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Predicting playing frequencies for clarinets: a comparison between numerical simulations and simplified analytical formulas

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When designing a wind instrument such as a clarinet it is interesting to be able to predict the playing frequencies. These depend on the resonance frequencies, deduced from the input impedance, but not only. Indeed, several control parameters such as the blowing pressure and reed opening have a significant influence on the playing frequency. Their role has to be taken into account in order to determine the playing frequencies. This paper presents a method to deduce these frequencies analytically from the different control parameters, using the input impedance curve. Four effects are known to alter the playing frequency and are examined separately: the flow rate due to the reed motion, the reed dynamics, the inharmonicity of the resonator, and the temperature gradient within the clarinet. The results for the first register of a particular clarinet are given and compared to simulated playing frequencies found using a numerical model.

I Introduction

Musical instruments are sophisticated tools which continue to evolve as technology does. More and more alterations and improvements are being made on these instruments to give the best possible sound and functioning. Nevertheless, there are still many complicated physical aspects of the instrument that are not yet understood, aspects which could help a musician play “better” or choose a better instrument.

The main objective of the present study is to understand what is the causes of the difference between the playing frequencies and the resonance frequencies. Traditional approaches by makers are based upon the assumption that these two frequencies are equal, up to a certain value of length correction. But this correction depends on the note, on the excitation parameters and on the higher resonances of the resonator. Having chosen a basic physical model, both analytical approximations and numerical simulations are calculated and compared.

In order to modify the playing frequency, the musician can vary multiple features, two of which include the mouth pressure and the lip force acting on the reed. The latter is difficult to introduce directly in a model, but it is related to the reed-opening size at rest, as well as to the reed damping and the lowest reed resonance frequency. These parameters can be to some extent determined experimentally, here however, values most often encountered in the literature are considered.

The paper studies the first register of the clarinet (first resonance frequency of each tube length) where the influence of the resonator on the playing frequency is much stronger than that of the instrumentalist and reed. In fact, in higher registers, the musician can significantly modify the frequency in order to play “in tune” (the definition of in tune is a very intricate issue and is not addressed in this paper). This paper examines three causes of discrepancy between playing and resonance frequencies: the flow rate due to the reed compliance, the reed dynamics, and the in-harmonicity of the resonator’s natural frequencies. Several formulas have been known for some time near the oscillation threshold [1]. However, the current work attempts to treat a wider range

of playing parameters.

This work follows that of two conference papers [2, 3]. The former treated solely the possibility of gathering the necessary analytical formulas in order to predict a reasonable playing frequency. The latter exploited these analytical formulas in order to create analytical tuning maps. As described, the current work uses a numerical simulation technique to validate the use of these analytical formulas.

Listed below, in Table 1, is a list of important symbols used in the paper. The paper begins in Section II by discussing characteristic equations which are widely cited in literature which describe the functioning of the clarinet. Also, this section provides measured values of the modal parameters of a level Bb clarinet, as inputs for the model. Next, in Section III the playing frequency is studied for the “ideal” case, where it is exactly equal to the resonance frequency, then for the non-ideal case. Analytical formulas for several different frequency corrections are proposed, where the different effects are assumed to be small and therefore can be studied independently. The simulations and comparisons between simulation and analytical formulae are discussed in Sections IV and V respectively. In Section VI the effect of a temperature gradient on the resonance frequencies is briefly studied, then in Section VII the total of all four effects is presented. The paper concludes in Section VIII.

II Basic equations

A Main equations

The classically used model of three equations is recalled in order to solve for the three unknowns p , u , and x : the pressure in the mouthpiece, the flow rate entering the instrument, and the displacement of the reed respectively. The variables are dimensionless[4] as the pressures are normalized by the closing pressure p_M ,

$$p_M = K_r H, \quad (1)$$

where K_r is the reed stiffness per unit area and H is the height of the reed channel at rest. The flow rates are normalized by the ratio p_M/Z_c , where $Z_c = \rho c/S$ is the characteristic impedance at the input of the tube (ρ is the air density, c the speed of sound, and S is the cross section area at the tube input). The reed displacement x is normalized by H : $x = -1$ is chosen for the beating reed, and $x = 0$ at rest.

The main dimensionless control parameters are the following: the parameter γ is the mouth pressure and the composite parameter ζ , is proportional to the maximum flow rate that can enter the tube, if w is the reed width: $\zeta = Z_c w H \sqrt{\frac{2}{\rho p_M}}$.

Using the Bernoulli equations, and some assumptions [5], the flow rate u_b due to the pressure difference across the reed opening is given by:

$$u_b = \zeta(1+x)\text{sign}(\gamma - p)\sqrt{|\gamma - p|} \text{ if } 1+x \geq 0 \quad (2)$$

$$\frac{dx(t)}{dt} = 0 \text{ if } 1+x \leq 0 \quad (3)$$

In the second case the reed beats against the reed table. Negative flow rate is assumed to not occur in the steady-state regime, for clarinet-like instruments[6]. The remaining equations are linear and are written in the frequency domain (notated with capital letters). The resonator is described by its input admittance $Y(\omega)$:

$$U_{tot}(\omega) = Y(\omega)P(\omega). \quad (4)$$

where $U_{tot}(\omega)$ is the total flow entering the resonator. The modal expansion of the dimensionless impedance $Z(\omega)$ can be written in a simplified way as follows:

$$Z(\omega) = j\omega \sum_n \frac{F_n}{\omega_n^2 - \omega^2 + j\omega\omega_n/Q_n}, \quad (5)$$

where the ω_n are the resonance frequencies, and Q_n are the quality factors. F_n are the “modal factors” obtained from the modal shapes calculated, at the input (For a perfect cylinder of length ℓ , F_n is equal to $2c/\ell$, and is independent of the rank of the resonance frequency). The reed motion is governed by the following equation:

$$\frac{1}{\omega_r^2} \frac{d^2x(t)}{dt^2} + \frac{q_r}{\omega_r} \frac{dx(t)}{dt} + x(t) = p(t) - \gamma \quad (6)$$

57 where $1/q_r$ is the quality factor and ω_r is the angular frequency of the first reed resonance.

58 **B Supplementary equation: Flow rate due to the reed movement**

The previous equations ignore the flow rate due to the reed movement, which is proportional to the velocity and negative or positive according to the orientation choice:

$$u_r = -\frac{Z_c S_r}{K_r} \frac{dx}{dt}, \quad (7)$$

where S_r is the reed area contributing to the flow rate (a value which is difficult to determine)¹. The effect of the reed movement is inversely proportional to the reed compliance[1]. The total flow entering the instrument is:

$$u_{tot} = u_b + u_r \quad (8)$$

At low frequencies, the reed dynamics can be ignored, and, within the non-beating reed conditions:

$$\frac{dp}{dt} = \frac{dx}{dt}. \quad (9)$$

59 Considering Eq. (7), the effect of the reed compliance is therefore equivalent to the compliance of
60 a volume V_{eq} :

$$u_r(\omega) = -Z_c \frac{V_{eq}}{\rho c^2} \frac{dp}{dt}, \text{ thus } V_{eq} = \frac{\rho c^2}{K_r} S_r. \quad (10)$$

Because of Eq. (8), it is a compliance in parallel with the input impedance, or an added air volume at the entry of the instrument, V_{eq} . Thus Eq. (4) can be modified as:

$$U_b(\omega) = \left[Y(\omega) + j \frac{\omega V_{eq}}{cS} \right] P(\omega) \quad (11)$$

Reed dynamics could be taken into account, by using Eq. (6), but its influence is very small, and in accordance with the assumption that the three effects mentioned in the introduction are independent, this correction is ignored. For convenience the notation $V_{eq} = S\Delta\ell_{eq}$ is often employed: if $\Delta\ell_{eq}$ is smaller than the wavelength, it is a simple length correction.

For a *beating reed*, the reed displacement is limited by the mouthpiece lay, therefore the flow rate is limited as well. When the flow rate is considered, the condition $dx/dt = 0$ if $x < -1$ (Eq. 3) replaces the classical condition $u_b = 0$ if $x < -1$ [4]. Hereafter the work done by Dalmont et al. [8], who published a satisfactory comparison between a simple model and experiment, is summarized. For a beating reed, the signal of the mouthpiece pressure is not far from a square signal, which is the exact shape when losses and reed dynamics are ignored. The pressure takes the following values: $-\gamma$ (when the reed beats), and $+\gamma$ (when the reed is open). For the first harmonic of frequency ω_1 , the amplitude of the mouthpiece pressure is therefore $2\gamma/\pi$. The reed displacement is expected to vary between -1 and 0 : thus its first harmonic has an amplitude of $1/\pi$, and $P(\omega_1) = 2\gamma X(\omega_1)$. Therefore:

$$\frac{dp}{dt} \simeq 2\gamma \frac{dx}{dt}. \quad (12)$$

Comparing this to Eq. (9) shows that this result is coherent with the case of non-beating reed: for the simplest theory, a lossless resonator while ignoring reed dynamics, the beating threshold is

69 given by $\gamma = 1/2$. Finally, both cases of non-beating and beating reed are considered by using the
70 equations:

$$U_b(\omega) = \left[Y(\omega) + j \frac{\omega \Delta \ell_{eq}}{c} \right] P(\omega) ; \quad (13)$$

$$\Delta \ell_{eq} = \frac{\Delta \ell_0}{G(\gamma)} ; \Delta \ell_0 = \frac{\rho c^2}{p_M} \frac{S_r}{S} H ; \quad (14)$$

$$G(\gamma) = 1 \text{ if } \gamma < 0.5 ; G(\gamma) = 2\gamma \text{ if } \gamma \geq 0.5 . \quad (15)$$

71 **C Extracted modal values**

The method of extraction of the modal parameters from the impedance plots is based upon a local optimization procedure (nonlinear least-square algorithm), and is not discussed here[9]. Here, for reference, Figure 1 represents the extracted modal frequencies (ω_p = playing frequencies). The modal factor F_n is roughly proportional to the fundamental frequency of the notes and is nearly independent of register (the values are almost equal for $n = 2$ and $n = 1$); this behavior is very similar to that of a perfect cylinder. These are the values used for both the analytical and numerical calculations throughout this article. These values are referred to as f_n s, F_n s, and Z_n s. Moreover, inharmonicity is defined by the following expression, if $\omega_n = 2\pi f_n$:

$$\omega_n = n\omega_1(1 + \eta_n), \text{ with } \eta_1 = 0. \quad (16)$$

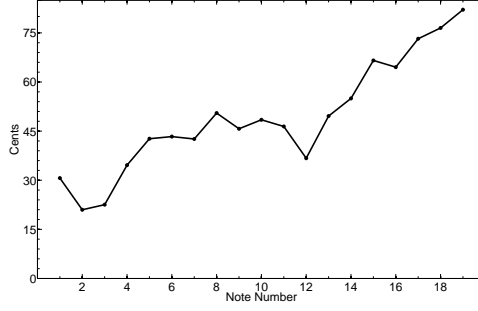


Figure 1: Difference, in cents between the extracted modal resonance frequencies and the tempered scale frequencies.

III Analytical Formulas for the Playing Frequency

A Playing frequency in the ideal case

When the reed flow rate and the reed dynamics are ignored ($\Delta\ell_{eq} = 0$, $p = x + \gamma$), the playing frequency at the oscillation threshold is given by

$$A = Y(\omega) \text{ with } A = \frac{\zeta(3\gamma - 1)}{2\sqrt{\gamma}} \quad (17)$$

A is the real coefficient for the linear term of the nonlinear characteristic $u(p)$. This implies that

$$\text{Im}[Y(\omega)] = 0, \quad (18)$$

thus for the first register $\omega = \omega_1$, and $A = \text{Re}[Y(\omega_1)] = A_{th}$.

$$\gamma_{th} \simeq \frac{1}{3} + \frac{2\text{Re}[Y(\omega_1)]}{3\sqrt{3}\zeta}. \quad (19)$$

Above the oscillation threshold, if the resonance frequencies are also harmonically related, the playing frequency remains the frequency found at the oscillation threshold, i.e. ω_1 . Also, when the reed dynamics are ignored, there is a static nonlinear characteristic which links the two variables

pressure p and flow rate u . Therefore it is possible to use the “reactive power rule” found by Boutillon for bowed instruments [10]:

$$\sum_n |P_n|^2 n \text{Im}[Y(n\omega_p)] = 0. \quad (20)$$

75 This equation, where $P_n = P(n\omega_p)$ is the amplitude of the n^{th} harmonic of the pressure, is one of
 76 the harmonic balance system of equations. If all the resonance frequencies are harmonically related
 77 to the first one, this equation is satisfied for $\omega_p = \omega_1$ regardless of the spectrum (or, equivalently,
 78 the excitation conditions). Thus the playing frequency does not change with the excitation level.
 79 The previous explanation seems to be trivial, but this is useful when studying the non-ideal case,
 80 treated as perturbation of the ideal one.

81 **B Approximations for the playing frequency in the non-ideal case**

With Equations (2) - (20) from the model, it is now possible to deduce approximations for the difference between natural frequencies and the playing frequencies f_p of the clarinet. The frequency difference Δf is sought:

$$\Delta f = f_p - f_1, \text{ with } f_1 = \omega_1/(2\pi). \quad (21)$$

If this value is small enough, the relative difference can be expressed in cents, as follows:

$$\frac{\Delta f}{f_1} = \frac{0.06}{100} \cdot N_{cents} \quad (22)$$

This is because one semi-tone (100 cents) corresponds to a ratio of 1.06. For a cylinder of length ℓ , a length correction can be defined as:

$$\frac{\Delta \ell}{\ell} = -\frac{\Delta f}{f_1}. \quad (23)$$

82 In what follows, the three effects are considered separately, assuming that the frequency shifts (or
83 the length corrections) can be simply added.

84 **1 Flow rate due to the reed movement**

If both the reed dynamics and the influence of higher order harmonics are ignored, the playing frequency is given by its value at the oscillation threshold[11]. Beginning from Equations (13) and (17):

$$Im \left[Y(\omega) + j \frac{\omega \Delta \ell_{eq}}{c} \right] = 0. \quad (24)$$

Given that the quality factor of a chosen impedance peak is high enough, only this peak is kept in the modal decomposition around the resonance frequency ω_n and the following approximation is valid ²:

$$Im [Y(\omega)] = \frac{2}{F_n} (\omega_p - \omega_n), \quad (25)$$

Therefore, for the first register, the solution of Eq. 24 is:

$$\omega_p = \frac{\omega_1}{1 + \frac{\Delta \ell_{eq} F_1}{2c}}. \quad (26)$$

For a perfect cylinder, if it is assumed that $\omega \Delta \ell / c \approx \tan (\omega \Delta \ell / c)$, the same result is found immediately:

$$\omega_p = \frac{\omega_1}{1 + \frac{\Delta \ell_{eq}}{\ell}}, \quad \text{with } \omega_1 = \frac{\pi c}{2\ell}, \quad (27)$$

85 and its condition of validity: $\omega_1 \Delta \ell / c \ll 1$. For this particular case, the effect of the flow rate can
86 be viewed as a simple length correction. Equations (13) and (26) are those proposed for this first
87 effect, for both a beating reed and a non-beating reed. Finally, it is written:

$$Ncents_{flow} = \frac{100}{0.06} \frac{\Delta f}{f} = -\frac{100}{0.06} \frac{F_1 \Delta \ell_0}{2G(\gamma)c} \quad (28)$$

88 2 Reed dynamics

When reed dynamics are considered $A \neq Y(\omega)$ at the threshold of oscillation. The study of the effect of the reed dynamics on the oscillation threshold (frequency and mouth pressure) has been done by Wilson and Beavers [13], and extended by Silva et al. [14] who added the effect of the reed flow rate. The results are valid for the case of strong reeds (e.g. organ reeds, with small reed damping), and weak reeds (e.g. woodwind reeds, with high damping by the lips). This method involves the linearization of Eq. (2), and solving the characteristic equation. Here the case of high damping, i.e. large q_r is considered. The result at the oscillation threshold is given by [15, 16]:

$$\omega_p = \omega_1 \left[1 - \frac{\zeta F_1 q_r}{2\sqrt{3} \omega_r} \right]. \quad (29)$$

89 For a perfect cylinder, this is again equivalent to a length correction, $\Delta\ell = q_r \zeta / (k_r \sqrt{3})$, where
 90 $k_r = \omega_r / c$. Notice that Nederveen tried to describe this effect, but he considered a reed with high
 91 damping and an infinite natural frequency [1].

This formula is valid at the threshold, therefore for a non-beating reed and a very small excitation pressure γ . Nevertheless, Kergomard and Gilbert [16], using the harmonic balance method analytically (limited to the first harmonic), found the following dependence on the excitation pressure:

$$\omega_p = \omega_1 \left[1 - \frac{\zeta F_1 q_r}{2\sqrt{3} \omega_r} \left[1 + \frac{3}{4}(\gamma - \gamma_{th}) \right] \right]. \quad (30)$$

92 This is the beginning of a series expansion above the oscillation threshold γ_{th} , consequently the
 93 formula is not necessarily valid at high values of γ : this formula is obtained for non-beating reeds
 94 only. The value of the oscillation threshold γ_{th} is given by Silva et al. [14], and a simplification is
 95 obtained by considering small values of $\theta = \omega / \omega_r$, using Eq. (19).

96 This effect can be transformed into a simple change in frequency, in cents, as well. This formula
 97 can be represented as follows:

$$\begin{aligned}
Ncents_{dynamics} &= \frac{100}{0.06} \frac{\Delta f}{f} \\
&= -\frac{100}{0.06} \cdot \frac{\zeta F_1}{2\sqrt{3}} \frac{q_r}{\omega_r} \left[1 + \frac{3}{4}(\gamma - \gamma_{th}) \right]
\end{aligned} \tag{31}$$

98 3 Effect of the inharmonicity of the resonator

99 If the resonance frequencies are not exactly harmonic, the playing frequency changes with the
100 level of excitation, because there is a weighting of the impedance peaks by the spectrum of the
101 mouthpiece pressure [17].

The method is valid for any given shape of the nonlinear characteristic $u = F(p)$, therefore for both non-beating and beating reed. In this case, because reed dynamics is ignored, this (static) characteristic exists ($x = p - \gamma$ in Eq. (6)). For clarinet-like instruments with weak inharmonicity (small η_n), the summation in Eq. (20) can be limited to odd harmonics. It is possible to use Eq. (25) near every resonance frequency, seeking the playing frequency in the form:

$$\omega_p = \omega_1(1 + \varepsilon). \tag{32}$$

At the first order in η_n and ε , Eq. (20) yields:

$$\sum_{n \text{ odd}} n^2 |P_n|^2 \frac{\varepsilon - \eta_n}{F_n} = 0. \tag{33}$$

For cylindrical instruments, because the modal factor F_n is almost independent of n (and equal to $2c/\ell$) and because $\eta_1 = 0$, the final result for the playing frequency is:

$$\varepsilon = \frac{\sum_{n \text{ odd} \geq 3} \eta_n d_n}{1 + \sum_{n \text{ odd} \geq 3} d_n} \text{ with } d_n = n^2 \left| \frac{P_n}{P_1} \right|^2. \tag{34}$$

(note that for a square signal $d_n = 1$ for every odd n). If the dependence of the spectrum with respect to the excitation pressure is known, it is possible to deduce the variation in the playing frequency. Approximate formulas for clarinet-like instruments were given by Kergomard et al [5]. The decrease of the higher harmonics is always faster than in the case of the square signal, therefore it is reasonable to search for an approximate formula by limiting the series to the third harmonic only. The ratio of the amplitude P_3/P_1 is:

$$\begin{aligned} \frac{P_3}{P_1} &= -\frac{1}{3} \frac{1}{1+z}; \quad z = \frac{Y_3 - Y_1}{A - Y_1}; \\ Y_n &= \frac{1}{Z_n} = \frac{\omega_n}{Q_n F_n}. \end{aligned} \quad (35)$$

Experimentally, there have been measurements of P_3/P_1 smaller than 1/3, but this formula is a good approximation for both non-beating and beating reeds. Finally a first order approximation is found:

$$\varepsilon = \frac{\eta_3}{1 + |1+z|^2}. \quad (36)$$

102 At the threshold $A = Y_1$, $\varepsilon = 0$ (the signal is sinusoidal), and Y_1 is real. For large excitation pres-
 103 sure, z tends to zero, and ε to $\eta_3/2$. Notice that because the reasoning is based upon a perturbation
 104 at the first order, the values of Y_3 and Y_1 can be determined without inharmonicity, i.e. they are
 105 real.

Refinements to this formula would be quite intricate (as an example the formula for the 5th harmonic is very complicated, see [5]). Nevertheless, this formula exhibits the sense of variation of the effect on inharmonicity of the second peak and doesn't necessarily warrant calculation beyond this. As before it is helpful to transform the frequency changes into a simpler value, in cents:

$$Ncents_{inharmonicity} = \frac{100}{0.06} \cdot \frac{\eta_3}{1 + |1+z|^2} \quad (37)$$

IV Numerical Simulations

In order to validate the results of the analytical models, synthesis model which is real-time compatible is chosen and based on one proposed by Guillemain et al.[9]. It is based on the physical model presented in Section II. This model provides a straightforward digital transposition of each part of the physical model. Many simplifications are made regarding the functioning of the reed and all parameters explicitly stated in this article are used when calculating the synthesized impedance peaks. The three equations (2) (5) and (6) are solved for the three dimensionless parameters p , u and x .

Equations 2 and 6 are discretized according to [9], which leads to an explicit computation scheme. The subscript “known” designs at each time sample n quantities that depend only on the past of the variables. Concerning the resonator, at each time sample n , each impedance mode k (Eq. 5) is discretized according to the invariance of the impulse response method after an estimation of modal parameters:

$$p_k(n) = b_{k0}u_{tot}(n) + V_{k_{known}}(p_k(n-y), u_{tot}(n-y), y > 0) \quad (38)$$

yielding:

$$p(n) = \sum_{k=1}^K p_k(n) = b_{M0}u_{tot}(n) + V_{known} \quad (39)$$

where

$$b_{M0} = \sum_{k=1}^K b_{k0} \quad \text{and} \quad V_{known} = \sum_{k=1}^K V_{k_{known}}. \quad (40)$$

Previous work [9] did not yet include the reed flow effect. In order to describe the beating reed a point where the velocity is zero (for a given pressure) is created, this is where the reed hits the table for the first time. The two sources of flow are then added together, $u_{tot}(n) = u_b(n) + u_r(n)$.

117 In the discrete time domain, Eqs. (3) and (6) are written as:

$$\text{if } 1 + x(n) \leq 0 \text{ then } x(n) = x(n - 1). \quad (41)$$

$$u_r(n) = -f_e \frac{\Delta \ell_0}{c} [x(n) - x(n - 1)], \quad (42)$$

118 where f_e is the sampling frequency. This leads to the new dimensionless coefficient $\lambda = -f_e \Delta \ell_0 / c$.
 119 For the simulations $\lambda = -0.7$ has been chosen: this value corresponds to $\Delta \ell_0 = 5.5$ mm, as ex-
 120 plained in the next section.

121 Here, unlike in the analytical formulas, the reed flow effect is valid above the beating reed
 122 regime because within the simulations there are a few added stipulations, such as the inclusion of
 123 the ability for the reed to touch the table of the mouthpiece and at that point have a velocity of zero
 124 as well as the ability to have negative flow ($1 + x(t) < 0$).

125 As a first step, the reed displacement discretization scheme (see [9]) is:

$$x(n) = F_{known}(x(n - y), p(n - y), y > 0); \quad (43)$$

$$\text{if } 1 + x(n) < 0, x(n) = x(n - 1); \quad (44)$$

$$u_r(n) = \lambda(x(n) - x(n - 1)); \quad (45)$$

$$W = \zeta(1 + x(n)). \quad (46)$$

126 As a second step, since W and u_r are known at time sample n , this leads to a set of two
 127 equations with two unknowns. Omitting the subscripts for time sample n and the known past for
 128 the sake of simplicity in the notations:

$$p = b_{M0}(u_b + u_r) + V; \quad (47)$$

$$u_b = W \text{sign}(\gamma - p) \sqrt{|\gamma - p|} \quad (48)$$

As a third step, these equations are then transformed into two second order polynomial equations in u corresponding to either positive or negative values of u_b , yielding the final solution:

$$u_{tot} = \frac{\text{sign}(\gamma - V - b_{M0}u_r)}{2}(-b_{M0}W^2 + \dots W\sqrt{(b_{M0}W)^2 + 4|\gamma - V - b_{M0}u_r|}) + u_r; \quad (49)$$

$$p_k = b_{k0}u_{tot} + V_k; \quad (50)$$

$$p = \sum_{k=1}^K p_k; \quad (51)$$

V Results: comparison between analytical formulas and simulation

The language used in this section refers to notes of the first register of the B \flat soprano clarinet with the numbers 1 - 19. Note 1 represents the lowest note on the B \flat clarinet, the fingered E which is a frequency near 146 Hz (considering the tempered scale). Whereas Note 19 represents the highest note in the first register, the fingered B \flat , 415 Hz.

The same MKS values of the following for each note are: vibrating surface area of the reed, $S_r = w \cdot \ell_r = 6.5 \cdot 10^{-5}$, where ℓ_r is the vibrating length of the reed and w is the width of the reed. The quality factor of the reed is $Q_r = 5$ (the inverse being $q_r = 0.2$), the resonance frequency of the reed is $f_r = 2400$ Hz, and $\zeta = 0.3$; they are reasonable choices for the playing parameters of the reed and the environment based on previous work by Wilson and Beavers [13] and Dalmont [18]. Many of these parameters are extremely difficult to measure in an experimental setting so it is best to choose well accepted values at this time. Concerning the effect of the reed movement, it is very difficult to estimate it from the knowledge of the reed area contributing to the flow rate, S_r .

Because of this an empirical value for the parameter $\Delta\ell_0$ of 5.5 mm is chosen: this allows to find a total length correction around 7 mm (see discussion below), in accordance with experimental results obtained by Dalmont et al. [18] Each of these formulas discussed in the previous sections led to a specific frequency correction which can be added together to become the calculated playing frequency of the clarinet.

In order to compare to the numerical and analytical results for each effect it was necessary to find a way to suppress individual effects within the simulation. To suppress the reed induced flow effect within the simulations it is sufficient to say that λ is zero. To suppress the reed dynamics a large value for ω_r and a low value for q_r may be chosen. This should be done carefully since too extreme a choice could force the clarinet to play in a different register. To suppress the effect of inharmonicity the second peak of the impedance spectrum is manually set to be at three times the fundamental.

Figures 2, 3 and 4 represent the frequency correction, in cents, for three different notes in the first register of the B \flat clarinet notes 1, 12 and 17. The figures compare the calculated playing frequency (either from the analytical formulas or the simulations) to the extracted modal resonance frequencies (the difference between these and the tempered scale is shown in Figure 1) for each particular choice of parameters. Notice that the frequency corrections are listed separately (reed induced flow, reed dynamics and inharmonicity) as well as totaled. Recall that, although the figures give a maximum value of $\gamma = 1$, the analytical reed dynamics Eq. (30) is only accurate until $\gamma \approx 0.5$ (this value corresponds to the simplest expression of the beating reed threshold). Each note of the clarinet exhibited a negative inharmonicity ($\eta_3 < 0$) though the two notes, 12 and 17 were chosen since they offer respectively the maximum and minimum values of inharmonicities possible for the first register of the instrument in question. Note 1 was chosen since it can exhibit the total resonator effects with no open tone holes.

Each figure shows the expected trends for each effect: the value for inharmonicity effect is zero at the threshold of oscillation (the lower limit in γ) and increases with increasing blowing pressure

172 (more-so for lower notes), the reed induced flow has the greatest effect on the playing frequency
 173 and the effect of the reed dynamics is small and does not depend very heavily on the value of the
 174 blowing pressure, γ . Further, the two latter effects increase with the fundamental frequency of
 175 the note, because at low frequencies and low excitation level, they are equivalent to a true length
 176 increase.

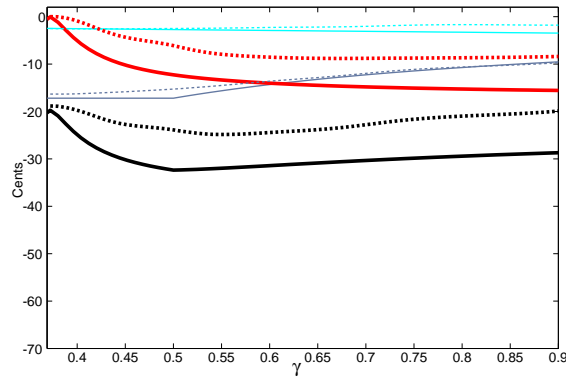


Figure 2: The frequency difference N_{cents} between the 1st impedance peak frequency (resonance frequency) and the playing frequency for Note 1 (fingering for E) of the clarinet. Note 1 values: $\zeta = 0.3$, $\eta_3 = -0.0201$, $F_1 = 1243$ Hz and $f_1 = 146$ Hz. Solid lines represent the analytical results, while dotted lines represent the numerical results. The grey thick line represents the inharmonicity effect, the grey thin line represents the reed flow effect, the light grey thin line represents the reed dynamics effect and finally, the black thick line is the total of all three of these effects.

177 For the numerical simulation, a question arises concerning the number of modes taken into ac-
 178 count. In our simulation, only two significant modes were used in the numerical simulations. This
 179 can affect the results especially for the lower notes where there are a large number of significant
 180 modes present, perhaps up to seven. Conversely, this is not as much of a problem for higher notes
 181 since there are perhaps only two or three significant modes present in the impedance spectrum.
 182 Convergence tests were run in order to verify that the use of two modes would produce a sufficient
 183 amount of accuracy. For Note 1, a perfect convergence is found when the number of modes is
 184 successively chosen to be 2, 4, and 6, and the result with 2 modes seems satisfactory for the effects
 185 of both reed flow and reed dynamics. Concerning the inharmonicity effect, the result with 6 modes

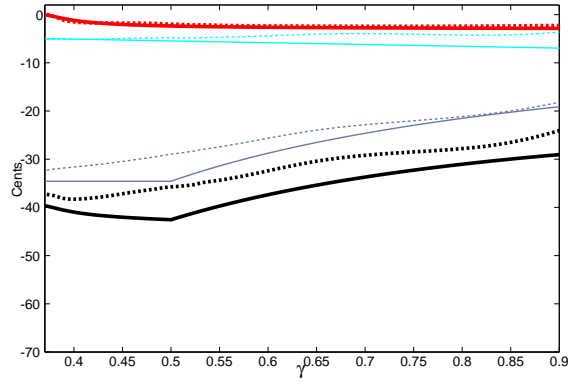


Figure 3: The frequency difference N_{cents} for Note 12 (fingering for E_b) of the clarinet. Note 12 values: $\zeta = 0.3$, $\eta_3 = -0.0036$, $F_1 = 2490$ Hz, $f_1 = 277$ Hz. The color / line scheme and the scale match that of Figure 2

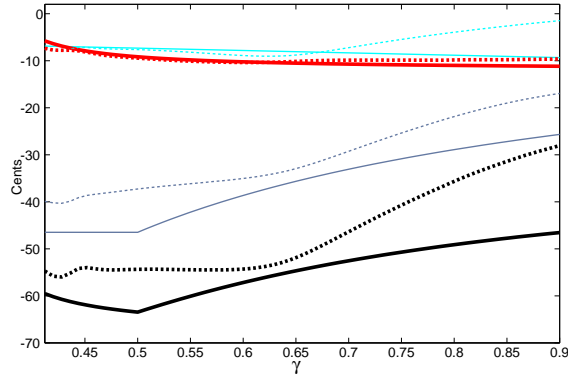


Figure 4: The frequency difference N_{cents} for Note 17 (fingering for A_b) of the clarinet. Note 17 values: $\zeta = 0.3$, $\eta_3 = -0.0142$, $F_1 = 3338$ Hz, $f_1 = 369$ Hz. The color / line scheme and the scale match that of Figure 2.

is closer (by 2 cents) to the analytical result than that with 2 modes. However this is not significant, because the addition of the 3rd mode (with frequency 5th times higher than the fundamental) in the analytical calculation for Note 1, following the work in [5], did not enhance accuracy. Actually the inharmonicity of the 3rd mode is negative ($\eta_5 = -0.052$) and the discrepancy between analytical and numerical result would increase when this mode is taken into account. In addition, an analytical approximation for the amplitude of the 5th harmonic is quite complicated to derive, and is not included in this work.

Another difficulty can come from the choice of the reed's (first) resonance frequency and has been found to be critical for higher notes: if this frequency is too low the simulated sound can play in another register.

Comparing the results from the analytical formulas and the simulations, the hope is that these results for each effect, as well as the total frequency difference will be as similar as possible in shape and value. A good benchmark is that the curves be less than 10 cents different since this is widely cited as a “Just Noticeable Difference”[19].

A few general comments can be made for the discrepancy between analytical and numerical results, for each of the three effects:

- For the inharmonicity effect, the general tendency is satisfactory, especially for higher notes. For Note 1 (see Fig. 2) , the order of magnitude of the discrepancy is 10 cents (see above), but the variation with the excitation pressure γ is well predicted. The discrepancy here for lower notes is found to be the most important of the present study, and explains the lower quality of the results for the total of the three effects for Note 1. Notice though, that [15, p. 447] showed that if the most basic approximation for this effect is considered, the order of magnitude should be near $\varepsilon = \eta/2$, which is indeed the case for the three notes studied.
- For the reed flow, the hypothesis leading to the analytical formula for a beating reed [8] is validated, because the discrepancy is limited to 2 or 3 cents. The decrease of the length

correction with increasing γ is very well predicted.

- For the reed dynamics, the effect is rather small as is the difference between analytical and numerical simulation. The slight frequency decrease just above the threshold is correctly predicted [16]. It is more surprising that the discrepancy remains small for a beating reed since no analytical formula was derived for this case (the formula for a non-beating reed is just continued).

Overall, for all notes in the first register of the clarinet, the validation of the analytical formula by the simulation is satisfactory. The two main delicate points remain the inharmonicity effect for the lower notes and the dynamics reed effect for the higher notes.

VI The temperature gradient effect

Until now, the calculations for a playing frequency are based on the resonance frequencies measured at $20^{\circ}C$, for both numerical and analytical calculations. It is intuitive that the air closest to the mouthpiece, having just left the air-column of the instrumentalist, would have a higher temperature than the air that will exit the instrument for the lowest note on the clarinet. In general, an average temperature over the instrument was used in predictions of playing frequencies since it was considered to not vary greatly with a change in note [20]. Previous research stated that the temperature could be seen as an average over the instrument since it does not vary greatly with a change in note [21, 22]. However, recent measurements by Noreland [23] show that these temperature differences can be as much as a 9° Celsius difference from the top of the barrel ($T_0 = 31^{\circ}C$) to the bottom of the bell ($22^{\circ}C$), as well as that the temperature profile is linear and nearly independent of note (i.e. the number of open holes). Thus the consideration of this effect is simple.

In order to take the temperature value and gradient into account, the frequency shift due to a temperature change of $11^{\circ}C$ (from $22^{\circ}C$ to $31^{\circ}C$), is first computed, then the effect of the

temperature decrease inside the instrument. Therefore first a correction to the final results is simply added which is equivalent to the effect of a $\Delta T = 11^\circ\text{C}$ change in temperature, nearly 33 cents.

The calculation for the effect of the gradient is similar to that for flute-like instruments done by Coltman [24]:

$$\Delta\ell = \int_0^\ell \frac{\delta\rho}{\rho} \sin^2(kx) dx \quad (52)$$

Using the measured linear temperature gradient from [23]:

$$T(x) = T_0[1 - \kappa \frac{x}{\ell}] \quad (53)$$

where x is the distance from the top of the barrel a useful analytical formula is found (from Eq. 52)) for calculating the equivalent length correction for the temperature gradient inside the clarinet:

$$\frac{\Delta\ell}{\ell} = (\frac{1}{4} + \frac{1}{\pi^2})\kappa \frac{\ell}{L} \quad (54)$$

where, in this work, $\kappa = 9/(T_0 + 273)$ (based on measurements in [23]), L is the total length of the instrument and ℓ is the equivalent length of the instrument at the particular note. In this calculation, the formula $k\ell = \frac{\pi}{2}$ is assumed. The specific effect of the CO_2 content and the percent humidity are not included in these calculations since they are small values and can be assumed to have a very small effect[25, 24, 26, 27]. More specifically Fuks states that the effect of gas changes can cause as much of a 20 cent decrease in fundamental frequency[26], however, this study, as with many others focused on a sustained note whereas we are only interested in a small portion of the playing frequency in the steady-state regime and not on the evolution of pitch throughout a sustained note.

The final results, in cents, for this effect is:

$$Ncents_{temperature} = \frac{100}{0.06} ((\frac{1}{4} + \frac{1}{\pi^2})\kappa \frac{\ell}{L} - \frac{1}{2} \frac{\Delta T}{(T_0 + 273)}). \quad (55)$$

VII Total of four effects

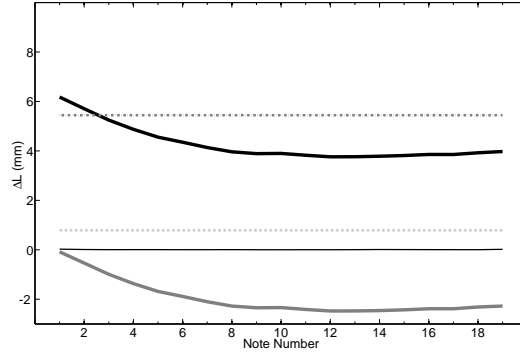


Figure 5: Analytical length corrections ($\Delta\ell$) values, in mm, as a function of note number, in the first register, at the threshold of oscillation. The black thin line represents the inharmonicity effect, the thin dashed line - reed dynamics, dark grey solid line (bottom) - temperature effect, thick dash-dotted line - reed flow effect and finally, the thick solid line is the total of all four effects. Realize that the plot for the inharmonicity effect is at zero for the $\gamma = \gamma_{th}$.

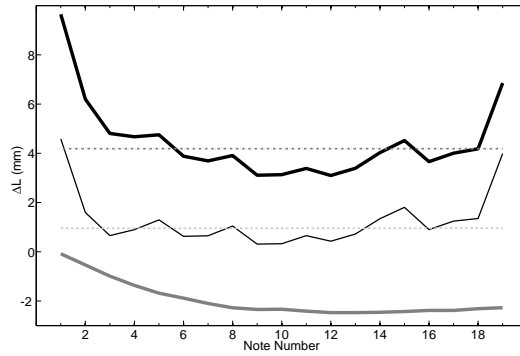


Figure 6: Analytical length corrections ($\Delta\ell$) values, in mm, as a function of note number, in the first register. The color / line scheme matches that of Figure 5. The data are represented for $\gamma = 0.65$.

Figure 5 and 6 show the total length corrections, in meters, as a function of note in the first register of the clarinet for two different values of γ . This total length correction is calculated by the following formula:

$$\Delta\ell_{total} = -\frac{\Delta f}{f}\ell, \text{ where } \ell = \frac{2c}{F_1}. \quad (56)$$

The value for ζ remained 0.3 and the value for γ was chosen to be either γ_{th} , the threshold of oscillation for each given note, or $\gamma = 0.65$. In Figure 5, notice that for $\gamma = \gamma_{th}$ the curve for the inharmonicity remains at zero, this is because the inharmonicity of the resonator has no effect at the threshold of oscillation. For the reed induced flow effect, as expected, a value near 5.5 mm is found. As required, the total magnitude of the length corrections for these notes corresponds well to the work done by Dalmont et al in [18] which showed length corrections totaling 7mm to 10mm (since this work was based on the use of an artificial mouth it does not consider the temperature effect which is discussed in section VI). The effect of inharmonicity is similar for each note creating a difference between analytical predictions and numerical simulations no larger than 15 cents for the total range of γ .

These figures show that the effect of the formulas that represent the reed induced flow effect is stable and offers the largest frequency shift of the three effects. This was to be expected based on the work in [8] and also shows the change in behavior above the beating reed regime. For the case of reed dynamics, not surprisingly, the effect is small for every note and nearly linear as a function of γ . It is obvious from these figures that the effects are very much dependent on note (length of resonator). This is intuitive for the clarinetists since as the effective length of the instrument is shortened, there are more factor influencing the playing frequency and therefore more compensation needed in order to play in-tune. Figure 7 shows the same total effects as in previous figures, but represents the frequency shift vs. the tempered scale.

VIII Conclusions

The analytical formulas presented here are computationally fast, accurate and can be used in a number of situations in order to study the playing frequency of the clarinet.

An important point to highlight is that the computation time for the full range of γ and ζ using the numerical simulations is about three hours for all notes in the first register. Using the

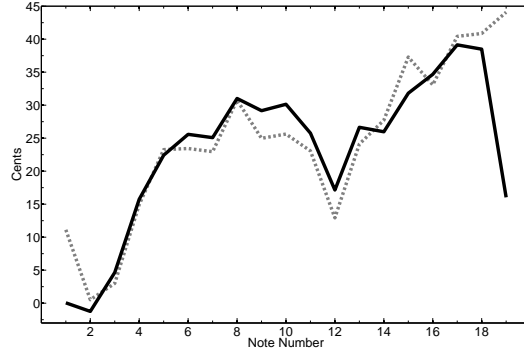


Figure 7: Total frequency shift vs. the tempered scale, in cents, as a function of note number, for the first register, where $\gamma = 0.65$ (black solid line) and γ_{th} (grey dotted line).

analytical formulas for the full range of control parameters can complete the task in less than two minutes. This is an extremely valuable decrease in computing time and makes this a valid and useful method for predicting clarinet playing frequencies. However, difficulties can occur because of the estimation uncertainty of the reed resonance frequency and the inharmonicity effect for the lowest notes.

The difference in predictions of the numerical simulations and analytical formulas are small and the results presented here are sufficient enough to warrant the sole use of the analytical formulas to predict playing frequencies. There are, nevertheless, certain aspects that were not taken into account in this work, including the nonlinearity of toneholes and the effect of the vocal tract on the playing frequency. The nonlinearity of the holes can be ignored because the particular clarinet studied employs undercut toneholes in order to avoid this effect [20]. Further, it is valid to ignore the effect of the vocal tract has been shown to have little effect on the lower register of the clarinet [28]. Finally, a limitation of this work comes when ignoring the effect of CO_2 and humidity and that the effect of temperature gradient is based only on the work by Noreland[23]. Taking detailed measurements which include not only the temperature profile but the CO_2 and humidity content as well could make the playing frequency predictions more accurate.

In order to use the analytical formulas (and simulations), a difficult task remains in determining

the main parameters (reed characteristics, etc.). Some of these could be deduced by using an artificial mouth, assuming a perfect cylinder. Obviously this task is even more difficult when attempting to measure with an instrumentalist. The important point is that these results can help to discern the inhomogeneity of tuning from the inhomogeneity of the modal parameters. Concerning this topic, the goal of this work joins that of the paper by Almeida et al [29], who used an experimental approach to show playing frequencies based on different input values. Comparing these results to measurements of the playing frequencies using an artificial mouth apparatus as well as from actual musician playing tests could validate the practical applications of the analytical formulas in the clarinet manufacturing process.

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Table 1: List of important symbols. x, p, y, γ and ζ are dimensionless.

Symbol	Definition	Symbol	Definition
p	pressure in the mouthpiece	x	reed displacement
p_M	reed closing pressure	H	reed opening
γ	dimensionless blowing pressure	ζ	dimensionless reed opening
S_r	reed vibrating surface area	S	cross sectional area of tube
ℓ_r	reed vibrating length	Z_c	characteristic impedance
ω_r	first reed resonance frequency	ω_n	modal frequencies
ω_p	playing frequency	u_r	reed flow rate
u_{tot}	total flow rate entering resonator	u_b	Bernoulli flow rate
V_{eq}	equivalent volume of reed flow	$\Delta\ell_0$	equivalent length correction due to reed flow at low frequencies and low pressures
$G(\gamma)$	empirical fit to reed flow curve	$\Delta\ell_{eq}$	$\Delta\ell_0/G(\gamma)$
$\Delta\ell$	length correction	L	total length of tube
Δf	frequency correction	N_{cents}	frequency difference in cents
ℓ	length of tube section		

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