A_k} u \in T_{\mathcal{K}_0}(\bar{u})$, for every $\lambda \in \Lambda_L(\bar{u})$ we have that $Q_1[\bar{u}, \lambda] \delta_{A_k} u \geq 0$. Also, by the continuity of the quadratic form $Q_2[\bar{u}, \lambda](\cdot)$ we get that $Q_2[\bar{u}, \lambda](\delta_{A_k} u) = O(\|\delta_{A_k} u\|_{2,2}^2)$, and so, using (5.10), we obtain the inequality

$$(5.11) \quad \int_{B_k} \delta_k \bar{H}[\lambda](t, x) dt dx \leq o(\|\delta_k u\|_{2,2}^2) + O(\|\delta_{A_k} u\|_{2,2}^2) = o(\|\delta_{B_k} u\|_{2,2}^2).$$

Using (3.23) we get that $\alpha \|\delta_{B_k} u\|_{2,2}^2 \leq o(\|\delta_{B_k} u\|_{2,2}^2)$ for some $\alpha > 0$, which is impossible. Thus, extracting a subsequence, we can assume that $\|\delta_{B_k} u\|_{2,2} = O(\|\delta_{A_k} u\|_{2,2})$. Now, let us set $\sigma_k := \|\delta_{A_k} u\|_{2,2}$, which can be assumed to be nonzero, and define $v_k := \delta_{A_k} u / \sigma_k$. Since $\|v_k\|_{2,2} = 1$, we have that up to some subsequence, v_k converges weakly in $L^{2,2}(Q)$ to some \bar{v} . Note that since $v^k \in T_{\mathcal{K}_0}(\bar{u})$ and $T_{\mathcal{K}_0}(\bar{u})$ is closed and convex in $L^{2,2}(Q)$, we obtain that $\bar{v} \in T_{\mathcal{K}_0}(\bar{u})$. Let us now prove that $\bar{v} \in C_{\mathcal{K}}(\bar{u})$. First note that by the results in [25, section 3],

$$(5.12) \quad J(u_k) - J(\bar{u}) = DJ(\bar{u})\delta_k u + O(\|\delta_k u\|_{2,2}^2) = DJ(\bar{u})\delta_k u + o(\|\delta_k u\|_{2,2}),$$

and, recalling (3.16) and using [25, Remark 3.4],

$$(5.13) \quad DJ(\bar{u})\delta_k u = \int_Q [\bar{\ell}_y(t, x)\zeta[\delta_k u] + \bar{\ell}_u(t, x)\delta_k u] dt dx + \int_\Omega \bar{\Phi}_y(x)\zeta[\delta_k u](T, x) dx,$$

where $\bar{\ell}_y(t, x) := \ell_y(t, x, \bar{y}(t, x), \bar{u}(t, x))$ and $\bar{\Phi}_y(x) = \Phi_y(x, \bar{y}(T, x))$. On the other hand, estimate (2.6) implies that

$$\|\zeta[\delta_{B_k} u]\|_{1,1} + \|\zeta[\delta_{B_k} u](T, \cdot)\|_1 = O(\|\delta_{B_k} u\|_{1,1}) = o(\|\delta_k u\|_{2,2}).$$

Therefore, since $\|\delta_k u\|_{2,2} = O(\|\delta_{A_k} u\|_{2,2})$, (5.9), (5.12), and (5.13) yield

$$\int_Q [\bar{\ell}_y(t, x)\zeta[\delta_{A_k} u] + \bar{\ell}_u(t, x)\delta_{A_k} u] dt dx + \int_\Omega \bar{\Phi}_y(x)\zeta[\delta_{A_k} u](T, x) dx \leq o(\|\delta_{A_k} u\|_{2,2}).$$

Dividing by σ_k we get that

$$\int_Q [\bar{\ell}_y(t, x)\zeta[v_k] + \bar{\ell}_u(t, x)v_k] dt dx + \int_\Omega \bar{\Phi}_y(x)\zeta[v_k](T, x) dx \leq o(1).$$

Using [25, Proposition 2.1(iii)] and passing to the limit we obtain that $DJ(\bar{u})\bar{v} \leq 0$. On the other hand, the same type of computations implies that

$$\int_\Omega (\bar{\Phi}_E)_y(x)\zeta[\bar{v}](T, x) dx = 0 \quad \text{and} \quad \int_\Omega (\bar{\Phi}_I)^j(x)\zeta[\bar{v}](T, x) dx \leq 0 \quad \forall j \in I(\bar{u}),$$

where $(\bar{\Phi}_E)_y(x) = (\Phi_E)_y(x, \bar{y}(T, x))$ with a similar notation for $(\bar{\Phi}_I)_y(x)$. The above relations, together with the fact that $\lambda \in \Lambda_L(\bar{u})$ and $\bar{v} \in T_{\mathcal{K}_0}(\bar{u})$, imply that $DJ(u)\bar{v} = 0$. Therefore, we have proved that $\bar{v} \in C_{\mathcal{K}}(\bar{u})$. By convexity of the Hamiltonian w.r.t. u , for all $\lambda \in \Lambda_L(\bar{u})$ one has $\delta_k \bar{H}[\lambda](t, x) \geq 0$ and $Q_1[\bar{u}, \lambda](\delta_{A_k} u) \geq 0$ and so

$$\int_{B_k} \delta_k \bar{H}[\lambda](t, x) dt dx + Q_1[\bar{u}, \lambda](\delta_{A_k} u) \geq 0.$$

Thus, we get with (5.10) that

$$(5.14) \quad Q_2[\bar{u}, \lambda](v_k) \leq o(1).$$

By assumption (iii), there exists $\lambda_{\bar{v}} \in \Lambda_L(\bar{u})$ such that $Q_2[\bar{u}, \lambda_{\bar{v}}](\bar{v}) \geq \alpha \|\bar{v}\|_{2,2}^2$. The structural assumption in (H3) implies that $Q_2[\bar{u}, \lambda_{\bar{v}}]$ is weakly lower semicontinuous (see, e.g., [25]) and so (5.14) yields,

$$0 \leq \alpha \|\bar{v}\|_{2,2}^2 \leq Q_2[\bar{u}, \lambda_{\bar{v}}](\bar{v}) \leq \liminf_{k \rightarrow \infty} Q_2[\bar{u}, \lambda_{\bar{v}}](v_k) \leq \limsup_{k \rightarrow \infty} Q_2[\bar{u}, \lambda_{\bar{v}}](v_k) \leq 0,$$

and so $\bar{v} = 0$ and $Q_2[\bar{u}, \lambda_{\bar{v}}](\bar{v}) = 0 = \lim_{k \rightarrow \infty} Q_2[\bar{u}, \lambda_{\bar{v}}](v_k)$. Using that $u \rightarrow \zeta[u]$ is continuous from $L^{2,2}(Q)$, endowed with the weak topology, to $L^{2,2}(Q)$, endowed with the strong topology (see, e.g., [25, Proposition 2.1(iii)]), (H3) implies that $v_k \rightarrow \bar{v} = 0$ strongly in $L^{2,2}(Q)$, contradicting $\|v_k\|_{2,2} = 1$. \square

Proof of Theorem 3.12. We only need to prove that (i) implies (ii). If \bar{u} is a strong local solution of (CP_1) satisfying (3.12), then in particular it is a weak solution satisfying (3.8). Assertion (ii) follows directly from [25, Theorem 5.2] and the assumption $C_{\mathcal{K}}(\bar{u}) = C_{\mathcal{K}}^s(\bar{u})$. \square

Proof of Theorem 3.13. The proof of this result is exactly the same as the previous one, the only difference being that instead of using [25, Theorem 5.2] we use [25, Theorem 5.7]. \square

Proof of Theorem 3.14. The proof of the equivalence between weak and strong minima satisfying the quadratic growth condition for problem (CP_1) is the same as the proof of the corresponding result for (CP_0) in Theorem 3.7, the only difference being that in order to write the second order necessary optimality condition for a weak minimum, Theorem 5.2 in [25] is used if condition (a) holds and Theorem 5.7 in [25] is used if condition (b) holds. Then, as in Theorem 3.7 the local Pontryagin principle together with the continuity assumption on data are used to obtain a global quadratic growth property for the Hamiltonian. Finally, the previous characterizations in Theorems 3.12 and 3.13 can be directly applied for cases (a) and (b), respectively. \square

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