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# GEOMETRY ON THE UTILITY SPACE\*

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We examine the geometrical properties of the space of expected utilities over a finite set of options, which is commonly used to model the preferences of an agent. We focus on the case where options are assumed to be symmetrical *a priori*. Specifically, we prove that the only Riemannian metric that respects the geometrical properties and the natural symmetries of the utility space is the round metric.

JEL code: D81.

## I INTRODUCTION

### *I.A Motivation*

In order to represent an agent's preferences over options in a context of uncertainty, a simple and elegant model was formally defined by Von Neumann et Morgenstern: expected utilities. For any agent, a numerical utility is associated to each option; the utility of a lottery over the options is computed as an expected value. See [Von Neumann and Morgenstern \(1944\)](#); [Fishburn \(1970\)](#); [Kreps \(1990\)](#); [Mas-Colell et al. \(1995\)](#).

For a great deal of applications, options are financial rewards or quantities of one or several goods. As a consequence, there is a natural structure over the space of options: the structure of the real line or  $\mathbb{R}^m$ , at least as a topological, or even metrical, vector space. In such cases, it is natural to study, for example, possible derivatives of the utility with respect to the reward: monotony, concavity or convexity (risk attitude), etc.

However, for some other applications, such as voting systems, options can be, for example, candidates for a position or possible decisions about a given

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question. Without additional information, we may consider that options are symmetrical *a priori*. In that case, to the best of our knowledge, little attention has been paid to the geometrical properties of the utility space.

Under the assumptions of Von Neumann–Morgenstern theorem (that will be recalled), an agent’s utility vector is defined up to two constants; choosing a specific normalization is arbitrary. In particular, utilities of two distinct agents are essentially incomparable without an additional normalization assumption. Geometrically, the utility space has the properties of a projective space.

*A priori*, this space has no natural metric. It is a pity, because it is generally convenient to work in a space with as much structure as possible. More specifically, with a metric, it is quite easy to endow the space with various probability laws, in order to model the “culture” of a population. In this paper, our goal is to give a better geometrical understanding of the utility space and to show the limited ways of endowing it with a natural metric.

### ***I.B Contributions***

Firstly, we investigate various geometrical properties of the utility space:

- We show that the utility space has a projective structure and may be seen as a quotient of the dual of the space of pairs of lotteries over the candidates;
- We naturally define an inversion operation, that corresponds to reversing preferences while keeping their intensities, and a summation operation, that is proved to preserve unanimous preferences;
- We remark that the utility space has a pathological topology when keeping the indifference point but a spherical topology when removing it.

Secondly, since the utility space is a variety, it is a natural wish to endow it with a Riemannian metric. We show that the only Riemannian representation that preserves the natural projective properties and the *a priori* symmetry between the candidates is the round metric.

### ***I.C Plan***

The rest of the paper is organized as follows. Von Neumann–Morgenstern theorem is recalled in section II. Several properties of this model are studied: duality with the space of pairs of lotteries in section III, inversion and summation operators in section IV, topology in section V. In section VI, it is shown what Riemannian metrics respect the natural symmetries of this space. Finally, in section VII, formulas are given that allow to use this spherical representation for applications.

In the appendix, the reader will find tables recalling the main notations of this paper.

## II VON NEUMANN–MORGENSTERN MODEL

In this section, we define some notations and we recall Von Neumann and Morgenstern theorem, which allows to represent an agent's preferences over probabilistic options by the means of an expected utility form; see [Von Neumann and Morgenstern \(1944\)](#).

We first formalize the available options and lotteries over them.

**Notations II.1** (candidates and lotteries). For any  $k \in \mathbb{N}$ , we denote  $\llbracket 1, k \rrbracket$  the integer interval from 1 to  $k$ .

Let  $m \in \mathbb{N} \setminus \{0\}$  and  $\mathcal{C}_m = \llbracket 1, m \rrbracket$ . Elements in  $\mathcal{C}_m$  represent mutually exclusive options called *candidates*.

Let  $\mathcal{L}_m = \{(L_1, \dots, L_m) \in \mathbb{R}_+^m \text{ s.t. } \sum_{j=1}^m L_j = 1\}$ . It is the set of *lotteries* over the candidates in  $\mathcal{C}_m$ .

For any two lotteries  $L = (L_1, \dots, L_m)$  and  $M = (M_1, \dots, M_m)$ , for any  $\alpha \in [0, 1]$ , we naturally define their reduced compound lottery as a barycenter:

$$\alpha L + (1 - \alpha)M = \left( \alpha L_1 + (1 - \alpha)M_1, \dots, \alpha L_m + (1 - \alpha)M_m \right).$$

The preferences of an agent over the space of lotteries are represented by a binary relation  $\succsim$  over  $\mathcal{L}_m$ . Let  $\prec$  be the strict relation associated to  $\succsim$ , defined as:  $L \prec M \Leftrightarrow L \succsim M$  and not  $M \succsim L$ .

We also need to introduce a few additional notations.

**Notations II.2** (linear algebra). We denote  $\vec{1}$  the vector whose  $m$  coordinates are 1 and  $J$  the  $m \times m$  matrix whose  $m^2$  coefficients are 1.

When  $E$  is a part of a vector space, we denote  $\text{vect}(E)$  the linear span of  $E$ .

The canonical inner product of  $\vec{u}$  and  $\vec{v}$  is denoted  $\langle \vec{u} | \vec{v} \rangle$ . The canonical Euclidean norm of  $\vec{u}$  is denoted  $\|\vec{u}\|$ .

**Definition II.3** (completeness). We say that relation  $\succsim$  is *complete* if and only if  $\forall (L, M) \in \mathcal{L}_m^2$ :

$$L \succsim M \text{ or } M \succsim L.$$

**Definition II.4** (transitivity). We say that relation  $\succsim$  is *transitive* if and only if  $\forall (L, M, N) \in \mathcal{L}_m^3$ :

$$L \succsim M \text{ and } M \succsim N \Rightarrow L \succsim N.$$

**Definition II.5** (archimedeaness). We say that relation  $\succsim$  is *archimedean* if and only if  $\forall (L, M, N) \in \mathcal{L}_m^3$ :

$$L \prec M \text{ and } M \prec N \Rightarrow \exists \varepsilon \in ]0, 1[ \text{ s.t. } (1 - \varepsilon)L + \varepsilon N \prec M \prec \varepsilon L + (1 - \varepsilon)N.$$

**Definition II.6** (independence of irrelevant alternatives). We say that  $\succsim$  is *independent of irrelevant alternatives* if and only if  $\forall (L, M, N) \in \mathcal{L}_m^3, \forall \alpha \in ]0, 1[$ :

$$L \prec M \Rightarrow \alpha L + (1 - \alpha)N \prec \alpha M + (1 - \alpha)N.$$

**Definition II.7** (utility vector). Let  $\succsim$  be a binary relation over  $\mathcal{L}_m$ . Let  $\vec{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$ .

We say that  $\vec{u}$  is a *utility vector* representing  $\succsim$  if and only, for any two lotteries  $L = (L_1, \dots, L_m)$  and  $M = (M_1, \dots, M_m)$ :

$$L \succsim M \Leftrightarrow \sum_{j=1}^m L_j u_j \leq \sum_{j=1}^m M_j u_j.$$

**Theorem II.8** (Von Neumann and Morgenstern). *Let  $\succsim$  be a binary relation over  $\mathcal{L}_m$ .*

*The following conditions are equivalent.*

1. *The binary relation  $\succsim$  is complete, transitive, archimedean and independent of irrelevant alternatives.*
2. *There exists  $\vec{u} \in \mathbb{R}^m$  a utility vector representing  $\succsim$ .*

*When they are met,  $\vec{u}$  is defined up to an additive constant and a positive multiplicative constant: if  $\vec{u} \in \mathbb{R}^m$  is a utility vector representing  $\succsim$ , then  $\vec{v} \in \mathbb{R}^m$  is a utility vector representing  $\succsim$  if and only if  $\exists a \in ]0, +\infty[, \exists b \in \mathbb{R}$  s.t.  $\vec{v} = a\vec{u} + b\vec{1}$ .*

For a proof, see [Von Neumann et al. \(2007\)](#), [Mas-Colell et al. \(1995\)](#) or [Kreps \(1990\)](#).

Discussing whether completeness, transitivity and archimedeaness assumptions are experimentally valid is out of the scope of this paper: see [Fishburn \(1988\)](#); [Mas-Colell et al. \(1995\)](#). Let us note that, while keeping a sheer simplicity, this model is sophisticated enough to deal with relative intensities of the agent's preferences. This gives more information than just a total preorder over the candidates.

**Definition II.9** (utility space). Let  $\approx$  be the equivalence relation defined by  $\forall (\vec{u}, \vec{v}) \in (\mathbb{R}^m)^2$ :

$$\vec{u} \approx \vec{v} \Leftrightarrow \exists a \in ]0, +\infty[, \exists b \in \mathbb{R} \text{ s.t. } \vec{v} = a\vec{u} + b\vec{1}.$$

We call *utility space over  $m$  candidates*, and we denote  $\mathcal{U}_m$ , the quotient space  $\mathbb{R}^m / \approx$ .

We call *canonical projection* from  $\mathbb{R}^m$  to  $\mathcal{U}_m$  the function:

$$\tilde{\pi} : \begin{array}{l} \mathbb{R}^m \rightarrow \mathcal{U}_m \\ \vec{u} \rightarrow \{\vec{v} \in \mathbb{R}^m \text{ s.t. } \vec{v} \approx \vec{u}\}. \end{array}$$

For any  $\vec{u} \in \mathbb{R}^m$  and  $\tilde{u} = \tilde{\pi}(\vec{u})$ , we denote without ambiguity  $\succsim_{\tilde{u}}$  the binary relation over  $\mathcal{L}_m$  represented by  $\vec{u}$  in the sense of theorem II.8 (Von Neumann–Morgenstern).

### III DUALITY

In this section, we remark that the utility space is a dual of the space of pairs of lotteries. This will be especially helpful to demonstrate theorem [IV.9](#), characterizing the summation operator that will be defined in section [IV](#).

In the example represented in figure [I](#), we consider  $m = 3$  candidates and  $\vec{u} = (\frac{5}{3}; -\frac{1}{3}; -\frac{4}{3})$ . The great triangle is the space of lotteries  $\mathcal{L}_m$ , that is, the simplex defined by  $\sum L_i = 1$  and  $\forall i, L_i \geq 0$ . Hatchings are the agent's indifference lines: she is indifferent between any pair of lotteries on the same line (see [Mas-Colell et al. \(1995\)](#), section 6.B). Here, we have drawn a utility vector  $\vec{u}$  that is in the plane of the great triangle and orthogonal to the indifference lines. This is not mandatory, since  $\vec{u}$  can be arbitrarily chosen in  $\tilde{u}$ , but it is a quite natural choice, since the component of  $\vec{u}$  in the direction  $\vec{1}$ , orthogonal to the simplex, has no meaning in terms of preferences.

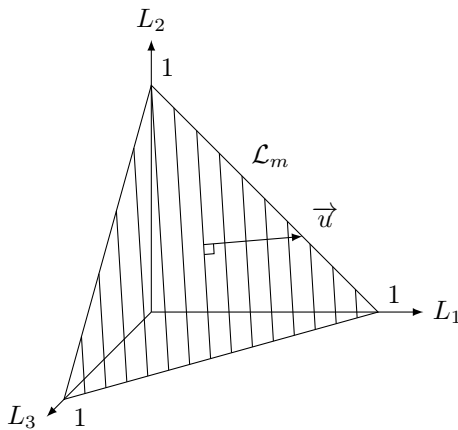


Figure I: Space of lotteries for 3 candidates.

The utility vector  $\vec{u}$  may be seen as a (uniform) gradient of preference: at each point, it reveals in what directions one can find lotteries that are preferred by the agent. However, only the direction of  $\vec{u}$  is important, whereas its norm has no specific meaning; as a consequence, the utility space is not exactly a dual space, but rather a quotient of a dual space, as we will see more formally.

**Definition III.1** (bipoint). For any two lotteries  $L = (L_1, \dots, L_m)$  and  $M = (M_1, \dots, M_m)$ , we call *bipoint* from  $L$  to  $M$  the vector in  $\mathbb{R}^m$ :

$$\vec{LM} = (M_1 - L_1, \dots, M_m - L_m).$$

**Definition III.2** (tangent polytope and hyperplane). We call *tangent polytope* of  $\mathcal{L}_m$  the set  $\mathcal{T}$  of bipoints of  $\mathcal{L}_m$ :

$$\mathcal{T} = \{\vec{LM}, (L, M) \in \mathcal{L}_m^2\}.$$

We call *tangent hyperplane* of  $\mathcal{L}_m$ :

$$\mathcal{H} = \{(\Delta_1, \dots, \Delta_m) \in \mathbb{R}^m \text{ s.t. } \sum_{j=1}^m \Delta_j = 0\}.$$

**Remark III.3** (geometrical interpretation). In figure I,  $\mathcal{T}$  is the set of the bipoints of the great triangle, seen as a part of a vector space (whereas  $\mathcal{L}_m$  is a part of an affine space). Hyperplane  $\mathcal{H}$  is the whole vector plan containing  $\mathcal{T}$ .

**Lemma III.4** ( $\mathcal{T}$  generates  $\mathcal{H}$ ).  $\forall \vec{\Delta} \in \mathcal{H}, \exists \overrightarrow{LM} \in \mathcal{T}, \exists \lambda \in ]0, +\infty[ \text{ s.t. } \vec{\Delta} = \lambda \overrightarrow{LM}$ .

*Proof.* Polytope  $\mathcal{T}$  contains a neighborhood of the origin in vector space  $\mathcal{H}$ .  $\square$

**Definition III.5** (linear forms on  $\mathcal{H}$ ). Let  $\mathcal{H}^*$  be the dual space of  $\mathcal{H}$ , that is, the set of linear forms on  $\mathcal{H}$ . Let  $\vec{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$ .

We call *linear form associated to*  $\vec{u}$  the following element of  $\mathcal{H}^*$ :

$$\langle \vec{u} | : \begin{array}{ccc} \mathcal{H} & \rightarrow & \mathbb{R} \\ \vec{\Delta} = (\Delta_1, \dots, \Delta_m) & \rightarrow & \langle \vec{u} | \vec{\Delta} \rangle = \sum_{j=1}^m u_j \Delta_j. \end{array}$$

We call *positive half-hyperplane associated to*  $\vec{u}$ :

$$\vec{u}^{+\mathcal{H}} = \{\vec{\Delta} \in \mathcal{H} \text{ s.t. } \langle \vec{u} | \vec{\Delta} \rangle \geq 0\}.$$

**Proposition III.6** (preferences expressed in terms of linear forms). *Let  $(L, M) \in \mathcal{L}_m^2$  and  $\vec{u} \in \mathbb{R}^m$ . Then:*

$$L \lesssim_{\vec{u}} M \Leftrightarrow \langle \vec{u} | \overrightarrow{LM} \rangle \geq 0 \Leftrightarrow \overrightarrow{LM} \in \vec{u}^{+\mathcal{H}}.$$

*Proof.* This follows easily from the definitions.  $\square$

**Proposition III.7** (utility space and linear forms). *For any  $(f, g) \in (\mathcal{H}^*)^2$ , we denote  $f \approx g$  if and only  $\exists a \in ]0, +\infty[ \text{ s.t. } g = af$ . We denote  $\tilde{\pi}(f) = \{g \in \mathcal{H}^* \text{ s.t. } f \approx g\}$ .*

*For any  $(\vec{u}, \vec{v}) \in (\mathbb{R}^m)^2$ , we have:*

$$\vec{u} \approx \vec{v} \Leftrightarrow \langle \vec{u} | \approx \langle \vec{v} |.$$

*The following application is a bijection:*

$$\Theta : \begin{array}{ccc} \mathcal{U}_m & \rightarrow & \mathcal{H}^* / \approx \\ \tilde{\pi}(\vec{u}) & \rightarrow & \tilde{\pi}(\langle \vec{u} |). \end{array}$$

*Proof.*  $\vec{u} \approx \vec{v}$

$$\begin{aligned} &\Leftrightarrow \exists a \in ]0, +\infty[, \exists b \in \mathbb{R} \text{ s.t. } \vec{v} - a\vec{u} = b\vec{1} \\ &\Leftrightarrow \exists a \in ]0, +\infty[ \text{ s.t. } \vec{v} - a\vec{u} \text{ is orthogonal to } \mathcal{H} \\ &\Leftrightarrow \exists a \in ]0, +\infty[ \text{ s.t. } \forall \vec{\Delta} \in \mathcal{H}, \langle \vec{v} | \vec{\Delta} \rangle = a \langle \vec{u} | \vec{\Delta} \rangle \\ &\Leftrightarrow \langle \vec{u} | \approx \langle \vec{v} |. \end{aligned}$$

The implication  $\Rightarrow$  proves that  $\Theta$  is correctly defined: indeed, if  $\tilde{\pi}(\vec{u}) = \tilde{\pi}(\vec{v})$ , then  $\tilde{\pi}(\langle \vec{u} |) = \tilde{\pi}(\langle \vec{v} |)$ . The implication  $\Leftarrow$  ensures that  $\Theta$  is injective. Finally,  $\Theta$  is obviously surjective.  $\square$

**Remark III.8** (interpretation). Hence, the utility space may be seen as a quotient of the dual  $\mathcal{H}^*$  of the tangent space  $\mathcal{H}$  of the lotteries  $\mathcal{L}_m$ . A utility vector may be seen, up to a positive constant, as a linear form over  $\mathcal{H}$  that reveals, for any point in the space of lotteries, in what directions the agent can find lotteries that she prefers.

## IV INVERSION AND SUMMATION OPERATORS

As a quotient of  $\mathbb{R}^m$ , the utility space inherits natural operations from  $\mathbb{R}^m$ : inversion and summation. We will see that both these quotient operators have an intuitive meaning regarding preferences.

**Definition IV.1** (inversion). In  $\mathcal{U}_m$ , the *inversion* operator is defined as:

$$- : \left| \begin{array}{l} \mathcal{U}_m \quad \rightarrow \quad \mathcal{U}_m \\ \tilde{\pi}(\vec{u}) \quad \rightarrow \quad \tilde{\pi}(-\vec{u}). \end{array} \right.$$

**Remark IV.2** (the inversion operator is a bijection). The inversion operator is correctly defined and it is a bijection. Indeed,  $\tilde{\pi}(\vec{u}) = \tilde{\pi}(\vec{v})$  if and only if  $\tilde{\pi}(-\vec{u}) = \tilde{\pi}(-\vec{v})$ .

**Remark IV.3** (interpretation in terms of preferences). Considering the additive inverse amounts to reverting the agent's preferences, without modifying their relative intensities.

Now, we want to push the summation operator from  $\mathbb{R}^m$  to the quotient  $\mathcal{U}_m$ . The idea is quite simple and general: considering  $\tilde{u}$  and  $\tilde{v}$  in  $\mathcal{U}_m$ , their antecedents are taken in  $\mathbb{R}^m$  thanks to  $\tilde{\pi}^{-1}$ , the sum is computed in  $\mathbb{R}^m$ , then the result is converted back into the quotient space  $\mathcal{U}_m$ , thanks to  $\tilde{\pi}$ .

However, the result is not unique. Indeed, let us take arbitrary representatives  $\vec{u} \in \tilde{u}$  and  $\vec{v} \in \tilde{v}$ . In order to compute the sum, we can think of any representatives. So, possible sums are  $a\vec{u} + a'\vec{v} + b + b'$ , where  $a$  and  $a'$  are positive and where  $b + b'$  is any real number.

Converting back to the quotient, we can get for example  $\tilde{\pi}(2\vec{u} + \vec{v})$  and  $\tilde{\pi}(\vec{u} + 3\vec{v})$ , which are generally distinct. As a consequence, the set of outputs is not the utility space  $\mathcal{U}_m$ , but rather the set  $\mathcal{P}(\mathcal{U}_m)$  of its subsets.

This example shows how we could define the sum of two elements  $\tilde{u}$  and  $\tilde{v}$ . In order to be more general, we will define the sum of any number of elements of  $\mathcal{U}_m$ . Hence we also take  $\mathcal{P}(\mathcal{U}_m)$  as the set of inputs.

**Definition IV.4** (summation). On the subsets of  $\mathcal{U}_m$ , we define the *summation* operator as:

$$\sum : \left| \begin{array}{l} \mathcal{P}(\mathcal{U}_m) \quad \rightarrow \quad \mathcal{P}(\mathcal{U}_m) \\ A \quad \rightarrow \quad \left\{ \tilde{\pi} \left( \sum_{i=1}^n \vec{u}_i \right), n \in \mathbb{N}, (\vec{u}_1, \dots, \vec{u}_n) \in (\tilde{\pi}^{-1}(A))^n \right\}. \end{array} \right.$$



**Example IV.5** (sum of two utility vectors). Let us consider  $\mathcal{U}_4$ . In figure II, for the purpose of visualization, we represent its projection in  $\mathcal{H}$ , which is permitted by the choice of normalization constants  $b$ . Since  $\mathcal{H}$  is a 3-dimensional space, let  $(\vec{h}_1, \vec{h}_2, \vec{h}_3)$  be an orthonormal base of  $\mathcal{H}$ .

For two non-trivial utility vectors  $\tilde{u}$  and  $\tilde{v}$ , the choice of normalization multipliers  $a$  allow to choose representatives  $\vec{u}$  and  $\vec{v}$  whose Euclidean norm is 1.

In this representation, the sum  $\sum\{\tilde{u}, \tilde{v}\}$  consists of utilities corresponding to vectors  $a\vec{u} + a'\vec{v}$ , where  $a$  and  $a'$  are nonnegative. Indeed, we took a representation in  $\mathcal{H}$ , so all normalization constants  $b$  vanish. Moreover,  $a$ ,  $a'$  or both can be equal to zero because our definition allows to ignore  $\vec{u}$ ,  $\vec{v}$  or both. Up to taking representatives of unitary norm for non-trivial utility vectors, let us note that the sum  $\sum\{\tilde{u}, \tilde{v}\}$  can be represented by the dotted line and the point  $\vec{0}$  of total indifference.

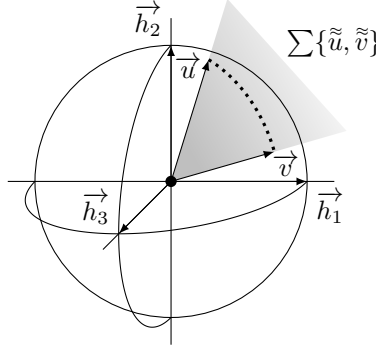


Figure II: Sum of two utility vectors in the utility space.

**Remark IV.6** (geometrical interpretation). Geometrically, the sum is the quotient of the convex hull of the inputs. Note that this convex hull is actually a convex cone. A bit later, we will see its interpretation in terms of preferences.

**Remark IV.7** (closed cone). Our convention is to consider the closed cone: for example,  $\tilde{u}$  fits in our definition. That would not be the case if we took only  $\tilde{\pi}(a\vec{u} + a'\vec{v} + b)$ , where  $a' > 0$ . In the same spirit,  $\tilde{\pi}(\vec{0})$  fits in our definition. Generally, that would not be the case if we took only  $\tilde{\pi}(a\vec{u} + a'\vec{v} + b)$ , where  $a > 0$  and  $a' > 0$ . The stated purpose of this convention is to have a concise wording for theorem IV.9 that follows.

**Proposition IV.8** (basic properties of the sum). *Let  $A \in \mathcal{P}(\mathcal{U}_m)$ . Then:*

1.  $\tilde{\pi}(\vec{0}) \in \sum A$ ;
2.  $A \subset \sum A$ .

*Proof.* 1. Consider  $n = 0$  in the definition.

2. For  $\tilde{u} \in A$ , if it sufficient to consider  $n = 1$  and  $\vec{u}_1 \in \tilde{\pi}^{-1}(\tilde{u})$  in order to prove that  $\tilde{u} \in \sum A$ . □

We now prove that, if  $A$  is the set of the utility vectors of a population, then  $\sum A$  is the set of utility vectors that respect the unanimous preferences of  $A$ .

**Theorem IV.9** (characterization of the sum). *Let  $A \in \mathcal{P}(\mathcal{U}_m)$  and  $\tilde{v} \in \mathcal{U}_m$ . The following conditions are equivalent.*

1.  $\tilde{v} \in \sum A$ .
2.  $\forall (L, M) \in \mathcal{L}_m^2: (\forall \tilde{u} \in A, L \succ_{\tilde{u}} M) \Rightarrow L \succ_{\tilde{v}} M$ .

*Proof.* Let  $\vec{v} \in \tilde{\pi}^{-1}(\tilde{v})$ . We have the following equivalences.

- $\forall (L, M) \in \mathcal{L}_m^2, (\forall \tilde{u} \in A, L \succ_{\tilde{u}} M) \Rightarrow L \succ_{\tilde{v}} M$ ,
  - $\forall \vec{LM} \in \mathcal{T}, (\forall \vec{u} \in \tilde{\pi}^{-1}(A), \langle \vec{u} | \vec{LM} \rangle \geq 0) \Rightarrow \langle \vec{v} | \vec{LM} \rangle \geq 0$ ,
  - $\forall \vec{\Delta} \in \mathcal{H}, (\forall \vec{u} \in \tilde{\pi}^{-1}(A), \langle \vec{u} | \vec{\Delta} \rangle \geq 0) \Rightarrow \langle \vec{v} | \vec{\Delta} \rangle \geq 0$  (cf. lemma III.4),
  - $\bigcap_{\vec{u} \in \tilde{\pi}^{-1}(A)} \vec{u}^{+\mathcal{H}} \subset \vec{v}^{+\mathcal{H}}$ ,
  - $\vec{v}$  is in the convex cone of  $\tilde{\pi}^{-1}(A)$ ,
  - $\tilde{v} \in \sum A$ .
- 

**Example IV.10** (a utility vector and its inverse). Let us consider a non null  $\tilde{u}$  and let us examine the sum of  $\tilde{u}$  and its additive inverse  $-\tilde{u}$ . Applying the definition, we see that the sum consists of  $\tilde{u}$ ,  $-\tilde{u}$  and  $\vec{0}$ .

However, we have just seen that the sum is the subset of utility vectors preserving the unanimous preferences over lotteries. Intuitively, we could think that, since  $\tilde{u}$  and  $-\tilde{u}$  seem to always disagree, any utility vector  $\tilde{v}$  respects the empty set of their common preferences; so, their sum should be the whole space.

But this is a false assumption. For example, let us consider  $\vec{u} = (1, 0, \dots, 0)$ . About any two lotteries  $L$  and  $M$ , inverse opinions  $\tilde{u}$  and  $-\tilde{u}$  agree if and only if  $L_1 = M_1$ : in that case, both of them are indifferent between  $L$  and  $M$ . Hence,  $\tilde{u}$  and  $-\tilde{u}$  do have a characteristic common feature: they pay attention to candidate 1 only, even if their opinions about her are diverging.

## V TOPOLOGY

In this section, we remark that the utility space has a pathological topology when keeping the indifference point, but a spherical topology when removing it. This will be used in section VI to demonstrate theorem VI.5, characterizing the suitable Riemannian metrics.

**Definition V.1** ( $T_0$  space). We say that a topological space is a  $T_0$  space or a *Kolmogorov space* if and only if, for any two distinct points, at least one of them has an open neighborhood not containing the other. See Guénard and Lelièvre (1985).

**Definition V.2** ( $T_1$  space). We say that a topological space is a  $T_1$  space if and only if, for any two distinct points, each of them has an open neighborhood not containing the other. See Guénard and Lelièvre (1985).

**Proposition V.3** (topology of the utility space). We assume  $m \geq 2$ . The utility space  $\mathcal{U}_m = \mathbb{R}^m / \approx$  is endowed with the quotient topology.

$\mathcal{U}_m \setminus \{\tilde{0}\}$  has the same topology as the sphere of dimension  $m-2$ . In particular it is a  $T_1$  space.

$\mathcal{U}_m$  is a  $T_0$  space but not a  $T_1$  space.

*Proof.* Let  $\mathcal{S}_{m-2} = \{\vec{u} = (u_1, \dots, u_m) \in \mathbb{R}^m \text{ s.t. } \sum_{j=1}^m u_j = 0, \sum_{j=1}^m u_j^2 = 1\}$ . Then:

$$\Theta : \begin{array}{l|l} \mathcal{S}_{m-2} & \rightarrow \mathcal{U}_m \setminus \{\tilde{0}\} \\ \vec{u} & \rightarrow \tilde{\pi}(\vec{u}) \end{array}$$

is a homeomorphism.

In  $\mathcal{U}_m$ , the only new case concerns  $\tilde{0}$  and any point  $\tilde{u} \neq \tilde{0}$ . Then  $\tilde{u}$  do have an open neighborhood not containing  $\tilde{0}$ . But  $\tilde{0}$  has a unique open neighborhood:  $\mathcal{U}_m$  in whole, which contains  $\tilde{u}$ .  $\square$

## VI RIEMANNIAN REPRESENTATION

Since the utility space is a variety, it is a natural desire to endow it with a Riemannian metric. In this section, we prove that there is a limited choice of metrics that are coherent with the natural properties of the space and with the *a priori* symmetry between the candidates.

**Definition VI.1** (line). We assume that  $m \geq 3$ . The image of a 2-dimensional subspace of  $\mathcal{H}$  in  $\mathcal{U}_m \setminus \{\tilde{0}\}$  by the canonical projection  $\tilde{\pi}$  is called a *line* in  $\mathcal{U}_m \setminus \{\tilde{0}\}$ .

**Remark VI.2** (connection with the sum operator). This notion is deeply connected to the summation operator defined in section IV: indeed, the sum of two non antipodal points in  $\mathcal{U}_m \setminus \{\tilde{0}\}$  is a segment of the line joining them, the shortest one in the sense of the round metric. We will show that this property holds true only for this metric.

**Notations VI.3** (round metric). The quotient  $\mathbb{R}^m/\text{vect}(\vec{1})$  is identified to  $\mathcal{H}$  and endowed with the inner product inherited from the canonical one of  $\mathbb{R}^m$ . The utility space  $\mathcal{U}_m \setminus \{\tilde{0}\}$  is identified to the unit sphere of  $\mathcal{H}$  and endowed with the induced Riemannian structure. Let us call  $\xi_0$  this Riemannian metric on  $\mathcal{U}_m \setminus \{\tilde{0}\}$ .

**Remark VI.4** (geometrical interpretation). In order to get an intuitive vision of this metric, one can represent any position  $\tilde{u}$  by a vector  $\vec{u}$  that verify:

$$\begin{cases} \sum u_i = 0, \\ \sum u_i^2 = 1. \end{cases}$$

We obtain a  $(m - 2)$ -dimensional sphere and we consider the metric induced by the canonical Euclidean metric of  $\mathbb{R}^m$ . That is, distances are measured on the surface of the sphere, using the restriction of the canonical inner product on each tangent space.

Let us remark that any inner product on  $\mathbb{R}^m$  induces a Riemannian metric on  $\mathcal{U}_m \setminus \{\tilde{0}\}$  by the same process and that any two of these metrics are isometric; however, they are not equal, except if these inner products are multiple of each other. The difference between isometry and equality is essentially the same as between two loops of equal length: there exists an application from one to the other that preserves distances along the curve, but they are not generally equal.

We will prove that for any  $m \geq 4$ , the spherical representation is the only one that is coherent with the natural properties of the space and that respects the *a priori* symmetry between candidates.

**Theorem VI.5** (Riemannian representation of the utility space). *We assume that  $m \geq 4$ . Let  $\xi$  be a Riemannian metric on  $\mathcal{U}_m \setminus \{\tilde{0}\}$ .*

*Conditions 1 and 2 are equivalent.*

1. (a) *The lines of  $\mathcal{U}_m \setminus \{\tilde{0}\}$  are geodesics of  $\xi$ ; and*  
 (b) *for any permutation  $\sigma$  of  $\llbracket 1, m \rrbracket$ , the action  $\Phi_\sigma$  induced on  $\mathcal{U}_m \setminus \{\tilde{0}\}$  by*

$$(u_1, \dots, u_m) \rightarrow (u_{\sigma(1)}, \dots, u_{\sigma(m)})$$

*is an isometry.*

2.  $\exists \lambda \in ]0, +\infty[$  s.t.  $\xi = \lambda \xi_0$ .

*Proof.* Since the implication  $2 \Rightarrow 1$  is obvious, we now prove  $1 \Rightarrow 2$ . The deep result behind this is a classical theorem of Beltrami, which dates back to the middle of the nineteenth century: see [Beltrami \(1866\)](#) and [Beltrami \(1869\)](#).

Beltrami's theorem precisely tells us that condition [1a](#) implies that  $\mathcal{U}_m \setminus \{\tilde{0}\}$  has constant curvature. Note that this result is in fact more subtle in dimension 2 (that is, for  $m = 4$ ) than in higher dimensions; see [Spivak \(1979a\)](#), Theorem 1.18 and [Spivak \(1979b\)](#), Theorem 7.2 for proofs.

Since  $\mathcal{U}_m \setminus \{\tilde{0}\}$  is a topological sphere, this constant curvature must be positive. Up to multiplying  $\xi$  by a constant, we can assume that this constant curvature is 1. As a consequence, there is an isometry  $\Psi : \mathcal{S}_{m-2} \rightarrow \mathcal{U}_m \setminus \{\tilde{0}\}$ , where  $\mathcal{S}_{m-2}$  is the unit sphere of  $\mathbb{R}^{m-1}$ , endowed with its usual round metric. The function  $\Psi$  obviously maps geodesic to geodesics, and by a standard projective geometry argument there is a linear map  $\Lambda : \mathbb{R}^{m-1} \rightarrow \mathcal{H}$  inducing  $\Psi$ , that is such that:

$$\Psi \circ \Pi = \Pi \circ \Lambda,$$

where  $\Pi$  denotes both projections  $\mathbb{R}^{m-1} \rightarrow \mathcal{S}_{m-2}$  and  $\mathcal{H} \rightarrow \mathcal{U}_m \setminus \{\tilde{0}\}$ . Using  $\Psi$  to push the canonical inner product of  $\mathbb{R}^{m-1}$ , we get that there exists an inner product  $(\vec{u}, \vec{v}) \rightarrow \phi(\vec{u}, \vec{v})$  on  $\mathcal{H}$  that induces  $\xi$ , in the sense that  $\xi$  is the Riemannian metric obtained by identifying  $\mathcal{U}_m \setminus \{\tilde{0}\}$  with the unit sphere defined in  $\mathcal{H}$  by  $\phi$  and restricting  $\phi$  to it.

The last thing to prove is that  $\phi$  is the inner product coming from the canonical one on  $\mathbb{R}^m$ . Note that hypothesis **1b** is mandatory, since any inner product on  $\mathcal{H}$  does induce on  $\mathcal{U}_m \setminus \{\tilde{0}\}$  a Riemannian metric satisfying **1a**.

Each canonical basis vector  $\vec{e}_j = (0, \dots, 1, \dots, 0)$  defines a point in  $\mathcal{U}_m \setminus \{\tilde{0}\}$  and a half-line  $\ell_j$  in  $\mathcal{H}$ . Condition **1b** ensures that these half-lines are permuted by isometries of  $(\mathcal{H}, \phi)$ . In particular, there are vectors  $\vec{u}_j \in \ell_j$  that have constant pairwise distance (for  $\phi$ ), so that they form a regular simplex and  $\sum_j \vec{u}_j = \vec{0}$ . By a standard projective geometry argument,  $\vec{u}_1, \dots, \vec{u}_{m-1}$  is up to multiplication by a scalar the unique basis of  $\mathcal{H}$  such that  $\vec{u}_j \in \ell_j$  and  $\sum_{j < m} \vec{u}_j \in -\ell_m$ .

Now consider the canonical inner product  $\phi_0$  on  $\mathcal{H}$  that comes from the canonical one on  $\mathbb{R}^m$ . Since permutations of coordinates are isometries, we get that the vectors  $\vec{v}_j = \Pi(\vec{e}_j)$  (where  $\Pi$  is now the orthogonal projection from  $\mathbb{R}^m$  to  $\mathcal{H}$ ) form a regular simplex for  $\phi_0$ , so that  $\sum_j \vec{v}_j = \vec{0}$ . It follows that  $\vec{u}_j = \lambda \vec{v}_j$  for some  $\lambda > 0$  and all  $j$ . We deduce that the  $\vec{u}_j$  form a regular simplex for both  $\phi$  and  $\phi_0$ , which must therefore be multiple from each other.  $\square$

**Remark VI.6** (utility space for three candidates). However, the implication **1**  $\Rightarrow$  **2** in the theorem is not true for  $m = 3$ . For each non null utility vector, let us consider its representative verifying  $\min(u_i) = 0$  and  $\max(u_i) = 1$ . This way,  $\mathcal{U}_m \setminus \{\tilde{0}\}$  is identified to edges of the unit cube in  $\mathbb{R}^3$ , as in figure **III**. Then,  $\mathcal{U}_m \setminus \{\tilde{0}\}$  is endowed with the metric induced on these edges by the canonical inner product on  $\mathbb{R}^3$ . Then conditions **1a** and **1b** of the theorem are met, but not condition **2**.

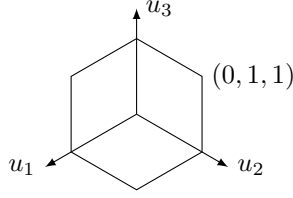


Figure III: A non-circular representation of the utility space for 3 candidates.

## VII RIEMANNIAN REPRESENTATION IN PRACTICE

The Riemann representation introduced in the previous section may be used for two main purposes.

- One may want to compute distances between two utility vectors.
- Once the space is endowed with a metric, it is endowed with a natural probability measure: the uniform measure in the sense of this metric. Hence, one can use the uniform measure itself or define other measures by their densities with respect to the uniform measure.

In this section, we detail how to deal with these two aspects in practice.

**Remark VII.1** (indifference point). Since the point  $\tilde{0}$  is a geometrical singularity, it is difficult to include it naturally in such a measure. If one wants to take it into account, the easiest way is to draw it with a given probability and to use some given measure over  $\mathcal{U}_m \setminus \{\tilde{0}\}$  in the other cases.

**Notations VII.2** (orthogonal projection on  $\mathcal{H}$ ). We denote  $D_0$  the matrix of the orthogonal projection on  $\mathcal{H}$ :

$$D_0 = \text{Id} - \frac{1}{m}J.$$

**Proposition VII.3** (distances in the utility space). Let  $(\vec{u}, \vec{v}) \in (\mathbb{R}^m \setminus \mathbb{R}\vec{1})^2$ . Then, in the sense of metric  $\lambda\xi_0$  of theorem VI.5, the distance between  $\tilde{u}$  and  $\tilde{v}$  is:

$$d(\tilde{u}, \tilde{v}) = \lambda \arccos \left\langle \frac{D_0 \vec{u}}{\|D_0 \vec{u}\|} \mid \frac{D_0 \vec{v}}{\|D_0 \vec{v}\|} \right\rangle.$$

*Proof.* First of all, we compute the projection  $\vec{u}'$  of  $\vec{u}$  on  $\mathcal{H}$ : we get  $\vec{u}' = D_0 \vec{u}$ . Then we normalize  $\vec{u}'$  in order to get it on the unit sphere of  $\mathcal{H}$ , that is,  $\vec{u}'' = \frac{1}{\|\vec{u}'\|} \vec{u}'$ . We compute  $\vec{v}''$  from  $\vec{v}$  in a similar way.

Now, we just have to compute the distance between  $\vec{u}''$  and  $\vec{v}''$  on the surface of the unit sphere and to multiply by  $\lambda$ : we get  $d(\tilde{u}, \tilde{v}) = \lambda \arccos \langle \vec{u}'' \mid \vec{v}'' \rangle$ .  $\square$

**Remark VII.4** (utilities have no absolute meaning). Hence, for any point  $\tilde{u}$  in the utility space, we use a canonical representative belonging to  $\mathcal{H}$  and whose Euclidean norm is equal to 1. For example, we might have the following representatives:

$$\begin{cases} \vec{u} = (0.71, -0.71, 0), \\ \vec{v} = (0.82, -0.41, -0.41). \end{cases}$$

Since  $v_1 > u_1$ , does it mean that an agent with preferences  $\tilde{v}$  “appreciates” candidate 1 more than an agent with preferences  $\tilde{u}$ ? We don’t say anything like this: it is well known that utilities belonging to two agents are essentially incomparable: see [Von Neumann and Morgenstern \(1944\)](#). Taking canonical representatives on the  $(m - 2)$ -dimensional sphere is only used to compute distances between two points in the utility space.

Regarding the other question considered in this section, how to draw vectors according to a uniform probability law over  $\mathcal{U}_m \setminus \{\tilde{0}\}$ , it is sufficient to use a uniform law on the unit sphere in  $\mathcal{H}$ .

To conclude about this spherical representation, here is an example of a law defined by its density. Given a vector  $\vec{u}_0$  in the unit sphere of  $\mathcal{H}$  and  $\kappa$  a nonnegative real number, the Von Mises–Fisher distribution of pole  $\vec{u}_0$  and concentration  $\kappa$  is defined by the following density with respect to the uniform law on the unit sphere in  $\mathcal{H}$ :

$$p(\vec{u}) = C_\kappa e^{\kappa \langle \vec{u} | \vec{u}_0 \rangle},$$

where  $C_\kappa$  is a normalization constant.

It can be proved that, given the mean resultant vector of a distribution over the sphere, Von Mises–Fisher distribution maximizes the entropy, in the same way that, in the Euclidean space, Gaussian distribution maximizes the entropy among laws with given mean and standard deviation. Hence, without additional information, it is the “natural” distribution that should be used. In order to draw data according to this law for simulation purposes, one may use Ulrich algorithm modified by Wood: see [Ulrich \(1984\)](#) and [Wood \(1994\)](#).

## VIII CONCLUSION

We have studied the geometrical properties of the classical model of expected utilities, introduced by Von Neumann and Morgenstern, when candidates are considered symmetrical *a priori*. We have remarked that the utility space may be seen as a dual of the space of lotteries, that inversion and summation operators inherited from  $\mathbb{R}^m$  have a natural interpretation in terms of preferences and that the space has a spherical topology when the indifference point is removed.

We have proved that the only Riemannian representation that respects the projective lines naturally defined by the summation operator and the symmetry between candidates is a round sphere.

All these considerations lay on the principle to add as little information as possible in the system, especially by respecting the *a priori* symmetry between

candidates. This does not imply that the spherical representation of the utility space  $\mathcal{U}_m$  is the most relevant one in order to study a specific situation. Indeed, as soon as one has additional information (for example, a model that places candidates in a political spectrum), it is natural to include it in the model. However, if one wishes, for example, to study a voting system in all generality, without focusing on its application in a specific field, it looks natural to consider a utility space with a metric as neutral as possible, like the one defined in this paper by the spherical representation.



## NOTATIONS

NON-ALPHABETICAL SYMBOLS	
$[\alpha, \beta[$	Real interval from $\alpha$ (included) to $\beta$ (excluded).
$\llbracket j, k \rrbracket$	Integer interval from $j$ to $k$ (both included).
$\vec{1}$	Vector whose $m$ coordinates are 1.
$\overrightarrow{LM} \in \mathcal{T}$	Bipoint $(M_1 - L_1, \dots, M_m - L_m)$ from $L \in \mathcal{L}_m$ to $M \in \mathcal{L}_m$ .
$\langle \vec{u}   \vec{v} \rangle$	Canonical inner product of $\vec{u}$ and $\vec{v}$ .
$\ \vec{u}\ $	Canonical Euclidean norm of $\vec{u}$ .
$\succsim$	Binary relation representing an agent's preferences over $\mathcal{L}_m$ .
$\prec$	Strict relation associated to $\succsim$ .
$\overset{\sim}{\underset{u}{\succsim}}$	Preferences over $\mathcal{L}_m$ represented by utility vector $\vec{u}$ .
$\vec{u} \approx \vec{v}$	Vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^m$ are two utility vectors representing the same preferences over $\mathcal{L}_m$ .
GREEK ALPHABET	
$\lambda \in \mathbb{R}$	A real number.
$\xi_0$	Metric on $\mathcal{U}_m \setminus \{\vec{0}\}$ induced by the canonical inner product on $\mathbb{R}^m$ .
$\tilde{\pi}(\vec{u})$	Equivalence class of $\vec{u}$ in the sense of $\approx$ . Function $\tilde{\pi} : \mathbb{R}^m \rightarrow \mathcal{U}_m$ is called <i>canonical projection</i> (quotient).
LATIN ALPHABET	
$a \in ]0, +\infty[$	Multiplicative normalization coefficient for the utilities of an agent.
$b \in \mathbb{R}$	Additive normalization coefficient for the utilities of an agent.
$\mathcal{C}_m$	Set $\llbracket 1, m \rrbracket$ of candidates.
$\mathcal{H}$	Tangent hyperplane of $\mathcal{L}_m$ , that is, orthogonal to $\vec{1}$ .
$\mathcal{H}^*$	Dual space of the hyperplane $\mathcal{H}$ .
$J$	Matrix of size $m \times m$ whose $m^2$ coefficients are 1.
$\mathcal{L}_m$	Set of lotteries over the candidates: $\{(L_1, \dots, L_m) \in \mathbb{R}_+^m \text{ t.q. } \sum_{j=1}^m L_j = 1\}$ .
$m \in \mathbb{N} \setminus \{0\}$	Number of candidates.
$\mathcal{T}$	Tangent polytope $\{\overrightarrow{LM}, (L, M) \in \mathcal{L}_m^2\}$ of $\mathcal{L}_m$ .
$\mathcal{U}_m$	Utility space $\mathbb{R}^m / \approx$ .
$\vec{u} \in \mathbb{R}^m$	A utility vector $(u_1, \dots, u_m)$ .
$\tilde{u} \in \mathcal{U}_m$	A utility vector (up to equivalence $\approx$ ).
$\vec{u}^{+\mathcal{H}}$	Positive half-hyperplane associated to $\vec{u}$ .
$\text{vect}(E)$	Linear span of $E$ , where $E$ is a part of a vector space.

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