Boundary singularities of positive solutions of quasilinear Hamilton-Jacobi equations

Marie-Françoise Bidaut-Véron, Marta Garcia-Huidobro, Laurent Véron

Boundary singularities of positive solutions of quasilinear Hamilton-Jacobi equations*

Marie-Françoise Bidaut-Véron
Marta Garcia-Huidobro
Laurent Véron

Abstract We study the boundary behaviour of the solutions of (E) $-\Delta_p u + |\nabla u|^q = 0$ in a domain $\Omega \subset \mathbb{R}^N$, when $N \geq p > q > p-1$. We show the existence of a critical exponent $q_* < p$ such that if $p-1 < q < q_*$ there exist positive solutions of (E) with an isolated singularity on $\partial\Omega$ and that these solutions belong to two different classes of singular solutions. If $q_* \leq q < p$ no such solution exists and actually any boundary isolated singularity of a positive solution of (E) is removable. We prove that all the singular positive solutions are classified according the two types of singular solutions that we have constructed.

2010 Mathematics Subject Classification. 35J62; 35J92; 35B40; 35A20 .

Key words. p-Laplace operator; singularities; spherical p-harmonic equations; Leray-Lions operators; Schauder fixed point theorem.

Contents

*To appear in Calc. Var. Part. Diff. Equ.

1 Introduction

Let $N \geq p > 1$, $q > p-1$ and $\Omega \subset \mathbb{R}^N$ ($N > 1$) be a C^2 bounded domain such that $0 \in \partial\Omega$. In this article we study the boundary behavior at 0 of nonnegative functions $u \in C^1(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ which satisfy

$$-\Delta_p u + |\nabla u|^q = 0 \qquad \text{in } \Omega, \tag{1.1}$$

where $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$. The two main questions we consider are as follows:

Q-1- Existence of positive solutions of (1.1).

Q-2- Description of positive solutions with an isolated boundary singularity at 0.

When $p = 2$ a fairly complete description of positive solutions of

$$-\Delta u + |\nabla u|^q = 0 \tag{1.2}$$

in Ω is provided by Nguyen-Phuoc and Véron [11]. In particular they prove the following series of results in the range of values $1 < q < 2$.

1- Any signed solution of (1.3) verifies the estimates

$$|\nabla u(x)| \leq c_{N,q}\left(d(x)\right)^{-\frac{1}{q-1}} \qquad \forall x \in \Omega, \tag{1.3}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$. As a consequence, if $u \in C(\overline{\Omega} \setminus \{0\})$ is a solution which vanishes on $\partial\Omega \setminus \{0\}$, it satisfies

$$|u(x)| \leq c_{q,\Omega} d(x)|x|^{-\frac{1}{q-1}} \qquad \forall x \in \Omega. \tag{1.4}$$

2- If $\frac{N+1}{N} \leq q < 2$ any positive solution of (1.3) in Ω which vanishes on $\partial\Omega \setminus \{0\}$ is identically 0. *An isolated boundary point is a removable singularity for* (1.2).

3- If $1 < q < \frac{N+1}{N}$ and $k > 0$ there exists a unique positive solution $u := u_k$ of (1.2) in Ω which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies $u(x) \sim c_N k P^\Omega(x, 0)$ as $x \to 0$, where P^Ω is the Poisson kernel in $\Omega \times \partial\Omega$.

4- If $1 < q < \frac{N+1}{N}$ there exists a unique positive solution u of (1.2) in the half-space $\mathbb{R}^N_+ := \{x = (x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > 0\}$ under the form $u(x) = |x|^{-\frac{2-q}{q-1}}\omega(|x|^{-1}x)$ which vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$. The function ω is the unique positive solution of

$$\begin{aligned} -\Delta'\omega + \left(\left(\tfrac{2-q}{q-1}\right)^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{q}{2}} - \lambda_{N,q}\omega = 0 \qquad &\text{in } S^{N-1}_+, \\ \omega = 0 \qquad &\text{in } \partial S^{N-1}_+, \end{aligned} \tag{1.5}$$

where S^{N-1} is the unit sphere of $\mathbb{R}^N$, $\partial S^{N-1}_+ = \partial\mathbb{R}^N_+ \cap S^{N-1}$, Δ' the Laplace-Beltrami operator and $\lambda_{N,q} > 0$ an explicit constant.

5- If $1 < q < \frac{N+1}{N}$ and u is a positive solution of (1.3) in Ω, which is continuous in $\overline{\Omega} \setminus \{0\}$ and vanishes on $\partial\Omega \setminus \{0\}$ the following dichotomy occurs:

(i) either $u(x) \sim |x|^{-\frac{2-q}{q-1}}\omega(|x|^{-1}x)$ as $x \to 0$,

(ii) or $u(x) \sim k c_N P^\Omega(x, 0)$ as $x \to 0$ for some $k \geq 0$.

The aim of this article is to extend to the quasilinear case $1 < p \leq N$ the above mentioned results. The following pointwise gradient estimate valid for any signed solution u of (1.1) has been proved in [3]: if $0 < p-1 < q$ there exists a constant $c_{N,p,q} > 0$ such that

$$|\nabla u(x)| \leq c_{N,p,q}(d(x))^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega. \tag{1.6}$$

As a consequence, any solution $u \in C^1(\overline{\Omega} \setminus \{0\}$ satisfies

$$|u(x)| \leq c_{p,q,\Omega} d(x)\, |x|^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega. \tag{1.7}$$

Concerning boundary singularities, the situation is much more complicated than in the case $p = 2$ and the threshold of critical exponent less explicit. We first consider the problem in $\mathbb{R}^N_+$. Assuming $p-1 < q \leq p$, separable solutions of (1.1) in $\mathbb{R}^N_+$ vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ can be looked for in spherical coordinates $(r, \sigma) \in \mathbb{R}^*_+ \times S^{N-1}$ (we denote $\mathbb{R}^*_+ = (0, \infty)$) under the form

$$u(x) = u(r, \sigma) = r^{-\beta_q}\omega(\sigma), \quad r > 0\,,\ \sigma \in S^{N-1}_+ := \{S^{N-1} \cap \mathbb{R}^N_+\}. \tag{1.8}$$

Then ω is solution of the following problem

$$\begin{aligned} -div'\left(\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla'\omega\right) - \beta_q\Lambda_{\beta_q}\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}}\omega \\ + \left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{q}{2}} = 0 \quad \text{in } S^{N-1}_+ \\ \omega = 0 \quad \text{on } \partial S^{N-1}_+, \end{aligned} \tag{1.9}$$

where

$$\beta_q = \frac{p-q}{q+1-p} \text{ and } \Lambda_{\beta_q} = \beta_q(p-1) + p - N, \tag{1.10}$$

and ∇' is the covariant derivative on S^{N-1} identified to the tangential gradient thanks to the canonical isometrical imbedding of S^{N-1} into $\mathbb{R}^N$, and div' the divergence operator acting on vector fields on S^{N-1}.

The existence of a positive solution to this problem cannot be separated from the problem of existence of *separable p-harmonic functions* which are p-harmonic in $\mathbb{R}^N_+$ which vanish on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and have the form $\Psi(x) = \Psi(r, \sigma) = r^{-\beta}\psi(\sigma)$ for some real number β. Necessarily such a ψ must satisfy

$$\begin{aligned} -div'\left(\left(\beta^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi\right) - \beta\Lambda_\beta\left(\beta^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi = 0 \qquad \text{in } S^{N-1}_+ \\ \psi = 0 \qquad \text{on } \partial S^{N-1}_+, \end{aligned} \tag{1.11}$$

where $\Lambda_\beta = \beta(p-1) + p - N$. We will refer to (1.11) as *the spherical p-harmonic eigenvalue problem*. The study of this problem has been initiated in the 2-dim case by Krol [8] ($\beta < 0$) and Kichenassamy and Véron [9] ($\beta > 0$). In this case ω satisfies a completely integrable second order differential equation. In the case where S^{N-1}_+ is replaced by a smooth domain $S \subset S^{N-1}$ with $N \geq 3$, Tolksdorf [14] proved the existence of a unique couple $(\tilde\beta_s, \tilde\psi_s)$ where $\tilde\beta_s < 0$ and $\tilde\psi_s$ has constant sign and is defined up to an homothety. Recently Porretta and Véron [12] gave a simpler and more general proof of the existence of two couples $(\tilde\beta_s, \tilde\psi_s)$ and $(\beta_{*\,s}, \psi_{*\,s})$ where $\beta_{*\,s} > 0$ and $\tilde\psi_s$ and $\psi_{*\,s}$ are positive solutions of (1.11) with

$\beta = \tilde{\beta}_s$ and $\beta = \beta_{*\,s}$ respectively and are unique up to a multiplication by a real number. When $p = 2$ this problem is an eigenvalue problem for the Laplace-Beltrami operator on a subdomain of S^{N-1}. If $S = S^{N-1}_+$, $\tilde{\beta}_s$ and $\beta_{*\,s}$ are respectively denoted by $\tilde{\beta}$ and β_* and accordingly $\tilde{\psi}_s$ and $\psi_{*\,s}$ by $\tilde{\psi}$ and ψ_*. Since $x \mapsto x_N$ is p-harmonic, $\tilde{\beta} = -1$. Except in the cases $N = 2$ where it is the positive root of some algebraic equation of degree 2, $p = 2$ where it is $N - 1$ and $p = N$ where it is 1, the value of β_* is unknown besides the straightforward estimate $\beta_* \geq \max\{1, \frac{N-p}{p-1}\}$. Using the fact that ψ_* depends only on the azimuthal variable and satisfies a differential equation, we prove in Appendix II the following new estimate:

Theorem A *Let $1 < p \leq N$.*
(i) If $2 \leq p \leq N$, then $\beta_* \leq \frac{N-1}{p-1}$ with equality only if $p = 2$ or N.
(ii) If $1 \leq p < 2$, then $\beta_* > \frac{N-1}{p-1}$.

The p-harmonic function $\Psi_*(x) = \Psi_*(r, \sigma) = r^{-\beta_*}\psi_*(\sigma)$ endows the role of a Poisson kernel. To this exponent β_* is associated the critical value q_* of q defined by $\beta_* = \beta_q$, or equivalently

$$q_* := \frac{\beta_*(p-1)+p}{\beta_*+1} = p - \frac{\beta_*}{\beta_*+1}. \tag{1.12}$$

The following result characterizes strong singularities.

Theorem B *Let $0 < p - 1 \leq N$, then*
(i) If $p - 1 < q < q_*$ problem (1.9) admits a unique positive solution ω_*.
(ii) If $q_* \leq q < p$ problem (1.9) admits no positive solution.

This critical exponent corresponds to the threshold of criticality for boundary isolated singularities.

Theorem C *Assume $q_* \leq q < p \leq N$. If $u \in C^1(\overline{\Omega} \setminus \{0\})$ is a nonnegative solution of (1.1) in Ω which vanishes on $\partial\Omega \setminus \{0\}$, it is identical zero.*

As in the case $p = 2$, there exist positive solutions (1.1) in Ω with weak boundary singularities which are characterized by their blow-up near the singularity. By opposition to the case $p = 2$ where existence is obtained by use of a weak formulation of the boundary value problem, combined with uniform integrability of the absorption term thanks to Poisson kernel estimates (see [11]), this approach cannot be performed in the case $p \neq 2$; the obtention of solutions with weak singularities necessitates a very long and delicate construction of subsolutions and supersolutions. Furthermore, when $p \neq N$, the construction is done only if Ω is locally an hyperplane near 0.

In the sequel we denote by $B_R(a)$ the open ball of center a and radius $R > 0$ and $B_R = B_R(0)$. We also set $B^+_R(a) := \mathbb{R}^N_+ \cap B_R(a)$, $B^+_R := \mathbb{R}^N_+ \cap B_R$, $B^-_R(a) := \mathbb{R}^N_- \cap B_R(a)$ and $B^-_R := \mathbb{R}^N_- \cap B_R$, where $\mathbb{R}^N_- := \{x = (x', x_N) : x' \in \mathbb{R}^{N-1}, x_N < 0\}$. If Ω is an open domain and $R > 0$, we put $\Omega_R = \Omega \cap B_R$
.

Theorem D *Let $\Omega \subset \mathbb{R}^N_+$ be a bounded domain such that $0 \in \partial\Omega$. Assume there exists $\delta > 0$ such that $\Omega_\delta = B^+_\delta$ and $0 < p - 1 < q < q_* < p \leq N$. Then for any $k > 0$ there exists a unique $u := u_k \in C^1(\overline{\Omega} \setminus \{0\})$, solution of (1.1) in Ω, vanishing on $\partial\Omega \setminus \{0\}$ and such that*

$$\lim_{\substack{x \to 0 \\ \frac{x}{|x|} \to \sigma \in S^{N-1}_+}} |x|^{\beta_*} u_k(x) = k\psi_*(\sigma). \tag{1.13}$$

Furthermore $\lim_{k\to\infty} u_k = u_\infty$ *and*

$$\lim_{\substack{x\to 0\\ \frac{x}{|x|}\to\sigma\in S^{N-1}_+}} |x|^{\beta_q} u_\infty(x) = \psi_*(\sigma). \tag{1.14}$$

When $p = N$, then $q_* = N - \frac{1}{2}$; in such a range of values we use the conformal invariance of Δ_N and prove that the previous result holds if Ω is *any* C^2 domain. Finally, the isolated singularities of positive solutions of (1.1) are completely described by the two types of singular solutions obtained in the previous theorem and we prove:

Theorem E *Let Ω be a bounded domain such that $0 \in \partial\Omega$. Assume there exists $\delta > 0$ such that $\Omega_\delta = B^+_\delta$ and $0 < p - 1 < q < q_* < p \le N$. If $u \in C^1(\overline{\Omega} \setminus \{0\})$ is a positive solution of (1.1) in Ω which vanishes on $\partial\Omega \setminus \{0\}$, then*

(i) either there exists $k \ge 0$ such that

$$\lim_{\substack{x\to 0\\ \frac{x}{|x|}\to\sigma\in S^{N-1}_+}} |x|^{\beta_*} u(x) = k\psi_*(\sigma); \tag{1.15}$$

(ii) or

$$\lim_{\substack{x\to 0\\ \frac{x}{|x|}\to\sigma\in S^{N-1}_+}} |x|^{\beta_q} u(x) = \psi_*(\sigma). \tag{1.16}$$

Acknowledgements This article has been prepared with the support of the MathAmsud collaboration program 13MATH-03 QUESP. The first two authors were supported by Fondecyt grant N°1110268. The authors are grateful to the referee for a careful reading of the manuscript.

2 A priori estimates

2.1 The gradient estimates and its applications

We recall the following estimate and its consequences which are proved in [3].

Proposition 2.1. *Assume $q > p - 1$ and u is a C^1 solution of (1.1) in a domain Ω. Then*

$$|\nabla u(x)| \le c_{N,p,q}(d(x))^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega. \tag{2.1}$$

The first application is a pointwise upper bound for solutions with isolated singularities.

Corollary 2.2. *Assume $q > p - 1 > 0$, $R^* > 0$ and Ω is a domain containing 0 such that $d(0) \ge 2R^*$. Then for any $x \in B_{R^*} \setminus \{0\}$, and $0 < R \le R^*$, any $u \in C^1(\Omega \setminus \{0\})$ solution of (1.1) in $\Omega \setminus \{0\})$ satisfies*

$$|u(x)| \le c_{N,p,q}\left||x|^{\frac{q-p}{q+1-p}} - R^{\frac{q-p}{q+1-p}}\right| + \max\{|u(z)| : |z| = R\}, \tag{2.2}$$

if $p \ne q$, and

$$|u(x)| \le c_{N,p}\left(\ln R - \ln|x|\right) + \max\{|u(z)| : |z| = R\}, \tag{2.3}$$

if $p = q$.

The second application corresponds to solutions with boundary blow-up. For $\delta > 0$ small enough we set $\Omega_\delta := \{z \in \Omega : d(z) < \delta\}$.

Corollary 2.3. *Assume $q > p - 1 > 0$, Ω is a bounded domain with a C^2 boundary. Then there exists $\delta_1 > 0$ which depends only on Ω such that any $u \in C^1(\Omega)$ solution of (1.1) in Ω satisfies*

$$|u(x)| \le c_{N,p,q} \left| (d(x))^{\frac{q-p}{q+1-p}} - \delta_1^{\frac{q-p}{q+1-p}} \right| + \max\{|u(z)| : d(z) = \delta_1\} \quad \forall x \in \Omega_{\delta_1} \tag{2.4}$$

if $p \neq q$, and

$$|u(x)| \le c_{N,p,q} \left(\ln \delta_1 - \ln d(x)\right) + \max\{|u(z)| : d(z) = \delta_1\} \quad \forall x \in \Omega_{\delta_1} \tag{2.5}$$

if $p = q$.

Remark. As a consequence of (2.4) there holds for $p > q > p - 1$

$$u(x) \le \left(c_{N,p,q} + K \max\{|u(z)| : d(z) \ge \delta_1\}\right) (d(x))^{\frac{q-p}{q+1-p}} \quad \forall x \in \Omega \tag{2.6}$$

where $K = (\mathrm{diam}(\Omega))^{\frac{p-q}{q+1-p}}$, with the standard modification if $p = q$.

As a variant of Corollary 2.3 the following upper estimate of solutions in an exterior domain will be used in the sequel.

Corollary 2.4. *Assume $q > p - 1 > 0$, $R > 0$ and $u \in C^1(B^c_{R_0})$ is any solution of (1.1) in $B^c_{R_0}$. Then for any $R > R_0$ there holds*

$$|u(x)| \le c_{N,p,q} \left| (|x| - R_0)^{\frac{q-p}{q+1-p}} - (R - R_0)^{\frac{q-p}{q+1-p}} \right| + \max\{|u(z)| : |z| = R\} \quad \forall x \in B^c_R \tag{2.7}$$

if $p \neq q$ and

$$|u(x)| \le c_{N,p,q} \left(\ln(|x| - R_0) - \ln(R - R_0)\right) + \max\{|u(z)| : |z| = R\} \quad \forall x \in B^c_R \tag{2.8}$$

if $p = q$.

Proof. The proof is a consequence of the identity

$$u(x) = u(z) + \int_0^1 \frac{d}{dt} u(tx + (1-t)z) dt = \int_0^1 \langle \nabla u(tx + (1-t)z), x - z \rangle dt$$

where $z = \frac{R}{|x|} x$. Since by (2.1)

$$|\nabla u(tx + (1-t)z)| \le C_{N,p,q} (t\,|x| + (1-t)R - R_0)^{-\frac{1}{q+1-p}},$$

(2.7) and $(2.8$) follow by integration. □

2.2 Boundary a priori estimates

The next result is the extension to isolated boundary singularities of a previous regularity estimate dealing with singularity in a domain proved in [3, Lemma 3.10].

Lemma 2.5. *Assume $p-1<q<p$, Ω is a bounded C^2 domain such that $0\in\partial\Omega$. Let $u\in C^1(\overline{\Omega}\setminus\{0\})$ be a solution of (1.1) in Ω which vanishes on $\partial\Omega\setminus\{0\}$ and satisfies*

$$|u(x)|\le\phi(|x|)\qquad\forall x\in\Omega, \tag{2.9}$$

where $\phi:\mathbb{R}_+^*\mapsto\mathbb{R}_+$ is continuous, nonincreasing and satisfies

$$\phi(rs)\le\gamma\phi(r)\phi(s)\quad\text{and}\quad r^{\frac{p-q}{q+1-p}}\phi(r)\le c, \tag{2.10}$$

for some $\gamma,c>0$ and any $r,s>0$. There exist $\alpha\in(0,1)$ and $c_1=c_1(p,q,\Omega)>0$ such that

$$\begin{array}{lll}(i) & |\nabla u(x)|\le c_1\phi(|x|)\,|x|^{-1} & \forall x\in\Omega,\\ (ii) & |\nabla u(x)-\nabla u(y)|\le c_1\phi(|x|)\,|x|^{-1-\alpha}\,|x-y|^{\alpha} & \forall x,y\in\Omega,\ |x|\le|y|\,.\end{array} \tag{2.11}$$

Furthermore

$$u(x)\le c_1\phi(|x|)\frac{d(x)}{|x|}\qquad\forall x\in\Omega. \tag{2.12}$$

Proof. For $\ell>0$, we set $\Omega^\ell:=\frac{1}{\ell}\Omega$. If $\ell\in(0,1]$ the curvature of $\partial\Omega^\ell$ remains uniformly bounded. As in [5, p 622], there exists $0<\delta_0\le1$ and an involutive diffeomorphism ψ from $\overline{B}_{\delta_0}\cap\overline{\Omega}^{\delta_0}$ into $\overline{B}_{\delta_0}\cap(\Omega^{\delta_0})^c$ which is the identity on $\overline{B}_{\delta_0}\cap\partial\Omega^{\delta_0}$ and such that $D\psi(\xi)$ is the symmetry with respect to the tangent plane $T_\xi\partial\Omega$ for any $\xi\in\partial\Omega\cap\overline{B}_{\delta_0}$. We extend any function v defined in $\overline{B}_{\delta_0}\cap\overline{\Omega}^{\delta_0}$ and vanishing on $\overline{B}_{\delta_0}\cap\partial\Omega^{\delta_0}$ into a function $\tilde v$ defined in $\overline{B}_{\delta_0}$ by

$$\tilde v(x)=\begin{cases} v(x) & \text{if } x\in\overline{B}_{\delta_0}\cap\overline{\Omega}^{\delta_0}\\ -v\circ\psi(x) & \text{if } x\in\overline{B}_{\delta_0}\cap(\Omega^{\delta_0})^c,\end{cases} \tag{2.13}$$

If $v\in C^1(\overline{B}_{\delta_0}\cap\overline{\Omega}^{\delta_0})$ is a solution of (1.1) in $B_{\delta_0}\cap\Omega^{\delta_0}$ which vanishes on $\partial\Omega^{\delta_0}\cap\overline{B}_{\delta_0}$, $\tilde v$ satisfies

$$-\sum_j\frac{\partial}{\partial x_j}\tilde A_j(x,\nabla\tilde v)+B(x,\nabla\tilde v)=0\quad\text{in } B_{\delta_0}. \tag{2.14}$$

As in [5, (2.37)] the A_j and B satisfy the following estimates

$$\begin{array}{ll}(i) & \tilde A_j(x,0)=0\\ (ii) & \displaystyle\sum_{i,j}\frac{\partial}{\partial\eta_i}\tilde A_j(x,\eta)\xi_i\xi_j\ge C_1\,|\eta|^{p-1}\,|\xi|^2\\ (iii) & \displaystyle\sum_{i,j}\left|\frac{\partial}{\partial\eta_j}\tilde A_j(x,\eta)\right|\le C_2\,|\eta|^{p-2}\,,\end{array} \tag{2.15}$$

and

$$|B(x,\eta)| \le C_3(1+|\eta|)^p, \tag{2.16}$$

where the C_j are positive constants. These estimates are the ones needed to apply Tolksdorf's result [15, Th 1,2]. There exists a constant C, such that for any ball $\overline{B}_{3R} \subset \overline{B}_{\delta_0}$, there holds

$$\|\nabla \tilde{v}\|_{L^\infty(B_R)} \le C, \tag{2.17}$$

where C depends on the constants C_k ($k=1,2,3$), N, p and $\|\tilde{v}\|_{L^\infty(B_{3R})}$. We define

$$\Phi_\ell[u](y) := u_\ell = \frac{1}{\phi(\ell)} u(\ell y) \qquad \forall y \in \Omega^\ell. \tag{2.18}$$

Then

$$|u_\ell(y)| \le \frac{\phi(\ell\,|y|)}{\phi(\ell)} \le \gamma \phi(|y|) \qquad \forall y \in \Omega^\ell \tag{2.19}$$

and

$$-\Delta_p u_\ell + (\ell^{\beta_q}\phi(\ell))^{q+1-p}\,|\nabla u_\ell|^q = 0 \qquad \text{in } \Omega^\ell. \tag{2.20}$$

Using formula (2.13) we extend u_ℓ into a function $\tilde{u}_\ell$ which satisfies

$$-\sum_j \frac{\partial}{\partial y_j}\tilde{A}_j(y,\nabla\tilde{u}_\ell) + (\ell^{\beta_q}\phi(\ell))^{q+1-p} B(y,\nabla\tilde{u}_\ell) = 0 \quad \text{in } B_{\delta_0}. \tag{2.21}$$

For $0 < |x| < \delta_0$ there exists $\ell \in (0,2)$ such that $\frac{\delta_0 \ell}{2} \le |x| \le \delta_0 \ell$. Then $y \mapsto \tilde{u}_\ell(y)$ with $y = \frac{x}{\ell}$ satisfies (2.21) in B_{δ_0} and $|\tilde{u}_\ell(y)| \le \gamma_* \phi(|y|)$ since ψ is a diffeomorphism and $D\psi(\xi) \in O(N)$ for any $\xi \in \partial\Omega \cap B_{\delta_0}$. The function $\tilde{u}_\ell$ remains bounded on any ball $B_{3R}(z) \subset \Gamma := \{y \in \mathbb{R}^N : \frac{\delta_0}{2} < |y| < \delta_0\}$, therefore $|\nabla \tilde{u}_\ell(y)| \le c$ for any $y \in B_R(z)$, for some constant $c > 0$. This implies

$$|\nabla u(x)| \le c\gamma_* \delta_0 \phi(\tfrac{2}{\delta_0})\phi(|x|)|x|^{-1} \quad \forall x \in \Omega \cap B_{\delta_0}, \tag{2.22}$$

which is (2.11)-(i). Moreover, by standard regularity estimates [10], there exists $\alpha \in (0,1)$ such that $|\nabla\tilde{u}_\ell(y) - \nabla\tilde{u}_\ell(y')| \le c\,|y-y'|^\alpha$ for all y and y' belonging to $B_R(z)$. This implies (2.11)-(ii).

Next we prove (2.12). Let $0 < \delta_1 \le \delta_0$ such that at any boundary point z there exist two closed balls of radius δ_1 tangent to $\partial\Omega$ at z and which are included in $\Omega \cup \{z\}$ and in $\overline{\Omega}^c \cup \{z\}$ respectively (δ_1 corresponds to the maximal radius of the interior and exterior sphere condition). Let $x \in \Omega$ such that $d(x) \le \delta_1$ (this is not a loss of generality) and z_x be the projection of x on $\partial\Omega$. We first assume that x does not belong to the cone $\Sigma_{\frac{\pi}{4}}$ with vertex 0, axis $-\mathbf{n}_0$, where $\mathbf{n}_0$ is the normal outward unit vector at 0, and angle $\frac{\pi}{4}$. Consider the path ζ from z_x to x defined by $\zeta(t) = tx + (1-t)z_x$ with $0 \le t \le 1$. Then

$$u(x) = \int_0^1 \frac{d}{dt} u\circ\zeta(t)dt = \int_0^1 \langle \nabla u \circ \zeta(t), x - z_x\rangle dt \tag{2.23}$$

Thus, by the Cauchy-Schwarz inequality, using (2.11),

$$|u(x)| \le c_1 d(x) \int_0^1 \frac{\phi(|\zeta(t)|)}{|\zeta(t)|}dt. \tag{2.24}$$

Since $x \notin \Sigma_{\frac{\pi}{4}}, \zeta(t) \notin \Sigma_{\frac{\pi}{4}}$ and there exists $c_2 > 0$ depending on Ω such that $c_2^{-1}|x| \leq |\zeta(t)| \leq c_2|x|$ for all $0 \leq t \leq 1$. Therefore $\phi(|\zeta(t)|) \leq \phi(c_2|x|) \leq \gamma\phi(c_2)\phi(|x|)$ by (2.10). This implies

$$|u(x)| \leq \gamma c_1 c_2 \phi(c_2) \frac{d(x)\phi(|x|)}{|x|} \tag{2.25}$$

by (2.12) whenever $x \notin \Sigma_{\frac{\pi}{4}}$. When $x \in \Sigma_{\frac{\pi}{4}}$ then $d(x) \leq |x| \leq c_3 d(x)$ where $c_3 > 0$ depends on the curvature of $\partial\Omega$. Then (2.9) combined with (2.10) implies the claim. □

Lemma 2.6. *Assume $p-1 < q \leq p$, Ω is a bounded C^2 domain such that $0 \in \partial\Omega$ and $R_0 = \max\{|z| : z \in \Omega\}$. If $u \in C(\overline{\Omega}\setminus\{0\}) \cap C^1(\Omega)$ is a positive solution of (1.1) which vanishes on $\partial\Omega\setminus\{0\}$, it satisfies*

$$u(x) \leq \begin{cases} c_2\left(|x|^{\frac{q-p}{q+1-p}} - R_0^{\frac{q-p}{q+1-p}}\right) & \text{if } q < p \\ (p-1)\ln\left(\frac{R_0}{|x|}\right) & \text{if } q = p \end{cases} \tag{2.26}$$

for all $x \in \Omega$, where $c_2 = c_2(p,q) > 0$.

Proof. For $\epsilon > 0$ we denote by $P_\epsilon : \mathbb{R} \mapsto \mathbb{R}_+$ the function defined by

$$P_\epsilon(r) = \begin{cases} 0 & \text{if } 0 \leq r \leq \epsilon \\ -\frac{r^4}{2\epsilon^3} + \frac{3r^3}{\epsilon^2} - \frac{6r^2}{\epsilon} + 5r - \frac{3\epsilon}{2} & \text{if } \epsilon < r < 2\epsilon \\ r - \frac{3\epsilon}{2} & \text{if } r \geq 2\epsilon, \end{cases} \tag{2.27}$$

and by u_ϵ the extension of $P_\epsilon(u)$ by zero outside Ω. There exists R_0 such that $\Omega \subset B_{R_0}$. Since $0 \leq P_\epsilon(r) \leq |r|$ and P_ϵ is convex, $u_\epsilon \in C(\mathbb{R}^N \setminus \{0\}) \cap W^{1,p}_{loc}(\mathbb{R}^N \setminus \{0\})$ and

$$-\Delta_p u_\epsilon + |\nabla u_\epsilon|^q \leq 0 \qquad \text{in } \mathbb{R}^N.$$

Let $R > R_0$. If $p-1 < q < p$

$$U_{\epsilon,R}(|x|) = c_2\left((|x|-\epsilon)^{\frac{q-p}{q+1-p}} - (R-\epsilon)^{\frac{q-p}{q+1-p}}\right) \qquad \text{in } B_R \setminus B_\epsilon, \tag{2.28}$$

with $c_2 = (p-q)^{-1}(q+p-1)^{\frac{q-p}{q+1-p}}$. Then $-\Delta_p U_{\epsilon,R} + |\nabla U_{\epsilon,R}|^q \geq 0$. Since u_ϵ vanishes on ∂B_R and is finite on ∂B_ϵ, it follows $u_\epsilon \leq U_{\epsilon,R}$. Letting successively $\epsilon \to 0$ and $R \to R_0$ yields to (2.26). If $q = p$ we take

$$U_{\epsilon,R}(|x|) = (p-1)\ln\left(\frac{R-\epsilon}{|x|-\epsilon}\right) \qquad \text{in } B_R \setminus B_\epsilon, \tag{2.29}$$

which turns out to be a supersolution of (1.1); the end of the proof is similar. □

As a consequence of Lemma 2.5 and Lemma 2.6, we obtain.

Corollary 2.7. *Let p, q Ω and u be as in Lemma 2.6. Then there exists a constant $c_3 = c_3(p,q,\Omega) > 0$ such that*

$$|\nabla u(x)| \leq c_3 |x|^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega \tag{2.30}$$

and

$$u(x) \leq c_3 d(x) |x|^{-\frac{1}{q+1-p}} \qquad \forall x \in \overline{\Omega}\setminus\{0\}. \tag{2.31}$$

Remark. If Ω is locally flat near 0, then estimates (2.30) and (2.31) are valid without any sign assumption on u. More precisely, if $\partial\Omega \cap B_{\delta_0} = T_0\partial\Omega \cap B_{\delta_0}$ we can perform the reflection of u through the tangent plane $T_0\partial\Omega$ to $\partial\Omega$ at 0 and the new function $\tilde{u}$ is a solution of (1.1) in $B_{\delta_0} \setminus \{0\}$. By Proposition 2.1, it satisfies

$$|\nabla \tilde{u}(x)| \le c_{N,p,q}\, |x|^{-\frac{1}{q+1-p}} \qquad \forall x \in B_{\frac{\delta_0}{2}} \setminus \{0\}. \tag{2.32}$$

Integrating this relation as in [3], we derive that for any $x \in B_{\frac{\delta_0}{2}} \cap \Omega$, there holds

$$|u(x)| \le \begin{cases} c_{N,p,q}\left(|x|^{-\beta_q} - (\frac{\delta_0}{2})^{-\beta_q}\right) + \max\{|u(z)| : |z| = \frac{\delta_0}{2}\} & \text{if } p \ne q \\ c_{N,p} \ln\left(\frac{\delta_0}{2|x|}\right) + \max\{|u(z)| : |z| = \frac{\delta_0}{2}\} & \text{if } p = q. \end{cases} \tag{2.33}$$

In the next result we allow the boundary singular set to be a compact set.

Proposition 2.8. *Let $p-1 < q < p$ and δ_1 as above. There exist $r^* \in (0, \delta_1]$ and $c_4 = c_4(N,p,q) > 0$ such that for any nonempty compact set $K \subset \partial\Omega$, $K \ne \partial\Omega$ and any positive solution $u \in C(\overline{\Omega} \setminus K) \cap C^1(\Omega)$ of (1.1) which vanishes on $\partial\Omega \setminus K$, there holds*

$$u(x) \le c_4 d(x) (d_K(x))^{-\frac{1}{q+1-p}} \qquad \forall x \in \partial\Omega \text{ s.t. } d(x) \le r^*, \tag{2.34}$$

where $d_K(x) = \text{dist}\,(x,K)$.

Proof. Step 1: Tangential estimates. Let $x \in \Omega$ such that $d(x) \le \delta_1$. We denote by $\sigma(x)$ the projection of x onto $\partial\Omega$, unique since $d(x) \le \delta_1$. Let $r\,, r', \tau > 0$ such that $\frac{3}{4}r < r' < \frac{7}{8}r$ and $0 < \tau \le \frac{r'}{2}$ and put $\omega_{\tau,x} = \sigma(x) + \tau \mathbf{n}_{\sigma(x)}$. Since $\partial\Omega$ is C^2, there exists $0 < r^* \le \delta_1$ depending on Ω such that $d_K(\omega_{\tau,x}) > \frac{7}{8}r$ whenever $d(x) \le r^*$. Let $a > 0$ and $b > 0$ to be specified later on; we define $\tilde{v}(s) = a(r'-s)^{\frac{q-p}{q+1-p}} - b$ and $v(y) = \tilde{v}(|y - \omega_{\tau,x}|)$ in $[0, r')$ and $B_{r'}(\omega_{\tau,x})$ respectively. Then

$$|\tilde{v}'|^{p-2}\left(|\tilde{v}'|^{q+2-p} - (p-1)\tilde{v}'' - \frac{N-1}{s}\tilde{v}'\right) = a^{p-1}\left(\frac{p-q}{q+1-p}\right)^{p-1}(r'-s)^{-\frac{q}{q+1-p}} X(s)$$

where

$$X(s) = \left(a\frac{p-q}{q+1-p}\right)^{q+1-p} - \frac{p-1}{q+1-p} - \frac{(N-1)(r'-s)}{s}.$$

For any $\tau \in (0, r')$ there exists $a > 0$ such that

$$\left(a\frac{p-q}{q+1-p}\right)^{q+1-p} \ge \frac{p-1}{q+1-p} + \frac{(N-1)(r'-s)}{s} \qquad \forall \tau \le s \le r'.$$

This implies

$$-\Delta_p v + |\nabla v|^q \ge 0 \qquad \text{in } B_{r'}(\omega_{\tau,x}) \setminus B_\tau(\omega_{\tau,x}). \tag{2.35}$$

Next we take $b = a(r'-\tau)^{\frac{q-p}{q+1-p}}$, thus $v = 0$ on $\partial B_\tau(\omega_{\tau,x})$. Clearly $B_\tau(\omega_{\tau,x}) \subset \overline{\Omega}^c$ since $\tau < \delta_1$. Therefore $v \ge 0 = u$ on $\partial\Omega \cap B_{r'}(\omega_{\tau,x})$ and $u \le v = \infty$ on $\Omega \cap \partial B_{r'}(\omega_{\tau,x})$. By the comparison principle, $v \ge u$ in $\Omega \cap B_{r'}(\omega_{\tau,x})$. In particular

$$u(x) \le v(x) \le a(r' - \tau - d(x))^{\frac{q-p}{q+1-p}} - a(r'-\tau)^{\frac{q-p}{q+1-p}}.$$

We take now $\tau = \frac{r'}{2}$ and $d(x) \leq \frac{r}{4}$ and we derive by the mean value theorem

$$u(x) \leq c_4' r'^{-\frac{1}{q+1-p}} d(x) = c_4' d(x)(d_K(x))^{-\frac{1}{q+1-p}}, \tag{2.36}$$

with $c_4' = c_4'(p,q) > 0$ Letting $r' \to \frac{7}{8}r$, we get (2.12).
Step 2: Global estimates. If $d(x) \geq \frac{1}{4} d_K(x)$, there holds

$$d(x)(d_K(x))^{-\frac{1}{q+1-p}} \geq 2^{-\frac{2}{q+1-p}} (d(x))^{\frac{q-p}{q+1-p}}.$$

Combining this inequality with (2.6) and obtain (2.34). □

Remark. Under the assumption of Proposition 2.8, it follows from the maximum principle that u is upper bounded in the set $\Omega'_{r^*} := \{x \in \Omega : d(x) > r^*\} = \Omega \setminus \overline{\Omega}_{r^*}$ by the solution w of

$$\begin{aligned} -\Delta_p w + |\nabla w|^q &= 0 && \text{in } \Omega_{r^*} \\ w &= c_4 d(x)(d_K(x))^{-\frac{1}{q+1-p}} && \text{in } \partial\Omega_{r^*}, \end{aligned} \tag{2.37}$$

and w itself is bounded by $d^* = \max\{cd(x)(d_K(x))^{-\frac{1}{q+1-p}} : d(x) = r^*\}$.

Next we prove a boundary Harnack inequality. We recall that δ_1 has been introduced at Corollary 2.3, and that the interior and exterior sphere conditions hold in the set $\{x \in \mathbb{R}^N : d(x) \leq \delta_1\}$.

Theorem 2.9. *Let $q > p-1$ and $0 \in \partial\Omega$. Then there exists $c_5 = c_5(N,p,q,\Omega) > 0$ such that for any positive solution $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}) \cap C^1(\Omega)$ of (1.1) in Ω, vanishing on $\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$, there holds*

$$\frac{u(y)}{c_5 d(y)} \leq \frac{u(x)}{d(x)} \leq c_5 \frac{u(y)}{d(y)} \tag{2.38}$$

for all $x, y \in B_{\frac{2\delta_1}{3}} \cap \Omega$ such that $\frac{1}{2}|x| \leq |y| \leq 2|x|$.

For proving Theorem 2.9 we need some intermediate lemmas. First we recall the following result from [1].

Lemma 2.10. *Assume that $a \in \partial\Omega$, $0 < r < \delta_1$ and $h > 1$ is an integer. There exists an integer N_0, depending only on δ_1, such that for any points x and y in $\Omega \cap B_{\frac{3r}{2}}(a)$ verifying $\min\{d(x), d(y)\} \geq r/2^h$, there exists a connected chain of balls $B_1, ..., B_j$ with $j \leq N_0 h$ such that*

$$\begin{aligned} &x \in B_1, y \in B_j, \quad B_i \cap B_{i+1} \neq \emptyset \text{ for } 1 \leq i \leq j-1 \\ &\text{and } 2B_i \subset B_{2r}(Q) \cap \Omega \text{ for } 1 \leq i \leq j. \end{aligned} \tag{2.39}$$

The next result is a standard Harnack inequality.

Lemma 2.11. *Assume $a \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2\delta_1}{3}}$ and $0 < r \leq |a|/4$. Let $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^1(\Omega)$ be a positive solution of (1.1) vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$. Then there exists a positive constant $c_6 > 1$ depending on N, p, q and δ_1 such that*

$$u(x) \leq c_6^h u(y), \tag{2.40}$$

for every $x, y \in B_{\frac{3r}{2}}(a) \cap \Omega$ such that $\min\{d(x), d(y)\} \geq r/2^h$ for some $h \in \mathbb{N}$.

Proof. For $\ell > 0$, we define $T_\ell[u]$ by

$$T_\ell[u](x) = \ell^{\frac{p-q}{q+1-p}} u(\ell x), \tag{2.41}$$

and we notice that if u satisfies (1.1) in Ω, then $T_\ell[u]$ satisfies the same equation in $\Omega^\ell := \ell^{-1}\Omega$. If we take in particular $\ell = |a|$, we can assume $|a| = 1$, thus the curvature of the domain $\Omega^{|a|}$ remains bounded. By Proposition 2.8

$$u(x) \le c_6' \quad \forall x \in B_{2r}(a) \cap \Omega \tag{2.42}$$

where c_6' depends on N, q, δ_1. Then we proceed as in [11], using Lemma 2.10 and internal Harnack inequality as quoted in [16, Corollary 10]. □

Since the solutions are Hölder continuous, the following statement holds as in [16, Theorem 4.2]:

Lemma 2.12. *Let the assumptions on a and u of Lemma 2.11 be fulfilled. If $b \in \partial\Omega \cap B_r(a)$ and $0 < s \le 2^{-1}r$, there exist two positive constants δ and c_7 depending on N, p, q and Ω such that*

$$u(x) \le c_7 \frac{|x-b|^\delta}{s^\delta} \max\{u(z) : z \in B_r(b) \cap \Omega\} \tag{2.43}$$

for every $x \in B_s(b) \cap \Omega$.

As a consequence we derive the following Carleson type estimate.

Lemma 2.13. *Assume $a \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2\delta_1}{3}}$ and $0 < r \le |a|/8$. Let $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^2(\Omega)$ be a positive solution of (1.1) vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$. Then there exists a constant c_8 depending only on N, p and q such that*

$$u(x) \le c_8 u(a - \tfrac{r}{2}\mathbf{n}_a) \quad \forall x \in B_r(a) \cap \Omega. \tag{2.44}$$

Proof. By Lemma 2.11 it is clear that for any integer h and $x \in B_r(a) \cap \Omega$ such that $d(x) \ge 2^{-h}r$, there holds

$$u(x) \le c_6^h u(a - \tfrac{r}{2}\mathbf{n}_a). \tag{2.45}$$

Therefore u satisfies inequality (2.43) as any Hölder continuous function does. The proof that the constant is independent of r and u is more delicate. It is done in [1, Lemma 2.4] for linear equations, but it is based only on Lemma 2.12 and a geometric construction, thus it is also valid in our case. □

Lemma 2.14. *Assume $a \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2\delta_1}{3}}$ and $0 < r \le |a|/8$. Let $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^2(\Omega)$ be a positive solution of (1.1) vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$. Then there exist $\alpha \in (0, 1/2)$ and $c_9 > 0$ depending on N, p and q such that*

$$\frac{1}{c_9}\frac{t}{r} \le \frac{u(b - t\mathbf{n}_b)}{u(a - \frac{r}{2}\mathbf{n}_a)} \le c_9 \frac{t}{r} \tag{2.46}$$

for any $b \in B_r(a) \cap \partial\Omega$ and $0 \le t < \frac{\alpha}{2}r$.

Proof. It is similar to the one of [11, Lemma 3.15]. □

Proof of Theorem 2.9. Assume $x \in B_{\frac{2\delta_1}{3}} \cap \Omega$ and set $r = \frac{|x|}{8}$.

Step 1: Tangential estimate: we suppose $d(x) < \frac{\alpha}{2} r$. Let $a \in \partial\Omega \backslash \{0\}$ such that $|a| = |x|$ and $x \in B_r(a)$. By Lemma 2.14,

$$\frac{8}{c_9} \frac{u(a - \frac{r}{2}\mathbf{n}_a)}{|x|} \leq \frac{u(x)}{d(x)} \leq 8c_9 \frac{u(a - \frac{r}{2}\mathbf{n}_a)}{|x|}. \tag{2.47}$$

We can connect $a - \frac{r}{2}\mathbf{n}_a$ with $-2r\mathbf{n}_0$ by m_1 (depending only on N) connected balls $B_i = B_{\frac{r}{4}}(x_i)$ with $x_i \in \Omega$ and $d(x_i) \geq \frac{r}{2}$ for every $1 \leq i \leq m_1$. It follows from (2.44) that

$$c_6^{-m_1} u(-2r\mathbf{n}_0) \leq u(a - \tfrac{r}{2}\mathbf{n}_a) \leq c_6^{m_1} u(-2r\mathbf{n}_0),$$

which, together with (2.47) leads to

$$\frac{1}{c_{10}} \frac{u(-2r\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq c_{10} \frac{u(-2r\mathbf{n}_0)}{|x|}, \tag{2.48}$$

with $c_{10} = 8c_9 c_6^{m_1}$.

Step 2: Internal estimate: we suppose $d(x) \geq \frac{\alpha}{2} r$. We can connect $-2r\mathbf{n}_0$ with x by m_2 (depending only on N) connected balls $B_i' = B_{\frac{\alpha r}{4}}(x_i')$ with $x_i' \in \Omega$ and $d(x_i') \geq \frac{\alpha}{2} r$ for every $1 \leq i \leq m_2$. By Harnack and Carleson inequalities (2.40) and (2.44) and since $\frac{\alpha}{4}|x| < d(x) \leq |x|$, we get

$$\frac{\alpha}{4c_6'^{m_2}} \frac{u(-2r\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq \frac{4c_6'^{m_2}}{\alpha} \frac{u(-2r\mathbf{n}_0)}{|x|}. \tag{2.49}$$

Step 3: End of proof. Suppose $\frac{|x|}{2} \leq s \leq 2|x|$, we can connect $-2r\mathbf{n}_Q$ with $-s\mathbf{n}_Q$ by m_3 (depending only on N) connected balls $B_i'' = B_{\frac{r}{2}}(x_i'')$ with $x_i'' \in \Omega$ and $d(x_i'') \geq r$ for every $1 \leq i \leq m_3$. This fact, jointly with (2.48) and (2.49), yields to

$$\frac{1}{c_{11}} \frac{u(-s\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq c_{11} \frac{u(-s\mathbf{n}_0)}{|x|} \tag{2.50}$$

where $c_{11} = c_{11}(N, q, \Omega)$. Finally, if $y \in B_{\frac{2r_0}{3}} \cap \Omega$ satisfies $\frac{|x|}{2} \leq |y| \leq 2|x|$, then by applying twice (2.50) we get (2.38) with $c_5 = c_{11}^2$. □

The following inequality is a consequence of Theorem 2.9.

Corollary 2.15. *Assume $q > p-1$ and $0 \in \partial\Omega$. Then there exists $c_{12} > 0$ depending on p, q and Ω such that for any positive solutions u_1, $u_2 \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^1(\Omega)$ of (1.1) in Ω, vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$, there holds*

$$\sup\left\{\frac{u_1(y)}{u_2(y)} : y \in B_r \setminus B_{\frac{r}{2}}\right\} \leq c_{12} \inf\left\{\frac{u_1(y)}{u_2(y)} : y \in B_r \setminus B_{\frac{r}{2}}\right\}. \tag{2.51}$$

3 Boundary singularities

3.1 Strongly singular solutions

In this section we consider the equation (1.1) in $\mathbb{R}^N_+$. We denote by $(r,\sigma)\in\mathbb{R}_+\times S^{N-1}$ the spherical coordinates in $\mathbb{R}^N$ and

$$S^{N-1}_+=\left\{(\sin\phi\sigma',\cos\phi):\sigma'\in S^{N-2},\phi\in[0,\frac{\pi}{2})\right\}.$$

If $v(x)=r^{-\beta}\omega(\sigma)$ satisfies (1.1) in $\mathbb{R}^N_+$ and vanishes on $\partial\mathbb{R}^N_+\setminus\{0\}$, then $\beta=\beta_q$ and ω is a solution of

$$\begin{aligned}-div'\left(\left(\beta_q^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla'\omega\right)-\beta_q\Lambda_{\beta_q}\left(\beta_q^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\omega\\ +\left(\beta_q^2\omega^2+|\nabla'\omega|^2\right)^{\frac{q}{2}}=0\quad\text{in } S^{N-1}_+\\ \omega=0\quad\text{on }\partial S^{N-1}_+.\end{aligned}\tag{3.1}$$

where β_q and Λ_{β_q} have been defined in (1.10). We denote by $(\beta_*,\psi_*)\in\mathbb{R}^*_+\times C^2(\overline{S}^{N-1}_+)$ the unique couple such $\max\psi_*=1$ with the property that the function $(r,\sigma)\mapsto r^{-\beta_*}\psi_*(\sigma)$ is positive, p-harmonic in $\mathbb{R}^N_+$ and vanishes on $\partial\mathbb{R}^N_+\setminus\{0\}$. Then $\psi_*=\psi$ satisfies

$$\begin{aligned}-div'\left(\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi\right)-\beta_*\Lambda_{\beta_*}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi=0\quad\text{in } S^{N-1}_+\\ \psi=0\quad\text{on }\partial S^{N-1}_+.\end{aligned}\tag{3.2}$$

Since the function ψ_* is unique it depends only on the azimuthal variable $\theta_{N-1}=\cos^{-1}(\frac{x_N}{|x|})$ (see Appendix II). Our first result is the following

Theorem 3.1. *If $q\geq q_*$, or equivalently $\beta_q\leq\beta_*$, there exists no positive solution to problem* (3.1).

Proof. Suppose such a solution ω exists and put $\theta=\beta_q/\beta_*$, then $0<\theta\leq 1$. Set $\eta=\psi^\theta$, where ψ is a positive solution of (3.2), and define the operator $\mathcal{T}$ by

$$\begin{aligned}\mathcal{T}(\eta)=-div'\left(\left(\beta_q^2\eta^2+|\nabla'\eta|^2\right)^{\frac{p-2}{2}}\nabla'\eta\right)-\beta_q\Lambda_{\beta_q}\left(\beta_q^2\eta^2+|\nabla'\eta|^2\right)^{\frac{p-2}{2}}\eta\\ +\left(\beta_q^2\eta^2+|\nabla'\eta|^2\right)^{\frac{q}{2}}.\end{aligned}\tag{3.3}$$

Since $\nabla\eta=\theta\psi^{\theta-1}\nabla\psi$,

$$\left(\beta_q^2\eta^2+|\nabla'\eta|^2\right)^{\frac{p-2}{2}}=\theta^{p-2}\psi^{(\theta-1)(p-2)}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}},$$

$$\left(\beta_q^2\eta^2+|\nabla'\eta|^2\right)^{\frac{p-2}{2}}\nabla'\eta=\theta^{p-1}\psi^{(\theta-1)(p-1)}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi,$$

therefore

$$\begin{aligned}\mathcal{T}(\eta)=&-\theta^{p-1}\psi^{(\theta-1)(p-1)}\mathrm{div}'\left(\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi\right)\\ &-\theta^{p-1}(\theta-1)(p-1)\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}|\nabla'\psi|^2\\ &-\beta_q\Lambda_{\beta_q}\theta^{p-2}\psi^{(\theta-1)(p-1)}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi+\theta^q\psi^{(\theta-1)q}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{q}{2}}.\end{aligned}$$

But $\beta_q\Lambda_{\beta_q}\theta^{p-2}=\beta_*\Lambda_{\beta_q}\theta^{p-1}\leq\beta_*\Lambda_{\beta_*}\theta^{p-1}$ since $\beta_q\leq\beta_*$. Using (3.2), we see that $\mathcal{T}(\eta)\geq 0$. Because Hopf Lemma is valid, there holds $\partial_{\mathbf{n}}\psi<0$ on ∂S^{N-1}_+. Since ω is C^1 in $\overline{S^{N-1}_+}$ and ψ is defined up to an homothety, there exists a smallest function ψ such that $\eta\geq\omega$, and the graphs of η and ω over $\overline{S^{N-1}_+}$ are tangent, either at some $\alpha\in S^{N-1}_+$, or only at a point $\alpha\in\partial S^{N-1}_+$. We put $w=\eta-\omega$. Then

$$\mathcal{T}(\eta)=\mathcal{T}(\eta)-\mathcal{T}(\omega)=\Phi(1)-\Phi(0), \tag{3.4}$$

where $\Phi(t)=\mathcal{T}(\omega_t)$ with $\omega_t=\omega+tw$.

We use local coordinates $(\sigma_1,...,\sigma_{N-1})$ on S^{N-1} near α. We denote by $g=(g_{ij})$ the metric tensor on S^{N-1} and by g^{jk} its contravariant components. Then, for any $\varphi\in C^1(S^{N-1})$,

$$|\nabla\varphi|^2=\sum_{j,k}g^{jk}\frac{\partial\varphi}{\partial\sigma_j}\frac{\partial\varphi}{\partial\sigma_k}=\langle\nabla\varphi,\nabla\varphi\rangle_g.$$

If $X=(X^1,...X^d)\in C^1(TS^{N-1})$ is a vector field, we lower indices by setting $X^\ell=\sum_i g^{\ell i}X_i$ and define the divergence of X by

$$div'_gX=\frac{1}{\sqrt{|g|}}\sum_\ell\frac{\partial}{\partial\sigma_\ell}\left(\sqrt{|g|}X^\ell\right)=\frac{1}{\sqrt{|g|}}\sum_{\ell,i}\frac{\partial}{\partial\sigma_\ell}\left(\sqrt{|g|}g^{\ell i}X_i\right).$$

We write $\Phi(t)=\Phi_1(t)+\Phi_2(t)+\Phi_3(t)$ where

$$\Phi_1(t)=-\beta_q\Lambda_{\beta_q}\left(\beta_q^2\omega_t^2+|\nabla'\omega_t|^2\right)^{\frac{p-2}{2}}\omega_t\ ,\quad \Phi_2(t)=\left(\beta_q^2\omega_t^2+|\nabla'\omega_t|^2\right)^{\frac{q}{2}}$$

and

$$\Phi_3(t)=-\mathrm{div}'\left(\left(\beta_{\mathrm{q}}^2\omega_{\mathrm{t}}^2+|\nabla'\omega_{\mathrm{t}}|^2\right)^{\frac{\mathrm{p}-2}{2}}\nabla'\omega_{\mathrm{t}}\right).$$

Then

$$\Phi_1(1)-\Phi_1(0)=-\sum_j a_j\frac{\partial w}{\partial\sigma_j}-bw\ \text{ and }\ \Phi_2(1)-\Phi_2(0)=\sum_j c_j\frac{\partial w}{\partial\sigma_j}+dw,$$

where

$$b=\beta_q\Lambda_{\beta_q}\left(\beta_q^2{\omega_t}^2+|\nabla\omega_t|^2\right)^{\frac{p}{2}-2}\left((p-1)\beta_q^2\omega_t^2+|\nabla\omega_t|^2\right),$$

$$a_j=(p-2)\beta_q\Lambda_{\beta_q}\left(\beta_q^2{\omega_t}^2+|\nabla\omega_t|^2\right)^{\frac{p}{2}-2}\omega_t\sum_k g^{jk}\frac{\partial\omega_t}{\partial\sigma_k},$$

$$d=q\beta_q^2\left(\beta^2{\omega_t}^2+|\nabla\omega_t|^2\right)^{\frac{q}{2}-1}\omega_t,$$

and

$$c_j=q\left(\beta_q^2{\omega_t}^2+|\nabla\omega_t|^2\right)^{\frac{q}{2}-1}\sum_k g^{jk}\frac{\partial\omega_t}{\partial\sigma_k}.$$

Furthermore

$$\Phi_3(1)-\Phi_3(0)=-(p-2)\mathrm{div}'\left(\left(\beta_q^2\omega_t^2+|\nabla'\omega_t|^2\right)^{\frac{p-4}{2}}\left(\beta_q^2\omega_t w+\langle\nabla'\omega_t,\nabla' w\rangle_g\right)\nabla'\omega_t\right)$$
$$-\mathrm{div}'\left(\left(\beta_q^2\omega_t^2+|\nabla'\omega_t|^2\right)^{\frac{p-2}{2}}\nabla' w\right).$$

Therefore we can write $\Phi(1)-\Phi(0)$ under the form

$$\Phi(1)-\Phi(0)=-\mathrm{div}'(A\nabla' w)+\langle B,\nabla' w\rangle_g+Cw:=\mathcal{L}w \tag{3.5}$$

where

$$\begin{aligned}\langle AX,X\rangle_g&=\left(\beta_q^2\omega_t^2+|\nabla'\omega_t|^2\right)^{\frac{p-4}{2}}\left((p-2)\langle\nabla'\omega_t,X\rangle_g^2+|\nabla'\omega_t|^2|X|^2\right)\\&\geq\left(\beta_q^2\omega_t^2+|\nabla'\omega_t|^2\right)^{\frac{p-4}{2}}\min\{1,p-1\}|\nabla'\omega_t|^2|X|^2.\end{aligned} \tag{3.6}$$

and B and C can be computed from the previous expressions. It is important to notice that $\beta_q^2\omega_t^2+|\nabla'\omega_t|^2$ is bounded between two positive constants m_1 and m_2 in $\overline{S_+^{N-1}}$. Thus the operator $\mathcal{L}$ is uniformly elliptic with bounded coefficients. Since w is nonnegative and either at some point α, $\nabla' w(\alpha)=0$ and $w(\alpha)>0$, or at some boundary point α where $w(\alpha)=0$ and $\partial_{\mathbf{n}}w(\alpha)<0$, it follows from the strong maximum principle or Hopf Lemma (see [7]) that $w=0$, contradiction. □

Theorem 3.2. *Assume $q<q_*$ or equivalently $\beta_q>\beta_*$. There exists a unique positive solution ω_* to problem* (3.1).

Proof. Existence. It will follow from [4]. Indeed problem (3.1) can be written under the form

$$\begin{aligned}\mathbf{A}(\omega):=-div'\,\mathbf{a}(\omega,\nabla'\omega)&=\mathbf{B}(\omega,\nabla'\omega)&&\text{in } S_+^{N-1}\\\omega&=0&&\text{on } \partial S_+^{N-1},\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}\mathbf{a}(r,\xi)&=\left(\beta_q^2r^2+|\xi|^2\right)^{\frac{p-2}{2}}\xi,\\\mathbf{B}(r,\xi)&=\beta_q\Lambda_{\beta_q}\left(\beta_q^2r^2+|\xi|^2\right)^{\frac{p-2}{2}}r-\left(\beta_q^2r^2+|\xi|^2\right)^{\frac{q}{2}}.\end{aligned} \tag{3.8}$$

The operator $\mathbf{A}$ is a Leray-Lions operator which satisfies the assumptions (1.6)-(1.8) of [4, Theorem 2.1], and the term $\mathbf{B}$ satisfies (1.9),(1.10) in the same article. Therefore the existence of a positive solution $\omega\in W_0^{1,p}(S_+^{N-1})\cap L^\infty(S_+^{N-1})$ is ensured whenever we can find a supersolution $\overline{\omega}\in W^{1,p}(S_+^{N-1})\cap L^\infty(S_+^{N-1})$ and a nontrivial subsolution $\underline{\omega}\in W^{1,p}(S_+^{N-1})$ of (3.7) such that

$$0\leq\underline{\omega}\leq\overline{\omega}\qquad\text{in } S_+^{N-1}. \tag{3.9}$$

First we note that $\eta=\eta_0$ is a supersolution if the positive constant η_0 is large enough. In order to find a subsolution, we set again $\eta=\psi^\theta$ with $\theta=\beta_q/\beta_*$ and ψ as in (3.2). Now $\theta>1$, thus $\eta\in W_0^{1,p}(S_+^{N-1})$. As above we have

$$\begin{aligned}\mathcal{T}(\eta)=&-\theta^{p-1}\psi^{(\theta-1)(p-1)}\mathrm{div}'\left(\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi\right)\\&-\theta^{p-1}(\theta-1)(p-1)\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}|\nabla'\psi|^2\\&-\beta_q\Lambda_{\beta_q}\theta^{p-2}\psi^{(\theta-1)(p-1)}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi+\theta^q\psi^{(\theta-1)q}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{q}{2}}.\end{aligned}$$

Now $\beta_q\Lambda_{\beta_q}\theta^{p-2}=\beta_*\Lambda_{\beta_q}\theta^{p-1}=\beta_*(\Lambda_{\beta_q}-\Lambda_{\beta_*})\theta^{p-1}+\beta_*\Lambda_{\beta_*}\theta^{p-1}$ and $\Lambda_{\beta_q}-\Lambda_{\beta_*}=(\beta_q-\beta_*)(p-1)=\beta_*(p-1)(\theta-1)$, hence

$$\begin{aligned}\mathcal{T}(\eta)=&-\theta^{p-1}\psi^{(\theta-1)(p-1)}\mathrm{div}'\left(\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi\right)\\&-\theta^{p-1}(\theta-1)(p-1)\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}|\nabla'\psi|^2\\&-\beta_*(\Lambda_{\beta_q}-\Lambda_{\beta_*})\theta^{p-1}\psi^{(\theta-1)(p-1)}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi\\&-\beta_*\Lambda_{\beta_*}\theta^{p-1}\psi^{(\theta-1)(p-1)}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi+\theta^q\psi^{(\theta-1)q}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{q}{2}}.\end{aligned}$$

Using the equation satisfied by ψ yields to the relation

$$\begin{aligned}\mathcal{T}(\eta)=&-\theta^{p-1}(\theta-1)(p-1)\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}|\nabla'\psi|^2\\&-\beta_*^2(p-1)(\theta-1)\theta^{p-1}\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi^2\\&+\theta^q\psi^{(\theta-1)q}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{q}{2}}\\=&-\theta^{p-1}(\theta-1)(p-1)\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p}{2}}\\&+\theta^q\psi^{(\theta-1)q}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{q}{2}}.\end{aligned}$$

If we replace $\eta:=\eta_1=\psi^\theta$ by $\eta:=\eta_m=(m\psi)^\theta$ in the above computation, the inequality $\mathcal{T}\eta_m)\le 0$ will be true provided

$$m^{\theta(q+1-p)}\psi^{(\theta-1)(q+1-p)+1}\le\theta^{p-1-q}(\theta-1)(p-1)\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-q}{2}},$$

which is satisfied if we choose m small enough so that $(m\psi)^\theta\le\eta_0$ and satisfying

$$m^{\theta(q+1-p)}\le\beta_*^{(\theta-1)(q+1-p)+1}\theta^{p-1-q}(\theta-1)(p-1)\frac{\min_{x\in S_+^{N-1}}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-q}{2}}}{\max_{x\in S_+^{N-1}}\psi^{(\theta-1)(q+1-p)+1}}.$$

Therefore $0<\eta_m\le\eta_0$ and standard regularity implies that the solution ω is C^1 in $\overline{S}_+^{N-1}$. Actually ω is C^∞ since the operator is not degenerate.

Uniqueness. We use the tangency method developed in the proof of Theorem 3.1. Assume ω_1 and ω_2 are two positive solutions of (3.2), then they are positive in S_+^{N-1} and $\partial_{\mathbf{n}}\omega_i<0$ on ∂S_+^{N-1}. Either the ω_i are ordered and $\omega_1\le\omega_2$, or their graphs intersect. In any case we can define

$$\tau=\inf\{s>1:s\omega_1\ge\omega_2\}.$$

We set $\omega^*=\tau\omega_1$. Then either the graphs of ω_2 and ω^* are tangent at some interior point α, or they are not tangent in S_+^{N-1}, $\partial_{\mathbf{n}}\omega^*\le\partial_{\mathbf{n}}\omega_2<0$ on ∂S_+^{N-1} and there exists $\alpha\in\partial S_+^{N-1}$ such that $\partial_{\mathbf{n}}\omega^*(\alpha)=\partial_{\mathbf{n}}\omega_2(\alpha)<0$. Furthermore $\mathcal{T}(\omega^*)\ge 0$. If we set $w=\omega^*-\omega_2$, then, as in Theorem 3.1,

$$-\mathrm{div}'(\tilde{A}\nabla'w)+\langle\tilde{B},\nabla'w\rangle_g+\tilde{C}w=\tilde{\mathcal{L}}w\ge 0$$

where

$$\begin{aligned}\langle \tilde{A}X, X\rangle_g &= \left(\beta_q^2\omega_t^2 + |\nabla'\omega_t|^2\right)^{\frac{p-4}{2}}\left((p-2)\langle\nabla'\omega_t, X\rangle_g^2 + |\nabla'\omega_t|^2|X|^2\right)\\ &\geq \left(\beta_q^2\omega_t^2 + |\nabla'\omega_t|^2\right)^{\frac{p-4}{2}}\min\{1, p-1\}|\nabla'\omega_t|^2|X|^2,\end{aligned} \tag{3.10}$$

in which $\omega_t = \omega_2 + t(\omega^* - \omega_2)$ and $t \in (0,1)$ is obtained by applying the mean value theorem and $\tilde{B}$ and $\tilde{C}$ are defined accordingly. Since $\tilde{\mathcal{L}}$ is uniformly elliptic and has bounded coefficients, it follows from the strong maximum principle that $w = 0$. Thus $\omega^* = \tau\omega_1 = \omega_2$ and $\tau = 1$ from the equation. This ends the proof. □

3.2 Removable boundary singularities

The following is the basic result for removability of isolated singularities. It is valid in the general case, but with a local geometric constraint.

Theorem 3.3. *Assume $q^* \leq q < p \leq N$, Ω is a C^2 bounded domain with $0 \in \partial\Omega$, such that $\Omega \cap B_\delta = B_\delta^+$ for some $\delta > 0$. If $u \in C^1(\overline{\Omega} \setminus \{0\})$ is a nonnegative solution of (1.1) in Ω which vanishes on $\partial\Omega \setminus \{0\}$, then it is identically 0.*

Proof. Step 1: Assume $\Omega \subset \mathbb{R}^N_+$. For $\epsilon > 0$, we set $\Omega'_\epsilon = \Omega \cap \overline{B^c_\epsilon}$ and $H_\epsilon = \mathbb{R}^N_+ \cap \overline{B^c_\epsilon}$. For $k, n \in \mathbb{N}_*$, $n \geq \operatorname{diam}(\Omega)$, we denote by $v_{k,n,\epsilon}$ ($n \in \mathbb{N}_*$) the solution of the problem

$$\begin{aligned} -\Delta_p v + |\nabla v|^q &= 0 && \text{in } H_\epsilon \cap B_n\\ v &= k\chi_{_{\mathbb{R}^N_+\cap\partial B_\epsilon}} && \text{on } \partial(H_\epsilon \cap B_n).\end{aligned} \tag{3.11}$$

If $k > c_2\epsilon^{\frac{q-p}{q+1-p}}$ for a suitable $c_2 = c_2(p,q) > 0$ (see Lemma 2.6), then $v_{k,n,\epsilon} \geq u$ in Ω'_ϵ. Moreover there holds $v_{k,n,\epsilon} \leq v_{k',n',\epsilon}$ for $n \leq n'$ and $k \leq k'$. Furthermore the function

$$U_{\epsilon,n}(x) = c_2\left((|x| - \epsilon)^{\frac{q-p}{q+1-p}} - (n-\epsilon)^{\frac{q-p}{q+1-p}}\right)$$

is a supersolution in $B_n \setminus B_\epsilon$, and there holds $v_{k,n,\epsilon} \leq U_{\epsilon,n}$. By monotonicity and standard a priori estimate, we obtain that $v_{k,n,\epsilon} \to v_\epsilon$ when $n, k \to \infty$ and that the function $v = v_\epsilon$ is solution of

$$\begin{aligned} -\Delta_p v + |\nabla v|^q &= 0 && \text{in } H_\epsilon\\ \lim_{|x|\to\epsilon} v(x) &= \infty\\ v &= 0 && \text{on } \partial\mathbb{R}^N_+ \cap \overline{B^c_\epsilon}.\end{aligned} \tag{3.12}$$

Furthermore

$$u(x) \leq v_\epsilon(x) \leq c_2(|x| - \epsilon)^{\frac{q-p}{q+1-p}} \quad \text{in } \Omega'_\epsilon. \tag{3.13}$$

The function v_ϵ may not be unique, however it is the minimal solution of the above problem since the $v_{k,n,\epsilon}$ is unique, and monotonicity in n and k holds. Actually, $v_\epsilon \leq v_{\epsilon'}$ if $0 \leq \epsilon \leq \epsilon'$. For $\ell > 0$, we recall that the transformation $v \mapsto T_\ell[v]$ defined by (2.41) leaves equation (1.1) invariant. As a consequence of the uniqueness of the approximations we have $T_\ell[v_{k,n,\epsilon}] = v_{\ell^{\frac{p-q}{q+1-p}}k,\ell^{-1}n,\ell^{-1}\epsilon}$, which implies

$$T_\ell[v_\epsilon] = v_{\ell^{-1}\epsilon}. \tag{3.14}$$

Letting $\epsilon \to 0$, we derive from the monotonicity with respect to ϵ and standard $C^{1,\alpha}$ estimates, that the following identity holds:

$$T_\ell[v_0] = v_0 \qquad \forall \ell > 0. \tag{3.15}$$

The function v_0 is a positive and separable solution of (1.1) in $\mathbb{R}^N_+$ which vanishes on $\partial\Omega\setminus\{0\}$. It follows from Theorem 3.1 that $v_0 = 0$, and so is u.

Step 2: The general case. We assume that $\Omega \cap B_\delta \subset \mathbb{R}^N_+$ and we denote by M the maximum of u on $\partial B_\delta \cap \Omega$. Then the function $(u-M)_+$ is a subsolution of (1.1) in $\Omega \cap B_\delta$ which vanishes on $\partial\Omega \cap B_\delta \setminus \{0\}$. By Step 1, it is dominated by v_0, which ends the proof. □

Remark. The previous result is valid if u is a subsolution with the same regularity. If u is no longer assumed to be nonnegative, only u^+ vanishes. Furthermore, the regularity of the boundary has not been used, but only the fact that Ω is locally contained into a half space to the boundary of which 0 belongs.

Remark. If no geometric assumption is made on $\partial\Omega$, we can prove that $u(x) = o(|x|^{-\beta_q})$ near 0. The next result shows that the removability holds if $q > q_*$.

Theorem 3.4. *Assume $q^* < q < p \le N$ and Ω is a C^2 bounded domain with $0 \in \partial\Omega$. If u is a nonnegative solution of (1.1) in Ω which belongs to $C^1(\overline{\Omega} \setminus \{0\})$ and vanishes on $\partial\Omega \setminus \{0\}$, it is identically 0.*

Proof. As it is proved in [12], for any smooth subdomain $S \subset S^{N-1}$, there exists a unique $\beta_{*\,s} > 0$ and $\psi^*_s > 0$, unique up to an homothety, such that $x \mapsto |x|^{-\beta_{*\,s}}\, \psi^*_s(|x|^{-1}\, x)$ is p harmonic in the cone $C_S = \{x \in \mathbb{R}^N \setminus \{0\} : |x|^{-1}\, x \in S\}$ and ψ^*_s satisfies

$$\begin{aligned} -div'\left(\left(\beta^2_{*\,s}\psi^{*\,2}_s + |\nabla'\psi^*_s|^2\right)^{\frac{p-2}{2}} \nabla'\psi^*_s\right) - \beta_{*\,s}\Lambda_{\beta_{*\,s}}\left(\beta^2_{*\,s}\psi^{*\,2} + |\nabla'\psi^*_s|^2\right)^{\frac{p-2}{2}} \psi^*_s = 0 \qquad &\text{in } S \\ \psi^*_s = 0 \qquad &\text{on } \partial S, \end{aligned} \tag{3.16}$$

Furthermore $S \subset \tilde{S} \subset S^{N-1}$ implies $\beta_{*\,\tilde{s}} \le \beta_{*\,s}$. Using the system of spherical coordinates defined in (6.5) in Appendix II, for $\epsilon > 0$ we denote by $S := S_\epsilon$ the spherical shell with vertex the north pole N and latitude angle $\theta_{N-1} \in [0, \frac{\pi}{2} + \epsilon]$. Because of uniqueness of $\beta_{*\,s}$, $\beta_{*\,s_\epsilon} \uparrow \beta_*$ as $\epsilon \to 0$. Therefore, if $q > q_*$, or equivalently $\beta_q < \beta_*$, there exists $\delta, \epsilon > 0$ such that $\Omega \cap B_\delta \subset C_{S_\epsilon} \cap B_\delta$ and $\beta_q < \beta_{*\,s_\epsilon}$. Since Theorem 3.1 is valid if S^{N-1}_+ is replaced by S_ϵ and $\beta_q < \beta_{*\,s_\epsilon}$ it follows that $u = 0$ as in the proof of Theorem 3.3, Steps 1 and 2. □

The next result, valid in the case $p = N$, is based upon the conformal invariance of the N-Laplacian. In this case the exponent β_* corresponding to the first spherical N-harmonic eigenvalue is equal to 1 and the corresponding spherical N-harmonic eigenfunction in S^{N-1}_+ is $x_N/\,|x|^2$.

Theorem 3.5. *Assume $N - \frac{1}{2} \le q < N$, Ω is a bounded domain and $0 \in \partial\Omega$ is such that there exists a ball $B \subset \Omega^c$ to the boundary of which 0 belongs. If u is a nonnegative solution of*

$$-\Delta_N u + |\nabla u|^q = 0 \qquad \text{in } \Omega, \tag{3.17}$$

which belongs to $C(\overline{\Omega} \setminus \{0\}) \cap W^{1,N}_0(\Omega \setminus \overline{B}_\epsilon(0))$ for any $\epsilon > 0$, it is identically 0.

Proof. We assume that the inward normal unit vector to $\partial\Omega$ at 0 is $\mathbf{e}_N = (0, 0, ..., 1)$ and that the ball $B = B_{\frac{1}{2}}(a)$ of center $a = -\frac{1}{2}\mathbf{e}_N$ and radius $\frac{1}{2}$ touches $\partial\Omega$ at 0 and is exterior to Ω (this can be assumed up to a rotation and a dilation). This is the consequence of the exterior sphere condition at the point 0. It is always valid if $\partial\Omega$ is C^2. We denote by $\mathcal{I}_\omega$ the inversion of center $\omega = -\mathbf{e}_N$ and power 1, i.e. $\mathcal{I}_\omega(x) = \omega + \frac{x-\omega}{|x-\omega|^2}$. Under this transformation, the complement of the ball $B_{\frac{1}{2}}(a)$, which contains Ω, is transformed into the half space $\mathbb{R}^N_-$ which contains the image $\tilde{\Omega}$ of Ω. Since u satisfies (3.17), $\tilde{u} = u \circ \mathcal{I}_\omega$ satisfies

$$-\Delta_N \tilde{u} + |x-\omega|^{2(q-N)} |\nabla \tilde{u}|^q = 0 \qquad \text{in } \tilde{\Omega}. \tag{3.18}$$

Furthermore since $0 = \mathcal{I}_\omega(0)$ and $\mathcal{I}_\omega$ is a diffeomorphism, $\tilde{u} \in C(\overline{\tilde{\Omega}} \setminus \{0\}) \cap C^1(\tilde{\Omega})$ and it vanishes on $\partial\tilde{\Omega} \setminus \{0\}$. Since $|x - \omega| \leq 1$ and $q < N$, $\tilde{u}$ is a subsolution for (3.17) in $\tilde{G}$. By Theorem 3.4, $\tilde{u} = 0$. □

3.3 Weakly singular solutions

The main result of this section is the following existence and uniqueness result concerning solutions of (1.1) with a boundary weak singularity. We recall that ψ_* is unique positive solution of (1.11) such that $\sup \psi_* = 1$. Our first result is valid for any $1 < p \leq N$ but it needs a geometric constraint on Ω.

Theorem 3.6. *Let $p - 1 < q < q_* < p \leq N$ and $\Omega \subset \mathbb{R}^N_+$ be a bounded C^2 domain such that $0 \in \partial\Omega$. Assume that there exists $\delta > 0$ such that $\Omega_\delta := \Omega \cap B_\delta = B^+_\delta$. Then for any $k > 0$ there exists a unique positive solution $u := u_k$ of (1.1) in Ω, which belongs to $C^1(\overline{\Omega} \setminus \{0\})$, vanishes on $\partial\Omega \setminus \{0\}$ and satisfies*

$$\lim_{x\to 0} \frac{u_k(x)}{\Psi_*(x)} = k \tag{3.19}$$

in the C^1-topology of S^{N-1}_+, where

$$\Psi_*(x) = |x|^{-\beta_*} \psi_*(|x|^{-1}x).$$

The proof of this theorem is long and difficult and requires a certain number of intermediate results.

Lemma 3.7. *Let the assumptions on p, q and Ω of Theorem 3.6 be satisfied. There exists a unique positive p-harmonic function Φ_* in Ω, which is continuous in $\overline{\Omega} \setminus \{0\}$, vanishes on $\partial\Omega \setminus \{0\}$ and satisfies*

$$\lim_{x\to 0} \frac{\Phi_*(x)}{\Psi_*(x)} = 1. \tag{3.20}$$

Proof. For $0 < \epsilon < \delta$ let v_ϵ be the unique nonnegative p-harmonic function in $\Omega \setminus \overline{B^+_\epsilon}$ which is continuous in $\overline{\Omega} \setminus B^+_\epsilon$, vanishes on $\partial\Omega \setminus B_\epsilon$ and achieves the value Ψ_* on $\partial B_\epsilon \cap \Omega$. Since $\Omega \subset \mathbb{R}^N_+$, $v_\epsilon \leq \Psi_*$ in $\Omega \setminus B^+_\epsilon$. Hence inequalities $0 < \epsilon < \epsilon' \leq \delta$ imply $v_\epsilon \leq v_{\epsilon'}$ in $\Omega \setminus \overline{B^+_{\epsilon'}}$. Because $\Psi_* \leq \delta^{-\beta_*}$, there holds

$$v_\epsilon + \delta^{-\beta_*} \geq \Psi_*, \tag{3.21}$$

in $\Omega \setminus B^+_\delta$. Since v_ϵ and Ψ_* coincide on ∂B^+_ϵ and vanish on $\partial\mathbb{R}^N_+ \cap (B^+_\delta \setminus B^+_\epsilon)$, (3.21) holds also in $B^+_\delta \setminus B^+_\epsilon$. Because $v_\epsilon \geq 0$ there holds

$$(\Psi_* - \delta^{-\beta_*})_+ \leq v_\epsilon \leq \Psi_* \quad \text{ in } \Omega \setminus B^+_\epsilon. \tag{3.22}$$

By a standard regularity result v_ϵ converges to a function Φ_* continuous in $\overline{\Omega}\setminus\{0\}$, p-harmonic in Ω such that

$$(\Psi_* - \delta^{-\beta_*})_+ \le \Phi_* \le \Psi_*$$

in Ω. Therefore (3.20) holds provided $\frac{x}{|x|}$ remains in a compact subset of S^{N-1}_+. Let us define a function $\tilde\phi_*$ by $\tilde\phi_*(x) = |x|^{\beta_*}\,\Phi_*(x)$, then $\tilde\phi_*(r,\sigma) \le \psi_*(\sigma)$ where $r = |x|$ and $\sigma = \frac{x}{|x|} \in S^{N-1}_+$. By standard $C^{1,\alpha}$ estimates, $\tilde\phi_*(r,.)$ is relatively compact in the $C(\overline{S^{N-1}_+})$-topology. Therefore the convergence of $\frac{\Phi_*(x)}{\Psi_*(x)}$ to 1 when x to 0 holds not only when $\frac{x}{|x|}$ remains in a compact subset of S^{N-1}_+, but uniformly on S^{N-1}_+, which implies (3.20). Uniqueness follows classically by (3.20) and the maximum principle. □

Lemma 3.8. *Let the assumptions on p, q and Ω of Theorem 3.6 be satisfied. If for some $k > 0$ there exists a solution u_k of (1.1) in Ω, which belongs to $C^1(\overline{\Omega}\setminus\{0\})$, vanishes on $\partial\Omega\setminus\{0\}$ and satisfies (3.19), then for any $k > 0$ there exists such a solution.*

Proof. We notice that for any $c < 1$ (resp $c > 1$), cu_k is a subsolution (resp. supersolution) of (1.1) in Ω. Let Φ_* be as in Lemma 3.7. If $c < 1$, the function $ck\Phi_*$ is a supersolution of (1.1) which vanishes on $\partial\Omega\setminus\{0\}$. Furthermore

$$\lim_{x\to 0}\frac{cu_k(x)}{\Psi_*(x)} = ck = \lim_{x\to 0}\frac{ck\Phi_*(x)}{\Psi_*(x)}.$$

Then there exists a solution u_{ck} of (1.1) in Ω which satisfies $cu_k \le u_{ck} \le ck\Phi_*$. If $c > 1$, we set $u^* := T_{c^\theta}[u_k]$, which means $u^*(x) = c^{\beta_q\theta}u_k(c^\theta\,x)$ with $\theta = (\beta_q - \beta_*)^{-1}$. Then u^* is a solution of (1.1) in $\Omega^{c^\theta} = \frac{1}{c^\theta}\Omega$. In particular, u^* satisfies the equation in $B^+_{\frac{\delta}{c^\theta}}(0)$. Since $c^\theta > 1$, $B^+_{\frac{\delta}{c^\theta}}(0) \subset B^+_\delta(0)$. Put $m = \max\{u^* : x \in \partial B^+_{\frac{\delta}{c^\theta}}(0)\}$. The function $(u^* - m)_+$, extended by 0 outside $B^+_{\frac{\delta}{c^\theta}}(0)$, is a subsolution of (1.1) in Ω. Furthermore it satisfies

$$\lim_{x\to 0}\frac{(u^*-m)_+(x)}{\Psi_*(x)} = ck,$$

uniformly on any compact subset of S^{N-1}_+. Therefore there exists a solution u_{ck} of (1.1) in Ω which satisfies $(u^*-m)_+ \le u_{ck} \le ck\Phi_*$, and in particular it vanishes on $\partial\Omega\setminus\{0\}$ and belongs to $C^1(\overline{\Omega}\setminus\{0\})$. By [13], u_{ck} is positive in Ω. Thus u_{ck} belongs to $C^{1,\alpha}(\overline{B^+_\delta}(0)\setminus\{0\})$ and satisfies

$$|x|^{\beta_*}\,|u_{ck}(x)| + |x|^{1+\beta_*}\,|\nabla u_{ck}(x)| + |x|^{1+\beta_*+\alpha}\sup_{\substack{|y|\le|x|\\ x\ne y}}\frac{|\nabla u_{ck}(x) - \nabla u_{ck}(y)|}{|x-y|^\alpha} \le M$$

by (2.11). Therefore the set of functions $\{r^{\beta_*+1}\nabla u_{ck}(r,.)\}_{r>0}$ is uniformly relatively compact in the topology of uniform convergence on $\overline{S}^{N-1}_+$. Since it converges to $ck\nabla'\psi_*$ uniformly on compact subsets of S^{N-1}_+ as $r\to 0$, this convergence holds in $C(\overline{S^{N-1}_+})$. This implies

$$\lim_{x\to 0}\frac{u_{ck}(x)}{\Psi_*(x)} = ck. \tag{3.23}$$

□

The next Lemma is the keystone of our construction. Its proof is very delicate and needs several intermediate steps.

Lemma 3.9. *Under the assumptions of Theorem 3.6 there exists a real number R_0 such that $0 < R_0 \leq \delta$ and a positive subsolution $\tilde{u}$ of (1.1) in $B^+_{R_0}$ which is Lipschitz continuous in $\overline{B^+_{R_0}} \setminus \{0\}$, vanishes on $\overline{B^+_{R_0}} \cap \partial\mathbb{R}^N_+ \setminus \{0\}$, is smaller than Ψ_* and satisfies*

$$\lim_{x\to 0} \frac{\tilde{u}(x)}{\Psi_*(x)} = 1. \tag{3.24}$$

Proof. The construction of the function $\tilde{u}$. We look for a subsolution under the form $\tilde{u} = \Psi_* - w$ for a suitable nonnegative function w.

Step 1: reduction of the problem. We use spherical coordinates for a C^1 function $u : x \mapsto u(x) = u(r,\sigma)$, $r = |x|$, $\sigma = \frac{x}{|x|}$. Then $\nabla u = u_r \mathbf{e} + r^{-1}\nabla' u$ where $\mathbf{e} = |x|^{-1} x$, $|\nabla u|^2 = u_r^2 + r^{-2}|\nabla' u|^2$ and $|\nabla u|^q = \left(u_r^2 + r^{-2}|\nabla' u|^2\right)^{\frac{q}{2}}$. The expression of the p-Laplacian in spherical coordinates is

$$\begin{aligned} -\Delta_p u = &-\left(\left(u_r^2 + r^{-2}|\nabla' u|^2\right)^{\frac{p-2}{2}} u_r\right)_r - \frac{N-1}{r}\left(u_r^2 + r^{-2}|\nabla' u|^2\right)^{\frac{p-2}{2}} u_r \\ &- \frac{1}{r^2} div'\left(\left(u_r^2 + r^{-2}|\nabla' u|^2\right)^{\frac{p-2}{2}} \nabla' u\right). \end{aligned}$$

Put $v(t,\sigma) = r^{\beta_*} u(r,\sigma)$ with $t = \ln r \in (-\infty, \ln\delta]$, then v satisfies

$$\begin{aligned} &\mathcal{Q}[v] := \\ &-\left(\left((v_t - \beta_* v)^2 + |\nabla' v|^2\right)^{\frac{p-2}{2}} (v_t - \beta_* v)\right)_t - div'\left(\left((v_t - \beta_* v)^2 + |\nabla' v|^2\right)^{\frac{p-2}{2}} \nabla' v\right) \\ &\quad + \Lambda_{\beta_*}\left((v_t - \beta_* v)^2 + |\nabla' v|^2\right)^{\frac{p-2}{2}} (v_t - \beta_* v) + e^{\nu t}\left((v_t - \beta_* v)^2 + |\nabla' v|^2\right)^{\frac{q}{2}} = 0 \end{aligned} \tag{3.25}$$

in $(-\infty, \ln\delta) \times S^{N-1}_+$ where $\nu = 1 - (q+1-p)(\beta_*+1) = 1 - \frac{\beta_*+1}{\beta_q+1} > 0$ and $\Lambda_{\beta_*} = \beta_*(p-1) + p - N$. Notice that ψ_* satisfies

$$-div'\left(\left(\beta_*^2\psi_*^2 + \left|\nabla'\psi_*\right|^2\right)^{\frac{p-2}{2}} \nabla'\psi_*\right) - \beta_*\Lambda_{\beta_*}\left(\beta_*^2\psi_*^2 + \left|\nabla'\psi_*\right|^2\right)^{\frac{p-2}{2}} \psi_* = 0, \tag{3.26}$$

hence it is a supersolution for (3.25). We look for a subsolution under the form

$$V(t,\sigma) = \psi_* - a(t) g(\psi_*)$$

where g is a continuous increasing function defined on $\mathbb{R}_+$, vanishing at 0 and smooth on $\mathbb{R}^*_+$ and $a(t) = e^{\gamma t}$ with $\gamma > 0$ to be chosen. Thus $a' = \gamma a$, $a'' = \gamma^2 a$, $V_t = -\gamma a g(\psi_*)$, $V_t - \beta_* V =$

$-\beta_*\psi_* + a(\beta_* - \gamma)g(\psi_*)$, $\nabla' V = (1 - ag'(\psi_*))\nabla'\psi_*$ and

$$\begin{aligned}(V_t - \beta_* V)^2 + |\nabla' V|^2 &= (-\beta_*\psi_* + a(\beta_* - \gamma)g(\psi_*))^2 + (1 - ag'(\psi_*))^2 |\nabla'\psi_*|^2 \\ &= \left(\beta_*^2\psi_*^2 + 2a\beta_*(\gamma - \beta_*)g(\psi_*)\psi_*\right) + (1 - 2ag'(\psi_*)) |\nabla'\psi_*|^2 + O(a^2 \|g(\psi)\|_{C^1}) \\ &= \beta_*^2\psi_*^2 + |\nabla'\psi_*|^2 + 2a\left(\beta_*(\gamma - \beta_*)\psi_* g(\psi_*) - g'(\psi_*) |\nabla\psi_*|^2\right) + O(a^2 \|g(\psi_*)\|_{C^1}).\end{aligned}$$

Therefore

$$\begin{aligned}&\left((V_t - \beta_* V)^2 + |\nabla' V|^2\right)^{\frac{p-2}{2}} \\ &= \left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} \left[1 + (p-2)a\frac{\beta_*(\gamma - \beta_*)\psi_* g(\psi_*) - g'(\psi_*) |\nabla\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}\right] \\ &\qquad + O(a^2 \|g(\psi)\|_{C^1}),\end{aligned}$$

and

$$\begin{aligned}&e^{\nu t}\left((V_t - \beta_* V)^2 + |\nabla' V|^2\right)^{\frac{q}{2}} \\ &= e^{\nu t}\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{q}{2}} \left[1 + qa\frac{\beta_*(\gamma - \beta_*)\psi_* g(\psi_*) - g'(\psi_*) |\nabla\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}\right] \\ &\qquad + O(e^{\nu t} a^2 \|g(\psi_*)\|_{C^1}),\end{aligned}$$

thus

$$\begin{aligned}&\left((V_t - \beta_* V)^2 + |\nabla' V|^2\right)^{\frac{p-2}{2}} (V_t - \beta_* V) \\ &\qquad = -\beta_*\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} \psi_* + a(\beta_* - \gamma)\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} g(\psi_*) \\ &\qquad\quad - a\beta_*(p-2)\frac{\beta_*(\gamma - \beta_*)\psi_* g(\psi_*) - g'(\psi_*) |\nabla\psi_*|^2}{(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2)^{\frac{4-p}{2}}}\psi_* + O(a^2 \|g(\psi_*)\|_{C^1}).\end{aligned}$$

Finally,

$$\begin{aligned}&-\left(\left((V_t - \beta_* V)^2 + |\nabla' V|^2\right)^{\frac{p-2}{2}} (V_t - \beta_* V)\right)_t \\ &\qquad = a\left[(\gamma^2 - \beta_*\gamma)\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} g(\psi_*)\right. \\ &\qquad \left. +\beta_*(p-2)\frac{\beta_*(\gamma^2 - \beta_*\gamma)\psi_* g(\psi_*) - \gamma g'(\psi_*) |\nabla\psi_*|^2}{(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2)^{\frac{4-p}{2}}}\psi_*\right] + O(a^2 \|g(\psi_*)\|_{C^2}).\end{aligned} \tag{3.27}$$

Since

$$\left((V_t-\beta_*V)^2+|\nabla' V|^2\right)^{\frac{p-2}{2}}\nabla' V=$$
$$\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}(1-ag'(\psi_*))\left[1+a(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_* g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}\right]\nabla'\psi_*$$
$$+O(a^2\,\|g(\psi_*)\|_{C^1})$$
$$=\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla'\psi_*$$
$$+a\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left[(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_* g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-g'(\psi_*)\right]\nabla'\psi_*$$
$$+O(a^2\,\|g(\psi_*)\|_{C^1}),$$

we get similarly

$$-div'\left(\left((V_t-\beta_*V)^2+|\nabla' V|^2\right)^{\frac{p-2}{2}}\nabla' V\right)=-div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla'\psi_*\right)$$
$$-a\,div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left[(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_* g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-g'(\psi_*)\right]\nabla'\psi_*\right)$$
$$+O(a^2\,\|g(\psi_*)\|_{C^2}).\tag{3.28}$$

Noting that

$$-div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla'\psi_*\right)\psi_*=\beta_*\Lambda_{\beta_*}\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\psi_*,\tag{3.29}$$

we obtain

$$e^{-\gamma t}\mathcal{Q}[V]$$
$$=\left[(\gamma^2-\beta_*\gamma)\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}g(\psi_*)+\beta_*(p-2)\frac{\beta_*(\gamma^2-\beta_*\gamma)\psi_* g(\psi_*)-\gamma g'(\psi_*)\,|\nabla\psi_*|^2}{(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2)^{\frac{4-p}{2}}}\psi_*\right]$$
$$-div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left[(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_* g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-g'(\psi_*)\right]\nabla'\psi_*\right)$$
$$-\Lambda_{\beta_*}\left((\gamma-\beta_*)\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}g(\psi_*)+\beta_*(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_* g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2)^{\frac{4-p}{2}}}\psi_*\right)$$
$$+e^{(\nu-\gamma)t}\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{q}{2}}\left[1+qa\frac{\beta_*(\gamma-\beta_*)\psi_* g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}\right]+O(a\,\|g(\psi_*)\|_{C^2}).\tag{3.30}$$

In this expression we have in particular

$$\begin{aligned}
-div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left[(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_* g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-g'(\psi_*)\right]\nabla'\psi_*\right)\\
=(p-1)div'\left[g'(\psi_*)\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla\psi_*\right]\\
-\beta_* div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-4}{2}}\left[(p-2)\beta_*\psi_* g'(\psi_*)+(p-2)(\gamma-\beta_*)g(\psi_*)\right]\psi_*\right)\\
=(p-1)g''(\psi_*)\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}|\nabla\psi_*|^2\\
+(p-1)g'(\psi_*)div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla\psi_*\right)\\
-(p-2)\beta_* div'\left[\frac{\left((\gamma-\beta_*)g(\psi_*)\psi_*+\beta_* g'(\psi_*)\psi_*^2\right)}{\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{4-p}{2}}}\nabla'\psi_*\right].
\end{aligned}\tag{3.31}$$

Using the equation (3.26) satisfied by ψ_*, it infers that

$$\begin{aligned}
-div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left[(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_* g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-g'(\psi_*)\right]\nabla'\psi_*\right)\\
=(p-1)\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left(g''(\psi_*)|\nabla'\psi_*|^2-\beta_*\Lambda_{\beta_*}g'(\psi_*)\psi_*\right)\\
-(p-2)\beta_* div'\left[\frac{\left((\gamma-\beta_*)g(\psi_*)\psi_*+\beta_* g'(\psi_*)\psi_*^2\right)}{\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{4-p}{2}}}\nabla'\psi_*\right].
\end{aligned}\tag{3.32}$$

Plugging this identity into the expression (3.30), we obtain after some simplifications

$$e^{-\gamma t}\mathcal{Q}[V]=\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}g(\psi_*)\mathcal{Q}_1[V]+e^{(\nu-\gamma)t}R[V]+O(a\,\|g(\psi_*)\|_{C^2}),\tag{3.33}$$

where

$$R[V]=e^{\nu t}\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{q}{2}}\left[1+q\frac{\beta_*(a'-\beta_* a)\psi_* g(\psi_*)-ag'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}\right],\tag{3.34}$$

and

$$\begin{aligned}\mathcal{Q}_1[V] &= (\gamma-\Lambda_{\beta_*})(\gamma-\beta_*)\left[1+(p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}\right]-(p-1)\beta_*\Lambda_{\beta_*}\frac{\psi_* g'(\psi_*)}{g(\psi_*)}\\ &\quad+\left[(p-4)\beta_*\Lambda_{\beta_*}\psi_*-2\Delta'\psi_*\right]\left(\gamma-\beta_*\left(1-\frac{\psi_* g'(\psi_*)}{g(\psi_*)}\right)\right)\frac{\beta_*\psi_*}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}\\ &-(p-2)\left[\frac{\psi_* g'(\psi_*)}{g(\psi_*)}\left((\beta_*+1)\gamma-\beta_*\Lambda_{\beta_*}+\beta_*\right)+\gamma-\beta_*+\beta_*\frac{\psi_*^2 g''(\psi_*)}{g(\psi_*)}\right]\frac{|\nabla'\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}\\ &\quad+(p-1)\frac{g''(\psi_*)}{g(\psi_*)}|\nabla'\psi_*|^2.\end{aligned} \tag{3.35}$$

In this expression the difficult term to deal with is $[(p-4)\beta_*\Lambda_{\beta_*}\psi_*-2\Delta'\psi_*]$ since it has not a prescribed sign. However $\Delta'\psi_*=O(\psi_*)$ by (6.19) in Appendix II.

Step 2: The perturbation method and the computation with $g(\psi_*)=\psi_*$. With such a choice of function g

$$\begin{aligned}\mathcal{Q}_1[V] &= (\gamma-\Lambda_{\beta_*})(\gamma-\beta_*)\left[1+(p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}\right]-(p-1)\beta_*\Lambda_{\beta_*}\\ &\quad-(p-2)\left[(\gamma-\Lambda_{\beta_*})\beta_*+2\gamma\right]\frac{|\nabla'\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}+\gamma\,O(\psi_*^2).\end{aligned} \tag{3.36}$$

Equivalently

$$\begin{aligned}\mathcal{Q}_1[V] &= \left[1+(p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}\right]\left(\gamma^2-(\Lambda_{\beta_*}+\beta_*)\gamma\right)\\ &\quad-\gamma\left[(p-2)(\beta_*+2)\frac{|\nabla'\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}+O(\psi_*^2)\right]\end{aligned}$$

and finally

$$\mathcal{Q}_1[V]=\left[1+(p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}\right]\gamma\left[\gamma-(\Lambda_{\beta_*}+\beta_*+(p-2)(\beta_*+2))+O(\psi_*^2)\right]. \tag{3.37}$$

Using the fact that $\beta_*>\frac{N-1}{p-1}$ if $1<p<2$ and $1<\beta_*<\frac{N-1}{p-1}$ if $2<p<N$ (see Theorem 6.1 in Appendix II), we have

$$\Lambda_{\beta_*}+\beta_*+(p-2)(\beta_*+2)\geq\begin{cases}\Lambda_{\beta_*}+\beta_*(p-1) & \text{if } p\geq 2\\ N+3(p-2)>N-3 & \text{if } 1<p<2.\end{cases} \tag{3.38}$$

When $N=2$, we have explicitly $\beta_*=\frac{1+2\sqrt{p^2-3p+3}}{3(p-1)}$ (see [9, Th 3.3]). Therefore for all $N\geq 2$ and $p>1$, there holds

$$\Lambda_{\beta_*}+\beta_*+(p-2)(\beta_*+2)>0. \tag{3.39}$$

We fix $\epsilon_0>0$ such that, whenever $\psi_*\leq\epsilon_0$, there holds

$$\Lambda_{\beta_*}+\beta_*+(p-2)(\beta_*+2)+O(\psi_*^2)>\frac{1}{2}\left(\Lambda_{\beta_*}+\beta_*+(p-2)(\beta_*+2)\right). \tag{3.40}$$

If we fix $\gamma_0 > 0$ such that

$$\gamma_0 < \min\left\{\frac{1}{2}\left(\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_*+2)\right), \nu, \beta_*\right\}, \tag{3.41}$$

we obtain

$$\mathcal{Q}_1[V] \leq -\min\{1, p-1\}\gamma m^2 \qquad \forall\, 0 < \gamma \leq \gamma_0, \tag{3.42}$$

whenever $\psi_* \leq \epsilon_0$, for some m depending only on p, q and N (through ψ_* and ν), which, in the same range of value of ψ_*, yields to

$$\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} g(\psi_*)\mathcal{Q}_1[V] \leq -c_{17}\psi_* \qquad \forall\, 0 < \gamma \leq \gamma_0, \tag{3.43}$$

for some $c_{17} > 0$ depending on N, p, q. This estimate is valid whatever is $p > 1$, but only in a neighborhood of $\psi_* = 0$. If we replace $g(\psi_*) = \psi_*$ by $g_k(\psi_*) = \psi_* e^{-k\psi_*}$ for $0 < k < 1$, and denote by $\mathcal{Q}_{1,k}[V]$ the corresponding expression of $\mathcal{Q}_1[V]$ which becomes now $\mathcal{Q}_{1,0}[V]$. We define similarly $\mathcal{Q}_k[V]$, and $\mathcal{Q}[V]$ becomes $\mathcal{Q}_0[V]$. Since $g_k'(\psi_*) = e^{-k\psi_*} - kg_k(\psi_*)$ and $g_k'' = -2ke^{-k\psi_*} + k^2 g_k(\psi_*)$, we obtain

$$\begin{aligned}\mathcal{Q}_{1,k}[V] = \mathcal{Q}_{1,0}[V] + k(p-1)\beta_*\Lambda_{\beta_*}\psi_* + (p-1)\left(-\frac{2k}{\psi_*} + k^2\right)|\nabla'\psi_*|^2 \\ + (2-p)\beta_*\left(-2k + k^2\right)\psi_* + O(\psi_*^2)\end{aligned} \tag{3.44}$$

Notice that $\nabla'\psi_*$ vanishes only at the North pole $\mathbf{e}_N$, thus there exists $k_0 \in (0, 1]$ such that

$$k(1-p)\beta_*\Lambda_{\beta_*}\psi_* + (p-1)\left(\frac{2k}{\psi_*} - k^2\right)\left|\nabla'\psi_*\right|^2 \geq \frac{1}{2}(2-p)_+\beta_*\left(-2k + k^2\right)\psi_* \qquad \forall k \leq k_0$$

whenever $\psi_* \leq \epsilon_0$ which yields to

$$\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} g_k(\psi_*)\mathcal{Q}_{1,k}[V] \leq -c_{18}k \qquad \forall k \leq k_0 \tag{3.45}$$

for some $c_{13} = c_{13}(N, p, q, \epsilon_0)$. There exists $c_{14} = c_{14}(N, p, q) > 0$ such that

$$\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{q}{2}}\left[1 + qe^{\gamma t}\frac{\beta_*(\gamma - \beta_*)\psi_* g_k(\psi_*) - g_k'(\psi_*)\left|\nabla\psi_*\right|^2}{\beta_*^2\psi_*^2 + \left|\nabla'\psi_*\right|^2}\right] \leq c_{14} \tag{3.46}$$

in $S_+^{N-1} \times (-\infty, \ln\delta]$. Moreover

$$O(a\left\|g(\psi_*)\right\|_{C^2}) \leq e^{\gamma t}\tilde{c}_k \tag{3.47}$$

for some $\tilde{c}_k = \tilde{c}_k(N, p, q) > 0$. We derive from (3.45)-(3.47)

$$e^{-\gamma t}\mathcal{Q}_k[V] \leq -c_{13}k + c_{14}e^{(\nu-\gamma)t} + e^{\gamma t}\tilde{c}_k \qquad \forall k \leq k_0 \tag{3.48}$$

Thus there exists $T_k \leq \ln\delta$ such that $\mathcal{Q}_k[V] \leq 0$, for all $t \leq T_k$ and provided $\psi_* \leq \epsilon_0$. This local estimate will be used in the construction of the subsolution when $p \geq 2$.

Step 3: The case $1 < p < 2$. Since the function ψ^* depends only on the azimuthal angle $\theta \in (0; \frac{\pi}{2}]$ we will write $\psi_*(\sigma) = \psi_*(\theta)$ and $\nabla' \psi_*(\sigma) = \psi_{*\theta}(\theta)\mathbf{n}$ where $\mathbf{n}$ is the downward unit vector tangent to S^{N-1} in the hyperplane going through σ and the poles. From (6.8),

$$(p-4)\beta_*\Lambda_{\beta_*}\psi_* - 2\Delta'\psi_* = (p-2)\left(\beta_*\Lambda_{\beta_*}\psi_* + 2\frac{\beta_*^2\psi_* + \psi_{*\theta\theta}}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\right), \tag{3.49}$$

since $\psi_{*\theta}^2 = |\nabla'\psi_*|^2$ and thus

$$\begin{aligned}((p-4)\beta_*\Lambda_{\beta_*}\psi_* - 2\Delta'\psi_*)\,\frac{\beta_*\gamma\psi_*}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \\ = (p-2)\gamma\left(\Lambda_{\beta_*}\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} + 2\beta_*\frac{\beta_*^2\psi_*^2 + \psi_{*\theta\theta}\psi_*}{(\beta_*^2\psi_*^2 + \psi_{*\theta}^2)^2}\right).\end{aligned} \tag{3.50}$$

From Theorem 6.1-Step 4 in Appendix II, we know that $\beta_*^2\psi_* + \psi_{*\theta\theta} \geq 0$, thus the contribution of this term to $\mathcal{Q}_1[V]$ is nonpositive. We replace this expression in $\mathcal{Q}_1[V]$ with $g(\psi_*) = \psi_*$ and obtain

$$\begin{aligned}\mathcal{Q}_1[V] &= (\gamma - \Lambda_{\beta_*})(\gamma - \beta_*)\left(1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\right) - \Lambda_{\beta_*}\beta_*(p-1) \\ &\quad + (p-2)\gamma\Lambda_{\beta_*}\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} - (p-2)\left((\beta_*+2)\gamma - \Lambda_{\beta_*}\beta_*\right)\frac{\psi_{*\theta}^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \\ &\quad + 2\beta_*(p-2)\frac{\beta_*^2\psi_*^2 + \psi_{*\theta\theta}\psi_*}{(\beta_*^2\psi_*^2 + \psi_{*\theta}^2)^2}\gamma \\ &\leq \gamma\left(1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\right)(\gamma - \Lambda_{\beta_*} - \beta_*) - (p-2)\gamma\frac{(\beta_*+2))\psi_{*\theta}^2 - \Lambda_{\beta_*}\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \\ &\leq \gamma\left(1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\right)\left(\gamma - \left(\Lambda_{\beta_*} + \beta_* + (p-2)\frac{(\beta_*+2)\psi_{*\theta}^2 - \Lambda_{\beta_*}\beta_*^2\psi_*^2}{(p-1)\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\right)\right).\end{aligned} \tag{3.51}$$

We can write

$$\begin{aligned}\Lambda_{\beta_*} + \beta_* + (p-2)\frac{(\beta_*+2)\psi_{*\theta}^2 - \Lambda_{\beta_*}\beta_*^2\psi_*^2}{(p-1)\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \\ = \frac{\left(\Lambda_{\beta_*} + (p-1)\beta_*\right)\beta_*^2\psi_*^2 + \left(\Lambda_{\beta_*} + \beta_*(p-1) + 2(p-2)\right)\psi_{*\theta}^2}{(p-1)\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \\ \geq c_{15}\left(\Lambda_{\beta_*} + \beta_*(p-1) + 2(p-2)\right)\end{aligned} \tag{3.52}$$

for some positive constant c_{15}. This expression $\Lambda_{\beta_*} + \beta_*(p-1) + 2(p-2)$ is always positive: obviously if $N \geq 3$ and by using the explicit expression of β_* if $N = 2$. Thus there exists γ_0 and $c_{16} > 0$ such that $\mathcal{Q}_1[V] < -c_{16}$ for $0 < \gamma \leq \gamma_0$. The perturbation method of Step 2, is valid in the whole range of values of ψ_* and we derive from (3.42)-(3.43) that (3.48) holds for all $k \leq k_0$ and $t \leq T_k$. Therefore $\mathcal{Q}_k[V] \leq 0$.

Step 4: The case $p \geq 2$. For $c > 0$ to be fixed and $\psi_* \geq \epsilon_0$, $\gamma \in (0, \gamma_0]$, we take $g(\psi_*) = c\psi_*^{1-\frac{\gamma}{\beta_*}}$. Then

we derive from (3.35):

$$\begin{aligned}\mathcal{Q}_1[V] &= (\gamma-\Lambda_{\beta_*})(\gamma-\beta_*)\frac{(p-1)\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-(p-1)\beta_*\Lambda_{\beta_*}\left(1-\frac{\gamma}{\beta_*}\right)\\ &\quad-(p-1)\frac{\gamma(\beta_*-\gamma)}{\beta_*^2}\psi_*^{-1-\frac{\gamma}{\beta_*}}|\nabla'\psi_*|^2-(p-2)(\beta_*-\gamma)(\gamma-\Lambda_{\beta_*})\frac{|\nabla'\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2} \qquad (3.53)\\ &=(1-p)\left[\gamma(\beta_*-\gamma)+\frac{\gamma(\beta_*-\gamma)}{\beta_*^2}\psi_*^{-1-\frac{\gamma}{\beta_*}}|\nabla'\psi_*|^2\right].\end{aligned}$$

For $k\le k_0$ we fix c such that $c\epsilon_0^{1-\frac{\gamma}{\beta_*}}=\epsilon_0e^{-k\epsilon_0}\Longleftrightarrow c=\epsilon_0^{\frac{\gamma}{\beta_*}}e^{-k\epsilon_0}$ and we define g by

$$g(\psi_*)=\min\left\{\psi_*e^{-k\psi_*},\epsilon_0^{\frac{\gamma}{\beta_*}}e^{-k\epsilon_0}\psi_*^{1-\frac{\gamma}{\beta_*}}\right\}=\begin{cases}\psi_*e^{-k\psi_*} & \text{if } 0\le\psi_*\le\epsilon_0\\ \epsilon_0^{\frac{\gamma}{\beta_*}}e^{-k\epsilon_0}\psi_*^{1-\frac{\gamma}{\beta_*}} & \text{if } \epsilon_0\le\psi_*\le 1,\end{cases} \qquad (3.54)$$

and we set $V(t,\sigma)=\psi^*(\sigma)-a(t)g(\psi_*(\sigma))$ with $(t,\sigma)\in(-\infty,T_k]\times S^{N-1}_+$ and define $\tilde u(r,\sigma)=r^{-\beta_*}(\psi^*(\sigma)-a(\ln r)g(\psi_*(\sigma)))$ accordingly for $(r,\sigma)\in(-\infty,e^{T_k}]\times S^{N-1}_+$. Since ψ_* is a decreasing function the coincidence set $\{\sigma\in S^{N-1}_+:\psi_*(\sigma)=\epsilon_0\}$ is a circular cone Σ_{θ_0} with vertex 0, axis $\mathbf{e}_N$ and angle θ_0. We set $R_0=e^{T_k}$

$$\begin{aligned}\Gamma_1&=\left\{x=(r,\theta)\in B^+_{R_0}:\theta_0<\theta<\tfrac{\pi}{2}\right\}=\left\{(r,\sigma)\in[0,R_0)\times S^{N-1}_+:0<\psi_*(\sigma)<\epsilon_0\right\},\\ \Gamma_2&=\left\{x=(r,\theta)\in B^+_{R_0}:0<\theta<\theta_0\right\}=\left\{(r,\sigma)\in[0,R_0)\times S^{N-1}_+:\epsilon_0<\psi_*(\sigma)<1\right\},\end{aligned}$$

and define

$$\begin{aligned}\tilde u(r,\sigma)&=r^{-\beta_*}\left(\psi_*(\sigma)-r^\gamma g(\psi_*(\sigma))\right)\\ &=\begin{cases}u_1(r,\sigma)=r^{-\beta_*}(1-r^\gamma e^{-k\psi_*(\sigma)})\psi_*(\sigma) & \text{if } (r,\theta)\in\Gamma_1\\ u_2(r,\sigma)=r^{-\beta_*}\left(1-r^\gamma\epsilon_0^{\frac{\gamma}{\beta_*}}e^{-k\epsilon_0}(\psi_*(\sigma))^{1-\frac{\gamma}{\beta_*}}\right)\psi_*(\sigma) & \text{if } (r,\theta)\in\Gamma_2.\end{cases}\end{aligned}$$

The function $\tilde u$ is a subsolution separately on Γ_1 and Γ_2 and is Lipschitz continuous in $\overline\Omega\setminus\{0\}$. If we denote by g_1 and g_2 the restriction of g to Γ_1 and Γ_2 respectively, that is to Ω_1 and Ω_2, then $g_1'(\epsilon_0)>g_2'(\epsilon_0)>0$. Let $\zeta\in C^1_c(B^+_{R_0})$ which vanishes in neighborhoods of 0 and $\partial B^+_{R_0}$, $\zeta\ge0$, then

$$\int_{\Gamma_i}|\nabla\tilde u|^{p-2}\nabla\tilde u.\nabla\zeta dx+\int_{\Omega_i}|\nabla\tilde u|^q\zeta dx\le\int_{\Sigma_{\theta_0}}|\nabla u_i|^{p-2}\partial_{\mathbf{n}_i}u_i\zeta dS, \qquad (3.55)$$

where $\mathbf{n}_i$ is the normal unit vector on Σ_{θ_0} outward from Γ_i. Actually, $\mathbf{n}_2=-\mathbf{n}_1=\mathbf{n}$ thus

$$\nabla\tilde u=\tilde u_r\mathbf{e}+r^{-\beta_*-1}(1-r^\gamma g'(\psi_*))\nabla'\psi_*=\tilde u_r\mathbf{e}+r^{-\beta_*-1}(1-r^\gamma g'(\psi_*))\psi_{*\theta}\,\mathbf{n}.$$

and on Σ_{θ_0},

$$\nabla\tilde u=\begin{cases}\tilde u_r\mathbf{e}-r^{-\beta_*-1}(1-r^\gamma g_1'(\epsilon_0))\psi_{*\theta}\,\mathbf{n} & \text{in } \Gamma_1\\ \tilde u_r\mathbf{e}+r^{-\beta_*-1}(1-r^\gamma g_2'(\epsilon_0))\psi_{*\theta}\,\mathbf{n} & \text{in } \Gamma_2\end{cases}$$

Therefore

$$|\nabla u_1|^{p-2}\,\partial_{\mathbf{n}_1} u_1 = -r^{-\beta_*-1}(1-r^\gamma g_1'(\epsilon_0))\left(\tilde{u}_r^2 + r^{-2\beta_*-2}(1-r^\gamma g_1'(\epsilon_0))^2\psi_{*\theta}^2\right)^{\frac{p-2}{2}}\psi_{*\theta} \quad \text{in } \Gamma_1$$

and

$$|\nabla u_2|^{p-2}\,\partial_{\mathbf{n}_2} u_2 = r^{-\beta_*-1}(1-r^\gamma g_2'(\epsilon_0))\left(\tilde{u}_r^2 + r^{-2\beta_*-2}(1-r^\gamma g_2'(\epsilon_0))^2\psi_{*\theta}^2\right)^{\frac{p-2}{2}}\psi_{*\theta} \quad \text{in } \Gamma_2.$$

By adding the two inequalities (3.55)

$$\int_\Omega |\nabla\tilde{u}|^{p-2}\,\nabla\tilde{u}.\nabla\zeta dx + \int_\Omega |\nabla\tilde{u}|^q\,\zeta dx \le \int_{\Sigma_{\theta_0}} \left(|\nabla u_1|^{p-2}\,\partial_{\mathbf{n_1}} u_1 + |\nabla u_2|^{p-2}\,\partial_{\mathbf{n_2}} u_2\right)\zeta dS. \tag{3.56}$$

By monotonicity of the function $X \mapsto \left(\tilde{u}_r^2 + X^2\right)^{\frac{p}{2}}$ and since

$$r^{-\beta_*-1}(1-r^\gamma g_2'(\epsilon_0)) \ge r^{-\beta_*-1}(1-r^\gamma g_1'(\epsilon_0)) \ge 0,$$

we derive

$$\begin{aligned} &r^{-\beta_*-1}(1-r^\gamma g_2'(\epsilon_0))\left(\tilde{u}_r^2 + r^{-2\beta_*-2}(1-r^\gamma g_2'(\epsilon_0))^2\psi_{*\theta}^2\right)^{\frac{p-2}{2}} \\ &\qquad \ge r^{-\beta_*-1}(1-r^\gamma g_1'(\epsilon_0))\left(\tilde{u}_r^2 + r^{-2\beta_*-2}(1-r^\gamma g_1'(\epsilon_0))^2\psi_{*\theta}^2\right)^{\frac{p-2}{2}} \end{aligned}$$

We derive that the right-hand side of (3.56) is nonpositive because $\psi_{*\theta} \le 0$, and therefore $\tilde{u}$ is a positive subsolution of (1.1) in $B_{R_0}^+$ dominated by Ψ_* and satisfying (3.24). □

Proof of Theorem 3.6. Let $M = \max\{\Psi_*(x) : x \in \partial B_{R_0}^+\}$, then $M = R_0^{-\beta_*}$. The function u^* defined by

$$u^*(x) = \begin{cases} (\tilde{u}(x) - M)_+ & \text{if } x \in B_{R_0}^+ \\ 0 & \text{if } x \in \Omega \setminus B_{R_0}^+, \end{cases}$$

is indeed a subsolution of (1.1) in whole Ω where it satisfies $u^* \le \Psi_*$ and it vanishes on $\partial\Omega \setminus \{0\}$. Since Φ_* is a positive p-harmonic function in Ω which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies (3.20), it is supersolution of (1.1) and therefore it dominates u^*. Therefore there exists a solution u of (1.1) in Ω which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies $u^* \le u \le \Phi_*$. This implies that (3.19) holds with $k = 1$ and we conclude with Lemma 3.8. This ends the proof of Lemma 3.9. □

When $p = N$ the statement of Theorem 3.6 holds without the flatness assumption on $\partial\Omega$. The proof of the next theorem is an easy adaptation to the one of Theorem 3.6, provided Lemma 3.7, Lemma 3.8 and Lemma 3.9 are modified accordingly.

Theorem 3.10. *Assume $N-1 < q < N - \frac{1}{2}$ and Ω be a bounded C^2 domain such that $0 \in \partial\Omega$. Then for any $k > 0$ there exists a unique positive solution $u := u_k$ of (3.17) in Ω, which belongs to $C^1(\overline{\Omega} \setminus \{0\})$, vanishes on $\partial\Omega \setminus \{0\}$ and satisfies uniformly with respect to $\sigma \in S_+^{N-1}$*

$$\lim_{\substack{x \to 0 \\ x/|x| \to \sigma}} |x|\,u_k(x) = k\psi_*(\sigma). \tag{3.57}$$

Since $p = N$, then $\beta_* = 1$ and $\psi_*(\sigma) = \frac{x_N}{|x|} = \cos\theta_{N-1}$ with the identification of σ and $\theta_{N-1} := \theta$. In a more intrinsic manner (3.57) can be written under the form

$$\lim_{\substack{x \to 0 \\ x \in \Omega}} |x|^2 \frac{u_k(x)}{d(x)} = k. \tag{3.58}$$

We recall that if $\omega \in \mathbb{R}^N$ and $\mathcal{I}_\omega$ denotes the inversion of center ω and power 1, i.e. $\mathcal{I}_\omega(x) = \omega + \frac{x-\omega}{|x-\omega|^2}$, then $\tilde{u} = u \circ \mathcal{I}_\omega$ satisfies (3.18).

Lemma 3.11. *Assume Ω be a bounded C^2 domain such that $0 \in \partial\Omega$. Then there exists a unique N-harmonic function Φ_* in Ω, which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies*

$$\lim_{\substack{x \to 0 \\ x/|x| \to \sigma}} |x|\, \Phi_*(x) = \psi_*(\sigma), \tag{3.59}$$

uniformly with respect to $\sigma \in S^{N-1}_+$.

Proof. Uniqueness is standard. Let $\omega = -\mathbf{e}_N \in \overline{\Omega}^c$, with the notations of the proof of Theorem 3.5, $\omega' = -\omega$, $a = -\frac{1}{2}\mathbf{e}_N$ and $a' = -a$. We can assume that the balls $B_{\frac{1}{2}}(a)$ and $B_{\frac{1}{2}}(a')$ are tangent to $\partial\Omega$ at 0 and respectively subset of Ω^c and Ω. The function $x \mapsto \Psi(x) = -\frac{x_N}{|x|^2}$ which is N-harmonic in $\mathbb{R}^N_-$ and vanishes on $\partial\mathbb{R}^N_- \setminus \{0\}$ is transformed by the inversion $\mathcal{I}_{\omega'}$ of center ω' and power 1 into the function $\Psi_{\omega'} = \Psi \circ \mathcal{I}_\omega$ which is positive and N-harmonic in $B_{\frac{1}{2}}(a')$ and vanishes on $\partial B_{\frac{1}{2}}(a') \setminus \{0\}$. The function $\hat{\Psi} = -\Psi$ which is N-harmonic in $\mathbb{R}^N_+$ and vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$ is transformed by the inversion $\mathcal{I}_\omega$ of center ω and power 1 into the function $\Psi_\omega = \hat{\Psi} \circ \mathcal{I}_\omega$ which is positive and N-harmonic in $B^c_{\frac{1}{2}}(a)$ and vanishes on $\partial B_{\frac{1}{2}}(a) \setminus \{0\}$. For $\epsilon > 0$ we denote by Φ_ϵ the solution of

$$\begin{array}{rll} -\Delta_N \Phi_\epsilon = 0 & & \text{in } \Omega \cap B^c_\epsilon \\ \Phi_\epsilon = 0 & & \text{in } (B^c_{\frac{1}{2}}(a') \cap \partial B_\epsilon) \cup (\partial\Omega \cap B^c_\epsilon) \\ \Phi_\epsilon = \Psi_{\omega'} & & \text{in } B_{\frac{1}{2}}(a') \cap \partial B_\epsilon. \end{array} \tag{3.60}$$

If $0 < \epsilon' < \epsilon$, $\Phi_{\epsilon'} \geq \Psi_{\omega'}$ in $B_{\frac{1}{2}}(a') \cap \partial B_\epsilon$, thus $\Phi_{\epsilon'} \geq \Phi_{\epsilon'}$ in $\Omega \cap B^c_\epsilon$. We also denote by $\hat{U}_\epsilon$ the solution of

$$\begin{array}{rll} -\Delta_N \hat{\Phi}_\epsilon = 0 & & \text{in } \Omega \cap B^c_\epsilon \\ \hat{\Phi}_\epsilon = 0 & & \text{in } \partial\Omega \cap B^c_\epsilon \\ \hat{\Phi}_\epsilon = \Psi_\omega & & \text{in } \Omega \cap \partial B^c_\epsilon. \end{array} \tag{3.61}$$

In the same way as above

$$0 < \epsilon' < \epsilon \Longrightarrow \hat{\Phi}_{\epsilon'} \leq \hat{\Phi}_\epsilon \quad \text{in } \Omega \cap \partial B^c_\epsilon$$

Using the explicit form of Ψ, $\mathcal{I}_\omega : x \mapsto \omega + \frac{x-\omega}{|x-\omega|^2}$ and $\mathcal{I}_{\omega'} : x \mapsto \omega' + \frac{x-\omega'}{|x-\omega'|^2}$ we see that

$$\Psi_{\omega'} \lfloor_{B_{\frac{1}{2}}(a') \cap \partial B_\epsilon} \leq \frac{1+\epsilon}{1-\epsilon} \Psi_\omega \lfloor_{B_{\frac{1}{2}}(a') \cap \partial B_\epsilon},$$

thus

$$\Phi_\epsilon \leq \frac{1+\epsilon}{1-\epsilon} \hat{\Phi}_\epsilon \quad \text{in } \Omega \cap B^c_\epsilon.$$

Letting $\epsilon \to 0$ we conclude that Φ_ϵ converges uniformly in $\overline{\Omega} \setminus \{0\}$ to Φ_* which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies (3.59). □

The proof of the next statement is similar to the one of Lemma 3.8 up to some minor modifications, so we omit it.

Lemma 3.12. *Let the assumptions on q and Ω of Theorem 3.10 be satisfied. If for some $k > 0$ there exists a solution u_k of (3.17) in Ω, which belongs to $C^1(\overline{\Omega} \setminus \{0\})$, vanishes on $\partial\Omega \setminus \{0\}$ and satisfies (3.57), then for any $k > 0$ there exists such a solution.*

Lemma 3.13. *Under the assumptions of Theorem 3.10 there exists a Lipschitz continuous nonnegative subsolution $\tilde{u}$ of (3.17) in Ω which vanishes on $\partial\Omega \setminus \{0\}$, is smaller than Φ_* and satisfies*

$$\lim_{\substack{x \to 0 \\ x/|x| \to \sigma}} |x|\, \tilde{u}(x) = \sigma, \tag{3.62}$$

uniformly with respect to $\sigma \in S^{N-1}_+$.

Proof. Let $\tau > 0$ to be fixed and let w be the solution of

$$-\Delta_N w + |\nabla w|^q = 0 \qquad \text{in } B_2^- \tag{3.63}$$

which vanishes on $\partial B_2^- \setminus \{0\}$ and satisfies

$$\lim_{\substack{x \to 0 \\ x/|x| \to \sigma}} |x| w(x) = \sigma \tag{3.64}$$

in the C^1-topology of S^{N-1}_-. Its existence follows from Theorem 3.6 and this function is dominated by the N-harmonic function Φ_* corresponding to this domain, obtained in Lemma 3.11. By $\mathcal{I}_{\omega'}$, the half-ball B_2^- is transform into the lunule $G = B_{\frac{1}{2}}(a') \setminus B_{\frac{2}{3}}(\frac{4}{3}\omega')$ and $\tilde{w} = w \circ \mathcal{I}_{\omega'}$ satisfies

$$-\Delta_N \tilde{w} + |x - \omega'|^{2(q-N)} |\nabla \tilde{w}|^q = 0 \qquad \text{in } G. \tag{3.65}$$

Since $|x - \omega'| \leq 1$ in G, $-\Delta_N \tilde{w} + |\nabla \tilde{w}|^q \leq 0$ in G. We extend $\tilde{w}$ by 0 in $\Omega \setminus G$ and the resulting function $\tilde{u}$ is a subsolution of (3.17) in Ω which vanishes on $\partial\Omega \setminus \{0\})$, is smaller than the N-harmonic function Φ_* obtained in Lemma 3.11, and satisfies (3.62). □

4 Classification of boundary singularities

We assume that $\Omega \subset \mathbb{R}^N$ is a C^2 domain and $0 \in \partial\Omega$. Furthermore, in order to avoid extremely technical computations, we shall assume either that $\partial\Omega$ is flat near 0 or $p = N$. We suppose that the tangent plane to $\partial\Omega$ at 0 is $\partial\mathbb{R}^N_+ = \{x = (x', 0)\}$ and the normal inward unit vector at 0 is $\mathbf{e}_N$, therefore $\mathbf{n} = -\mathbf{e}_N$ in the sequel. We denote by $\omega_{s^{N-1}_+}$ the unique positive solution of (3.1) in S^{N-1}_+ and by $U_{s^{N-1}_+}$ the corresponding singular solution of (1.1) in $\mathbb{R}^N_+$ defined by

$$U_{s^{N-1}_+}(x) = |x|^{-\beta_q}\, \omega_{s^{N-1}_+}(\frac{x}{|x|}). \tag{4.66}$$

We recall that ψ_* is the unique positive solution of (3.2) with maximum 1 and Ψ_* the corresponding p-harmonic function

$$\Psi_*(x) = |x|^{-\beta_*}\, \psi_*(\frac{x}{|x|}). \tag{4.67}$$

4.1 The case $1 < p < N$

The first statement points out the link between weak and strong singularities.

Proposition 4.1. *Under the assumptions of Theorem 3.6 there exists* $\lim_{k\to\infty} u_k = u_\infty$ *which is the unique element of* $C(\overline{\Omega}\setminus\{0\})\cap C^1(\Omega)$ *which vanishes on* $\partial\Omega\setminus\{0\}$*, satisfies* (1.1) *in* Ω *and*

$$\lim_{x\to 0}\frac{u_\infty(x)}{U_{s_+^{N-1}}(x)} = 1. \tag{4.68}$$

Proof. Uniqueness follows from (4.68) and the maximum principle. For existence, since the mapping $k \mapsto u_k$ is increasing and $u_k \le U_{s_+^{N-1}}$, there exists $\lim_{k\to\infty} u_k := u_\infty \le U_{s_+^{N-1}}$ and $u_\infty \in C(\overline{\Omega}\setminus\{0\})\cap C^1(\Omega)$. It vanishes on $\partial B_\delta^+\setminus\{0\}$ and satisfies (1.1) in B_δ^+. In order to take into account the domain B_δ^+ in the notations, we set $u_k = u_{k,\delta}$. Since the mapping $\delta \mapsto u_{k,\delta}$ is also increasing and $u_{k,\delta} \le k\Psi_*$, there also exists $\lim_{\delta\to\infty} u_{k,\delta} := u_{k,\infty} \le k\Psi_*$ Then, for all $\ell > 0$,

$$T_\ell[u_{k,\delta}](x) = \ell^{\beta_q} u_{k,\delta}(\ell x) = u_{k\ell^{\beta_q},\ell^{-1}\delta}(x). \tag{4.69}$$

Letting $k\to\infty$, we obtain

$$T_\ell[u_{\infty,\delta}](x) = \ell^{\beta_q} u_{\infty,\delta}(\ell x) = u_{\infty,\ell^{-1}\delta}(x), \tag{4.70}$$

and letting $\delta\to\infty$, we obtain

$$T_\ell[u_{\infty,\infty}](x) = \ell^{\beta_q} u_{\infty,\infty}(\ell x) = u_{\infty,\infty}(x). \tag{4.71}$$

This implies that

$$u_{\infty,\infty}(r,\sigma) = r^{-\beta_q}\omega'(\sigma), \tag{4.72}$$

and ω' is a positive solution of problem (3.1). Therefore $\omega' = \omega_{s_+^{N-1}}$ by Theorem 3.2. If we let $\ell\to 0$ in (4.69) and take $|x| = 1$, $x = \sigma$, we derive

$$\lim_{\ell\to 0}\ell^{\beta_q} u_{\infty,\delta}(\ell,\sigma) = \lim_{\ell\to 0} u_{\infty,\ell^{-1}\delta}(1,\sigma) = u_{\infty,\infty}(1,\sigma) = \omega_{s_+^{N-1}}(\sigma). \tag{4.73}$$

This convergence holds in $C^1(\overline{S_+^{N-1}})$ because of Lemma 2.5. This implies (4.68). □

The main classification result is as follows.

Theorem 4.2. *Assume* $1 < p < N$*,* $p-1 < q < q^*$ *and* $\partial\Omega\cap B_\delta = \{x = (x',0) : |x'| < \delta\}$*, for some* $\delta > 0$*. If* $u \in C(\overline{\Omega}\setminus\{0\})\cap C^1(\Omega)$ *is a positive solution of* (1.1) *in* Ω *which vanishes on* $\partial\Omega\setminus\{0\}$*, then we have the following alternative:*

(i) either there exists $k \ge 0$ *such that*

$$\lim_{x\to 0}\frac{u(x)}{\Psi_*(x)} = k, \tag{4.74}$$

(ii) or

$$\lim_{x\to 0}\frac{u(x)}{U_{s_+^{N-1}}(x)}=1. \tag{4.75}$$

Proof. Step 1. Assume

$$\liminf_{x\to 0}\frac{u(x)}{\Psi_*(x)}<\infty, \tag{4.76}$$

then we claim that (4.74) holds. We first note that if (4.76) holds, there also holds

$$\liminf_{x\to 0}\frac{u(x)}{u_1(x)}<\infty, \tag{4.77}$$

where u_1 is the solution of (1.1) obtained in Theorem 3.6 with $k=1$. If $\{x_n\}$ is converging to 0 and such that for some $k>0$

$$\liminf_{x\to 0}\frac{u(x)}{u_1(x)}=k=\lim_{n\to\infty}\frac{u(x_n)}{u_1(x_n)},$$

there also holds by the boundary Harnack inequality (2.38) applied to both u and u_1,

$$\frac{u(x_n)}{u_1(x_n)}=\frac{u(x_n)}{d(x_n)}\frac{d(x_n)}{u_1(x_n)}\geq c_5^{-2}\frac{u(x)}{u_1(x)}\quad \forall x \text{ s.t. } |x|=|x_n|\,.$$

This implies in particular

$$u(x)\leq c_5^2(k+\epsilon_n)u_1(x)\qquad \forall x \text{ s.t. } |x|=|x_n|$$

where $\{\epsilon_n\}$ is converging to 0_+, and by the comparison principle

$$u(x)\leq Ku_1(x)\qquad \forall x\in\mathbb{R}^N_+ \text{ s.t. } |x_n|\leq|x|\leq\frac{\delta}{2},$$

for some $K>0$ and all $n\in\mathbb{N}_*$. Therefore

$$\limsup_{x\to 0}\frac{u(x)}{u_1(x)}<\infty. \tag{4.78}$$

We can assume that $k\neq 0$, otherwise (4.74) holds with $k=0$ and actually u remains bounded near 0. As a consequence of the Hopf Lemma and C^1 regularity, there exists $K>0$ such that

$$u(x)\leq K\Psi_*(x)\qquad \forall x\in B^+_{\frac{\delta}{2}}. \tag{4.79}$$

Let $m=\max\{u(x):|x|=\delta\}$. For $0<\tau<\delta$ we denote by k_τ the minimum of the $\kappa>0$ such that $u(x)\leq\kappa\Psi_*(x)+m$ for $\tau\leq|x|\leq\delta$. Then $u(x)\leq k_\tau\Psi_*(x)+m$, and either the graphs of the mappings $u(.)$ and $k_\tau\Psi_*(.)+m$ are tangent at some $x_\tau\in B^+_\delta\setminus\overline{B}^+_\tau$, or they are tangent on the boundary of the domain, and the only possibility is that they are tangent on $|x|=\tau$. Since

$$|\nabla\Psi_*(x)|^2=|x|^{-2(\beta_*+1)}\,(\beta_*^2\psi_*^2+|\nabla\psi_*|^2),$$

it never vanishes. If we set $w = u - (k_\tau \Psi_*(x) + m)$, then

$$-\mathcal{L}w + |\nabla u|^q = 0 \tag{4.80}$$

where the operator

$$\mathcal{L} = \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$$

is uniformly elliptic in a neighborhood of x_τ (see [6, Lemma 1.3]). Furthermore $w \leq 0$ and $w(x_\tau) = 0$ by the strong maximum principle $\nabla u(x_\tau)$ must vanish, which contradicts the fact that $\nabla u(x_\tau) = \nabla w(x_\tau)$ by the tangency condition, and $\nabla w(x_\tau) \neq 0$. Therefore $|x_\tau| = \tau$ and $x_\tau \notin \partial\mathbb{R}^N_+$. If $\tau' < \tau$, $k_\tau \leq k_{\tau'}$, and we set $k = \lim_{\tau\to 0} k_\tau$, which is finite because of (4.79). There exists $\{\tau_n\}$ such that $\sigma_n := \tau^{-1}x_{\tau_n} \to \sigma_0$. Furthermore

$$r^{\beta_*}u(r,\sigma) \leq k_\tau\psi_*(\sigma) + mr^{\beta_*} \quad \text{if } \tau \leq r \leq \delta \quad \text{and } \tau^{\beta_*}u(\tau,\sigma_\tau) = k_\tau\psi_*(\sigma_\tau) + m\tau^{\beta_*}. \tag{4.81}$$

Put

$$u_\tau(x) = \tau^{\beta_*}u(\tau x) \tag{4.82}$$

Then

$$-\Delta_p u_\tau + \tau^{p-q-\beta_*(p+1-q)}\,|\nabla u_\tau|^q = 0 \qquad \text{in } B^+_{\frac{\delta}{\tau}} \setminus \{0\}$$

and, by (4.79),

$$0 \leq u_\tau(x) \leq K\,|x|^{-\beta_*} \qquad \text{in } B^+_{\frac{\delta}{2\tau}} \setminus \{0\}.$$

By Lemma 2.5, the set of functions $\{u_\tau(.)\}$ is relatively compact in the C^1_{loc} topology of $\overline{\mathbb{R}^N_+} \setminus \{0\}$. Therefore, as $q < q^*$, there exist a sequence $\{\tau'_n\} \subset \{\tau_n\}$ converging to 0, and a positive p-harmonic function v in $\mathbb{R}^N_+$, continuous in $\overline{\mathbb{R}^N_+}\setminus\{0\}$ and vanishing on $\partial\mathbb{R}^N_+\setminus\{0\}$, such that $u_{\tau'_n} \to v$, and v satisfies (4.79) in $\overline{\mathbb{R}^N_+} \setminus \{0\}$. By Theorem 5.1 in Appendix I, there exists k^* such that $v = k^*\Psi_*$. In particular,

$$\lim_{\tau'_n\to 0} u_{\tau'_n}(1,\sigma) = k^*\psi_*(\sigma) \tag{4.83}$$

in the $C^1(\overline{S^{N-1}_+})$ topology. Combining (4.81), $(4.82)$and (4.83) we conclude that $k^* = k$ and

$$\lim_{\tau'_n\to 0} \tau'^{\beta_*}_n u_{\tau'_n}(1,\sigma) = k\psi_*(\sigma) \tag{4.84}$$

Using Theorem 3.6, it is equivalent to

$$\lim_{\tau'_n\to 0} \frac{u(\tau'_n,\sigma)}{u_k(\tau'_n,\sigma)} = 1 \tag{4.85}$$

uniformly on S^{N-1}_+. For any $\epsilon > 0$, there exists $n_\epsilon > 0$ such that $n \geq n_\epsilon$ implies

$$u_{k-\epsilon}(\tau'_n,\sigma) \leq u(\tau'_n,\sigma) \leq u_{k+\epsilon}(\tau'_n,\sigma)$$

By comparison principle,

$$u_{k-\epsilon} \leq u \leq u_{k+\epsilon} + m \qquad \text{in } B^+_\delta \setminus B^+_{\tau'_n}, \tag{4.86}$$

and finally

$$u_{k-\epsilon} \le u \le u_{k+\epsilon} + m \qquad \text{in } B_\delta^+, \tag{4.87}$$

Since ϵ is arbitrary and using again Theorem 3.6, it implies

$$\lim_{r\to 0} \frac{u(r,\sigma)}{\Psi_*(r,\sigma)} = k, \tag{4.88}$$

locally uniformly on S^{N-1}. But since the convergence holds in $C^1(\overline{S_+^{N-1}})$, (4.74) follows.

Step 2. Assume

$$\lim_{x\to 0} \frac{u(x)}{\Psi_*(x)} = \infty. \tag{4.89}$$

For any $0 < \epsilon < \delta$ and $k > 0$, there holds

$$u_k(x) \le u(x) \le v_\epsilon(x) \qquad \text{in } B_\delta^+ \setminus B_\epsilon^+ \tag{4.90}$$

where v_ϵ has been defined in (3.12) and u_k is given by Theorem 3.6. Letting $\epsilon \to 0$, $k \to \infty$, and using Proposition 4.1, we derive

$$u_\infty(x) \le u(x) \le v_0(x) \qquad \text{in } B_\delta^+ \setminus \{0\}. \tag{4.91}$$

We have seen in Theorem 3.3 that v_0 is a separable solution of (1.1) in $\mathbb{R}^N_+$ which vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$, therefore $v_0(x) = U_{s_+^{N-1}}(x)$. This implies

$$u_\infty(x) \le u(x) \le |x|^{-\beta_q}\, \omega_{s_+^{N-1}}(\frac{x}{|x|}) \qquad \text{in } B_\delta^+ \setminus \{0\}. \tag{4.92}$$

We conclude using Proposition 4.1. □

4.2 The case $p = N$

When $p = N$, the assumption that $\partial\Omega$ is an hyperplane near 0 can be removed. The proof of the next results is based upon Theorem 3.10. The following result is the extension to the case $p = N$ of Proposition 4.1.

Proposition 4.3. *Under the assumptions of Theorem 3.10 there exists $\lim_{k\to\infty} u_k = u_\infty$ which is the unique element of $C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$ which satisfies (3.17) in Ω, vanishes on $\partial\Omega \setminus \{0\}$ and such that*

$$\lim_{x\to 0} \frac{u_\infty(x)}{U_{s_+^{N-1}}(x)} = 1. \tag{4.93}$$

Proof. We denote by u_k^Ω the unique positive solution of (3.17) satisfying (3.57) obtained in Theorem 3.6. Then

$$T_\ell[u_k^\Omega] = u_{\ell^{\beta_q-\beta_*}k}^{\Omega^\ell}, \tag{4.94}$$

because of uniqueness. We denote by $B := B_{\frac{1}{2}}(a)$ and $B' := B_{\frac{1}{2}}(a')$ the two balls tangent to $\partial\Omega$ at 0 respectively interior and exterior to Ω introduced in the proof of Lemma 3.11. Estimate (3.58) implies

$$u_k^{B'^c} \le u_k^\Omega \le u_k^B \tag{4.95}$$

the left-hand side inequality holding in Ω and the right-hand side one in B. Therefore

$$T_\ell[u_k^{B'^c}] := u_{\ell^{\beta_q-\beta_*}k}^{B'^c\,\ell} \le T_\ell[u_k^\Omega] \le T_\ell[u_k^B] := u_{\ell^{\beta_q-\beta_*}k}^{B^\ell}, \tag{4.96}$$

the domains of validity of these inequalities being modified accordingly. Using again (3.58) we obtain

$$T_{\ell'}[u_{k'}^{B'^c}] \le T_\ell[u_k^{B'^c}] \qquad \text{in } B'^{c\,\ell'}, \tag{4.97}$$

for any $0 < \ell' \le \ell$ and $\ell'^{\beta_q-\beta_*}k' \le \ell^{\beta_q-\beta_*}k$. In the same way

$$T_{\ell'}[u_{k'}^B] \ge T_\ell[u_k^B] \qquad \text{in } B^\ell, \tag{4.98}$$

for any $0 < \ell' \le \ell$ and $\ell'^{\beta_q-\beta_*}k' \ge \ell^{\beta_q-\beta_*}k$. Since $u_k^\Omega\ u_k^B$, $u_k^{B'^c}$ are increasing with respect to k, they converge respectively to $u_\infty^\Omega\ u_\infty^B$, $u_\infty^{B'^c}$ and there holds for any $\ell > 0$

$$T_\ell[u_\infty^{B'^c}] \le T_\ell[u_\infty^\Omega] \le T_\ell[u_\infty^B], \tag{4.99}$$

from (4.96) and

$$\begin{array}{lll} (i) & T_{\ell'}[u_\infty^{B'^c}] \le T_\ell[u_\infty^{B'^c}] & \text{in } B'^{c\,\ell'} \\ (ii) & T_{\ell'}[u_\infty^B] \ge T_\ell[u_\infty^B] & \text{in } B^\ell \end{array} \tag{4.100}$$

for any $0 < \ell' \le \ell$. Notice that , replacing ℓ by $\ell\ell'$ we can rewrite (4.99) as follows

$$T_{\ell'}[T_\ell[u_\infty^{B'^c}]] \le T_{\ell'}[T_\ell[u_\infty^\Omega]] \le T_{\ell'}[T_\ell[u_\infty^B]]. \tag{4.101}$$

Because of the monotonicity with respect to ℓ the following limits exist

$$U^{B'^c} = \lim_{\ell\to 0} T_\ell[u_\infty^{B'^c}] \quad \text{and} \quad U^B = \lim_{\ell\to 0} T_\ell[u_\infty^B]. \tag{4.102}$$

By Lemma 2.5 applied with $\phi(|x|) = |x|^{-\beta_q}$ and since there holds $u_\infty^B(x) \le c|x|^{-\beta_q}$ and $u_\infty^{B'}(x) \le c|x|^{-\beta_q}$, we derive

$$\begin{array}{lll} (i) & |\nabla T_\ell[u_\infty^B](x)| \le c_2|x|^{-\beta_q-1} & \forall x \in B^\ell \\ (ii) & |\nabla T_\ell[u_\infty^B](x) - \nabla T_\ell[u_\infty^B](y)| \le c_2|x|^{-\beta_q-1-\alpha}|x-y|^\alpha & \forall x,y \in B^\ell,\ |x| \le |y| \\ (iii) & T_\ell[u_\infty^B](x) \le c_2|x|^{-\beta_q-1}(\text{dist}\,(x,\partial B^\ell))^\alpha & \forall x \in B^\ell, \end{array} \tag{4.103}$$

and

$$\begin{array}{lll} (i) & |\nabla T_\ell[u_\infty^{B'^c}](x)| \le c_2|x|^{-\beta_q-1} & \forall x \in B'^{c\,\ell} \\ (ii) & |\nabla T_\ell[u_\infty^{B'^c}](x) - \nabla T_\ell[u_\infty^{B'^c}](y)| \le c_2|x|^{-\beta_q-1-\alpha}|x-y|^\alpha & \forall x,y \in B'^{c\,\ell},\ |x| \le |y| \\ (iii) & T_\ell[u_\infty^{B'^c}](x) \le c_2|x|^{-\beta_q-1}(\text{dist}\,(x,\partial B'^{c\,\ell}))^\alpha & \forall x \in B'^{c\,\ell}. \end{array} \tag{4.104}$$

Thus the sets of functions $\{T_\ell[u_\infty^B]\}$ and $\{T_\ell[u_\infty^{B'}]\}$ are equicontinuous in the C^1-loc topology and by uniqueness, the limit in (4.102) below holds in this topology. Hence $U^{B'^c}$ and U^{B^c} are positive solutions of (3.17) in $\mathbb{R}^N_+$ which vanish on $\partial\mathbb{R}^N_+ \setminus \{0\}$. Furthermore $U^{B'^c} \le U^{B^c}$ Since for any $\ell, \ell' > 0$,

$T_{\ell'}[T_\ell[u_\infty^{B'^c}]] = T_{\ell\ell'}[u_\infty^{B'^c}]$, it follows $T_{\ell'}[U^{B'^c}] = U^{B'^c}$ and in the same way $T_{\ell'}[U^B] = U^B$. This means that U^B and $U^{B'^c}$ are self-similar solutions of (3.17) in $\mathbb{R}^N_+$ and they vanish on $\partial\mathbb{R}^N_+ \setminus \{0\}$. Hence

$$U^B = U^{B'^c} = U_{S^{N-1}_+}. \tag{4.105}$$

Applying again Lemma 2.5 to u_∞^Ω with $\phi(|x|) = |x|^{-\beta_q}$ we have

$$\begin{array}{lll} (i) & |\nabla T_\ell[u_\infty^\Omega](x)| \le c_2|x|^{-\beta_q-1} & \forall x \in \Omega^\ell \\ (ii) & |\nabla T_\ell[u_\infty^\Omega](x) - \nabla T_\ell[u_k^\Omega](y)| \le c_2|x|^{-\beta_q-1-\alpha}|x-y|^\alpha & \forall x,y \in \Omega^\ell,\ |x| \le |y| \\ (iii) & T_\ell[u_\infty^\Omega](x) \le c_2|x|^{-\beta_q-1}(\text{dist}\,(x,\partial\Omega^\ell))^\alpha & \forall x \in \Omega^\ell. \end{array} \tag{4.106}$$

This implies that the set of functions $\{T_\ell[u_\infty^\Omega]\}_\ell$ is equicontinuous in the C^1-loc topology of $\mathbb{R}^N_+$ and there exists a sequence $\{\ell_n\} \to 0$ and a function U such that $T_{\ell_n}[u_\infty^\Omega] \to U^\Omega$ in this topology of $\mathbb{R}^N_+$, and U is a positive solution of (3.17) in $\mathbb{R}^N_+$ which vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$. From (4.99) and (4.105) there holds $U^\Omega = U_{S^{N-1}_+}$ and therefore

$$\lim_{\ell\to 0} T_\ell[u_\infty^\Omega] = U_{S^{N-1}_+}. \tag{4.107}$$

This implies (4.93) and

$$\lim_{r\to 0} r^{\beta_q} u_\infty^\Omega(r,\sigma) = \omega_{S^{N-1}_+}(\sigma) \tag{4.108}$$

uniformly on compact subsets of S^{N-1}_+. □

Up to minor modifications the proof of the next classification theorem is similar to the one of Theorem 4.2.

Theorem 4.4. *Assume $N-1 < q < N - \frac{1}{2}$ If $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$ is a positive solution of (3.17) in Ω which vanishes on $\partial\Omega \setminus \{0\}$, then we have the following alternative:*

(i) either there exists $k \ge 0$ such that (4.74) holds,

(ii) or (4.75) holds.

5 Appendix I: Positive p-harmonic functions in a half space

In this section we prove the following rigidity result.

Theorem 5.1. *Assume $1 < p \le N$ and $u \in C^1(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+} \setminus \{0\})$ is a positive p-harmonic function which vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and such that $|x|^{\beta_*} u(x)$ is bounded. Then there exists $k \ge 0$ such that*

$$u(x) = k\Psi_*(x) \qquad \forall x \in \mathbb{R}^N_+. \tag{5.1}$$

Proof. Since $|x|^{\beta_*} u(x)$ is bounded, $|x|^{\beta_*+1} \nabla u(x)$ is also bounded and there exists $m > 0$ such that $u(x) \le m\Psi_*(x)$ in B^+_δ. We denote by k the infimum of the $c > 0$ such that $u(x) \le c\Psi_*(x)$. Then

$$0 \le u(x) \le k\Psi_*(x) \qquad \forall x \in \mathbb{R}^N_+ \setminus \{0\} \tag{5.2}$$

and we assume that $k>0$ otherwise $u=0$. Assume that the graphs over $\mathbb{R}^N_+$ of the functions $x\mapsto u(x)$ and $x\mapsto k\Psi_*(x)$ are tangent at some point $x_0\in\mathbb{R}^N_+$ or $x_0\in\partial\mathbb{R}^N_+\setminus\{0\}$. Since $\nabla\Psi_*$ never vanishes in $\overline{\mathbb{R}}^N_+\setminus\{0\}$ it follows from the strong maximum principle or Hopf Lemma that $u=k\Psi_*$. If the two graphs are not tangent in $\overline{\mathbb{R}}^N_+\setminus\{0\}$, either they are asymptotically tangent at 0, or at ∞.
(i) In the first case there exists two sequences $\{k_n\}$ increasing to k and $\{x_n\}\subset\mathbb{R}^N_+$ converging to zero such that $\frac{u(x_n)}{\Psi_*(x_n)}=k_n$. We set $r_n=|x_n|$ and $u_{r_n}(x)=r_n^{\beta_*}u(r_nx)$. Then u_{r_n} is p-harmonic and positive and $0<u_{r_n}(x)\le k\,|x|^{-\beta_*}\,\psi_*(\frac{x}{|x|})$; therefore

$$|\nabla u_{r_n}(x)|\le C\,|x|^{-\beta_*-1}\ \text{ and }\ \left|\nabla u_{r_n}(x)-\nabla u_{r_n}(x')\right|\le C\,|x|^{-\beta_*-1-\alpha}\left|x-x'\right|^{\alpha}\tag{5.3}$$

for $0<|x|\le|x'|$ and some constants $C>0$ and $\alpha\in(0,1)$. Up to a subsequence, we can assume that u_{r_n} converges to some U in the C^1_{loc} topology of $\overline{\mathbb{R}}^N_+\setminus\{0\}$ and $\frac{x_n}{r_n}\to\xi\in S^{N-1}_+$. The function U is p-harmonic and positive in $\mathbb{R}^N_+$ and satisfies $0\le U\le k\Psi_*$ in $\mathbb{R}^N_+$ and $U(\xi)=k\Psi_*(\xi)$ if $\xi\in S^{N-1}_+$ or $U_{x_N}(\xi)=k\Psi_{*\,x_N}(\xi)$ if $\xi\in\partial S^{N-1}_+$. It follows from the strong maximum principle or Hopf Lemma that $U=k\Psi_*$. Therefore $u_{r_n}\to k\Psi_*$ and in particular

$$\lim_{r_n\to0}\frac{r_n^{\beta_*}u(r_n,\sigma)}{\psi_*(\sigma)}=k\quad\text{uniformly on }\ S^{N-1}_+.\tag{5.4}$$

For any $\epsilon>0$, there exists $n_\epsilon\in\mathbb{N}_*$ such that for $n\ge n_\epsilon$, $(k-\epsilon)\Psi_*(x)\le u(x)\le(k+\epsilon)\Psi_*(x)$ if $|x|=r_n$. This implies $(k-\epsilon)\Psi_*(x)\le u(x)\le(k+\epsilon)\Psi_*$ for $|x|\ge r_n$ and therefore in $\mathbb{R}^N$. Since ϵ is arbitrary, we deduce that $u=k\Psi_*$.
(ii) if the two graphs are tangent at infinity, there exist two sequences $\{k_n\}$ increasing to k and $\{x_n\}$ such that $r_n=|x_n|\to\infty$ with $u(x_n)=k_n\Psi_*(x_n)$ and

$$\lim_{r_n\to\infty}\frac{r_n^{\beta_*}u(r_n,\sigma)}{\psi_*(\sigma)}=k\quad\text{uniformly on }\ S^{N-1}_+.\tag{5.5}$$

Therefore we look at the supremum of the $c>0$ such that $u\ge c\Psi_*$. If the set of such c is empty, it would mean that

$$\inf_{x\in\mathbb{R}^N_+}\frac{u(x)}{\Psi_*(x)}=0.$$

Clearly, if this infimum is achieved at some point, the strong maximum principle or Hopf Lemma imply $u\equiv0$, contradicting (5.5), and this relation prevents also this infimum be achieved at infinity. We are left with the case where there exists a sequence $\{z_n\}\subset\mathbb{R}^N_+$, converging to 0, such that

$$\lim_{n\to\infty}\frac{u(z_n)}{\Psi_*(z_n)}=0.\tag{5.6}$$

By boundary Harnack inequality [2, th 2.11], there exists $c>0$ such that

$$c^{-1}\frac{u(z)}{\Psi_*(z)}\le\frac{u(z_n)}{\Psi_*(z_n)}\le c\frac{u(z)}{\Psi_*(z)}\quad\forall z\in\mathbb{R}^N_+\text{ s.t. }|z|=|z_n|\tag{5.7}$$

Combining (5.6) and (5.7), we derive that

$$\lim_{n\to\infty}\sup_{|z|=|z_n|}\frac{u(z)}{\Psi_*(z)}=0, \tag{5.8}$$

Denoting by ϵ_n the supremum in the above relation, we obtain that $u\le\epsilon_n\Psi_*$ in $\mathbb{R}^N_+\setminus B_{\epsilon_n}$ and finally $u=0$, contradiction. Thus we are left with the case where there exists $k'\in(0,k]$ which is the supremum of the $c>0$ such that $u\ge c\Psi_*$. In particular $u\ge k'\Psi_*$. Remembering that $u\le k\Psi_*$ we get $k=k'$, which implies $u=k\Psi_*$.

Next we assume that $k'<k$. Clearly the graphs of u and $k'\Psi_*$ cannot be tangent in $\overline{\mathbb{R}}^N_+$, because of strong maximum principle or Hopf Lemma. They cannot be tangent at infinity because of (5.5). Therefore there exist two sequences $\{k'_n\}$ increasing to k' and $\{x'_n\}\subset\mathbb{R}^N_+$ converging to 0 such that $\frac{u(x'_n)}{\Psi_*(x'_n)}=k'_n$. As in case (i) we obtain that

$$\lim_{r'_n\to 0}\frac{r'^{\beta_*}_n u(r'_n,\sigma)}{\psi_*(\sigma)}=k'\quad\text{uniformly on } S^{N-1}_+, \tag{5.9}$$

where $r'_n=|x'_n|$, and finally derive that $u=k'\Psi_*$, a contradiction with (5.5). Therefore $k=k'$, which ends the proof. □

Remark. In the case $p=N$ the result holds under the weaker assumption $\lim_{|x|\to\infty}u(x)=0$. This is due to the fact that this condition implies by regularity

$$\lim_{|x|\to\infty}\frac{u(x)}{\omega_{s^{N-1}_+}(\frac{x}{|x|})}=0$$

and therefore

$$u(x)\le m\Psi_*(x)\quad\forall x\text{ s.t. }|x|\ge 1,$$

where $m=\max_{|x|=1}\frac{u(x)}{\omega_{s^{N-1}_+}(\frac{x}{|x|})}$. Using the inversion $x\mapsto\frac{x}{|x|^2}$, we obtain that the estimate $u\le m\Psi_*$ holds $\mathbb{R}^N$, and we conclude by Theorem 5.1.

Remark. We conjecture that the rigidity result holds under the mere condition

$$\lim_{|x|\to\infty}|x|^{-\tilde\beta}\,u(x)=0, \tag{5.10}$$

were $\tilde\beta$ is the (positive) exponent corresponding to the regular spherical p-harmonic function under the form

$$\tilde\Psi=|x|^{\tilde\beta}\,\tilde\psi(\frac{x}{|x|}), \tag{5.11}$$

see [14], [12]. Note that $\tilde\beta=1$ when $p=N$.

6 Appendix II: Estimates on β_*

When $N = 2$ and $1 < p \leq 2$, it is proved in [9] that

$$\beta_* = \frac{3 - p + 2\sqrt{p^2 - 5p + 7}}{3(p-1)}. \tag{6.1}$$

Up to now no estimate is known when $N > 2$ except in the cases $p = 2$ where $\beta_* = N - 1$ and $p = N$ where $\beta_* = 1$, besides the classical one

$$\beta_* > \frac{N-p}{p-1}, \tag{6.2}$$

valid when $p < N$. In this section we prove the following result

Theorem 6.1. *Assume $1 < p < N$. Then the following estimates hold:*

$$1 < p < 2 \Longrightarrow \beta_* > \frac{N-1}{p-1}, \tag{6.3}$$

$$2 < p < N \Longrightarrow \max\left\{1, \frac{N-p}{p-1}\right\} < \beta_* < \frac{N-1}{p-1}. \tag{6.4}$$

Remark. It is worth noticing that when $p = 2$ or $p = N$, there holds $\beta_* = \frac{N-1}{p-1}$.

Proof of Theorem 6.1. We consider the following set of spherical coordinates in $\mathbb{R}^N_+$ with $x = (x_1, ..., x_N)$

$$\begin{array}{ll} x_1 & = r\sin\theta_{N-1}\sin\theta_{N-2}...\sin\theta_2\sin\theta_1 \\ x_2 & = r\sin\theta_{N-1}\sin\theta_{N-2}...\sin\theta_2\cos\theta_1 \\ \vdots & \\ x_{N-1} & = r\sin\theta_{N-1}\cos\theta_{N-2} \\ x_N & = r\cos\theta_{N-1} \end{array} \tag{6.5}$$

with $\theta_1 \in [0, 2\pi]$ and $\theta_k \in [0, \pi]$ for $k = 2, ..., N-2$ and $\theta_{N-1} \in [0, \frac{\pi}{2}]$. Under this representation, a solution ω of (3.2) verifies

$$\begin{array}{l} -\dfrac{1}{\sin^{N-2}\theta_{N-1}}\left[\sin^{N-2}\theta_{N-1}\left(\beta_*^2\omega^2 + \omega_{\theta_{N-1}}^2 + \dfrac{1}{\sin^2\theta_{N-1}}|\nabla_{\theta'}\omega|^2\right)^{\frac{p-2}{2}}\omega_{\theta_{N-1}}\right]_{\theta_{N-1}} \\ -\dfrac{1}{\sin^2\theta_{N-1}}div'_{\theta'}\left[\sin^{N-2}\theta_{N-1}\left(\beta_*^2\omega^2 + \omega_{\theta_{N-1}}^2 + \dfrac{1}{\sin^2\theta_{N-1}}|\nabla_{\theta'}\omega|^2\right)^{\frac{p-2}{2}}\nabla_{\theta'}\omega\right] \\ = \beta_*\Lambda_{\beta_*}\left[\sin^{N-2}\theta_{N-1}\left(\beta_*^2\omega^2 + \omega_{\theta_{N-1}}^2 + \dfrac{1}{\sin^2\theta_{N-1}}|\nabla_{\theta'}\omega|^2\right)^{\frac{p-2}{2}}\omega\right] \end{array} \tag{6.6}$$

where $\nabla_{\theta'}$ and $div'_{\theta'}$ denotes respectively the spherical gradient the divergence in variables $\theta' = (\theta_1, ..., \theta_{N-2})$ parameterizing S^{N-2} and Λ_{β_*} is defined in Introduction. If ω is the unique positive solution of (3.2)

(up to homothety), it depends only on θ_{N-1} and is C^∞. For simplicity we set $\theta_{N-1}=\theta\in[0,\frac{\pi}{2}]$ and $\omega=\omega(\theta)$ satisfies

$$\begin{aligned}-\frac{1}{\sin^{N-2}\theta}\left[\sin^{N-2}\theta\left(\beta_*^2\omega^2+\omega_\theta^2\right)^{\frac{p-2}{2}}\omega_\theta\right]_\theta=\beta_*\Lambda_{\beta_*}\left[\sin^{N-2}\theta\left(\beta_*^2\omega^2+\omega_\theta^2\right)^{\frac{p-2}{2}}\omega\right]\\ \text{in } (0,\tfrac{\pi}{2})\\ \omega(\tfrac{\pi}{2})=0\,,\ \omega_\theta(0)=0.\end{aligned}\tag{6.7}$$

Step 1: The eigenvalue identity. Equation (6.7) can also be written under the form

$$-\omega_{\theta\theta}-(N-2)\cot\theta\,\omega_\theta-(p-2)\frac{\beta_*^2\omega+\omega_{\theta\theta}}{\beta_*^2\omega^2+\omega_\theta^2}\omega_\theta^2=\beta_*\Lambda_{\beta_*}\omega.\tag{6.8}$$

By multiplying (6.8) by $\cos\theta\sin^{N-2}\theta$ and then integrating over $(0,\frac{\pi}{2})$ we obtain

$$-\int_0^{\frac{\pi}{2}}\left(\omega_{\theta\theta}+(N-2)\cot\theta\,\omega_\theta\right)\cos\theta\sin^{N-2}\theta d\theta=(N-1)\int_0^{\frac{\pi}{2}}\omega\cos\theta\sin^{N-2}\theta d\theta.$$

Noticing that

$$\beta_*\Lambda_{\beta_*}+1-N=(p-1)\left(\beta_*-\frac{N-1}{p-1}\right)(\beta_*+1)$$

we derive

$$\begin{aligned}(2-p)\int_0^{\frac{\pi}{2}}\frac{\beta_*^2\omega+\omega_{\theta\theta}}{\beta_*^2\omega^2+\omega_\theta^2}\omega_\theta^2\omega\cos\theta\sin^{N-2}\theta d\theta\\ =(p-1)\left(\beta_*-\frac{N-1}{p-1}\right)(\beta_*+1)\int_0^{\frac{\pi}{2}}\omega\cos\theta\sin^{N-2}\theta d\theta.\end{aligned}\tag{6.9}$$

Step 2: Elliptic coordinates and reduction. Writing $\omega(\theta)=\omega(0)+a\theta^2+o(\theta^2)$, $\omega_\theta(\theta)=2a\theta+o(\theta)$ and $\omega_{\theta\theta}(\theta)=2a+o(1)$, then $-Na=\beta_*\Lambda_{\beta_*}$. This implies that ω is decreasing near 0. It is immediate that it cannot have a local minimum in $(0,\frac{\pi}{2})$, therefore it remains decreasing in the whole interval. We parameterize the ellipse

$$E_r=\{(x,y):x>0,\,y<0,\,x^2+\beta_*^{-2}y^2=r^2\}$$

by setting

$$\omega=r\cos\phi\ \text{ and }\ -\omega_\theta=\beta_*r\sin\phi\ \text{ with }\ \phi=\phi(\theta)\ \text{ and }\ r=r(\theta).$$

The functions r and ϕ are C^2. Hence $r_\theta\cos\phi-r\sin\phi\phi_\theta=-\beta_*r\sin\phi$, then $r_\theta\cos\phi=(\phi_\theta-\beta_*)r\sin\phi$ and $r_\theta=(\phi_\theta-\beta_*)r\tan\phi$. Plugging this into (6.8), we derive

$$-\left((p-1)\frac{r_\theta}{r}+\phi_\theta\cot\phi+(N-2)\cot\theta\right)+\Lambda_{\beta_*}\cot\phi=0,\tag{6.10}$$

and finally

$$(p-1)(\phi_\theta-\beta_*)\tan\phi+(\phi_\theta-\Lambda_{\beta_*})\cot\phi=(2-N)\cot\theta.\tag{6.11}$$

Step 3: Estimates on ϕ_θ. We can write (6.11) under the equivalent form

$$(p-1)(\phi_\theta-\beta_*)\tan^2\phi+\phi_\theta-\Lambda_{\beta_*}=(2-N)\frac{\cos\theta}{\cos\phi}\frac{\sin\phi}{\sin\theta}. \tag{6.12}$$

Since

$$\lim_{\theta\to 0}\frac{\sin\phi}{\sin\theta}=\lim_{\theta\to 0}\frac{\cos\phi}{\cos\theta}\phi_\theta=\phi_\theta(0),$$

we derive $\phi_\theta(0)-\Lambda_{\beta_*}=(2-N)\phi_\theta(0)$ and thus $\phi_\theta(0)=\dfrac{\Lambda_{\beta_*}}{N-1}$. Similarly, the expansion of $\phi(\theta)$ near $\theta=\frac{\pi}{2}$ yields to $\phi_\theta(\frac{\pi}{2})=\beta_*$. Since $p<N$, $\Lambda_{\beta_*}/(N-1)<\beta_*$. We claim now that

$$\phi_\theta(\theta)\le\beta_* \qquad \forall\theta\in(0,\tfrac{\pi}{2}). \tag{6.13}$$

If $\Lambda_{\beta_*}\le\beta_*$, then

$$(2-N)\cot\theta=(p-1)(\phi_\theta-\beta_*)\tan\phi+(\phi_\theta-\Lambda_{\beta_*})\cot\phi\ge((p-1)\tan\phi+\cot\phi)(\phi_\theta-\beta_*)$$

thus (6.13) holds.
Next we assume $\beta_*<\Lambda_{\beta_*}$. It means $0<(p-2)\beta_*-(N-p)$ and thus $p>2$. We claim that

$$\beta_*\le\frac{N-2}{p-2}. \tag{6.14}$$

We proceed by contradiction and assume

$$\beta_*>\frac{N-2}{p-2}. \tag{6.15}$$

Then

$$(p-2)\left(\beta_*^2-\frac{N-p}{p-2}\beta_*-\frac{N-2}{p-2}\right)=(p-2)\,(\beta_*+1)\left(\beta_*-\frac{N-2}{p-2}\right)>0.$$

Equivalently

$$\beta_*(\Lambda_{\beta_*}-\beta_*)>N-2.$$

Since

$$\lim_{\theta\to\frac{\pi}{2}}\cot\theta\tan\phi=\lim_{\theta\to\frac{\pi}{2}}\frac{\cos\theta}{\cos\phi}=\lim_{\theta\to\frac{\pi}{2}}\frac{\sin\theta}{\phi_\theta\sin\phi}=\frac{1}{\beta_*}$$

and

$$\begin{aligned}(p-1)(\phi_\theta(\theta)-\beta_*)\tan^2\phi&=\Lambda_{\beta_*}-\phi_\theta(\theta)+(2-N)\frac{\cos\theta}{\cos\phi}\frac{\sin\phi}{\sin\theta}\\&=\frac{1}{\beta_*}\left(\beta_*(\Lambda_{\beta_*}-\beta_*)+2-N\right)+o(1),\end{aligned} \tag{6.16}$$

thus, if (6.15) holds there exists $\epsilon>0$ such that $\phi_\theta(\theta)>\beta_*$ for any $\theta\in[\frac{\pi}{2}-\epsilon,\frac{\pi}{2})$. Since $\phi_\theta(0)<\beta_*$, there exists $\bar\theta\in(0,\frac{\pi}{2})$ such that $\phi_\theta(\bar\theta)=\beta_*$ and $\phi_{\theta\theta}(\bar\theta)\ge 0$. We compute $\phi_{\theta\theta}$ and get

$$(p-1)\phi_\theta(\phi_\theta-\beta_*)\sec^2\phi+((p-1)\tan\phi+\cot\phi)\,\phi_{\theta\theta}-\phi_\theta(\phi_\theta-\Lambda_{\beta_*})\csc^2\phi=(N-2)\csc^2\theta$$

Hence, at $\theta = \bar\theta$

$$\phi_{\theta\theta}(\bar\theta)\left((p-1)\tan\phi(\bar\theta)+\cot\phi(\bar\theta)\right) = \beta_*(\beta_*-\Lambda_{\beta_*})\csc^2\phi(\theta)+(N-2)\csc^2\bar\theta$$

From (6.11),

$$\cot\phi(\bar\theta) = \frac{N-2}{\Lambda_{\beta_*}-\beta_*}\cot\bar\theta$$

Therefore

$$\begin{aligned} A(\bar\theta) &:= \phi_{\theta\theta}(\bar\theta)\left((p-1)\tan\phi(\bar\theta)+\cot\phi(\bar\theta)\right)\\ &= \left(1+\left(\frac{N-2}{\Lambda_{\beta_*}-\beta_*}\right)^2\cot^2\bar\theta\right)\beta_*(\beta_*-\Lambda_{\beta_*})+(N-2)(1+\cot^2\bar\theta)\\ &= \beta_*(\beta_*-\Lambda_{\beta_*})+N-2-\left(\frac{(N-2)^2}{\Lambda_{\beta_*}-\beta_*}+2-N\right)\cot^2\bar\theta\\ &= -(p-2)(\beta_*+1)\left(\beta_*-\frac{N-2}{p-2}\right)-\frac{N-2}{\Lambda_{\beta_*}-\beta_*}\left(\beta_*(N-1)-\Lambda_{\beta_*}\right)\cot^2\bar\theta\\ &< 0, \end{aligned} \tag{6.17}$$

using (6.15) and the fact that $N > p$. This is a contradiction, thus (6.14) holds.

Next, if $\beta_* < \frac{N-2}{p-2}$, it follows from (6.16) that there exists $\epsilon > 0$ such that $\phi_\theta < \beta_*$ in $[\frac{\pi}{2}-\epsilon, \frac{\pi}{2})$. If (6.13) is not true, there exist $0 < \theta_1 < \theta_2 < \frac{\pi}{2}-\epsilon$ such that $\phi_\theta(\theta_1) = \phi_\theta(\theta_2) = \beta_*$, $\phi_{\theta\theta}(\theta_1) \geq 0$, $\phi_{\theta\theta}(\theta_2) \leq 0$. Using the equation satisfied by $\phi_{\theta\theta}$, we obtain for $i = 1, 2$,

$$A(\theta_i) = (2-p)(\beta_*+1)\left(\beta_*-\frac{N-2}{p-2}\right)-\frac{N-2}{\Lambda_{\beta_*}-\beta_*}(\beta_*(N-1)-\Lambda_{\beta_*})\cot^2\theta_i. \tag{6.18}$$

On one hand $A(\theta_2) \leq 0 \leq A(\theta_1)$, and on the other

$$A(\theta_2)-A(\theta_1) = \frac{N-2}{\Lambda_{\beta_*}-\beta_*}(\beta_*(N-1)-\Lambda_{\beta_*})(\cot^2\theta_1-\cot^2\theta_2) > 0,$$

since cot is decreasing in $(0, \frac{\pi}{2})$, $\cot^2\theta_1 > \cot^2\theta_2$, a contradiction. Therefore $\phi_\theta \leq \beta_*$ in $(0, \frac{\pi}{2})$.

Finally, if $\beta_* = \frac{N-2}{p-2}$ and the maximum of ϕ_θ on $[0, \frac{\pi}{2})$ is larger than β_* and achieved at some $\bar\theta < \frac{\pi}{2}$ the exists $\theta_1 < \bar\theta$ such that $\phi_\theta(\theta_1) = \beta_*$ and $\phi_{\theta\theta}(\theta_1) \geq 0$. In that case

$$0 \leq A(\theta_1) = -\frac{N-2}{\Lambda_{\beta_*}-\beta_*}\left(\beta_*(N-1)-\Lambda_{\beta_*}\right)\cot^2\theta_1 < 0$$

which is again a contradictions.

Step 4: End of the proof. Since $r^2 = \beta_*^2\omega^2+\omega_\theta^2$, $r_\theta = r(\phi_\theta-\beta_*)\tan\phi$, we have

$$rr_\theta = \left(\beta_*^2\omega+\omega_{\theta\theta}\right)\omega_\theta = r(\phi_\theta-\beta_*)\tan\phi.$$

Since $\omega_\theta < 0$ on $(0, \frac{\pi}{2})$, it follows from Step 3 that $\beta_*^2\omega+\omega_{\theta\theta} \geq 0$ and thus

$$\int_0^{\frac{\pi}{2}} \frac{\beta_*^2\omega+\omega_{\theta\theta}}{\beta_*^2\omega^2+\omega_\theta^2}\omega_\theta^2\omega\cos\theta\sin^{N-2}\theta d\theta > 0,$$

since the integrand cannot be identically 0. The conclusion follows from (6.9). □

Remark. Since $\omega_\theta(\frac{\pi}{2}) = -c^2 < 0$, it follows $\omega(\theta) = -\omega_\theta(\theta)\cot\theta + O(\frac{\pi}{2} - \theta)$ as $\theta \to \frac{\pi}{2}$, and from the eigenfunction equation (6.8)

$$\frac{\beta_*^2\omega + \omega_{\theta\theta}}{\beta_*^2\omega^2 + \omega_\theta^2}\omega_\theta^2 = (\beta_*^2\omega + \omega_{\theta\theta})(1 + o(1)).$$

Therefore

$$-(p-1)\omega_{\theta\theta} = (\beta_*\Lambda_{\beta_*} + (p-2)\beta_*^2 + 2 - N)\omega(1 + o(1)) \quad \text{as } \theta \to \frac{\pi}{2}$$

and since $\Delta'\omega := \omega_{\theta\theta} + (N-2)\cot\theta\,\omega_\theta$

$$-\Delta'\omega = \frac{\beta_*(\beta_*(2p-3) + p - N) + (p-2)(N-2)}{p-1}\omega(1 + o(1)) \quad \text{as } \theta \to \frac{\pi}{2}.$$

Because ω is C^∞ we obtain finally

$$\left|\Delta'\omega\right| \le c\omega, \tag{6.19}$$

for some $c > 0$.

References

[1] Bauman P., *Positive solutions of elliptic equations in nondivergence form and their adjoints* Ark. Mat. **22**, 153-173 (1984).

[2] Bidaut-Véron M.F., Borghol R., Véron L., *Boundary Harnack inequality and a priori estimates of singular solutions of quasilinear elliptic equations*, Calc. Var. Part. Diff. Equ. **27**, 159-177 (2006).

[3] Bidaut-Véron M.F., Garcia Huidobro M., Véron L., *Local and global behaviour of solutions of quasilinear Hamilton-Jacobi equations*, J. Funct. Anal. **267**, 3294-3331 (2014).

[4] Boccardo L., Murat F., Puel J., *Résultats d'existence pour certains problèmes elliptiques quasilinéaires,* Ann. Scuola. Norm. Sup. Pisa **11** (2) 213-235 (1984).

[5] Borghol R., Véron L., *Boundary singularities of solutions of N-harmonic equations with absorption*, J. Funct. Anal. **241**, 611-637 (2006).

[6] Friedman A., Véron L., *Singular Solutions of Some Quasilinear Elliptic Equations,*, Arch. Rat. Mec. Anal. **96**, 258-287 (1986).

[7] Gilbarg D., Trudinger N., *Elliptic Partial Differential Equations of Second Order*, Grundlehren der mathematischen Wissenschaften **224**, 2nd ed. Berlin, Heidelberg, New-York, Tokyo, Springer Verlag (1983).

[8] Kroll, I. N., *The behaviour of the solutions of a certain quasilinear equation near zero cusps of the boundary*, Proc. Steklov Inst. Math. **125**, 140-146 (1973).

[9] Kichenassamy S., Véron L., *Singular solutions of the p-Laplace equation*, Math. Ann. **275**, 599-615 (1986).

[10] Lieberman G., *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Part. Diff. Eq. **16**, 311-361 (1991).

[11] Nguyen Phuoc T., Véron L., *B*oundary singularities of solutions to elliptic viscous Hamilton-Jacobi equations, J. Funct. An. **263**, 1487-1538 (2012).

[12] Porretta A., Véron L., *Separable p-harmonic functions in a cone and related quasilinear equations on manifolds*, J. Europ. Math. Soc. **11**, 1285-1305 (2009).

[13] Pucci P., Serrin S., Zou H., *A strong maximum principle and a compact support principle for singular elliptic inequalities*, J. Math. Pures Appl. **78**, 769- 789 (1999).

[14] Tolksdorf P., *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Part. Diff. Equ. **8**, 773-817 (1983).

[15] Tolksdorf P., *Regularity for a More General Class of Quasilinear Elliptic Equations*, J. Diff. Equ. **51**, 126-150 (1984).

[16] Trudinger N., *On Harnack type inequalities and their applications to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20**, 721-747 (1967).