

Ray Chaos in a Photonic Crystal - Supplementary Materials

Emmanuel Rousseau and Didier Felbacq
¹Université Montpellier 2,
 Laboratoire Charles Coulomb UMR 5221,
 F-34095, Montpellier, France

I. GAUSSIAN BEAM PROPAGATION

We have compute the propagation of a laser beam through a sample of 7×7 cylinders of the photonics crystal. The Maxwell equations are exactly solved and the result is presented on Fig:1. The parameters used in the simulation are:

- optical index $n = 1.5$
- wavelength $\lambda = 500 \text{ nm}$
- cylinders radius $R = 20\lambda$
- photonic crystal period $T = 5R$

This simulation indicates that the laser beam can be modeled by rays, a ray being the path of the light energy. The initial gaussian beam is split into many light rays during propagation. It also indicates that the field distribution is dominated by the transmitted rays as a result of the low optical index of the cylinders.

II. RAY DYNAMICS

The ray dynamics is splitted in two part. At the n -step, an incoming ray impact on a cylinder. It is first refracted at the dioptrie air/glass then propagate into the cylinder and finally is refracted at the dioptrie glass/air. This path leads as seen on Fig:1 to the point B_p . $(\theta_{B_p}, \theta_{B_p}^1)$ can be linked to $(\theta_{A_p}, \theta_{A_p}^1)$ by the set of equations:

$$\theta_{B_p}^1 = -\theta_{A_p}^1 \quad (1a)$$

$$\theta_{B_p}^1 = \theta_{A_p} + 2 \arcsin \left(\frac{\sin(\theta_{A_p}^1)}{n} \right) \quad (1b)$$

where n is the optical index of the cylinders.

From the point B_p the ray propagate in air until it intersects a new cylinder defining then the point A_{p+1} . The couple $(\theta_{A_{p+1}}, \theta_{A_{p+1}}^1)$ is found with the help of equations:

$$f_B(x) = \tan(\alpha) [x - R \cos(\theta_{B_p})] + R \sin(\theta_{B_p}) \quad (2a)$$

$$(x_{A_{p+1}} - x_{p+1}^c)^2 + (f_B(x_{A_{p+1}}) - y_{p+1}^c)^2 = R^2 \quad (2b)$$

with α the angle between the horizontal axis and the ray. It verifies: $\alpha = \theta_{B_p} + \theta_{B_p}^1 = \theta_{A_{p+1}} + \theta_{A_{p+1}}^1$. (x_{p+1}^c, y_{p+1}^c) are the cylinder coordinates at the $(p+1)^{th}$ collision.

The Jacobian J_p of the total transformation $(\theta_{A_p}, \theta_{A_p}^1) \rightarrow (\theta_{A_{p+1}}, \theta_{A_{p+1}}^1)$ is given by $J_p = J_p^{AB} \cdot J_p^{BA'}$

where $J_p^{BA'}$ is the Jacobian of the transformation $(\theta_{B_p}, \theta_{B_p}^1) \rightarrow (\theta_{A_{p+1}}, \theta_{A_{p+1}}^1)$

$$J_p^{BA'} = \begin{pmatrix} m_n - 1 & m_n \\ 2 - m_p & 1 - m_p \end{pmatrix} \quad (3)$$

with

$$m_p = -\frac{1}{\cos[\theta_{B_p} + \theta_{B_p}^1 - \theta_{A_{p+1}}]} \frac{l_{jump}}{R} \quad (4)$$

where R is the cylinder radius and l_{jump} the distance between the points B_p and A_{p+1} .

J_p^{AB} is the Jacobian of transformation $(\theta_{A_p}, \theta_{A_p}^1) \rightarrow (\theta_{B_p}, \theta_{B_p}^1)$

$$J_p^{AB} = \begin{pmatrix} 1 & \frac{2}{n} \frac{\cos[\theta_{A_p}^1]}{\sqrt{1 - \frac{\sin^2[\theta_{A_p}^1]}{n^2}}} \\ 0 & -1 \end{pmatrix} \quad (5)$$

In the Gauss conditions all Jacobians become independent of n and the Jacobian of the total transformation J reduced to:

$$J_{Gauss} = \begin{pmatrix} \frac{2}{n}t^* + 1 - t^* & 2\left(\frac{n-1}{n}\right) - t^*\left(\frac{n-2}{n}\right) \\ -t^* & 1 - t^* \end{pmatrix} \quad (6)$$

where $t^* = T/R$ is the normalized period of the crystal (T is the crystal period, R is the cylinders radius).

The eigenvalues of the Jacobian in the Gauss conditions are given by:

$$\Lambda_{\pm} = 1 + \frac{t^*}{n} - t^* \pm \frac{1}{n} \sqrt{t^*(n-1)[(n-1)t^* - 2n]} \quad (7)$$

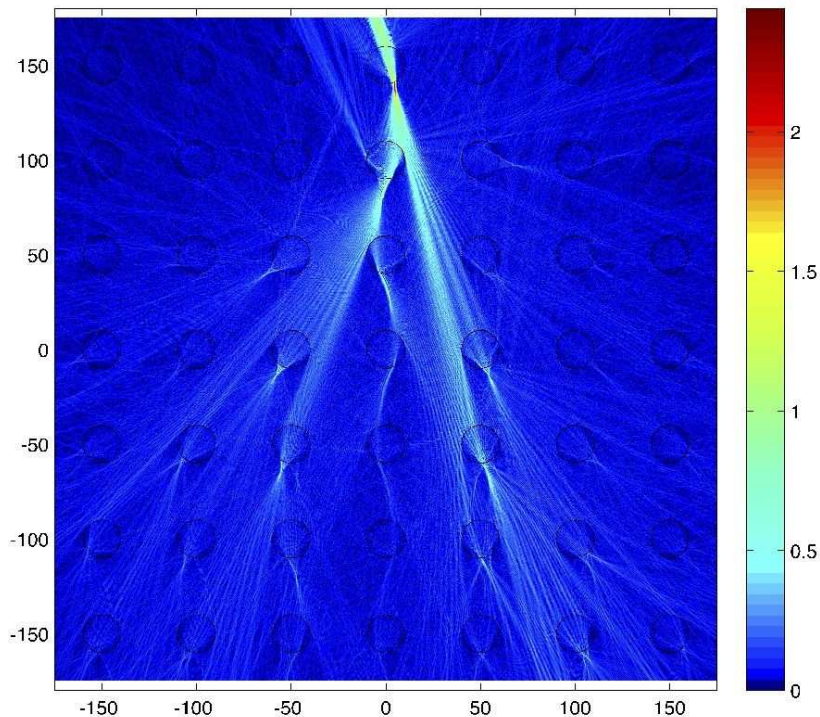


Figure 1: Propagation of a laser beam into a portion of the photonic crystal.

III. PERIOD OF THE BALLISTIC TRAJECTORIES IN THE GAUSS CONDITIONS AND QUASI-PERIODIC TRAJECTORIES;

From Eq:7 it is possible to find the period $P(t^*)$ of the ballistic trajectories in the Gauss conditions. It can be defined as the number of cylinders after which the couple $(\theta_{A_{p+P(t^*)}}, \theta_{A_{p+P(t^*)}^1})$ is equal to $(\theta_{A_p}, \theta_{A_p^1})$. When they are complexes the eigenvalues read:

$$\Lambda_{\pm} = 1 + \frac{t^*}{n} - t^* \pm i \frac{1}{n} \sqrt{t^*(n-1)[2n - (n-1)t^*]}$$

Their modulus is 1 (the map is area preserving). In the eigenbase the map is given by:

$$\begin{pmatrix} \theta_{A_{p+P(t^*)}} \\ \theta_{A_{p+P(t^*)}^1} \end{pmatrix}_{EB} = \begin{pmatrix} e^{i \arg[\Lambda_+]} & 0 \\ 0 & e^{i \arg[\Lambda_-]} \end{pmatrix}^{P(t^*)} \begin{pmatrix} \theta_{A_p} \\ \theta_{A_p^1} \end{pmatrix}_{EB}$$

where EB refers to the eigenbase

Finally we find that the period of the ballistic trajectories in the Gauss conditions is:

$$P(t) = \frac{2\pi}{\arctan\left[\frac{\sqrt{t^*(n-1)[2n - (n-1)t^*]}}{n - t^*(n-1)}\right]} \quad (8)$$

It is plotted in Fig:2 where we compare numerical results in the Gauss conditions and the analytical formula. We can notice an excellent agreement between the numerical results (green dots in Fig:2) and the analytical result (plain line in Fig:2).

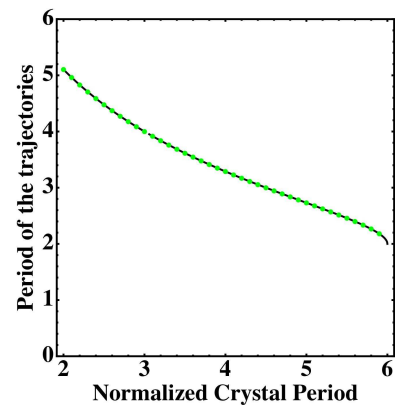


Figure 2: Period of the ballistic trajectories versus the normalized crystal period $t^* = T/R$. Dots are numerical results whereas the plain line is the analytical result eq.(8).

The period of the oscillations $P(t^*)$ is a real number whereas the number of cylinders is of course an integer number. This leads to quasi-periodicity. As a matter of fact, in the eigenbase, for example θ_{A_n} writes $\theta_{A_n} = \theta_{A_0} e^{2i\pi \frac{n}{P(t^*)}}$ where θ_{A_0} is the initial value. If we now

write $P(t^*) = I(t^*) - F(t^*)$ where $I(t^*)$ is the nearest integer from $P(t^*)$ and $F(t^*)$ the fractional part. An easy way to understand the origin of quasi-periodicity is to assume that $F(t^*) \ll I(t^*)$. Then we can write $\theta_{A_n} = \theta_{A_0} e^{2i\pi \frac{n}{I(t^*)}} (1 + 2i\pi \frac{nF(t^*)}{I(t^*)^2})$. The first term $e^{2i\pi \frac{n}{I(t^*)}}$ is a periodic function. The second term $(1 + 2i\pi \frac{nF(t^*)}{I(t^*)^2})$ leads to a slipping in the phase space and the trajectory appears as quasi-periodic. It is illustrated by the Fig:3 for $t^* = 3.2$ and $t^* = 4.2$. The period is $P(3.2) = 3.84$ then $I(3.2) = 4$ and $F(3.2) = 0.16$ and $P(4.2) = 3.17$ with $I(4.2) = 3$ and $F(4.2) = -0.17$. After respectively 4 or 3 steps the point $A_{n+P(t^*)}$ is close to the point A_n but shift from a quantity due to $F(t^*)$ which leads to quasi-periodicity.

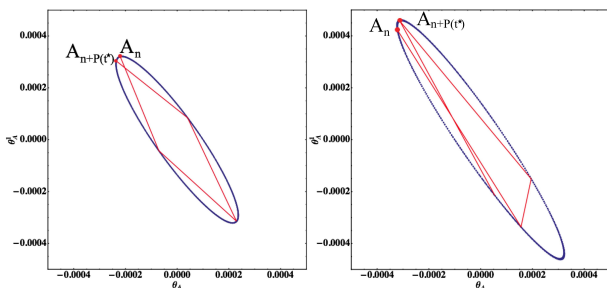


Figure 3: Phase space for $t^* = 3.2$ and $t^* = 4.2$ in Gauss conditions. Periods are respectively $P(3.2) = 3.84$ and $P(3.2) = 3.17$. This leads to ellipsis in the phase space and quasi-periodicity of the ray trajectory in the crystal.

IV. EXISTENCE OF THE ISLANDS OF STABILITY FOR $t^* < \frac{2n}{n-1}$

Ballistic trajectories in the crystal are characterized by closed curves in the phase space (stability islands). Stability islands on their own are characterized by complex eigenvalues of the Jacobian. Thus ballistic trajectories can propagate around the crystal axes ($\theta_A = 0, \pi/2 \text{ mod } [\pi]$) if $t^* < \frac{2n}{n-1}$ and around the bisectrices ($\theta_A = \pi/4 \text{ mod } [\pi]$) if $t^* \sqrt{2} < \frac{2n}{n-1}$. This give the threshold above which islands of stability no more exist and above which the motion of light in the crystal is only chaotic.

V. LYAPUNOV EXPONENT BEHAVIOR AT LARGE t^*

Here we show that the Lyapunov exponent grows asymptotically as $\lambda \sim \ln(t^*)$.

For large normalized period of the crystal $t^* \gg 1$ collisions are uncorrelated. So the Jacobian become independent of the step number. Moreover the incident angle θ_A^1 is close to $0 \text{ mod } \pi$. Then J^{AB} reduces to:

$$J^{AB} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (9)$$

and m_n reduces to $m = -\frac{l_{jump}}{R}$. The distance between two collisions is given by the mean free path $l_{jump} = \bar{l} = \frac{1}{n^d \sigma} = \frac{(t^*)^2 R}{2}$ where n^d is the density of disks $n^d \sim 1/T^2$ (T is the crystal period) and σ the cross section $\sigma = 2R$.

The Jacobian J of the total transformation $J = J^{AB} \cdot J^{BA'}$ is:

$$J = \begin{pmatrix} m-1 & m \\ m-2 & m-1 \end{pmatrix} \quad (10)$$

The eigenvalues are given by: $\Lambda_{\pm} = \frac{Tr(J) \pm \sqrt{Tr(J)^2 - 4}}{2}$. Within the assumption $t^* \gg 1$ the Lyapunov exponent is $\lambda = \ln(\Lambda_+) \sim \ln(t^*)$. So asymptotically it grows as $\ln(t^*)$. Of course the same dependence is found if expanding the positive eigenvalue of the Jacobian Eq:7.

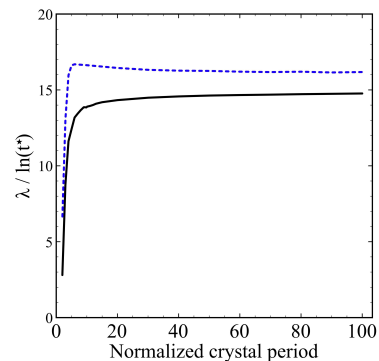


Figure 4: Lyapunov exponent divided by $\ln(t^*)$ versus the normalized crystal period t^* for the refractive Lorentz gas (plain line) and the reflective Lorentz gas (dashed line).

VI. EXPERIMENTAL RESULTS

We have performed experiments to illustrate that light propagation in the photonic crystal is highly sensitive to the initials conditions. We shine a photonic crystal made of 20×20 glass cylinders with a laser beam (cylinder radius ≈ 2 mm, photonic crystal period $T=1.1$ cm leading to $t^* = 2.2$, glass refractive index $n=1.78$, laser wavelength $\lambda=473$ nm). The beam diameter is on the order of 2 mm. It corresponds to a collection of light rays. The dynamics of each of them is described by our model. We start with initial conditions $\theta = 24^\circ$ and $\theta_1 = 24^\circ$. In that case, initial conditions are such that all the laser beam follow a regular path (see Fig: 5)-a). It oscillates with a period close to 4 in agreement with the value given

by formula 8 $P(t^* = 2.2, n = 1.78) = 4.09$. We only increase the incident angle by 3° ($\theta_1 = 37^\circ$). The laser beam starts to follow a regular path (see Fig: 5)-b) but leave the crystal axis and finally diffuse in the entire crys-

tal. By increasing again the initial conditions $\theta_1 = 34^\circ$, the laser beam splits after the first cylinder in several beams that diffuse in the photonic crystal.

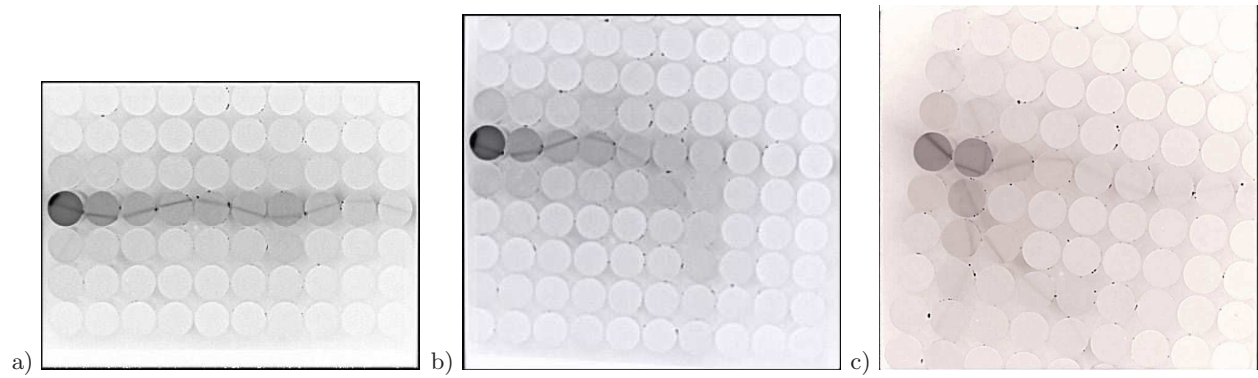


Figure 5: Experimental results showing propagation of a laser beam in the refractive Lorentz gas. The cylinder radius is $R = 0,5 \text{ cm}$, the crystal period is $T = 1.1 \text{ cm}$ leading to $t = 2.2$. a) The initial conditions are $\theta = 24^\circ$ and $\theta_1 = 24^\circ$. The laser beam follows a regular path oscillating back and forth around the crystal axis. b) The initial conditions are $\theta = 24^\circ$ and $\theta_1 = 27^\circ$. The laser beam starts to follow a regular path but after hitting 4 cylinders it leaves the crystal axis and diffuses in the entire crystal. c) The initial conditions are $\theta_0 = 24^\circ$ and $\theta_0^1 = 34^\circ$. The laser beam splits in several beams that diffuse in the photonic crystal.