Asymptotic behavior of mixed power variations and statistical estimation in mixed models

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Statistical estimation by power variations in mixed models

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Abstract: In this paper we study asymptotic behaviour of power variations of a linear combination of independent Wiener process and fractional Brownian motion. This results are applied to construct consistence parameter estimators and approximate confidence intervals in mixed models.

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1. Introduction

Fractional Brownian motion (fBm) is frequently used to model short- and long-range dependence. By definition, fBm with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $\left\{B_t^H, t \geq 0\right\}$ with the covariance function

$$\mathsf{E}\left[\, B_t^H B_s^H \,\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

For $H > 1/2$ fBm has a property of long-range dependence, for $H < 1/2$ it has short-range dependence and, in fact, is counterpersistent, i.e. its increments are negatively correlated. For $H = 1/2$ fBm is a standard Wiener process.

Two important properties of fBm are stationary increments and self-similarity. However, this properties restrict applications of fBm, so recently so-called multifractional processes gained huge attention. Multifractionality can consist both in dependence of memory depth and regularity of process on the time instance and on the time scale. In this paper, we are dealing with the latter kind of multifractionality, where the properties of the process depend on the size of time

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interval, on which the process is considered. In other words, we are considering processes, which are not inherently self-similar. A simplest approach is to consider a linear combination of fBm's with different Hurst parameters.

Here we will concentrate on the case where we have only two fBm's and one of them has Hurst parameter equal to $1/2$, simply put, it is a Wiener process. So we consider a process

$$M_t^H = aB_t^H + bW_t, t \geq 0 \tag{1}$$

where a and b are some non-zero coefficients. Such mixed models and their applications where considered in many papers, see [1, 6, 11, 16, 17].

The main aim of this paper is statistical identification of model (1), i.e. the statistical estimation of the model parameters. The principal attention will be given to the estimation of H, though we will give some comments on estimation of a and b.

In a "pure" fBm case, there exist several methods to estimate the Hurst parameter, an extensive overview of which is given in [7]. The most popular methods are based on quadratic and, more generally, higher power variations of the process. A huge literature is devoted to such questions, we will cite only few: asymptotic behaviour of power variations and, more generally, of non-linear transformations of stationary Gaussian sequences is studied in [4, 10, 13, 20], and stochastic estimation for fBm and multifractional processes with the help of power variations, in [3, 8, 9, 12, 14, 21]. Weighted power variations serving similar purposes for stochastic differential equations driven by fBm, were studied in [18, 19].

In this paper we will use power variations of process (1) for estimation of parameters. Related questions for mixed models were already considered in [5, 11, 15, 23], but we use different techniques here. We remark that, in contrast to the pure fractional case, there is no self-similarity property in mixed model (1), so we cannot directly apply the results of [4, 10, 13, 20] on asymptotic behaviour of sums of transformed stationary Gaussian sequences. For this reason we need to study asymptotic behaviour as $n \to \infty$ of "mixed" power variations of the form

$$\sum_{k=0}^{n-1} \left(W_{(k+1)/n} - W_{k/n}\right)^p \left(B^H_{(k+1)/n} - B^H_{k/n}\right)^r, \tag{2}$$

involving increments of independent fBm B^H and Wiener process W, where $p \geq 0, r \geq 0$ are fixed integer parameters. We obtain that the behaviour of such power variations depends on parity of p and r and on the value of the Hurst parameter H.

The paper is organized as follows. Section 2 contains results on the asymptotic behaviour of mixed power variations. These results are used in Section 3 to construct strongly consistent estimators of parameters H, a, b in model (1) and asymptotic confidence intervals for H. Finally, in Section 4 we present simulation results to illustrate quality of the estimators provided.

2. Asymptotic behaviour of mixed power variations

Let $W = \{W_t, t \geq 0\}$ be a standard Wiener process and $B^H = \{B^H_t, t \geq 0\}$ be an independent of W fractional Brownian motion with Hurst parameter $H \in (0,1)$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$.

For a function $X\colon [0,1] \to \mathbb{R}$ and integer $n \geq 1$, $k = 0, 1, \ldots, n-1$ we denote $\Delta^n_k X = X_{(k+1)/n} - X_{k/n}$. In this section we will study the asymptotic behavior as $n \to \infty$ of the following mixed power variations

$$\sum_{k=0}^{n-1} (\Delta^n_k W)^p \left(\Delta^n_k B^H\right)^r,$$

where $p \geq 0$, $r \geq 0$ are fixed integer numbers. Thanks to self-similarity of B^H and W, the sequence $\left\{(n^H \Delta^n_k B^H, n^{1/2}\Delta^n_k W), 0 \leq k \leq n-1\right\}$ is equivalent in law to $\{(\xi_k, \zeta_k), 0 \leq k \leq n-1\}$, where $\{\xi_k, k \geq 0\}$ is a sequence of independent standard Gaussian random variables, $\{\zeta_k, k \geq 1\}$ is independent of $\{\xi_k, k \geq 1\}$ stationary sequence of standard Gaussian variables with covariance

$$\rho_H(m) = \mathsf{E}\left[\,\zeta_0\zeta_m\,\right] = \mathsf{E}\left[\,B^H_1(B^H_{m+1} - B^H_m)\,\right] = \frac{1}{2}\left(|m+1|^{2H} + |m-1|^{2H}\right) - |m|^{2H}.$$

Therefore, by ergodic theorem,

$$n^{rH+p/2-1} \sum_{k=0}^{n-1} (\Delta^n_k W)^p \left(\Delta^n_k B^H\right)^r \to \mu_p \mu_r, \ n \to \infty, \tag{3}$$

a.s., where for integer $m \geq 0$

$$\mu_m = \mathsf{E}\left[\,N(0,1)^m\,\right] = (m-1)!!\mathbf{1}_{m \text{ is even}}$$

is the mth moment of the standard Gaussian law. So it is natural to study the following centered sums

$$S^{H,p,r}_n = \sum_{k=0}^{n-1} \left(n^{rH+p/2} (\Delta^n_k W)^p \left(\Delta^n_k B^H\right)^r - \mu_p \mu_r\right).$$

The following theorem summarizes limit behaviour of $S^{H,p,r}_n$. We remark that some (but not all) of results can be obtained from the limit theorems for stationary Gaussian sequences of vectors, see e.g. [2]. However, we believe that our approach (using one-dimensional limit theorems) is more accessible and leads quicker to the desired results.

Theorem 2.1. *If p and r are even, then*

- *for $H \in (0, 3/4)$*

$$n^{-1/2} S^{H,p,r}_n \Rightarrow N(0, \sigma^2_{H,r}\mu^2_p + \sigma^2_{p,r}), \ n \to \infty, \tag{4}$$

where

$$\sigma^2_{H,r} = \sum_{l=1}^{r/2} \frac{(l!)^2}{(2l)!((r-2l)!!)^2} \sum_{m=-\infty}^{\infty} \rho_H(m)^{2l}, \quad \sigma^2_{p,r} = \mu_{2r}\left(\mu_{2p} - \mu_p^2\right);$$

- *for* $H = 3/4$

$$\frac{S_n^{3/4,p,r}}{\sqrt{n\log n}} \Rightarrow N(0, \sigma^2_{3/4,r}\mu_p^2 + \sigma^2_{r,p}),\ n \to \infty, \tag{5}$$

where $\sigma_{3/4,r} = 3r(r-1)/4$*;*

- *for* $H \in (3/4, 1)$

$$n^{1-2H} S_n^{H,p,r} \Rightarrow \zeta_{H,p,r},\ n \to \infty, \tag{6}$$

where $\zeta_{H,p,r}$ *is a special "Rosenblatt" random variable.*

If p *is odd, then for any* $H \in (0,1)$

$$n^{-1/2} S_n^{H,p,r} = n^{rH-1/2} R_n k_n \Rightarrow N(0, \sigma^2_{p,r}) = N(0, \mu_{2p}\mu_{2r}). \tag{7}$$

If p *is even and* r *is odd, then*

- *for* $H \in (0, 1/2]$

$$n^{-1/2} S_n^{H,p,r} \Rightarrow N(0, \sigma^2_{p,r} + \mu_p^2\sigma^2_{H,r}),\ n \to \infty, \tag{8}$$

where

$$\sigma^2_{H,r} = \sum_{l=0}^{(r-1)/2} \frac{(r!)^2}{(2l+1)!((r-2l-1)!!)^2} \sum_{m=-\infty}^{\infty} \rho_H(m)^{2l+1};$$

- *for* $H \in (1/2, 1)$

$$n^{-H} S_n^{H,p,r} \Rightarrow N(0, \mu_p^2\mu^2_{r+1}),\ n \to \infty. \tag{9}$$

Proof. We study different cases in the same order as they appear in the formulation.

Assume first that p and r are even. The principal idea in this case is to rewrite mixed power variation as

$$S_n^{H,p,r} = S'_n + S''_n,$$

where

$$\begin{aligned} S'_n &= n^{rH} \sum_{k=0}^{n-1} \left(\Delta_k^n B^H\right)^r \left(n^{p/2}\left(\Delta_k^n W\right)^p - \mu_p\right), \\ S''_n &= \mu_p \sum_{k=0}^{n-1} \left(n^{rH}\left(\Delta_k^n B^H\right)^r - \mu_r\right), \end{aligned} \tag{10}$$

and to apply known results concerning asymptotic behaviour of S''_n, since it contains only fractional Brownian motion, and consider S'_n conditionally on the fractional Brownian motion. In what follows we realize this idea.

For $H \in (0, 3/4)$ write

$$n^{-1/2} S_n^{H,p,r} = n^{-1/2} S'_n + n^{-1/2} S''_n.$$

According to [4], for r even, $H \in (0, 3/4)$,

$$n^{-1/2} \sum_{k=0}^{n-1} \left(n^{rH} \left(\Delta_k^n B^H \right)^r - \mu_r \right) \Rightarrow N(0, \sigma_{H,r}^2), \tag{11}$$

as $n \to \infty$. Consequently, $n^{-1/2} S''_n \Rightarrow N(0, \sigma_{H,r}^2 \mu_p^2)$, $n \to \infty$. Further,

$$n^{-1/2} S'_n = n^{rH-1/2} R_n k_n,$$

where

$$\begin{aligned} R_n &= \frac{1}{k_n} \sum_{k=0}^{n-1} \lambda_{k,n} \left(n^{p/2} \left(\Delta_k^n W \right)^p - \mu_p \right) \\ k_n &= \left((\mu_{2p} - \mu_p^2) \sum_{k=0}^{n-1} \lambda_{k,n}^2 \right)^{1/2}, \ \lambda_{k,n} = \left(\Delta_k^n B^H \right)^r. \end{aligned} \tag{12}$$

Since B^H is uniformly continuous a.s., $\max_{1 \le k \le n} \lambda_{k,n} \to 0, n \to \infty$ a.s. Thus, taking into account independence of B^H and W and an evident fact that $\mathsf{var}\, R_n = 1$, we get by CLT that the conditional distribution of R_n given B^H converges to standard normal distribution as $n \to \infty$ a.s. Further, from the ergodic theorem

$$n^{2rH-1} \sum_{k=0}^{n-1} (\Delta_k^n B^H)^{2r} \to \mu_{2r}, \ n \to \infty,$$

a.s., hence

$$n^{rH-1/2} k_n \to \left(\mu_{2r} (\mu_{2p} - \mu_p^2) \right)^{1/2}, n \to \infty \text{ a.s.}$$

So by Slutsky's theorem, the conditional distribution of S'_n given B^H converges to $N(0, \sigma_{p,r}^2)$ a.s., that is, for any $t \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} e^{itx} \mathsf{P}(S'_n \in dx \mid B^H) \to e^{-t^2 \sigma_{p,r}^2/2} \tag{13}$$

a.s. as $n \to \infty$. Now write

$$\mathsf{E} \left[e^{it(S'_n + S''_n)} \right] = \mathsf{E} \left[\mathsf{E} \left[e^{it(S'_n + S''_n)} \middle| B^H \right] \right] = \mathsf{E} \left[\int_{\mathbb{R}} e^{itx} \mathsf{P} \left(S'_n \in dx \middle| B^H \right) e^{itS''_n} \right],$$

whence

$$\left| \mathsf{E} \left[e^{it(S'_n + S''_n)} \right] - e^{-t^2(\sigma_{H,r}^2 \mu_p^2 + \sigma_{p,r}^2)/2} \right| \le E_1 + E_2,$$

where

$$E_1 = \left| \mathsf{E} \left[\left(\int_{\mathbb{R}} e^{itx} \mathsf{P}\left(S'_n \in dx \middle| B^H\right) - e^{-t^2\sigma^2_{p,r}/2} \right) e^{itS''_n} \right] \right| \to 0,\ n \to \infty$$

by (13) and dominated convergence;

$$E_2 = e^{-t^2\sigma^2_{p,r}} \left| \mathsf{E}\left[e^{itS''_n} \right] - e^{-t^2\sigma^2_{H,r}\mu_p^2/2} \right| \to 0,\ n \to \infty$$

by (11). It follows that

$$n^{-1/2} S_n^{H,p,r} \Rightarrow N(0, \sigma^2_{H,r}\mu_p^2 + \sigma^2_{p,r}),\ n \to \infty,$$

as required in this case.

In the case where $H = 3/4$, we have by [4]

$$\frac{1}{\sqrt{n \log n}} \sum_{k=0}^{n-1} \left(n^{rH} \left(\Delta_k^n B^H\right)^r - \mu_r \right) \Rightarrow N(0, \sigma^2_{3/4,r}),$$

as $n \to \infty$, whence (5) can be deduced using the same reasoning as above.

For $H \in (3/4, 1)$, write

$$n^{1-2H} S_n^{H,p,r} = n^{1-2H} S'_n + n^{1-2H} S''_n = n^{1-2H+rH} R_n k_n + n^{1-2H} S''_n,$$

where R_n, k_n, S''_n are defined above. As before, $R_n \Rightarrow N(0,1)$ conditionally given B^H as $n \to \infty$ a.s. However, this time $n^{1-2H+rH} k_n \to 0$, $n \to \infty$ a.s., since $n^{rH-1/2} k_n$ has a finite limit and $n^{3/2-2H} \to 0$, $n \to \infty$. Therefore, $n^{1-2H} S'_n \to 0$, $n \to \infty$. Further, according to [10], see also [13, 20],

$$n^{1-2H} S''_n \Rightarrow \zeta_{H,p,r},\ n \to \infty,$$

where $\zeta_{H,p,r}$ is a "Rosenblatt" random variable. Thus, we get (6) using Slutsky's theorem. This finishes the case where p and r are even.

Now assume that p or r is odd. In this case $S_n^{H,p,r}$ has a form

$$S_n^{H,p,r} = \sum_{k=0}^{n-1} n^{rH+p/2} \left(\Delta_k^n W\right)^p \left(\Delta_k^n B^H\right)^r.$$

Write

$$n^{-1/2} S_n^{H,p,r} = n^{rH-1/2} R_n k_n + Z_n,$$

where R_n and k_n are defined above, $Z_n = n^{rH-1/2}\mu_p \sum_{k=0}^{n-1} \left(\Delta_k^n B^H\right)^r$. As before, given B^H, $n^{rH-1/2} R_n k_n \Rightarrow N(0, \sigma^2_{p,r})$, $n \to \infty$, a.s.

Now if p is odd, we have $Z_n^2 = 0$ irrespective of value of H, whence (7) immediately follows.

Further, assume that p is even and r odd. For $H \in (0, 1/2]$, we have by [4]

$$n^{rH-1/2} \sum_{k=0}^{n-1} \left(\Delta_k^n B^H\right)^r \Rightarrow N(0, \sigma^2_{H,r}),\ n \to \infty.$$

Therefore, $Z_n \Rightarrow N(0, \mu_p^2 \sigma_{H,r}^2)$, $n \to \infty$. Arguing as in deriving of (4), we get (8). For $H \in (1/2, 1)$, it follows from [10] that

$$n^{(r-1)H} \sum_{k=0}^{n-1} \left(\Delta_k^n B^H\right)^r \Rightarrow N(0, \mu_{r+1}^2),\ n \to \infty, \tag{14}$$

so $n^{1/2-H} Z_n \Rightarrow N(0, \mu_p^2 \mu_{r+1}^2)$, $n \to \infty$ But

$$n^{(r-1)H} R_n k_n = n^{1/2-H} n^{rH-1/2} R_n k_n \to 0,\ n \to \infty,$$

whence (9) follows. The proof is now complete. □

3. Statistical estimation in mixed model

Now we turn to the question of parametric estimation in the mixed model

$$M_t^H = aB_t^H + bW_t,\ t \in [0, T], \tag{15}$$

where a, b are non-zero numbers, which we assume to be positive, without loss of generality. Out primary goal is to construct a strongly consistent estimator for the Hurst parameter H, given a single observation of M^H.

It is well-known (see [6]) that for $H \in (3/4, 1)$ the measure induced by M^H in $C[0, T]$ is equivalent to that of bW. Therefore, the property of almost sure convergence in this case is independent of H. Consequently, no strongly consistent estimator for H based on a single observation of M^H exists.

In this section we denote $\Delta_k^n X = X_{Tk/n} - X_{T(k-1)/n}$ and

$$V_n^{H,p,r} = \sum_{k=0}^{n-1} (\Delta_k^n W)^p \left(\Delta_k^n B^H\right)^r.$$

Also for a real sequence $\{a_n, n \geq 1\}$, non-zero sequence $\{b_n, n \geq 1\}$, and a sequence $\{\xi_n, n \geq 1\}$ of random variables, we will abbreviate

$$\xi_n \approx a_n + b_n N(0, 1)$$

whenever the following two assertions hold:

- either $\xi_n \sim a_n$ as $n \to \infty$ a.s. or $a_n \equiv 0$;
- $(\xi_n - a_n)/b_n \Rightarrow N(0, 1)$ as $n \to \infty$.

3.1. Statistical estimation based on quadratic variation

Consider quadratic variation of M^H, i.e.

$$V_n^{H,2} = \sum_{k=0}^{n-1} \left(\Delta_k^n M^H\right)^2.$$

The following lemma contains information on asymptotic behaviour of $V_n^{H,2}$, which is needed to construct estimators.

Lemma 3.1. *1. For $H \in (0, 1/2)$,*

$$V_n^{H,2} \approx a^2 T^{2H} n^{1-2H} + b^2 T + a^2 T^{2H} n^{1/2-2H} \sigma_{H,2} N(0,1). \tag{16}$$

2. For $H \in (1/2, 1)$

$$V_n^{H,2} \approx b^2 T + a^2 T^{2H} n^{1-2H} + \sqrt{2} b^2 T n^{-1/2} N(0,1). \tag{17}$$

3. For $H = 1/2$

$$V_n^{H,2} \approx (a^2 + b^2) T + \sqrt{2}(a^2 + b^2) T n^{-1/2} N(0,1).$$

Proof. Write

$$V_n^{H,2} = a^2 V_n^{H,0,2} + 2ab V_n^{H,1,1} + b^2 V_n^{H,2,0}. \tag{18}$$

By law of large numbers

$$V_n^{H,2,0} \to T, \ n \to \infty.$$

Therefore, we have

$$V_n^{H,2,0} \approx T + \sqrt{2} T n^{-1/2} N(0,1)$$

by the classical CLT.

This gives immediately the result in the case $H = 1/2$, since in this case $M_t^H = \sqrt{a^2 + b^2}\widetilde{W}_t$, where $\widetilde{W}$ is a standard Wiener process.

Further, by ergodic theorem and (11), for $H \in (0, 3/4)$

$$V_n^{H,0,2} \approx n^{1-2H} T^{2H} + n^{1/2-2H} T^{2H} \sigma_{H,2} N(0,1),$$

for $H = 3/4$

$$V_n^{H,0,2} \approx T^{3/2} n^{-1/2} + \frac{3}{2} T^{3/2} n^{-1} \sqrt{\log n}\, N(0,1),$$

and for $H \in (3/4, 1)$

$$V_n^{H,0,2} \approx T^{2H} n^{-1} \zeta_H,$$

where ζ_H is a Rosenblatt random variable.

Finally, from (7),

$$V_n^{H,1,1} \approx n^{-H} T^{H+1/2} N(0,1).$$

The required statement now follows by comparing the orders of terms in (18). □

As we see, the asymptotic behaviour of $V_n^{H,2}$ depends on whether $H < 1/2$ or not. Let us consider these cases individually (we omit the case $H = 1/2$, which is of minor interest).

3.1.1. $H \in (0, 1/2)$

For $H \in (0, 1/2)$ we have

$$V_n^{H,2} \sim a^2 T^{2H} n^{1-2H}, \ n \to \infty, \tag{19}$$

a.s., which allows to estimate the Hurst parameter H consistently. Precisely, we have the following result.

Proposition 3.1. *For $H \in (0, 1/2)$, the following statistics*

$$\widehat{H}_k = \frac{1}{2}\left(1 - \frac{1}{k}\log_2 V_{2^k}^{H,2}\right)$$

and

$$\widetilde{H}_k = \frac{1}{2}\left(\log_2 \frac{V_{2^k}^{H,2}}{V_{2^{k+1}}^{H,2}} + 1\right)$$

are strongly consistent estimators of the Hurst parameter H.

Proof. Write from (16)

$$\log_2 V_{2^k}^{H,2} = \log_2\left(a^2 T^{2H} 2^{k(1-2H)}\right) + \log_2\left(1 + \frac{b^2}{a^2} T^{1-2H} 2^{-k(1-2H)} + 2^{-k/2}\zeta_k\right),$$

where the sequence $\{\zeta_k, k \geq 1\}$ is bounded in probability. Hence we have

$$\begin{gathered}\log_2 V_{2^k}^{H,2} = 2\log_2 a + 2H\log_2 T + (1-2H)k \\ +O(2^{-k(1-2H)}) + O_P(2^{-k/2}), k \to \infty,\end{gathered} \tag{20}$$

where O is the standard big-O notation, while $\xi_k = O_P(\alpha_k)$, $k \to \infty$, means that the sequence $\{\xi_k/\alpha_k, k \geq 1\}$ is bounded in probability. In particular,

$$\log_2 V_{2^k}^{H,2} \sim 2\log_2 a + 2H\log_2 T + (1-2H)k, k \to \infty,$$

whence the result immediately follows. □

Remark 3.1. At the first sight, there is no clear advantage of $\widehat{H}_k$ or $\widetilde{H}_k$. But a careful analysis shows that $\widetilde{H}_k$ is better. Indeed, from (20) it is easy to see that

$$\widehat{H}_k = H - \frac{\log_2 a + H\log_2 T}{k} + o(k^{-1}), k \to \infty,$$

while

$$\widetilde{H}_k = H + O(2^{-k(1-2H)}) + O_P(2^{-k/2}), k \to \infty. \tag{21}$$

Now it is absolutely clear that $\widetilde{H}_k$ performs much better (unless one hits the jackpot by having $aT^H = 1$).

Asymptotic confidence intervals for H and consistent estimates for a and b Asymptotic expansion (16) allows to construct confidence intervals for H. In view of (16), we should distinguish two cases: $H \in (0, 1/4)$ and $H \in [1/4, 1)$.

In order to proceed, we should first give a consistent estimate for a. The previous remark enables us to do this.

Proposition 3.2. *For $H \in (0, 1/4)$, the statistic*

$$\widetilde{a}_k^2 = 2^{k(2\widetilde{H}_k - 1)} T^{-2\widetilde{H}_k} V_{2^k}^{H,2}$$

is a consistent estimator of a^2.

Proof. Observe that

$$\frac{\widetilde{a}_k^2}{a^2} \sim 2^{k(\widetilde{H}_k - H)} T^{k(H - \widetilde{H}_k)} \to 1, \ k \to \infty,$$

in probability, since $k(\widetilde{H}_k - H) \to 0$, $k \to \infty$, in probability by (21). Therefore, we have the desired statement. □

The construction of asymptotic confidence intervals is different in cases $H \in (0, 1/4)$ and $H \in [1/4, 1)$. Let us first concentrate on the case $H \in (0, 1/4)$. In this case

$$V_n^{H,2} \approx a^2 T^{2H} n^{1-2H} + a^2 T^{2H} n^{1/2-2H} \sigma_{H,2} N(0, 1).$$

Now in order to construct a confidence interval for H, observe that

$$2^{1-2H} V_{2^k}^{H,2} - V_{2^{k+1}}^{H,2} \approx a^2 T^{2H} \sigma'_H 2^{k(1/2-2H)} N(0, 1), \tag{22}$$

where the constant σ'_H can be computed explicitly in terms of H, however, the expression is extremely cumbersome; thus we give only a less explicit expression in terms of B^H:

$$\sigma'_H = \left(\rho'_{H,1} + 2 \sum_{k=1}^{\infty} \rho'_{H,k} \right)^{1/2},$$

where

$$\rho'_{H,k} = \mathsf{E}\Bigg[\left(2^{1-2H} \left(B_1^H \right)^2 - \left(B_{1/2}^H \right)^2 - \left(B_1^H - B_{1/2}^H \right)^2 \right)$$
$$\times \left(2^{1-2H} \left(B_{k+1}^H - B_k^H \right)^2 - \left(B_{k+1/2}^H - B_k^H \right)^2 - \left(B_{k+1}^H - B_{k+1/2}^H \right)^2 \right) \Bigg].$$

Now, it follows from Slutsky's theorem that

$$2^{k(2H-1/2)} \frac{2^{1-2H} V_{2^k}^{H,2} - V_{2^{k+1}}^{H,2}}{\widetilde{a}_k^2 T^{2H} \sigma'_H} \Rightarrow N(0, 1), \ k \to \infty,$$

whence the construction of asymptotic confidence interval is obvious. However, this leads to solving highly complicated equation involving quite unpleasant function σ'_H. A better idea is to replace H with $\widetilde{H}_k$:

$$2^{k(2\widetilde{H}_k-1/2)}\frac{2^{1-2H}V^{H,2}_{2^k}-V^{H,2}_{2^{k+1}}}{\widetilde{a}^2_k T^{2\widetilde{H}_k}\sigma'_{\widetilde{H}_k}}=2^{k/2}\frac{2^{1-2H}V^{H,2}_{2^k}-V^{H,2}_{2^{k+1}}}{\sigma'_{\widetilde{H}_k}V^{H,2}_{2^k}}\Rightarrow N(0,1),\ k\to\infty.$$

Hence, we have the following result, where z_α is the standard normal $(1-\alpha/2)$-quantile; for well-definiteness we denote

$$\log_{2+}x=\begin{cases}\log_2 x, & x>0,\\ 0, & x\le 0.\end{cases}$$

Proposition 3.3. *Let $H\in(0,1/4)$. The interval $[H_{\min},H_{\max}]$, where*

$$H_{\min}=\frac{1}{2}\left(1+\log_2 V^{H,2}_{2^k}-\log_2\left(V^{H,2}_{2^{k+1}}+z_\alpha\sigma'_{\widetilde{H}_k}V^{H,2}_{2^k}2^{-k/2}\right)\right),$$
$$H_{\max}=H_{\min}\vee\frac{1}{2}\left(1+\log_2 V^{H,2}_{2^k}-\log_{2+}\left(V^{H,2}_{2^{k+1}}-z_\alpha\sigma'_{\widetilde{H}_k}V^{H,2}_{2^k}2^{-k/2}\right)\right),$$

is an asymptotic confidence interval for H of level $1-\alpha$.

Remark 3.2. As it is known from [22], for $H\in(0,1/4)$ the measure induced by M^H in $C[0,T]$ is equivalent to that of aB^H. So the results are essentially the same in this case as for fractional Brownian motion alone. Another important consequence that in this case it is not possible to estimate b consistently.

Now let us move to the case $H\in(1/4,1/2)$. (We omit the case $H=1/4$ for two reasons: considered similarly to $H\in(1/4,1/2)$, and it is hard to distinguish it statistically from the case where $H\ne 1/4$.) The construction of confidence interval is more involved in this case, since the second term in (16) cannot be omitted. However, the idea is the same: one needs to remove non-random terms from the (19). To be more precise, we have the following result, where

$$U^{H,2}_k=V^{H,2}_{2^k}-V^{H,2}_{2^{k+1}};$$

and the asymptotic variance is

$$\sigma''_H=\left(\rho''_{H,1}+2\sum_{k=1}^{\infty}\rho''_{H,k}\right)^{1/2},$$
$$\rho''_{H,k}=\mathsf{E}\Big[\Big(c_H s_{0,1}-(c_H+1)\left(s_0^{1/2}+s_{1/2}^{1/2}\right)+s_0^{1/4}+s_{1/4}^{1/4}+s_{1/2}^{1/4}+s_{3/4}^{1/4}\Big)$$
$$\times\Big(c_H s_k^1-(c_H+1)\left(s_k^{1/2}+s_{k+1/2}^{1/2}\right)+s_k^{1/4}+s_{k+1/4}^{1/4}+s_{k+1/2}^{1/4}+s_{k+3/4}^{1/4}\Big)\Big]$$

with $s_t^h=\left(B^H_{t+h}-B^H_t\right)^2$, $c_H=2^{1-2H}$.

Proposition 3.4. *Let $H \in [1/4, 1/2)$. The interval $[H_{\min}, H_{\max}]$, where*

$$H_{\min} = \frac{1}{2}\left(1 + \log_{2+}\left(-U_k^{H,2}\right) - \log_{2+}\left(-U_{k+1}^{H,2} + z_\alpha \sigma''_{\widetilde{H}_k} V_{2^k}^{H,2} 2^{-k/2}\right)\right),$$
$$H_{\max} = H_{\min} \vee \frac{1}{2}\left(1 + \log_{2+}\left(-U_k^{H,2}\right) - \log_{2+}\left(-U_{k+1}^{H,2} - z_\alpha \sigma''_{\widetilde{H}_k} V_{2^k}^{H,2} 2^{-k/2}\right)\right),$$

is an asymptotic confidence interval for H of level $1-\alpha$.

Proof. It follows from [4] that

$$2^{1-2H} V_{2^k}^{H,2} - \left(1 + 2^{1-2H}\right) V_{2^{k+1}}^{H,2} + V_{2^{k+2}}^{H,2} \approx a^2 \sigma''_H T^{2H} 2^{(1/2-2H)k} N(0,1).$$

Thus, by Slutsky's theorem, we have the weak convergence

$$2^{k/2} \frac{2^{1-2H} V_{2^k}^{H,2} - \left(1 + 2^{1-2H}\right) V_{2^{k+1}}^{H,2} + V_{2^{k+2}}^{H,2}}{\sigma''_{\widetilde{H}_k} V_{2^k}^{H,2}} \Rightarrow N(0,1),\ k \to \infty, \tag{23}$$

whence the result follows. □

Now we turn to estimation of b. In order to proceed, we will need another consistent estimator of H.

Proposition 3.5. *For $H \in (0, 1/2)$, the statistic*

$$\widetilde{H}_k^{(2)} = \frac{1}{2}\left(\log_{2+} \frac{U_k^{H,2}}{U_{k+1}^{H,2}} + 1\right)$$

is a consistent estimator of H, moreover,

$$\widetilde{H}_k^{(2)} = H + O_P(2^{-k/2}),\ k \to \infty.$$

Remark 3.3. Despite $\widetilde{H}_k^{(2)}$ has asymptotically a better rate of approximation that $\widetilde{H}_k$ for $H \in (1/4, 1/2)$, we still do not recommend to use it, as the asymptotic variance is high; it's practically useless for $k \le 10$.

Proof. From (19) and [4] it follows that

$$U_k^{H,2} \approx a^2 T^{2H} \left(2^{1-2H} - 1\right) 2^{k(1-2H)} + a^2 T^{2H} 2^{k(1/2-2H)} \sigma'''_H N(0,1)$$

with some constant σ'''_H, which is of no importance now. In particular, $U_k^{H,2}$ is eventually negative, and we can write

$$\begin{aligned} \log_2(-U_k^{H,2}) &= 2\log_2 a + 2H \log_2 T \\ &+ \log_2\left(1 - 2^{1-2H}\right) + k(1-2H) + O_P(2^{-k/2}). \end{aligned} \tag{24}$$

Thus,

$$\log_2 \frac{U_k^{H,2}}{U_{k+1}^{H,2}} = 2H - 1 + O_P(2^{-k/2}),\ k \to \infty,$$

whence the assertion follows. □

Now we are able to construct a consistent estimator for b. However, the variance of this estimator is so huge that it is practically useless (see Section 4).

Proposition 3.6. *For $H \in (1/4, 1/2)$ the statistic*

$$\widetilde{b}_k^2 = \frac{2^{1-2\widetilde{H}_k^{(2)}} V_{2^k}^{H,2} - V_{2^{k+1}}^{H,2}}{(2^{1-2\widetilde{H}_k^{(2)}} - 1)T}$$

is a consistent estimator of b^2.

Proof. Define

$$\widehat{b}_k^2 = \frac{2^{1-2H} V_{2^k}^{H,2} - V_{2^{k+1}}^{H,2}}{\left(2^{1-2H} - 1\right)T}.$$

It easily follows from (16) that $\widehat{b}_k^2 \to b^2$, $k \to \infty$. So it is enough to show that $\widetilde{b}_k^2 - \widehat{b}_k^2 \to 0$, $k \to \infty$. To this end, write

$$\widetilde{b}_k^2 - b^2 = \frac{\left(2^{1-2\widetilde{H}_k^{(2)}} - 2^{1-2H}\right) V_{2^k}^{H,2}}{\left(2^{1-2\widetilde{H}_k^{(2)}} - 1\right)T}$$
$$+ T^{-1} \left(2^{1-2H} V_{2^k}^{H,2} - V_{2^{k+1}}^{H,2}\right) \left(\left(2^{1-2\widetilde{H}_k^{(2)}} - 1\right)^{-1} - \left(2^{1-2H} - 1\right)^{-1}\right).$$

Obviously, the second term converges to zero. To study the first one, write

$$\left(2^{1-2\widetilde{H}_k^{(2)}} - 2^{1-2H}\right) V_{2^k}^{H,2} \sim -2^{2-2H} (\widetilde{H}_k^{(2)} - H) a^2 T^{2H} 2^{k(1-2H)} \log 2$$
$$= 2^{k(1-2H)} O_P(2^{-k/2}),\ k \to \infty,$$

whence the consistency of $\widetilde{b}_k^2$ follows. □

3.1.2. $H \in (1/2, 3/4)$

Now we move to the case $H \in (1/2, 1)$. In view of (17), both $\widehat{H}_k$ and $\widetilde{H}_k$ converge to $1/2$ for $H \in [1/2, 1)$, so they are not suitable for estimating H.

In order to estimate H, the trick is to cancel out the first term of asymptotic expansion (17). The most natural way to accomplish this is to consider

$$U_k^{H,2} := V_{2^k}^{H,2} - V_{2^{k+1}}^{H,2} = 2 \sum_{j=0}^{2^k - 1} \Delta_{2j}^{2^{k+1}} M^H \Delta_{2j+1}^{2^{k+1}} M^H. \tag{25}$$

By (17),

$$U_k^{H,2} \sim a^2 T^{2H} 2^{(1-2H)k} \left(1 - 2^{1-2H}\right), k \to \infty \tag{26}$$

a.s., whence we can easily construct estimators exactly as in the case $H \in (0, 1/2)$.

Proposition 3.7. *For $H \in (1/2, 3/4)$, statistics*

$$\widehat{H}_k^{(2)} = \frac{1}{2}\left(1 - \frac{1}{k}\log_{2+} U_k^{H,2}\right)$$

and

$$\widetilde{H}_k^{(2)} = \frac{1}{2}\left(\log_{2+} \frac{U_k^{H,2}}{U_{k+1}^{H,2}} + 1\right)$$

are strongly consistent estimators of the Hurst parameter H.

We will see in Section 4 that $\widetilde{H}_k^2$ performs very poorly, and $\widehat{H}_k^2$ performs much better, despite having a worse asymptotic rate of convergence. For the values of H close to $1/2$, both estimators are quite poor, and it is better to estimate H by regressing several values of $\log_2 U_k^{H,2}$ on k.

Asymptotic confidence intervals for H and consistent estimates for a and b To construct a confidence interval, it is not hard to check (since we are not going to use this result, we will not give a proof) that

$$2^{1-2H}U_k^{H,2} - U_{k+1}^{H,2} \approx b^2 T\sqrt{2^{3-4H}+1}\; 2^{-k/2-1/2}N(0,1).$$

In contrast to the case $H \in (0, 1/2)$, where one has to replace H by its estimate, this result can be used immediately to construct asymptotic confidence intervals for H (though one needs to replace b^2T by $V_{2^k}^{H,2}$). Indeed, one can write

$$\mathsf{P}\left(2^{k/2+1/2}\frac{2^{1-2H}U_k^{H,2} - U_{k+1}^{H,2}}{V_{2^k}^{H,2}\sqrt{2^{3-4H}+1}} \in [-z_\alpha, z_\alpha]\right) \to 1-\alpha,\ k\to\infty, \tag{27}$$

where z_α is the standard Gaussian $(1-\alpha/2)$-quantile. Thus, the construction of asymptotic confidence interval is reduced to solving quadratic inequalities and taking logarithms. Unfortunately, even for such large values as $\alpha = 0.1$ and $k = 20$ these confidence intervals are useless: approximation (27) is very imprecise, and, what is more dramatic, they usually contain the whole interval $(1/2, 3/4)$.

Estimating b is quite straightforward now: it is easy to verify that

$$\widehat{b}_k^2 = T^{-1}V_{2^k}^{H,2}$$

is a consistent estimator b^2. We did not manage to construct a reasonable variance-based estimator for a. Similar to estimating b in the case $H \in (0, 1/2)$, this requires to construct an estimate of H with approximation rate $o(k^{-1})$. However, none of the estimators we presented has this order. One should take a much finer partition for this, moreover, the number of points should be depending on H, which is practically impossible.

3.1.3. $H \in (3/4, 1)$

As we have already mentioned in the beginning of this section, in this case it is impossible to make conclusions about the value of H. In particular, in this case we have

$$V_n^{H,2} \approx b^2 T + \sqrt{2} b^2 T n^{-1/2} N(0,1),$$

so the behaviour of $V_n^{H,2}$ is essentially the same as that of the quadratic power variation of Wiener process

$$V_n^2 = \sum_{k=0}^{n-1} \left(\Delta_k^n W\right)^2.$$

This confirms that we cannot make any conclusions about H from the asymptotic behaviour of quadratic variation.

Nevertheless, we will study the behaviour of quadratic variation in this case in more detail in order to be able to distinguish between the cases $H < 3/4$ and $H > 3/4$ statistically.

Define

$$Z_k = \frac{2^{k/2}}{b^2 T} U_k^{H,2}.$$

Proposition 3.8. *For $H \in (3/4, 1)$, the sequence $(Z_k, Z_{k+1}, \dots)$ converges in distribution as $k \to \infty$ to a sequence $(\zeta_1, \zeta_2, \dots)$ of independent standard Gaussian variables.*

Remark 3.4. We emphasize a sharp contrast with the case $H \in (1/2, 3/4)$, where the sequence $\{Z_k, k \geq 1\}$ has a positive limit in view of (26), hence, it eventually becomes positive. This clearly gives a possibility to distinguish statistically between cases $H \in (1/2, 3/4)$ and $H \in (3/4, 1)$. (See 4.1.3 for comparative simulations.)

Proof. Define

$$\xi_k = \frac{2^{k/2}}{\sqrt{2}T}\left(V_{2^k}^2 - T\right) = \frac{2^{-k/2}}{\sqrt{2}T} \sum_{i=0}^{2^k-1} \left(2^k \left(\Delta_i^{2^k} W\right)^2 - T\right).$$

By CLT, $\xi_k \Rightarrow N(0,1)$, $k \to \infty$, so we need to study the collective behaviour. To this end, observe that the vector $(\xi_k, \xi_{k+1}, \dots, \xi_{k+m})$ can be represented as a sum of independent vectors

$$(\xi_k, \xi_{k+1}, \dots, \xi_{k+m}) = \sum_{i=0}^{2^k-1} \zeta_{k,i},$$

where the jth coordinate of $\zeta_{k,i}$, $j = 0, 1, 2, \dots, m$, is

$$\zeta_{k,i,j} = \frac{2^{-k/2}}{\sqrt{2}T} \sum_{l=0}^{2^j-1} \left(2^{2(k+j)} \left(\Delta_{l+i2^j}^{2^{k+j}} W\right)^2 - T\right).$$

(We simply group terms on the intervals of the partition $\{i2^{-k}, i = 0, \dots, 2^k\}$.) Therefore, we can apply a vector CLT and deduce that for every $m \geq 0$ the vector $(\xi_k, \xi_{k+1}, \dots, \xi_{k+m})$ converges in distribution to an $(m+1)$-dimensional centered Gaussian vector as $k \to \infty$. Consequently, the sequence $(\xi_k, \xi_{k+1}, \xi_{k+2}, \dots)$ converges to a centered stationary Gaussian sequence as $k \to \infty$.

Taking into account the fact that the asymptotic behaviour of $V_n^{H,2}$ and V_n^2 is the same (up to the factor b^2), we obtain that

$$Z_k = \frac{2^{k/2}}{b^2 T} U_k^{H,2}$$

with $U_k^{H,2}$ is defined in (25), also converges to a centered stationary Gaussian sequence. It is straightforward to check that the limit covariance is that of an i.i.d. standard Gaussian sequence, whence the result follows. □

3.1.4. $H = 3/4$

For $H = 3/4$, an analogue of Proposition 3.8 can be proved, that is, $(Z_k, Z_{k+1}, \dots)$ converges in distribution as $k \to \infty$ to a sequence $(\zeta_1, \zeta_2, \dots)$ of independent Gaussian variables with unit variance. However, since

$$V_n^{3/4,2} \approx b^2 T + a^2 T^{3/2} n^{-1/2} + \sqrt{2} b^2 T n^{-1/2} N(0,1),$$

the limiting stationary distribution now has a positive mean, namely, $\mathsf{E}[\zeta_1] = a^2 b^{-2} T^{1/2}(1 - 2^{-1/2})$. As long as this value depends on how big is a compared to b, we might be unable to distinguish this case from $H > 3/4$. On the other hand, if b is small relative to a, it might be hard to distinguish this case from $H < 3/4$.

3.2. Statistical estimation using 4th power variation

It was mentioned in the previous section that the performance of quadratic variation estimators in the case $H \in (1/2, 3/4)$ is not very satisfactory. One could try to improve it by considering quartic variation of M^H

$$V_n^{H,4} := \sum_{k=0}^{n-1} \left(\Delta_k^n M^H\right)^4 = \sum_{i=0}^{4} \binom{4}{i} a^i b^{4-i} V_n^{H,4-i,i}.$$

Theorem 3.1. *Statistics*

$$\widehat{H}_k^{(4)} = -\frac{1}{2k} \log_{2+} U_k^{H,4}$$

and

$$\widetilde{H}_k^{(4)} = \frac{1}{2} \log_{2+} \frac{U_k^{H,4}}{U_{k+1}^{H,4}}$$

are strongly consistent estimators of the Hurst parameter $H \in (1/2, 3/4)$ in the mixed model (15).

Remark 3.5. Both these estimators are quite poor. Regression of several values of $\log_2 U_k^{H,4}$ on k leads to a much better estimator. However, as numerical experiments in Section 4 suggest, it is better to use the quadratic variation based estimators (although they are not apparently very efficient as well).

Remark 3.6. It is clear that (28) also allows to construct approximate confidence intervals for H. One should be careful with the third term in (28): for $H \in (5/8, 3/4)$ it is smaller than the Gaussian fluctuation, so can be omitted, while for $H \in (1/2, 5/8]$ it does affect a confidence interval. However, a priori a value of H is not known, of course, so in case where the value of estimator is close to $5/8$ one should take a partition sufficiently fine in order to determine whether $H > 5/8$ or not.

Proof. By LLN and CLT,

$$V_n^{H,4,0} \approx 3T^2 n^{-1} + 4\sqrt{6}T^2 n^{-3/2} N(0,1).$$

Using (7), we obtain

$$V_n^{H,3,1} \approx T^{H+3/2}\sqrt{\mu_6\mu_2}\, n^{-H-1} N(0,1) = \sqrt{15}T^{H+3/2} n^{-1-H} N(0,1).$$

Further, from (4)

$$V_n^{H,2,2} \approx T^{2H+1} n^{-2H} + \sqrt{\sigma_{H,2}^2\mu_2^2 + 6}\, T^{2H+1} n^{-2H-1/2} N(0,1).$$

Finally,

$$V_n^{H,1,3} \approx \sqrt{15}T^{3H+1/2} n^{-3H} N(0,1)$$

by (7) and

$$V_n^{H,0,4} \approx 3T^{4H} n^{1-4H} + T^{4H} n^{1/2-4H} \sigma_{H,4} N(0,1),$$

by the ergodic theorem and (11). Collecting all the terms and omitting the terms of lower order, we get

$$V_n^{H,4} \approx 3T^2 b^4 n^{-1} + 6T^{2H+1} a^2 b^2 n^{-2H} + 3T^{4H} n^{1-4H} a^4 + 4\sqrt{6}T^2 b^4 n^{-3/2} N(0,1). \tag{28}$$

Hence, the assertion follows. □

3.3. Estimate of Hurst parameter for known a and b

When the scale coefficients a and b are known, the estimation procedure significantly simplifies, and the quality of estimators is improved. Since we already know asymptotic expansions of power variations, we immediately proceed to statistical conclusions.

3.3.1. $H \in (0, 1/2)$

Proposition 3.9. *For $H \in (0, 1/2)$ and known a, the statistic*

$$\widehat{H}_k(a) = \frac{k + 2\log_2 a - \log_2 V_{2^k}^{H,2}}{2(k - \log_2 T)}$$

is a strongly consistent estimator of H.

Using the asymptotic expansion (24), we can construct another estimator of H, which has an asymptotically better rate of convergence, namely, we can define $\widetilde{H}_k(a)$ as a solution from the interval $(0, 1/2)$ of the equation

$$\log_2(-U_k^{H,2}) = 2\log_2 a + 2\widetilde{H}_k(a)\log_2 T + \log_2\left(1 - 2^{1-2\widetilde{H}_k(a)}\right) + k(1 - 2\widetilde{H}_k(a)).$$

We can also construct asymptotic confidence interval using the knowledge of a and b; there is no big advantage of this knowledge in the case $H \in (0, 1/4)$, but for $H \in (1/4, 1/2)$ the asymptotic confidence interval will be better.

Proposition 3.10. *Let $H \in (0, 1/2)$, a, b are known. The interval $[H_{\min}(a, b), H_{\max}(a, b)]$, where*

$$H_{\min}(a,b) = \frac{1}{2}\left(1 + \log_{2+} Y_k(b) - \log_{2+}\left(Y_{k+1}(b) + z_\alpha \sigma'_{\widehat{H}_k(a)} V_{2^k}^{H,2} 2^{-k/2}\right)\right),$$

$$H_{\max}(a,b) = H_{\min}(a,b) \vee \frac{1}{2}\left(1 + \log_{2+} Y_k(b) - \log_{2+}\left(Y_{k+1}(b) - z_\alpha \sigma'_{\widehat{H}_k(a)} V_{2^k}^{H,2} 2^{-k/2}\right)\right),$$

$$Y_k(b) = V_{2^k}^{H,2} - b^2 T.$$

is an asymptotic confidence interval for H of level $1 - \alpha$.

Proof. We have

$$2^{1-2H} V_{2^k}^{H,2} - V_{2^{k+1}}^{H,2} \approx \left(2^{1-2H} - 1\right) b^2 T + a^2 T^{2H} \sigma'_H 2^{k(1/2-2H)} N(0,1),$$

therefore by Slutsky's theorem,

$$2^{k/2} \frac{2^{1-2H} V_{2^k}^{H,2} - V_{2^{k+1}}^{H,2} - \left(2^{1-2H} - 1\right) b^2 T}{\sigma'_{\widehat{H}_k(a)} V_{2^k}^{H,2}} \Rightarrow N(0,1),\ k \to \infty, \tag{29}$$

as required. □

3.3.2. $H \in (1/2, 3/4)$

In this case the knowledge of a and b allows us we can improve on estimation of H by utilising (17).

Proposition 3.11. *If $H \in (1/2, 3/4)$ and a, b are known, the statistic*

$$\widehat{H}_k^{(2)}(a,b) = \frac{k + 2\log_2 a - \log_2\left(V_{2^k}^{H,2} - b^2 T\right)}{2(k - \log_2 T)}$$

is a strongly consistent estimator of H.

If only a is known, we can still use (26) to construct an estimator for H. This requires solving a non-linear equation

$$\log_2 U_k^{H,2} = 2\log_2 a + 2\widetilde{H}_k^{(2)}(a)\log_2 T + (1-2\widetilde{H}_k^{(2)}(a))k + \log_2\left(1-2^{1-2\widetilde{H}_k^{(2)}(a)}\right)$$

on the interval $(1/2, 3/4)$.

Knowing coefficient b also allows to construct more efficient confidence intervals (those constructed using (27) are pretty useless).

Proposition 3.12. *Let $H \in (1/2, 3/4)$, b is known. The interval $[H_{\min}(b), H_{\max}(b)]$, where*

$$H_{\min}(b) = \frac{1}{2}\left(1 + \log_{2+} A_k(b) - \log_{2+}\left(B_k(b) + \sqrt{B_k(b)^2 - A_k(b)C_k(b)}\right)\right),$$

$$H_{\max}(b) = H_{\min}(b) \vee \frac{1}{2}\left(1 + \log_{2+} A_k(b) - \log_{2+}\left(B_k(b) - \sqrt{B_k(b)^2 - A_k(b)C_k(b)}\right)\right),$$

$$A_k(b) = Y_k(b)^2 - 4D_k(b), \quad B_k(b) = 2Y_k(b)Y_{k+1}(b) - D_k(b),$$
$$C_k(b) = Y_{k+1}(b)^2 - 2D_k(b), \quad D_k(b) = z_\alpha^2 b^4 T^2 2^{-k-1},$$

is an asymptotic confidence interval for H of level $1-\alpha$.

Proof. Write

$$2^{k/2}\frac{2^{1-2H}\left(V_{2^k}^{H,2} - b^2T\right) - \left(V_{2^{k+1}}^{H,2} - b^2T\right)}{bT\sqrt{2^{3-4H} - 2^{1-2H} + 1}} \Rightarrow N(0,1),\ k\to\infty. \tag{30}$$

Then the required result follows by solving the quadratic inequality for 2^{1-2H} in

$$\mathsf{P}\left(2^{k/2}\frac{2^{1-2H}\left(V_{2^k}^{H,2} - b^2T\right) - \left(V_{2^{k+1}}^{H,2} - b^2T\right)}{b^2T\sqrt{2^{3-4H} - 2^{1-2H} + 1}} \in [-z_\alpha, z_\alpha]\right) \to 1-\alpha,\ k\to\infty.$$

□

4. Simulations

In each procedure we take $T = 3$, $a = b = 1$, $n = 2^{20}$ and use the circulant method to simulate values of B^H on the uniform partition $\{iT/n, i = 0, 1, \dots, n\}$ of $[0, T]$. The simulation takes about 6 seconds on Intel Core i5-3210M processor, computing all estimators takes 10–500 milliseconds.

4.1. Estimators based on quadratic variation

4.1.1. $H \in (0, 1/2)$

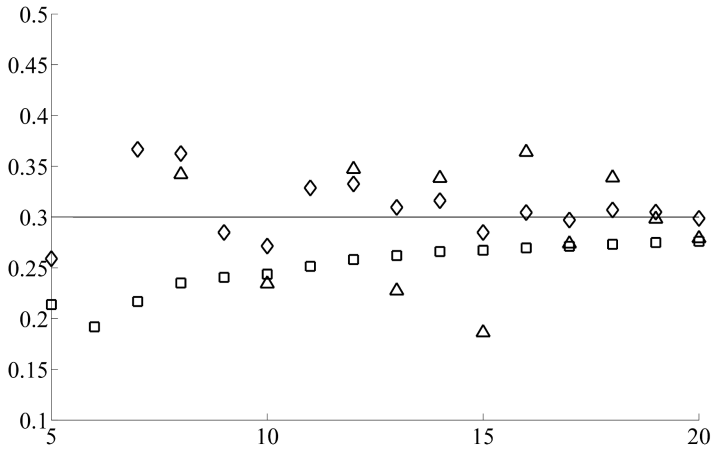
The following table contains estimators of Hurst parameter H and 99 % approximate confidence interval $[H_{\min}, H_{\max}]$ for it, the values of H range from

0.05 to 0.5 with step 0.05. For each value of H we also estimate coefficient a, and for H between 0.25 and 0.5 we give value of estimator for b. (N/A means that $\widetilde{b}^2_{20}$ is negative; * means theoretical value.)

H	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$\widehat{H}_{20}$	.046	.092	.1382	.1841	.2302	.276	.3215	.3656	.4054
$\widetilde{H}_{20}$	.0509	.1017	.1489	.1993	.2511	.3018	.3525	.407	.4605
$\widetilde{H}^{(2)}_{20}$	.0566	.106	.1449	.1922	.2547	.2919	.3534	.375	.4397
$H_{\min}$	.0448	.0932	.1311	.1768	.2371	.2708	.3269	.335	.3551
$H_{\max}$	.0686	.119	.159	.208	.2728	.3136	.3809	.417	$.5^*$
$\widetilde{a}_{20}$	1.019	1.023	.985	.991	1.014	1.027	1.043	1.135	1.293
$\widetilde{b}_{20}$						2.627	N/A	1.823	1.241

We see from the table that the estimator $\widetilde{H}_k$ has consistently the best performance: the relative error is within 2 % for all values but 0.45, where the relative error is 2.33 %.

The estimator $\widehat{H}_k$ underestimates all values of H by around 8 %, which seems a lot for such number of computations. Finally, the relative error of $\widetilde{H}^{(2)}_k$ ranges from 1 % for $H = 0.35$ to 6.25 % for $H = 0.4$ and 13.2 % for $H = 0.05$. This hints at a large variance of the estimator $\widetilde{H}^{(2)}_k$, which is supported by the following figure. It contains values of estimator $\widetilde{H}_k$ (lozenges), $\widehat{H}_k$ (squares) and $\widetilde{H}^{(2)}_k$ (triangles), $k = 5, \dots, 20$, $H = 0.3$. The values of $\widetilde{H}^{(2)}_k$ outside $[0.1, 0.5]$ were dropped. It is clearly visible that the performance of $\widetilde{H}^{(2)}_k$ is better than that of $\widetilde{H}_k$, despite the former has asymptotically better rate of convergence to H.



For sake of completeness, we present some values of estimators for $H > 0.5$.

H	0.5	0.6	0.7	0.8	0.9
$\widehat{H}_{20}$	.4353	.4577	.4601	.4603	.4604
$\widetilde{H}_{20}$	.5021	.507	.5027	.4996	.4997

As expected, the values of $\widetilde{H}_{20}$ are close to 0.5.

4.1.2. $H \in (1/2, 3/4)$

The following table contains estimators $\widehat{H}^{(2)}_{20}$ and $\widetilde{H}^{(2)}_{20}$ of Hurst parameter H, and estimator $\widehat{b}_{20}$ of the coefficient b. The values of H range from 0.525 to 0.725 with step 0.025. We also give a "regression" estimator $\bar{H}^{(2)}$. It is obtained in the following way: we consider linear regression of $\{\log_{2+} U^{H,2}_j, j = m, m+1, \dots, 20\}$ on $\{m, m+1, \dots, 20\}$, where $m = 11, 12, \dots, 15$, and take the best regression (in terms of the coefficient of determination). If $\bar{r}^{(2)}$ is the coefficient of the best linear regression, we set $\bar{H}^{(2)} = (1 - \bar{r}^{(2)})/2$. We do not give values of the estimator $\widehat{a}_{20}$ and confidence intervals in view of their poor quality. (In the table, N/A means an estimator value outside the theoretical range $(1/2, 3/4)$.)

H	0.525	0.55	0.575	0.6	0.625	0.65	0.675	0.7	0.725
$\widehat{H}^{(2)}_{20}$	.6061	.604	.6096	.627	.639	.6447	.6629	.7013	.6931
$\widetilde{H}^{(2)}_{20}$	N/A	.5559	.5341	.6308	N/A	N/A	.7241	N/A	N/A
$\bar{H}^{(2)}$	.5668	.5797	.6083	.6041	.6116	N/A	N/A	N/A	N/A
$\widehat{b}_{20}$	1.229	1.123	1.065	1.033	1.017	1.009	1.005	1.002	1.001

As we can see, when the value of estimator $\widehat{H}^{(2)}_{20}$ is greater than .625, it is quite reliable, while other two estimators are not. In other cases $\bar{H}^{(2)}$ performs slightly better.

4.1.3. $H \in (1/2, 3/4)$

Below are values of $\{[10^4 U^{H,2}_k], k = 11, 12, \dots, 20\}$ for $H = 0.7$ and $H = 0.8$. The difference is clearly visible: for $H = 0.7$ the sequence is positive, while for $H = 0.8$ there is a plenty of negative values.

$H = 0.7$	869	649	523	3	260	18	78	98	53	50
$H = 0.8$	665	−620	482	−475	8	−29	−104	−71	−78	−28

4.2. Estimators based on quartic variation

4.2.1. $H \in (1/2, 3/4)$

The following table contains estimators $\widehat{H}_{20}^{(4)}$ and $\widetilde{H}_{20}^{(4)}$ of Hurst parameter H, the values of H range from 0.525 to 0.725 with step 0.025. We also give a "regression" estimator $\bar{H}^{(4)}$. It is obtained in the following way: we consider linear regression of $\{\log_{2+} U_j^{H,4}, j = m, m+1, \dots, 20\}$ on $\{m, m+1, \dots, 20\}$, where $m = 11, 12, \dots, 17$, and take the best regression (in terms of the coefficient of determination). If $\bar{r}^{(4)}$ is the coefficient of the best linear regression, we set $\bar{H}^{(4)} = -\bar{r}^{(4)}/2$. ($N/A$ means an estimator value outside the theoretical range $(1/2, 3/4)$.)

H	0.525	0.55	0.575	0.6	0.625	0.65	0.675	0.7	0.725
$\widehat{H}_{20}^{(4)}$	N/A	N/A	.5103	.5224	.5303	.5644	N/A	N/A	.5802
$\widetilde{H}_{20}^{(4)}$	N/A	.6227	.6098	.5736	N/A	N/A	N/A	N/A	N/A
$\bar{H}^{(4)}$	.5321	.5606	.6189	.6783	.6378	.7425	N/A	N/A	N/A

We see that the estimators based on the quartic variation are quite useless and definitely worse than those based on the quadratic variation; only the regression estimator is reliable for H close to 0.5.

4.3. Estimation when a and b are known

4.3.1. $H \in (0, 1/2)$

Below are values of estimators $\widehat{H}_{20}$ and $\widetilde{H}_{20}$ for H from 0.05 to 0.45 with the step 0.05. Since the estimates are very close to the real values, we give a rescaled difference. We can see that $\widetilde{H}_{20}$ has more stable performance, while $\widehat{H}_{20}$ has worse (but still very good) quality for H close to $1/2$.

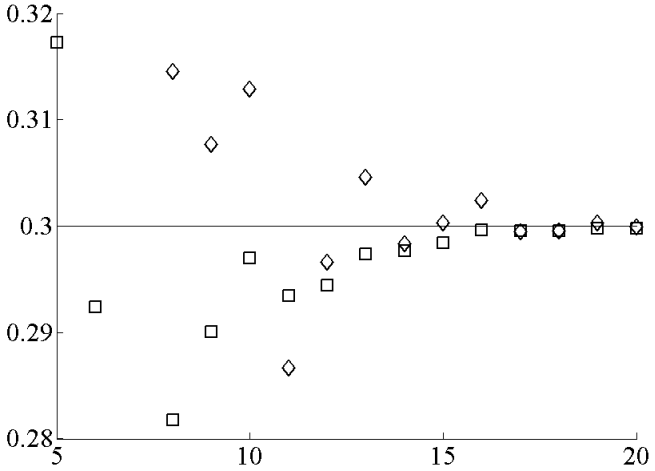
H	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$10^5(\widehat{H}_{20}(a) - H)$	.66	-2.5	2.3	-1.3	.36	19	190	300	960
$10^5(\widetilde{H}_{20}(a) - H)$	-220	-22	51	50	-70	60	11	1.1	47

Further we give asymptotic 99 % confidence intervals for $H \in [1/4, 1/2)$ using (29).

H	0.25	0.3	0.35	0.4	0.45
$H_{\min}(a, b)$	.2469	.2972	.3473	.3956	.4457
$H_{\max}(a, b)$	.2538	.3037	.3537	.402	.453

We see that the intervals are smaller than those given by (3.4).

The picture below illustrates the quality of estimators $\widetilde{H}_k(a)$ (lozenges) and $\widehat{H}_k(a)$ (squares), $k = 5, \dots, 20$, $H = 0.3$.



4.3.2. $H \in (1/2, 3/4)$

The following table contains estimators $\widehat{H}^{(2)}_{20}(a,b)$ and $\widetilde{H}^{(2)}_{20}(a)$ of Hurst parameter H. The values of H range from 0.525 to 0.725 with step 0.025. We also give 99 % confidence intervals constructed with the help of (30). Starting from $H = 0.65$, these confidence intervals already contain whole values from 1/2 to 3/4, so we give 80 % confidence intervals (marked by †). As before, by * we denote the theoretical bound.

H	0.525	0.55	0.575	0.6	0.625	0.65	0.675	0.7	0.725
$\widehat{H}^{(2)}_{20}(a,b)$	.5249	.55004	.57496	.5995	.6262	.6487	.6736	.6927	.712
$\widetilde{H}^{(2)}_{20}(a)$	.5538	.5547	.5734	.5962	.6336	.6422	.6642	.6701	.683
$H_{\min}(b)$	.5203	.539	.5586	.571	.5404	$.609^\dagger$	$.5951^\dagger$	$.6251^\dagger$	$.5785^\dagger$
$H_{\max}(b)$	.5301	.557	.5944	.639	.6785	$.732^\dagger$	$.75^*$	$.75^*$	$.75^*$

We see that $\widehat{H}^{(2)}_{20}(a,b)$ outperforms $\widetilde{H}^{(2)}_{20}(a)$, but the advantage of the latter is that it uses only knowledge of a. The confidence intervals are acceptable for $H \le 0.6$ and pretty useless for bigger values.

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