How unique is Lovász's theta function?

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***Abstract*—The famous Lovász's ϑ function is computable in polynomial time for every graph, as a semi-definite program (Grötschel, Lovász and Schrijver, 1981 [5]). The chromatic number and the clique number of every perfect graph G are computable in polynomial time, since they are equal to $f_\vartheta(G) = \vartheta(\overline{G})$. Despite numerous efforts since the last three decades, recently stimulated by the Strong Perfect Graph Theorem (Chudnovsky, Robertson, Seymour and Thomas, 2006 [2]), no combinatorial proof of this result is known.**

In this work, we try to understand why the "key properties" of Lovász's ϑ function make it so "unique". We introduce an infinite set of convex functions, which includes the clique number ω and f_ϑ. This set includes a sequence of linear programs which are monotone increasing and converging to f_ϑ. We provide some evidences that f_ϑ is the unique function in this setting allowing to compute the chromatic number of perfect graphs in polynomial time.

***Keywords*—semi-definite programming;theta function.**

I. Introduction

Berge introduced perfect graphs [1] in the early sixties, motivated from Shannon's problem of finding the zero-error capacity of a discrete memoryless channel [13]. A graph G is a *perfect* graph if and only if $\omega(G') = \chi(G')$ holds for all induced subgraphs $G' \subseteq G$ (where the order of a largest clique of G is its *clique number* $\omega(G)$, and the least number of colors required to assign different colors to adjacent nodes is its *chromatic number* $\chi(G)$).

Berge conjectured that a graph G is perfect if and only if its complement $\overline{G}$ is perfect (the complement $\overline{G}$ has the same nodes as G, but two nodes are adjacent in $\overline{G}$ if and only if they are non-adjacent in G). This was proved by Lovász [9], who gave two short and elegant proofs.

A further conjecture of Berge was proved by Chudnovsky et al. [2] who characterized perfect graphs as precisely the graphs without chordless cycles C_{2k+1} with $k \geq 2$, termed *odd holes*, or their complements, the *odd antiholes* $\overline{C}_{2k+1}$.

Perfect graphs have been extensively studied and turned out to be an interesting and important class of graphs with a rich structure. Most notably, the two in general hard to compute graph parameters $\omega(G)$ and $\chi(G)$ can be determined in polynomial time if G is perfect [4].

The latter result relies on the following polyhedral characterization of perfect graphs. The *stable set polytope* $\mathrm{STAB}(G)$ is defined as the convex hull of the incidence vectors of all stable sets of G.

A canonical relaxation of $\mathrm{STAB}(G)$ is the *clique constraint stable set polytope*

$$\mathrm{QSTAB}(G) = \{\mathbf{x} \in \mathbb{R}_+^{|V|} : \sum_{i \in Q} x_i \leq 1,\ Q \subseteq G \text{ clique}\}.$$

We have $\mathrm{STAB}(G) \subseteq \mathrm{QSTAB}(G)$ in general and equality for perfect graphs [3] only. However, solving the stable set problem for a perfect graph G by maximizing a linear objective function over $\mathrm{QSTAB}(G)$ does not work directly [4], but only via a detour involving a geometric representation of graphs [10] and the resulting *theta-body* $\mathrm{TH}(G)$ introduced by Lovász et al. [6].

An orthonormal representation of a graph $G = (V, E)$ is a sequence $(\mathbf{u_i} : i \in V)$ of $|V|$ unit-length vectors $\mathbf{u_i} \in \mathbb{R}^N$, where N is some positive integer, such that $\mathbf{u_i}^T \mathbf{u_j} = 0$ for all $ij \notin E$. For any orthonormal representation of G and any additional unit-length vector $\mathbf{c} \in \mathbb{R}^N$, the corresponding orthonormal representation constraint is $\sum_{i \in V} (\mathbf{c}^T \mathbf{u_i})^2 x_i \leq 1$. $\mathrm{TH}(G)$ denotes the convex set of all vectors $\mathbf{x} \in \mathbb{R}_+^{|V|}$ satisfying all orthonormal representation constraints for G. For any graph G, we have

$$\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{QSTAB}(G).$$

The key property of $\mathrm{TH}(G)$ is that, for any graph G, the optimization problem

$$\vartheta(G) = \max\{\mathbb{1}^T \mathbf{x} : \mathbf{x} \in \mathrm{TH}(G)\}$$

can be solved in polynomial time [4]. This deep result relies on the fact that $\vartheta(G)$ can be characterized in many equivalent ways, e.g., as the

- optimum value of a semidefinite program,
- largest eigenvalue of a certain set of symmetric matrices,
- maximum value of a function involving orthonormal representation constraints,

see [5] for further details.

For perfect graphs, $\mathrm{STAB}(G)$ and $\mathrm{TH}(G)$ coincide which allows to compute the clique number by $\omega(G) = \vartheta(\overline{G})$ and the chromatic number by $\chi(G) = \omega(G)$ for perfect graphs G in polynomial time.

Denote by f_ϑ the function defined by $f_\vartheta(G) = \vartheta(\overline{G})$ for every graph G. We shall call f_ϑ "the theta function", though it is actually the usual theta function applied to the complement of the input graph. Then f_ϑ satisfies the three assertions:

P_1) f_ϑ is computable in polynomial time for any graph G;

P_2) f_ϑ is monotonic with respect to homomorphism: if G is homomorphic to H then $f_\vartheta(G) \leq f_\vartheta(H)$;

P_3) f_ϑ is strictly monotonic on cliques: for every integer $i \geq 1$, $f_\vartheta(K_i) < f_\vartheta(K_{i+1})$ and the difference has a polynomial space encoding.

Graph homorphisms is a crucial concept in this paper as it has a prominent role with respect to clique and chromatic number. Recall that a graph G is said to be homomorphic to H if there is a mapping from the nodes of G to the nodes of H, preserving adjacency. Then the clique number (resp. the chromatic number) of a graph G is equal to the biggest (resp. smallest) integer k such that K_k (resp. G) is homomorphic to G (resp. K_k).

The proof that the chromatic number of perfect graphs is computable in polynomial time relies on the three main properties introduced above. Indeed, take any real function g satisfying P_1, P_2 and P_3. Let G be a perfect graph with clique number ω and chromatic number χ: G is homomorphic to $K_\chi = K_\omega$ and K_ω is homomorphic to G. From property P_2, it follows that $g(G) = g(K_\omega)$. Let n be the number of nodes of G. From property P_1, we may compute $g(G)$, $g(K_1), \ldots, g(K_n)$ in polynomial time. From property P_3, there is a unique index, say k, such that $g(G) = g(K_k)$ and we may determine it in polynomial time. Thus $\omega = k$ is computable in polynomial time.

Notice that is is easy to get functions satisfying two of the properties P_1, P_2 and P_3. Indeed, any constant function satisfies P_1 and P_2 (but not P_3), the function returning the number of nodes of a graph satisfies P_1 and P_3 (but not P_2), the clique number satisfies P_2 and P_3 (but not P_1).

However, there does not seem to be many functions satisfying P_1, P_2 and P_3, though f_ϑ is not the unique one, as some of its variants, such as the vectorial chromatic number [7] and the strong vectorial chromatic number [12], for instance, also satisfy these three properties.

The purpose of this work (which continues the considerations presented in [11]) is to investigate "how unique" the theta function is, by considering a more general setting, based on some convex supersets of SDP matrices.

The paper is organized as follows:

- In the second section, we define for every set of reals X including $\{0,1\}$, a real function f_X. We give the basic properties of every function f_X, and establish that $f_{\{0,1\}} = \omega$ and $f_{\mathbb{R}} = f_\vartheta$.
- In the third section, we study functions f_X, such that X is infinite.
- In the fourth section, we focus on the case of X being finite and exhibit a sequence of linear programs monotone increasing and converging to f_ϑ.

The results of sections 2 and 3 are the content of the third section of [11].

II. Notations and basic properties

Let $\{0,1\} \subseteq X \subseteq \mathbb{R}$. For every graph $G = (V, E)$ with at least one edge, denote by n its number of nodes and by $f_X(G)$ the value $1 - \frac{1}{s}$ where s is the optimum of the following program:

$$\begin{aligned} \min \quad & s \\ s.t. \quad & \exists M \in \mathcal{M}_X \\ & M \text{ is symmetric} \\ & M_{ii} = 1, \forall i \in V \\ & M_{ij} = s, \forall ij \in E \end{aligned}$$

where $\mathcal{M}_X$ is defined as the following set of matrices:

$$\mathcal{M}_X = \{M \in \mathbb{R}^{V \times V}, \text{ s.t. }, \mathbf{u}^T M \mathbf{u} \geq 0, \forall \mathbf{u} \in X^V\}$$

If G does not have any edge, we let $f_X(G) = 1$. If M is a matrix of $\mathcal{M}_X$, we say that M is feasible. A feasible matrix which yields the value $f_X(G)$ is called optimal.

Here are some basic observations, for every graph G:

- $f_{\mathbb{R}}(G) = \vartheta(\overline{G})$ (Lovász's theta function [10]), and thus $f_{\mathbb{R}}$ is computable in polynomial time with given accuracy;
- if $X \subseteq X'$ then $\mathcal{M}_{X'} \subseteq \mathcal{M}_X$ and thus $f_X(G) \leq f_{X'}(G)$.
- for every $\lambda \in \mathbb{R}^+$, $f_{\lambda X}(G) = f_X(G)$ as $\mathcal{M}_{\lambda X} = \mathcal{M}_X$.

Table 1 presents some numerical values $f_X(G)$ for some small graphs G and the sets X in $\{\{0,1\}, \{-1,0,1\}, \{-2,-1,0,1,2\}, \mathbb{R}\}$.

	{0,1}	{-1,0,1}	{-2,-1,0,1,2}	$\mathbb{R}$
$C_9 = K_{9/4}$	2	2.061		2.064
$C_7 = K_{7/3}$	2	2.103	2.1096	2.1099
$C_5 = K_{5/2}$	2	2.200	2.231	2.236
$K_{8/3}$	2	2.333		2.343
$K_{11/4}$	2	2.3996		2.408
$K_{10/3}$	3	3.125		3.167
$\overline{C_7} = K7/2$	3	3.222	3.294	3.318
$K_{11/3}$	3	3.400		3.452
$\overline{C_9} = K_{9/2}$	4	4.231		4.360
Petersen	2	2.5	2.5	2.5
$\overline{\text{Petersen}}$	4	4	4	4
C_5 + 1 multiplied node	2	2.210526		2.236

TABLE I

Some numerical result for f_X,
$X \in \{\{0,1\}, \{-1,0,1\}, \{-2,-1,0,1,2\}, \mathbb{R}\}$

Lemma 1. *$\mathcal{M}_X$ is a convex cone and a superset of the set of semi-definite positive matrices of size $n \times n$.*

We first compute the value f_X for cliques:

Lemma 2. *$f_X(K_i) = i$ for every i.*

It follows from Lemma 2 that every function f_X satisfies property P_3. We now establish in the following lemma that every function f_X partially satisfies property P_2.

Lemma 3. *If H is a subgraph of G then $f_X(H) \le f_X(G)$.*

This implies the so-called sandwich-property:

Corollary 4. $\omega(G) \le f_X(G) \le \vartheta(\overline{G}) \le \chi(G)$

Proof: Due to Lemma 2 and Lemma 3, we have $\omega(G) \le f_X(G)$. Furthermore, $f_X(G) \le \vartheta(\overline{G})$ by definition of $\mathcal{M}_X$. ■

III. X INFINITE: THE ROLES OF THE CLIQUE NUMBER AND THE THETA FUNCTION

Multiplying a node v of a graph G means to replace v by a stable set S such that all nodes in S have the same neighbors in G as the original node v. Thus, multiplying a node of a graph G gives a homomorphically equivalent graph H. Hence if X is a set of reals such that f_X satisfies the monotonic property P_2, then $f_X(G) = f_X(H)$. Thus $f_{\{-1,0,1\}}$ does not satisfy P_2 as multiplying a node of a C_5 yields a different value (see Table II). Therefore, additional constraints are needed for sets X in order to ensure that property P_2 is fulfilled. We next show that being closed with respect to addition is such a sufficient condition:

Lemma 5. *Assume that X is closed with respect to addition. If G is homomorphic to H then $f_X(G) \le f_X(H)$ (monotonic property).*

If X contains 0 and positive reals only then f_X is the clique number:

Lemma 6. *For every graph G, $f_{\mathbb{R}^+}(G) = \omega(G)$.*

As an obvious consequence of Lemma 6, we get:

Corollary 7. *$f_{\{0,1\}}$ is NP-hard to compute.*

Due to Lemma 6, the base set X has to have one negative element, say -1, in order to get a function f_X which is different from the clique number. If we apply the requirement of Lemma 5 to get a function satisfying the monotonic property then X contains all integers. We next establish that this implies that f_X has to be the theta function:

Lemma 8. *If $\mathbb{Z} \subseteq X$ or $[-1,1] \subseteq X$ then $f_X(G) = \vartheta(\overline{G})$ for every graph G.*

These results show that the clique number and f_ϑ are two prominent functions when X is infinite: we do not know whether there is a function f_X, with X infinite, distinct of the clique number and f_ϑ.

IV. X FINITE: A SEQUENCE OF LINEAR PROGRAMS CONVERGING TO THE THETA FUNCTION

For every positive integer k, let X_k denote the set of integers $\{-k, -(k-1), \ldots, -1, 0, 1, \ldots, k-1, k\}$, and f_k be the function f_{X_k}. Notice that for every graph G with n nodes, the value $f_k(G)$ is the output of a linear program with exponentially many constraints (approximatively $(2k+1)^n$ constraints). Furthermore, $f_k(G)$ is a rational for every k and graph G (and thus distinct of f_ϑ). The sequence $f_k(G)_{k\ge 1}$ is an increasing sequence, as $X_{k-1} \subsetneq X_k$ (for every $k \ge 2$).

Hence we have, for every graph G,

$$\omega(G) \le f_1(G) \le f_2(G) \le \ldots \le f_k(G) \le f_\vartheta(G).$$

We establish that

$$\lim_k f_k(G) = f_\vartheta(G)$$

holds for every graph G as a consequence of the following lemma and Lemma 8:

Lemma 9. *Let $Y_1 \subset Y_2 \subset \cdots$ be a monotonous chain of subsets containing $\{0,1\}$ and set $Y = \cup_k Y_k$. For every graph G we have*

$$f_Y(G) = \lim_k f_{Y_k}(G).$$

Due to Lemma 9, the sequence f_k is converging to f_ϑ.

Notice that for graphs G such that $f_1(G) = f_\vartheta(G)$ (e.g. perfect graphs) then $f_1(G) = f_2(G) = \ldots = f_k(G)$ holds for every k. We do not know whether the sequence is strictly increasing for graphs G such that $f_1(G) \ne f_\vartheta(G)$, but suspect that it is. In particular, computer experiments suggest that $f_k(C_5) < f_{k+1}(C_5)$ for every positive integer k.

We believe that none of the functions f_k is monotonic with respect to homorphisms but were not yet able to prove it.

In Lemma 5, the set X is assumed to be closed with respect to addition, a property which is satisfied by none of the sets X_k. We used this assumption in the proof of Lemma 5 by constructing an optimal matrix for a graph G with a duplicated node: the construction consists of duplicating one row and one column.

The next lemma shows that if f_k is monotonic with respect to homorphism, then every optimal matrix for a graph is such a matrix with "one duplicated row and one duplicated column". This suggests that X_k has "somehow" to be closed with respect to addition, a contradiction.

Lemma 10. *Let H be a circulant graph (that is a Cayley on a cyclic group) and let G be obtained from H by duplicating a node. If $f_k(G) = f_k(H)$ then every optimal matrix of M is obtained from an optimal matrix of H by duplicating one row and column.*

V. CONCLUDING REMARKS

Our study seems to indicate that the clique number function and the theta function are the only functions in our setting that satisfy the monotonic requirement with respect to homomorphism (property P_2). Hence in this sense, the theta function is really unique, since it is also computable in polynomial time (property P_1).

As of the sandwich property, we point out that it holds even if the monotonic property is not satisfied (Corollary 4): there are many different functions f_X in between the clique and the chromatic number, all of them being a lower bound for the theta function.

For further works, it is worth to notice that the numerical values presented in Table II suggest that the function $f_{\{-1,0,1\}}$ gives already good lower bounds for the theta function.

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