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REAL HARMONIZABLE MULTIFRACTIONAL STABLE PROCESS AND ITS LOCAL PROPERTIES

MARCO DOZZI AND GEORGIY SHEVCHENKO

ABSTRACT. A real harmonizable multifractional stable process is defined, its Hölder continuity and localizability are proved. The existence of local time is shown and its regularity is established.

INTRODUCTION

Fractional processes are one of the main tools for modeling the phenomena of long-range dependence in natural sciences, financial mathematics, telecommunication networks etc. Due to the role played by Gaussian distribution, the most popular and the most intensively investigated fractional process is the fractional Brownian motion B^H , a centered Gaussian process with the covariance function $\mathbf{E} [B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$. The parameter $H \in (0, 1)$ is called the Hurst parameter and measures the smoothness of trajectories of the process (it is approximately the Hölder exponent of the process) and the “depth of memory” of the process (for $H > 1/2$ the process exhibits the property of long-range dependence).

From the point of view of possible applications, there are two main drawbacks of fractional Brownian motion. The first one comes from the Gaussian distribution, which has extremely light tails, though many

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data coming from applications are heavy-tailed. The second one is the homogeneity of increments that does not allow to model processes having different regularity and different time dependence properties at different time instances. A related problem is a self-similarity property, which briefly means that the properties of the process are the same under each scale. However, the absence of such property is apparent in many cases and mostly evident in stock price processes: long-term data is much smoother than wild intraday quotes.

The light tails problem is worked around usually by considering fractional stable processes. In contrast to the Gaussian case, where the covariance structure determines whole distribution of a process, so there is essentially one fractional process, in stable case there are many of them: linear fractional stable process, harmonizable fractional stable process, Liouville stable process etc. (See book [10] for an extensive review of different fractional processes.)

In turn, the homogeneity problem is solved by considering multifractional processes. Recently, several multifractional extensions of fractional Brownian motion were defined, based on different representations of the fractional Brownian motion: moving average (linear) multifractional Brownian motion [8], Volterra multifractional Brownian motion [9], harmonizable multifractional Brownian motion [1].

In this paper, we consider a process called *real harmonizable multifractional stable process* which has the both properties of heavy tails and multifractionality, which can be regarded both as a multifractional generalization of a harmonizable fractional stable process and as a stable generalization of harmonizable multifractional Brownian motion, and can be used to improve models involving either kind of processes.

Our main interest in this paper is in path properties of this process: continuity, existence and joint continuity of local times. For fractional harmonizable stable process continuity was proved in [3] and local times properties were considered in [11].

The paper is organized as follows. In Section 1 we give necessary pre-requisites on stable distributions and local times. Section 2 focuses on path properties of the process considered: almost sure continuity and localizability. Section 3 is devoted to existence and properties of local times.

1. PRE-REQUISITES

1.1. Stable random variables and processes. In this paper we focus only on symmetric α -stable (S α S) random variables with $\alpha \in (1, 2)$. We recall that a random variable ξ is called S α S with a scale parameter s^α if it has a characteristic function

$$\mathbb{E} \left[e^{i\lambda\xi} \right] = e^{-|\sigma\lambda|^\alpha}.$$

An important tool to construct stable random variables is *independently scattered rotationally invariant complex S α S random measure* with the Lebesgue control measure, which is a complex-valued σ -additive random measure $M = M_\alpha$ on \mathbb{R} defined by the following properties.

- (1) (Rotationally invariant complex S α S) for any Borel set $A \subset \mathbb{R}$ and any $\theta \in \mathbb{R}$ the distribution of $e^{i\theta} M(A)$ is the same as of $M(A)$, and $\operatorname{Re} M(A)$ is S α S with the scale parameter $\lambda(A)$.
- (2) (*Independently scattered*) for any disjoint Borel sets $A_1, \dots, A_n \subset [0, \infty)$ the values $M(A_1), \dots, M(A_n)$ are independent.
- (3) For any Borel set $A \subset \mathbb{R}$ $M(-A) = \overline{M(A)}$.

For a function $f : \mathbb{R} \rightarrow \mathbb{C}$ such that

$$(1.1) \quad f(-x) = \overline{f(x)} \text{ for all } x \in \mathbb{R}$$

and

$$\|f\|_{L^\alpha(\mathbb{R})}^\alpha = \int_{\mathbb{R}} |f(x)|^\alpha dx < \infty$$

it is possible to define a stochastic integral

$$\int_{\mathbb{R}} f(x)M(dx),$$

which appears to be a real S α S random variable with the scale parameter $\|f\|_{L^\alpha(\mathbb{R})}^\alpha$.

In other words, stochastic integral gives an isometry between the space of S α S real random variables spanned by the measure M with the norm

$$\|\xi\|_\alpha = c(\xi) = -\log \mathbf{E} [e^{i\xi}]$$

and the subspace of $L^\alpha(\mathbb{R})$ consisting of functions satisfying (1.1), i.e. having adjoint values at symmetric points.

We end this subsection with the so-called LePage representation of processes given as transformations of S α S random measure. For details see [6, 3].

Assume we have a measurable function $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{C}$ such that for each $t \geq 0$ the function $f(t, \cdot)$ satisfies (1.1) and belongs to $L^\alpha(\mathbb{R})$. Define a process $\{X_t, t \geq 0\}$ by

$$(1.2) \quad X_t = \int_{\mathbb{R}} f(t, x)M(dx).$$

The next proposition is a slight modification of [4], the proof is exactly the same as there with a slight adjustment for the property $M(-dx) = \overline{M(dx)}$ in our case, so we skip it.

Theorem 1.1. *Let φ be arbitrary probability density on \mathbb{R} equivalent to the Lebesgue measure. Also let $\{\Gamma_k, k \geq 1\}$, $\{\xi_k, k \geq 1\}$, $\{g_j, j \geq 1\}$ be three independent sets of random variables, such that*

- $\{\Gamma_k, k \geq 1\}$ is a sequence of arrivals of Poisson process with unit intensity;
- $\{\xi_k, k \geq 1\}$ is a sequence of independent random variables with density φ ;
- $\{g_k, k \geq 1\}$ are independent rotationally invariant complex Gaussian with $\mathbb{E}[|\operatorname{Re} g_k|^\alpha] = 1$.

Then the process $\{X_t, t \geq 0\}$ defined by (1.2) has the same finite-dimensional distributions as the process

$$(1.3) \quad X'_t = C_\alpha \operatorname{Re} \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} f(t, \xi_k) g_k,$$

where $C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x dx\right)^{1/\alpha}$, and this series converges almost surely for each t .

1.2. Local times. Marco

2. DEFINITION AND PATHWISE PROPERTIES OF REAL HARMONIZABLE STABLE PROCESS

Let M be an independently scattered rotationally invariant complex S α S measure on \mathbb{R} defined in Subsection 1.1.

Throughout the paper we will denote by C any constant, which does not depend on any variables, unless otherwise is stated. Of course, C may change from line to line.

Recall that a real harmonizable fractional stable process with Hurst parameter H is defined as

$$(2.1) \quad Z^H(t) = \int_{\mathbb{R}} \frac{e^{itx} - 1}{|x|^{1/\alpha+H}} M(dx).$$

A multifractional generalization of this definition consists, naturally, in letting the Hurst parameter depend on t .

Definition 2.1. A real harmonizable multifractional stable process (rhmsp) with Hurst function $H(t)$ and a stability parameter α is defined as

$$(2.2) \quad X(t) = \int_{\mathbb{R}} \frac{e^{itx} - 1}{|x|^{1/\alpha+H(t)}} M(dx).$$

Clearly, $X(t) = Z^{H(t)}(t)$. We assume that $0 < \hat{H} = \inf_t H(t) \leq H(t) \leq \sup_t H(t) = \check{H} < 1$.

2.1. Norm estimates for the increments.

Lemma 2.2. For all $H_1, H_2 \in (\hat{H}, \check{H})$ it holds

$$\|Z_t^{H_1} - Z_t^{H_2}\|_{\alpha} \leq C |H_1 - H_2|, \quad t \in [0, T].$$

Proof. Write

$$\begin{aligned} \|Z_t^{H_1} - Z_t^{H_2}\|_{\alpha}^{\alpha} &= \int_{\mathbb{R}^d} |e^{itx} - 1|^{\alpha} |x|^{-1} \left| |x|^{-H_1} - |x|^{-H_2} \right|^{\alpha} dx \\ &\leq C \int_{\mathbb{R}} (1 \wedge |x|)^{\alpha} |x|^{-1} |\log |x||^{\alpha} (|x|^{-\alpha H_1} \vee |x|^{-\alpha H_2}) |H_1 - H_2|^{\alpha} dx \\ &= C |H_1 - H_2|^{\alpha} \left(\int_{|x|<1} |x|^{\alpha(1-\hat{H})-1} |\log |x||^{\alpha} dx + \int_{|x|>1} |x|^{-1-\alpha\check{H}} |\log |x||^{\alpha} dx \right) \\ &\leq C |H_1 - H_2|, \end{aligned}$$

whence we have the assertion. \square

We assume that H is Hölder continuous with order greater than \check{H} , i.e., there exists $\gamma > \check{H}$ s.t. for all $t, s \geq 0$

$$|H(t) - H(s)| \leq C|t - s|^\gamma.$$

Lemma 2.3. *There exist positive constants $C_1, C_2 > 0$ such that for any $H \in [\hat{H}, \check{H}]$ one has*

$$C_1|t - s|^H \leq \|Z_t^H - Z_s^H\|_\alpha \leq C_2|t - s|^H$$

locally uniformly in s, t .

Proof. Write

$$\begin{aligned} \|Z_t^H - Z_s^H\|_\alpha^\alpha &= \int_{\mathbb{R}} |e^{itx} - e^{isx}|^\alpha |x|^{-1-\alpha H} dx \leq C \int_{\mathbb{R}} (1 \wedge |t - s| |x|)^\alpha |x|^{-1-\alpha H} dx \\ &= C \left(|t - s|^\alpha \int_{|x| < 1/|t-s|} |x|^{\alpha(1-H)-1} dx + \int_{|x| > 1/|t-s|} |x|^{-1-\alpha H} dx \right) \\ &\leq C (|t - s|^{\alpha-\alpha(1-H)} + |t - s|^{\alpha H}) = C |t - s|^{\alpha H}. \end{aligned}$$

To prove the lower bound, observe that there exist positive constants c_1, c_2 such that $|e^{iy} - 1| > c_1 |y|$ for $|y| < c_2$ and write

$$\begin{aligned} \|Z_t^H - Z_s^H\|_\alpha^\alpha &\geq \int_{|x| < c_2/|t-s|} |e^{i(t-s)x} - 1|^\alpha |x|^{-1-\alpha H} dx \\ &\geq C |t - s|^\alpha \int_{|x| < c_2/|t-s|} |x|^{\alpha(1-H)-1} dx = C |t - s|^{\alpha H}. \end{aligned}$$

□

Lemmata 2.2 and 2.3 imply the following

Proposition 2.4. *There exist $\delta, C_1, C_2 > 0$ s.t. for rhmsp X given by*

(2.2) *and $|t - s| < \delta$ it holds*

$$(2.3) \quad C_1 |t - s|^{\hat{H}(t,s)} \leq \|X(t) - X(s)\|_\alpha \leq C_2 |t - s|^{\check{H}(t,s)}, \quad t, s \in [0, T]$$

where $\hat{H}(t, s) = \min_{[t,s]} H(u)$, $\check{H}(t, s) = \max_{[t,s]} H(u)$.

Proof. Let $\check{H}(t, s) = H(\check{t})$, $\hat{H}(t, s) = H(\hat{t})$.

$$\begin{aligned} \|X(t) - X(s)\|_\alpha &\leq \left\| Z_t^{H(t)} - Z_t^{\check{H}(t,s)} \right\|_\alpha + \left\| Z_s^{H(t)} - Z_s^{\check{H}(t,s)} \right\|_\alpha + \left\| Z_t^{\check{H}(t,s)} - Z_s^{\check{H}(t,s)} \right\|_\alpha \\ &\leq |H(t) - H(\check{t})| + |H(s) - H(\check{t})| + C|t - s|^{\hat{H}(t,s)} \leq C|t - s|^\gamma + C|t - s|^{\hat{H}(t,s)} \end{aligned}$$

Since $\hat{H}(t, s) < \gamma$, we get the upper bound.

The left hand one is proved similarly:

$$\begin{aligned} C|t - s|^{\hat{H}(t,s)} &\leq \left\| Z_t^{\hat{H}(t,s)} - Z_s^{\hat{H}(t,s)} \right\|_\alpha \\ &\leq \|X(t) - X(s)\|_\alpha + \left\| Z_t^{H(t)} - Z_t^{\hat{H}(t,s)} \right\|_\alpha + \left\| Z_s^{H(t)} - Z_s^{\hat{H}(t,s)} \right\|_\alpha \\ &\leq \|X(t) - X(s)\|_\alpha + |H(t) - H(\hat{t})| + |H(s) - H(\hat{t})| \\ &\leq \|X(t) - X(s)\|_\alpha + C|t - s|^\gamma. \end{aligned}$$

□

2.2. Hölder continuity of rhmsp. In this subsection we prove a Hölder continuity of rhmsp. Our argument is a slight modification of the one found in [3] for harmonizable fractional stable motion.

Theorem 2.5. *The rhmsp X has a version, which is almost surely Hölder continuous of any order $\kappa < \hat{H}$ and moreover almost surely satisfies*

$$\sup_{\substack{t, s \in [0, T] \\ |t - s| < \delta}} |X(t) - X(s)| = o(\delta^{\hat{H}} |\log \delta|^{1/\alpha + 1/2 + \varepsilon}), \quad \delta \rightarrow 0+,$$

for all $T, \varepsilon > 0$.

Proof. Let $T > 0$ be fixed and throughout this proof $t, s \in [0, T]$.

We use the LePage representation (1.3). To simplify the notation we write this representation for the process X itself rather than for its

version:

$$X_t = C_\alpha \operatorname{Re} \sum_{k \geq 1} \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} f(t, \xi_k) g_k,$$

where $f(t, x) = (e^{itx} - 1) |x|^{-1/\alpha - H(t)}$, $\varphi(x) = K_\eta |x|^{-1} |\log |x||^{-1-\eta}$, $\eta > 0$ is arbitrary but fixed, K_η is a normalizing constant.

Conditionally on Γ and ξ , it has the Gaussian distribution, so

$$\mathbb{E} [(X_t - X_s)^2 \mid \Gamma, \xi] = C_\alpha^2 \sum_{k \geq 1} \Gamma_k^{-2/\alpha} \varphi(\xi_k)^{-2/\alpha} |f(t, \xi_k) - f(s, \xi_k)|^2 \leq Ca(u),$$

where

$$a(z) = \sum_{k \geq 1} \Gamma_k^{-2/\alpha} \varphi(\xi_k)^{-2/\alpha} \sup_{|t-s| < u} |f(t, \xi_k) - f(s, \xi_k)|^2.$$

Write

$$\begin{aligned} & \sup_{|t-s| < u} |f(t, \xi_k) - f(s, \xi_k)|^2 \\ \leq & \sup_{|t-s| < u} |(e^{itx} - e^{isx})|x|^{-1/\alpha - H(t)} + \sup_{|t-s| < u} |e^{isx} - 1| |x|^{-1/\alpha} |x^{-H(t)} - x^{-H(s)}| \\ & \leq C(u|x| \wedge 1) |x|^{-1/\alpha} (|x|^{-\hat{H}} \vee |x|^{-\check{H}}) \\ & + C(|x| \wedge 1) |x|^{-1/\alpha} (|x|^{-\hat{H}} \vee |x|^{-\check{H}}) |\log |x|| \sup_{|t-s| < u} |H(t) - H(s)| \\ & \leq C |x|^{-1/\alpha} (|x|^{-\hat{H}} \vee |x|^{-\check{H}}) ((u|x| \wedge 1) + (|x| \wedge 1) |\log |x|| u^\gamma). \end{aligned}$$

Keeping this estimate in mind, take now the expectation $\mathbb{E}_\xi [a(z)]$ with respect to the variables ξ only:

$$\mathbb{E}_\xi [a(z)] \leq CS(\Gamma)(I_1 + I_2),$$

where

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}} |x|^{-2/\alpha} (|x|^{-2\hat{H}} \vee |x|^{-2\check{H}}) (u|x| \wedge 1)^2 \varphi^{1-2/\alpha}(x) dx \\
&= 2 \int_0^\infty x^{-1} (x^{-2\hat{H}} \vee x^{-2\check{H}}) (ux \wedge 1)^2 |\log x|^{(1+\eta)(2/\alpha-1)} dx \\
&\leq C u^{2\hat{H}} \int_0^\infty z^{-1} (z^{-2\hat{H}} \vee z^{-2\check{H}}) (|z| \wedge 1)^2 |\log(z/u)|^{(1+\eta)(2/\alpha-1)} dz \\
&\leq C u^{2\hat{H}} |\log u|^{(1+\eta)(2/\alpha-1)}, \\
I_2 &= u^{2\gamma} \int_{\mathbb{R}} |x|^{-2/\alpha} (|x|^{-2\hat{H}} \vee |x|^{-2\check{H}}) (|x| \wedge 1)^2 |\log |x||^2 \varphi^{1-2/\alpha}(x) dx \\
&= 2u^{2\gamma} \int_0^\infty x^{-1} (x^{-2\hat{H}} \vee x^{-2\check{H}}) (x \wedge 1)^2 |\log x|^{(1+\eta)(2/\alpha-1)+2} dx \leq C u^{2\gamma}, \\
S(\Gamma) &= \sum_{k \geq 1} \Gamma_k^{-2/\alpha} < \infty \quad \text{a.a. } \Gamma,
\end{aligned}$$

where the last is true owing to the fact that $\Gamma_j/j \rightarrow 1$, $j \rightarrow \infty$, almost surely by the strong law of large numbers, and $2/\alpha > 1$. Therefore

$$\mathbb{E}_\xi [a(z)] \leq C(\Gamma) u^{2\hat{H}} |\log u|^{(1+\eta)(2/\alpha-1)}$$

almost surely.

Define $b(z) = u^{2\hat{H}} |\log u|^{2(1+\eta)/\alpha}$. We have

$$\mathbb{E}_\xi \left[\sum_{n \geq 1} \frac{a(2^{-n})}{b(2^{-n})} \right] \leq C(\Gamma) \sum_{n \geq 1} n^{-1-\eta},$$

so for almost all ξ, Γ we have $a(2^{-n})/b(2^{-n}) \rightarrow 0$, $n \rightarrow \infty$. It is easy to see that $b(2t) \leq Cb(t)$, and $a(z)$ is increasing, so from the last convergence we get $a(u)/b(u) \rightarrow 0$, $u \rightarrow 0+$, or $a(u) = o_{\xi, \Gamma}(u^{2\hat{H}} |\log u|^{2(1+\eta)/\alpha})$.

So we have

$$\mathbb{E} [(X_t - X_s)^2 | \Gamma, \xi] = o_{\xi, \Gamma}(u^{2\hat{H}} |\log u|^{2(1+\eta)/\alpha}), \quad u \rightarrow 0+.$$

Now recall once more that X is Gaussian given ξ and Γ , so by Lemma 1 of [3]

$$\sup_{|t-s|<\delta} |X_t - X_s| = o_\omega(\delta^{\hat{H}} |\log \delta|^{1/\alpha+\eta/\alpha+1/2}), \quad \delta \rightarrow 0+,$$

whence we get the statement of the theorem. \square

2.3. Localizability of rhmsp. We start this section by giving Falconer's notion of localizability.

Definition 2.6. Process X is called H -localizable at a point t with the local version Y if

$$(2.4) \quad \left\{ \frac{1}{\delta^H} (X_{t+\delta u} - X_t), u \geq 0 \right\} \xrightarrow{fdd} \{Y_u, u \geq 0\}, \quad \delta \rightarrow 0+.$$

(Here \xrightarrow{fdd} stands for the convergence of finite-dimensional distributions.)

It is called strongly H -localisable at a point t if in (2.4) the convergence is in the sense of the distribution on the path space.

Some authors use the term *local asymptotic self-similarity* for localizability, which reflects the fact that the local version Y is an H -self-similar process.

Theorem 2.7. *The rhmsp X is localizable at any point t with local version being real harmonizable fractional stable process with Hurst parameter $H(t)$.*

Proof. Define

$$Y_t^\delta(u) = \frac{1}{\delta^{H(t)}} (X(t + \delta u) - X(t)).$$

We will assume throughout that $\delta < 1$.

For $u_1, \dots, u_n > 0$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ denote $s_k = t + \delta u_k$ and write

$$\begin{aligned} & -\log \mathbb{E} \left[\exp \left\{ i \sum_{k=1}^n \lambda_k Y_t^\delta(u_k) \right\} \right] \\ = & -\log \mathbb{E} \left[\exp \left\{ \frac{i}{\delta^{H(t)}} \int_{\mathbb{R}} \frac{1}{|x|^{1/\alpha}} \sum_{k=1}^n \lambda_k \left[\frac{e^{is_k x} - 1}{|x|^{H(s_k)}} - \frac{e^{itx} - 1}{|x|^{H(t)}} \right] M(dx) \right\} \right] \\ = & \left\{ \frac{1}{\delta^{\alpha H(t)}} \int_{\mathbb{R}} \frac{1}{|x|} \left| \sum_{k=1}^n \lambda_k \left[\frac{e^{is_k x} - 1}{|x|^{H(s_k)}} - \frac{e^{itx} - 1}{|x|^{H(t)}} \right] \right|^\alpha dx \right\}. \end{aligned}$$

Now estimate the integrand multiplied by $|x|$:

$$\begin{aligned} & \left| \sum_{k=1}^n \lambda_k \left[\frac{e^{is_k x} - 1}{|x|^{H(s_k)}} - \frac{e^{itx} - 1}{|x|^{H(t)}} \right] \right|^\alpha \leq C \sum_{k=1}^n |\lambda_k|^\alpha \left| \frac{e^{is_k x} - 1}{|x|^{H(s_k)}} - \frac{e^{itx} - 1}{|x|^{H(t)}} \right|^\alpha \\ \leq & C \sum_{k=1}^n |\lambda_k|^\alpha \left(|e^{is_k x} - 1|^\alpha \left| |x|^{-H(s_k)} - |x|^{-H(t)} \right|^\alpha + |x|^{-\alpha H(t)} |e^{is_k x} - e^{itx}|^\alpha \right) \\ \leq & C \sum_{k=1}^n |\lambda_k|^\alpha \left(|\log |x||^\alpha |x|^{-\alpha \theta} |H(s_k) - H(t)|^\alpha (1 \wedge |x|)^\alpha + |x|^{-\alpha H(t)} |s_k - t|^\alpha (1 \wedge |x|)^\alpha \right) \\ \leq & C \sum_{k=1}^n |\lambda_k|^\alpha (|x|^{-\alpha \hat{H}} \vee |x|^{-\alpha \tilde{H}}) (1 \wedge |x|)^\alpha [|s_k - t|^{\alpha \gamma} |\log |x||^\alpha + |s_k - t|^\alpha] \\ \leq & C \sum_{k=1}^n |\lambda_k|^\alpha \delta^{\alpha \gamma} (|x|^{\alpha(1-\hat{H})} \mathbb{1}_{|x|<1} + |x|^{-\alpha \tilde{H}} \mathbb{1}_{|x|>1}) (1 + |\log |x||^\alpha) \\ \leq & C \delta^{\alpha \gamma} (|x|^{\alpha(1-\hat{H})} \mathbb{1}_{|x|<1} + |x|^{-\alpha \tilde{H}} \mathbb{1}_{|x|>1}) (1 + |\log |x||^\alpha). \end{aligned}$$

We remark that the constants here depend only on α , t , s_1, \dots, s_n and $\lambda_1, \dots, \lambda_n$.

Now

$$\begin{aligned} & \frac{1}{\delta^{\alpha H(t)}} \frac{1}{|x|} \left| \sum_{k=1}^n \lambda_k \left[\frac{e^{is_k x} - 1}{|x|^{H(s_k)}} - \frac{e^{itx} - 1}{|x|^{H(t)}} \right] \right|^\alpha \\ \leq & C (|x|^{\alpha(1-\hat{H})-1} \mathbb{1}_{|x|<1} + |x|^{-1-\alpha \tilde{H}} \mathbb{1}_{|x|>1}) (1 + |\log |x||^\alpha), \end{aligned}$$

which is integrable over \mathbb{R} . Hence by the dominated convergence theorem

$$\begin{aligned}
& - \lim_{\delta \rightarrow 0^+} \log \mathbb{E} \left[\exp \left\{ i \sum_{k=1}^n \lambda_k Y_t^\delta(u_k) \right\} \right] \\
&= \int_{\mathbb{R}} \frac{1}{|x|} \lim_{\delta \rightarrow 0^+} \left\{ \frac{1}{\delta^{\alpha H(t)}} \left| \sum_{k=1}^n \lambda_k \left[\frac{e^{i s_k x} - 1}{|x|^{H(s_k)}} - \frac{e^{i t x} - 1}{|x|^{H(t)}} \right] \right|^\alpha dx \right\} \\
&= \int_{\mathbb{R}} \frac{1}{|\delta x|} \left| \sum_{k=1}^n \lambda_k \lim_{\delta \rightarrow 0^+} \frac{1}{\delta^{H(t)}} \left[\frac{e^{i(t+\delta u_k)x} - 1}{|x|^{H(t+\delta u_k)}} - \frac{e^{i t x} - 1}{|x|^{H(t)}} \right] \right|^\alpha d(\delta x) \\
&= \int_{\mathbb{R}} \frac{1}{|y|} \left| \sum_{k=1}^n \lambda_k \lim_{\delta \rightarrow 0^+} \left[\frac{e^{i u_k y} - e^{-i t y / \delta}}{|y|^{H(t+\delta u_k)}} \delta^{H(t+\delta u_k) - H(t)} - \frac{1 - e^{-i t y / \delta}}{|y|^{H(t)}} \right] \right|^\alpha dy \\
&= \int_{\mathbb{R}} \frac{1}{|y|} \left| \sum_{k=1}^n \lambda_k \lim_{\delta \rightarrow 0^+} \left[\frac{e^{i u_k y} - 1}{|y|^{H(t)}} + (e^{i t u_k} - e^{-i t y / \delta}) R_\delta \right] \right|^\alpha dy,
\end{aligned}$$

where

$$R_\delta = \frac{1}{|y|^{H(t+\delta u_k)}} \delta^{H(t+\delta u_k) - H(t)} - \frac{1}{|y|^{H(t)}} = \frac{1}{|y|^{H(t)}} \left[\left(\frac{\delta}{|y|} \right)^{H(t+\delta u_k) - H(t)} - 1 \right].$$

Estimate

$$\begin{aligned}
\left| \log \left(\frac{\delta}{|y|} \right)^{H(t+\delta u_k) - H(t)} \right| &\leq |H(t+\delta u_k) - H(t)| (|\log \delta| + |\log |y||) \\
&\leq C \delta^\gamma (|\log \delta| + |\log |y||) \rightarrow 0, \quad \delta \rightarrow 0^+.
\end{aligned}$$

Thus $R_\delta \rightarrow 0$, $\delta \rightarrow 0^+$.

Finally,

$$(2.5) \quad \lim_{\delta \rightarrow 0^+} \log \mathbb{E} \left[\exp \left\{ i \sum_{k=1}^n \lambda_k Y_t^\delta(u_k) \right\} \right] = - \int_{\mathbb{R}} \frac{1}{|y|} \left| \sum_{k=1}^n \lambda_k \left[\frac{e^{i u_k y} - 1}{|y|^{H(t)}} \right] \right|^\alpha dy,$$

which is exactly the logarithm of the characteristic function of $Z^{H(t)}(u_1), \dots, Z^{H(t)}(u_n)$, as required. \square

Remark 2.8. By using the same kind of argument as the one used in the proof of continuity, it is possible to prove tightness of laws of

processes on the space of continuous paths and whence derive a strong localizability.

3. LOCAL TIMES FOR RMHSP

3.1. Properties of the local time. We start this section by showing the existence and square integrability of a local time.

Proposition 3.1. *The rhmsp X has a square integrable local time $L(t, x)$.*

Proof. According to [2], it is enough to check the following “condition (\mathcal{H})”: there exists $\rho > 0$ and $H \in (0, 1)$ and $\psi \in L^1(\mathbb{R})$ such that

$$(3.1) \quad \mathbb{E} \left[\exp \left\{ i\lambda (X(t) - X(s)) \right\} \right] \leq \psi(\lambda |t - s|^H).$$

But

$$\begin{aligned} \mathbb{E} \left[\exp \left\{ i\lambda (X(t) - X(s)) \right\} \right] &= \exp \left\{ -|\lambda|^\alpha \|X(t) - X(s)\|_\alpha^\alpha \right\} \\ &\leq \exp \left\{ -C |\lambda|^\alpha \|t - s\|^{\alpha \hat{H}(t,s)} \right\} \leq \exp \left\{ -C |\lambda|^\alpha \|t - s\|^{\alpha \hat{H}} \right\}, \end{aligned}$$

whence we have (3.1) with $\psi = \exp \{-|x|^\alpha\}$, $H = \hat{H}$. \square

In order to prove further properties, we need

Definition 3.2 ([7]). A stable random process X is $\|\cdot\|_\alpha$ *locally non-deterministic* (LND) on \mathbf{T} if

- (L1) $\|X(t)\|_\alpha > 0$ for all $t \in \mathbf{T}$;
- (L2) $\|X(t) - X(s)\|_\alpha > 0$ for all sufficiently close distinct $s, t \in \mathbf{T}$;
- (L3) for any $n > 1$ there exists C_n s.t. for any $t_1 < t_2 < \dots < t_n \in \mathbf{T}$ sufficiently close together one has

$$(3.2) \quad \|X(t_n) - \text{span} \{X(t_1), \dots, X(t_{n-1})\}\|_\alpha \geq C_n \|X(t_n) - X(t_{n-1})\|.$$

In [7] it is shown that the local non-determinism property is equivalent to the property of $\|\cdot\|_\alpha$ *locally approximately independent increments*, which consists of properties (L1), (L2) above and

(L3a) for any $n > 1$ there exists C_n s.t. for any $t_1 < t_2 < \dots < t_n \in \mathbf{T}$ sufficiently close together and any $a_1, \dots, a_n \in \mathbb{R}$ one has

$$(3.3) \quad \begin{aligned} & \left\| a_1 X(t_1) + \sum_{k=1}^{n-1} a_k (X(t_{k+1}) - X(t_k)) \right\|_\alpha \\ & \geq C_n \left(\|a_1 X(t_1)\|_\alpha + \sum_{k=1}^{n-1} \|a_k (X(t_{k+1}) - X(t_k))\|_\alpha \right). \end{aligned}$$

Theorem 3.3. *For any $\varepsilon > 0$ the rhmsp X is LND on $[\varepsilon, T]$.*

Proof. The main difficulty is to prove property (L3) of LND, as property (L1) is obvious and property (L2) follows from (2.3).

We proceed in two steps.

Step I. We prove LND for a modification of rhmsp X defined by

$$(3.4) \quad Y(t) = \int_{\mathbb{R}} (1 - e^{-itx})(ix)^{-H(t)-1/\alpha} M(dx),$$

where

$$(ix)^{-K} = |x|^{-K} e^{i\pi K \operatorname{sign} x/2}.$$

The Fourier transform of the function $f_Y(t, x) = (1 - e^{-itx})(ix)^{-H(t)-1/\alpha}$ (w.r.t. the second variable) on $L^\alpha(\mathbb{R})$ is

$$(3.5) \quad \widehat{f}_Y(t, x) = \frac{1}{\Gamma(H(t) + 1/\alpha)} \left((t-x)_+^{H(t)-1/\beta} - (-x)_+^{H(t)-1/\beta} \right),$$

where $\beta = \alpha/(\alpha - 1)$ is the exponent adjoint to α , see Lemma A.2.

In order to check property (L3) for Y , we have to find a good lower bound to

$$\left\| Y(t_n) - \sum_{k=1}^{n-1} u_k Y(t_k) \right\|_\alpha = \left\| f_Y(t_n, \cdot) - \sum_{k=1}^{n-1} u_k f_Y(t_k, \cdot) \right\|_{L^\alpha(\mathbb{R})}.$$

It is fortunately given by the Hausdorff-Young inequality:

$$\begin{aligned} \left\| f_Y(t_n, \cdot) - \sum_{k=1}^{n-1} u_k f_Y(t_k, \cdot) \right\|_{L^\alpha(\mathbb{R})} &\geq C \left\| \widehat{f}_Y(t_n, \cdot) - \sum_{k=1}^{n-1} u_k \widehat{f}_Y(t_k, \cdot) \right\|_{L^\beta(\mathbb{R})} \\ &\geq C \left\| \widehat{f}_Y(t_n, \cdot) \right\|_{L^\beta([t_{n-1}, t_n])} = C \left(\frac{1}{\Gamma(H(t) + 1/\alpha)^\beta} \int_{t_{n-1}}^{t_n} (t_n - x)^{\beta H(t) - 1} \right)^{1/\beta} \\ &\geq C (t_n - t_{n-1})^{H(t)} \geq C \|X(t_n) - X(t_{n-1})\|_\alpha \end{aligned}$$

for t_n and t_{n-1} close enough. (We have used the fact that $\widehat{f}_Y(t_k, x)$ vanishes on $[t_n, t_{n-1}]$ for $k < n$ in the middle, and inequality (2.3) in the last step.) But it is straightforward to check (see a much stronger statement below in the Step 2) that

$$\begin{aligned} \|X(t_n) - X(t_{n-1})\|_\alpha &\geq \|Y(t_n) - Y(t_{n-1})\|_\alpha - C |H(t_n) - H(t_{n-1})| \\ &\geq \|Y(t_n) - Y(t_{n-1})\|_\alpha - C |t_n - t_{n-1}|^\gamma \geq C \|Y(t_n) - Y(t_{n-1})\|_\alpha, \end{aligned}$$

which gives the desired LND property.

Step 2. Here we show how the property of locally asymptotically independent increments for Y implies that for X . Denote $f_X(t, x) = (e^{it_{k+1}x} - 1) |x|^{-H(t)-1/\alpha}$ and write for $0 < t_1 < t_2 < \dots < t_n < T$ and $a_1, \dots, a_n \in \mathbb{R}$ (we put $t_0 = 0$ for the sake of simplicity)

$$\begin{aligned} \left\| \sum_{k=1}^{n-1} a_k (X(t_{k+1}) - X(t_k)) \right\|_\alpha &= \left\| \sum_{k=0}^{n-1} a_k (f_X(t_{k+1}, \cdot) - f_X(t_k, \cdot)) \right\|_{L^\alpha(\mathbb{R})} \\ &= \left\| e^{i\pi(H(t_1)+1/\alpha)\text{sign } x/2} \sum_{k=0}^{n-1} a_k (f_X(t_{k+1}, \cdot) - f_X(t_k, \cdot)) \right\|_{L^\alpha(\mathbb{R})} \\ &\geq \left\| \sum_{k=0}^{n-1} a_k (f_Y(t_{k+1}, \cdot) - f_Y(t_k, \cdot)) \right\|_{L^\alpha(\mathbb{R})} - \sum_{k=0}^{n-1} \|a_k (\Delta(t_{k+1}, \cdot) - \Delta(t_k, \cdot))\|_{L^\alpha(\mathbb{R})} \\ &= \left\| \sum_{k=0}^{n-1} a_k (Y(t_{k+1}) - Y(t_k)) \right\|_\alpha - \sum_{k=0}^{n-1} \|a_k (\Delta(t_{k+1}, \cdot) - \Delta(t_k, \cdot))\|_{L^\alpha(\mathbb{R})} \end{aligned}$$

where

$$\begin{aligned}\Delta(t, x) &= e^{i\pi(H(t_1)+1/\alpha)\text{sign } x/2} f_X(t, x) - f_Y(t, x) \\ &= e^{i\pi\text{sign } x/(2\alpha)} (e^{i\pi H(t_1)\text{sign } x/2} - e^{i\pi H(t)\text{sign } x/2}) f_X(t, x).\end{aligned}$$

Write

$$\begin{aligned}& \|\Delta(t_{k+1}, \cdot) - \Delta(t_k, \cdot)\|_{L^\alpha(\mathbb{R})} \\ & \leq \|(e^{i\pi H(t_{k+1})\text{sign } x/2} - e^{i\pi H(t_k)\text{sign } x/2}) f_X(t_{k+1}, \cdot)\|_{L^\alpha(\mathbb{R})} \\ & + \|(e^{i\pi H(t_k)\text{sign } x/2} - e^{i\pi H(t_1)\text{sign } x/2}) (f_X(t_{k+1}, \cdot) - f_X(t_k, \cdot))\|_{L^\alpha(\mathbb{R})} \\ & \leq C |H(t_{k+1}) - H(t_k)| \|f_X(t_{k+1}, \cdot)\|_{L^\alpha(\mathbb{R})} \\ & + C |H(t_{k+1}) - H(t_1)| \|f_X(t_{k+1}, \cdot) - f_X(t_k, \cdot)\|_{L^\alpha(\mathbb{R})} \\ & \leq C |t_{k+1} - t_k|^\gamma \|X(t_{k+1})\|_\alpha + C |t_{k+1} - t_1|^\gamma \|X(t_k) - X(t_k + 1)\|_\alpha \\ & = o(\|X(t_{k+1}) - X(t_k)\|_\alpha), \quad |t_n - t_1| \rightarrow \infty.\end{aligned}$$

Thanks to LND property of Y we can write

$$\left\| \sum_{k=0}^{n-1} a_k (Y(t_{k+1}) - Y(t_k)) \right\|_\alpha \geq \sum_{k=0}^{n-1} \|a_k (Y(t_{k+1}) - Y(t_k))\|_\alpha,$$

and similarly to the above we write

$$\begin{aligned}\|Y(t_{k+1}) - Y(t_k)\|_\alpha &\geq \|X(t_{k+1}) - X(t_k)\|_\alpha - \|\Delta(t_{k+1}, \cdot) - \Delta(t_k, \cdot)\|_{L^\alpha(\mathbb{R})} \\ &\geq \|X(t_{k+1}) - X(t_k)\|_\alpha - o(\|X(t_{k+1}) - X(t_k)\|_\alpha),\end{aligned}$$

so finally

$$\begin{aligned} & \left\| \sum_{k=1}^{n-1} a_k (X(t_{k+1}) - X(t_k)) \right\|_{\alpha} \\ & \geq \sum_{k=1}^{n-1} |a_k| \left(\|X(t_{k+1}) - X(t_k)\|_{\alpha} - o(\|X(t_{k+1}) - X(t_k)\|_{\alpha}) \right) \\ & \geq C_n \sum_{k=1}^{n-1} |a_k| \|X(t_{k+1}) - X(t_k)\|_{\alpha} \end{aligned}$$

for $|t_n - t_1|$ small enough. \square

Thanks to [7, Theorem 4.1] and estimates for the norms of increments of rmhsp X we have the following result.

Theorem 3.4. *The local time $L(t, x)$ of the rmhsp X is jointly continuous in (t, x) for $t > 0$, moreover, for any $\kappa < (1/\check{H} - 1)/2$ it is κ -Hölder continuous in x .*

APPENDIX A. FOURIER TRANSFORM

In this appendix we compute the Fourier transform which is used by many authors, however, we were not able to find a rigorous derivation. Below we define the Fourier transform

$$\widehat{f}(u) = \int_{\mathbb{R}} e^{iux} f(x) dx,$$

and use the notation $x_+ = x \vee 0$.

Lemma A.1. *For $h \in (1, 2)$, $t > 0$ the Fourier transform of*

$$f_{h,t}(x) = (1 - e^{-itx})(-ix)^{-h} = (1 - e^{-itx}) |x|^{-h} e^{i\pi h \operatorname{sign} x/2}$$

is

$$\widehat{f_{h,t}}(u) = \frac{2\pi}{\Gamma(h)} \left((t - u)_+^{h-1} - (-u)_+^{h-1} \right).$$

Proof. First note that $z^{-h} = \frac{1}{\Gamma(h)} \int_0^\infty e^{-vz} v^{h-1} dv$ is analytic for $\operatorname{Re} z > 0$. So it follows from the operational calculus that for any $a > 0$

$$\frac{v_+^{h-1}}{\Gamma(h)} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{vz} z^{-h} dz = -\frac{1}{2\pi} \int_{-\infty-ai}^{\infty-ai} e^{-ivy} (-iy)^{-h} dy,$$

where we have changed the variable $z \rightarrow iy$. Plugging $v = -u$ and $v = t - u$ to this identity, we get

$$(A.1) \quad \frac{1}{\Gamma(h)} \left((t-u)_+^{h-1} - (-u)_+^{h-1} \right) = \frac{1}{2\pi} \int_{-\infty-ai}^{\infty-ai} e^{iuy} (1 - e^{-ity}) (-iy)^{-h} dy,$$

Now let in this integral $y = x - ai$, $x \in \mathbb{R}$ and estimate for $a \in (0, 1)$ the integrand as

$$(A.2) \quad \begin{aligned} |e^{iuy} (1 - e^{-ity}) (-iy)^{-h}| &= e^{au} |1 - e^{-at-itx}| |y|^{-h} \leq C(u) (t|x - ai| \wedge 1) |y|^{-h} \\ &= C(u) (t|y|^{-h+1} \wedge |y|^{-h}) \leq C(u) (t|x|^{-h+1} \wedge |x|^{-h}), \end{aligned}$$

which is integrable due to the assumption $h \in (1, 2)$. So letting $a \rightarrow 0+$ in (A.1) yields the desired result by the dominated convergence theorem. \square

By the Hausdorff-Young inequality (see [5, Theorem 5.7]), for $\alpha \in [1, 2]$ the Fourier transform from $L^1(\mathbb{R}) \cap L^\alpha(\mathbb{R})$ can be extended to a bounded linear operator $\mathcal{F}_\alpha: L^\alpha(\mathbb{R}) \rightarrow L^\beta(\mathbb{R})$, where $\beta = \alpha/(\alpha - 1)$ is the exponent adjoint to α . We will call this map a *Fourier transform on $L^\alpha(\mathbb{R})$* , and we emphasize once more its boundedness due to the Hausdorff-Young inequality:

$$(A.3) \quad \|\mathcal{F}_\alpha f\|_{L^\beta(\mathbb{R})} \leq C_\alpha \|f\|_{L^\alpha(\mathbb{R})}.$$

The following lemma is an $L^\alpha(\mathbb{R})$ analogue of Lemma A.1.

Lemma A.2. For $\alpha \in (1, 2)$, $h \in (1/\alpha, 1 + 1/\alpha)$ and $t > 0$ the Fourier transform on L^α of

$$f_{h,t}(x) = (1 - e^{-itx})(-ix)^{-h} = (1 - e^{-itx}) |x|^{-h} e^{i\pi h \operatorname{sign} x/2}$$

is

$$\mathcal{F}_\alpha f_{h,t}(u) = \frac{2\pi}{\Gamma(h)} \left((t-u)_+^{h-1} - (-u)_+^{h-1} \right).$$

Proof. Repeat the proof of the previous lemma to inequality (A.2) and raise it to the power α :

$$|e^{iuy}(1 - e^{-ity})(-iy)^{-h}|^\alpha \leq C(u)^\alpha (t|x|^{\alpha(1-h)} \wedge |x|^{-\alpha h}),$$

which is integrable for $h \in (1/\alpha, 1 + 1/\alpha)$. So the integrands in (A.1) converge as $a \rightarrow 0+$ in $L^\alpha(\mathbb{R})$ to $f_{h,t}(x)$ by the dominated convergence theorem, hence by continuity of \mathcal{F}_α on $L^\alpha(\mathbb{R})$ we get the statement of the lemma. \square

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