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Marie-Françoise Bidaut-Véron, Marta Garcia-Huidobro, Laurent Véron

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# Boundary singularities of positive solutions of quasilinear Hamilton-Jacobi equations\*

Marie-Françoise Bidaut-Véron  
Marta Garcia-Huidobro  
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**Abstract** We study the boundary behaviour of the solutions of (E)  $-\Delta_p u + |\nabla u|^q = 0$  in a domain  $\Omega \subset \mathbb{R}^N$ , when  $N \geq p > q > p - 1$ . We show the existence of a critical exponent  $q_* < p$  such that if  $p - 1 < q < q_*$  there exist positive solutions of (E) with an isolated singularity on  $\partial\Omega$  and that these solutions belong to two different classes of singular solutions. If  $q_* \leq q < p$  no such solution exists and actually any boundary isolated singularity of a positive solution of (E) is removable. We prove that all the singular positive solutions are classified according the two types of singular solutions that we have constructed.

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*Key words.*  $p$ -Laplace operator; singularities; spherical  $p$ -harmonic equations; Leray-Lions operators; Schauder fixed point theorem.

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## 1 Introduction

Let  $N \geq p > 1$ ,  $q > p - 1$  and  $\Omega \subset \mathbb{R}^N$  ( $N > 1$ ) be a  $C^2$  bounded domain such that  $0 \in \partial\Omega$ . In this article we study the boundary behavior at 0 of nonnegative functions  $u \in C^1(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$  which satisfy

$$-\Delta_p u + |\nabla u|^q = 0 \quad \text{in } \Omega, \quad (1.1)$$

where  $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ . The two main questions we consider are as follows:

**Q-1-** Existence of positive solutions of (1.1).

**Q-2-** Description of positive solutions with an isolated boundary singularity at 0.

When  $p = 2$  a fairly complete description of positive solutions of

$$-\Delta u + |\nabla u|^q = 0 \quad (1.2)$$

in  $\Omega$  is provided by Nguyen-Phuoc and Véron [11]. In particular they prove the following series of results in the range of values  $1 < q < 2$ .

**1-** Any signed solution of (1.3) verifies the estimates

$$|\nabla u(x)| \leq c_{N,q} (d(x))^{-\frac{1}{q-1}} \quad \forall x \in \Omega, \quad (1.3)$$

where  $d(x) = \operatorname{dist}(x, \partial\Omega)$ . As a consequence, if  $u \in C(\overline{\Omega} \setminus \{0\})$  is a solution which vanishes on  $\partial\Omega \setminus \{0\}$ , it satisfies

$$|u(x)| \leq c_{q,\Omega} d(x) |x|^{-\frac{1}{q-1}} \quad \forall x \in \Omega. \quad (1.4)$$

**2-** If  $\frac{N+1}{N} \leq q < 2$  any positive solution of (1.3) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$  is identically 0. An isolated boundary point is a removable singularity for (1.2).

**3-** If  $1 < q < \frac{N+1}{N}$  and  $k > 0$  there exists a unique positive solution  $u := u_k$  of (1.2) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies  $u(x) \sim c_N k P^\Omega(x, 0)$  as  $x \rightarrow 0$ , where  $P^\Omega$  is the Poisson kernel in  $\Omega \times \partial\Omega$ .

**4-** If  $1 < q < \frac{N+1}{N}$  there exists a unique positive solution  $u$  of (1.2) in the half-space  $\mathbb{R}_+^N := \{x = (x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > 0\}$  under the form  $u(x) = |x|^{-\frac{2-q}{q-1}} \omega(|x|^{-1}x)$  which vanishes on  $\partial\mathbb{R}_+^N \setminus \{0\}$ . The function  $\omega$  is the unique positive solution of

$$\begin{aligned} -\Delta' \omega + \left( \left( \frac{2-q}{q-1} \right)^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{q}{2}} - \lambda_{N,q} \omega &= 0 & \text{in } S_+^{N-1}, \\ \omega &= 0 & \text{in } \partial S_+^{N-1}, \end{aligned} \quad (1.5)$$

where  $S^{N-1}$  is the unit sphere of  $\mathbb{R}^N$ ,  $\partial S_+^{N-1} = \partial\mathbb{R}_+^N \cap S^{N-1}$ ,  $\Delta'$  the Laplace-Beltrami operator and  $\lambda_{N,q} > 0$  an explicit constant.

**5-** If  $1 < q < \frac{N+1}{N}$  and  $u$  is a positive solution of (1.3) in  $\Omega$ , which is continuous in  $\overline{\Omega} \setminus \{0\}$  and vanishes on  $\partial\Omega \setminus \{0\}$  the following dichotomy occurs:

(i) either  $u(x) \sim |x|^{-\frac{2-q}{q-1}} \omega(|x|^{-1}x)$  as  $x \rightarrow 0$ ,

(ii) or  $u(x) \sim k c_N P^\Omega(x, 0)$  as  $x \rightarrow 0$  for some  $k \geq 0$ .

The aim of this article is to extend to the quasilinear case  $1 < p \leq N$  the above mentioned results. The following pointwise gradient estimate valid for any signed solution  $u$  of (1.1) has been proved in [3]: if  $0 < p - 1 < q$  there exists a constant  $c_{N,p,q} > 0$  such that

$$|\nabla u(x)| \leq c_{N,p,q} (d(x))^{-\frac{1}{q+1-p}} \quad \forall x \in \Omega. \quad (1.6)$$

As a consequence, any solution  $u \in C^1(\bar{\Omega} \setminus \{0\})$  satisfies

$$|u(x)| \leq c_{p,q,\Omega} d(x) |x|^{-\frac{1}{q+1-p}} \quad \forall x \in \Omega. \quad (1.7)$$

Concerning boundary singularities, the situation is much more complicated than in the case  $p = 2$  and the threshold of critical exponent less explicit. We first consider the problem in  $\mathbb{R}_+^N$ . Assuming  $p - 1 < q \leq p$ , separable solutions of (1.1) in  $\mathbb{R}_+^N$  vanishing on  $\partial\mathbb{R}_+^N \setminus \{0\}$  can be looked for in spherical coordinates  $(r, \sigma) \in \mathbb{R}_+^* \times S^{N-1}$  (we denote  $\mathbb{R}_+^* = (0, \infty)$ ) under the form

$$u(x) = u(r, \sigma) = r^{-\beta_q} \omega(\sigma), \quad r > 0, \quad \sigma \in S_+^{N-1} := \{S^{N-1} \cap \mathbb{R}_+^N\}. \quad (1.8)$$

Then  $\omega$  is solution of the following problem

$$\begin{aligned} -\operatorname{div}' \left( (\beta_q^2 \omega^2 + |\nabla' \omega|^2)^{\frac{p-2}{2}} \nabla' \omega \right) - \beta_q \Lambda_{\beta_q} (\beta_q^2 \omega^2 + |\nabla' \omega|^2)^{\frac{p-2}{2}} \omega \\ + (\beta_q^2 \omega^2 + |\nabla' \omega|^2)^{\frac{q}{2}} = 0 \quad \text{in } S_+^{N-1} \\ \omega = 0 \quad \text{on } \partial S_+^{N-1}, \end{aligned} \quad (1.9)$$

where

$$\beta_q = \frac{p-q}{q+1-p} \quad \text{and} \quad \Lambda_{\beta_q} = \beta_q(p-1) + p - N, \quad (1.10)$$

and  $\nabla'$  is the covariant derivative on  $S^{N-1}$  identified to the tangential gradient thanks to the canonical isometrical imbedding of  $S^{N-1}$  into  $\mathbb{R}^N$ , and  $\operatorname{div}'$  the divergence operator acting on vector fields on  $S^{N-1}$ .

The existence of a positive solution to this problem cannot be separated from the problem of existence of *separable  $p$ -harmonic functions* which are  $p$ -harmonic in  $\mathbb{R}_+^N$  which vanish on  $\partial\mathbb{R}_+^N \setminus \{0\}$  and have the form  $\Psi(x) = \Psi(r, \sigma) = r^{-\beta} \psi(\sigma)$  for some real number  $\beta$ . Necessarily such a  $\psi$  must satisfy

$$\begin{aligned} -\operatorname{div}' \left( (\beta^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \nabla' \psi \right) - \beta \Lambda_\beta (\beta^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \psi = 0 \quad \text{in } S_+^{N-1} \\ \psi = 0 \quad \text{on } \partial S_+^{N-1}, \end{aligned} \quad (1.11)$$

where  $\Lambda_\beta = \beta(p-1) + p - N$ . We will refer to (1.11) as *the spherical  $p$ -harmonic eigenvalue problem*. The study of this problem has been initiated in the 2-dim case by Krol [8] ( $\beta < 0$ ) and Kichenassamy and Véron [9] ( $\beta > 0$ ). In this case  $\omega$  satisfies a completely integrable second order differential equation. In the case where  $S_+^{N-1}$  is replaced by a smooth domain  $S \subset S^{N-1}$  with  $N \geq 3$ , Tolksdorf [14] proved the existence of a unique couple  $(\tilde{\beta}_s, \tilde{\psi}_s)$  where  $\tilde{\beta}_s < 0$  and  $\tilde{\psi}_s$  has constant sign and is defined up to an homothety. Recently Porretta and Véron [12] gave a simpler and more general proof of the existence of two couples  $(\tilde{\beta}_s, \tilde{\psi}_s)$  and  $(\beta_{*s}, \psi_{*s})$  where  $\beta_{*s} > 0$  and  $\tilde{\psi}_s$  and  $\psi_{*s}$  are positive solutions of (1.11) with

$\beta = \tilde{\beta}_s$  and  $\beta = \beta_{*s}$  respectively and are unique up to a multiplication by a real number. When  $p = 2$  this problem is an eigenvalue problem for the Laplace-Beltrami operator on a subdomain of  $S^{N-1}$ . If  $S = S_+^{N-1}$ ,  $\tilde{\beta}_s$  and  $\beta_{*s}$  are respectively denoted by  $\tilde{\beta}$  and  $\beta_*$  and accordingly  $\tilde{\psi}_s$  and  $\psi_{*s}$  by  $\tilde{\psi}$  and  $\psi_*$ . Since  $x \mapsto x_N$  is  $p$ -harmonic,  $\tilde{\beta} = -1$ . Except in the cases  $N = 2$  where it is the positive root of some algebraic equation of degree 2,  $p = 2$  where it is  $N - 1$  and  $p = N$  where it is 1, the value of  $\beta_*$  is unknown besides the straightforward estimate  $\beta_* \geq \max\{1, \frac{N-p}{p-1}\}$ . Using the fact that  $\psi_*$  depends only on the azimuthal variable and satisfies a differential equation, we prove in Appendix II the following new estimate:

**Theorem A** *Let  $1 < p \leq N$ .*

- (i) *If  $2 \leq p \leq N$ , then  $\beta_* \leq \frac{N-1}{p-1}$  with equality only if  $p = 2$  or  $N$ .*
- (ii) *If  $1 \leq p < 2$ , then  $\beta_* > \frac{N-1}{p-1}$ .*

The  $p$ -harmonic function  $\Psi_*(x) = \Psi_*(r, \sigma) = r^{-\beta_*} \psi_*(\sigma)$  endows the role of a Poisson kernel. To this exponent  $\beta_*$  is associated the critical value  $q_*$  of  $q$  defined by  $\beta_* = \beta_q$ , or equivalently

$$q_* := \frac{\beta_*(p-1) + p}{\beta_* + 1} = p - \frac{\beta_*}{\beta_* + 1}. \quad (1.12)$$

The following result characterizes strong singularities.

**Theorem B** *Let  $0 < p - 1 \leq N$ , then*

- (i) *If  $p - 1 < q < q_*$  problem (1.9) admits a unique positive solution  $\omega_*$ .*
- (ii) *If  $q_* \leq q < p$  problem (1.9) admits no positive solution.*

This critical exponent corresponds to the threshold of criticality for boundary isolated singularities.

**Theorem C** *Assume  $q_* \leq q < p \leq N$ . If  $u \in C^1(\bar{\Omega} \setminus \{0\})$  is a nonnegative solution of (1.1) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ , it is identical zero.*

As in the case  $p = 2$ , there exist positive solutions (1.1) in  $\Omega$  with weak boundary singularities which are characterized by their blow-up near the singularity. By opposition to the case  $p = 2$  where existence is obtained by use of a weak formulation of the boundary value problem, combined with uniform integrability of the absorption term thanks to Poisson kernel estimates (see [11]), this approach cannot be performed in the case  $p \neq 2$ ; the obtention of solutions with weak singularities necessitates a very long and delicate construction of subsolutions and supersolutions. Furthermore, when  $p \neq N$ , the construction is done only if  $\Omega$  is locally an hyperplane near 0.

In the sequel we denote by  $B_R(a)$  the open ball of center  $a$  and radius  $R > 0$  and  $B_R = B_R(0)$ . We also set  $B_R^+(a) := \mathbb{R}_+^N \cap B_R(a)$ ,  $B_R^- := \mathbb{R}_+^N \cap B_R$ ,  $B_R^-(a) := \mathbb{R}_-^N \cap B_R(a)$  and  $B_R^- := \mathbb{R}_-^N \cap B_R$ , where  $\mathbb{R}_-^N := \{x = (x', x_N) : x' \in \mathbb{R}^{N-1}, x_N < 0\}$ . If  $\Omega$  is an open domain and  $R > 0$ , we put  $\Omega_R = \Omega \cap B_R$ .

**Theorem D** *Let  $\Omega \subset \mathbb{R}_+^N$  be a bounded domain such that  $0 \in \partial\Omega$ . Assume there exists  $\delta > 0$  such that  $\Omega_\delta = B_\delta^+$  and  $0 < p - 1 < q < q_* < p \leq N$ . Then for any  $k > 0$  there exists a unique  $u := u_k \in C^1(\bar{\Omega} \setminus \{0\})$ , solution of (1.1) in  $\Omega$ , vanishing on  $\partial\Omega \setminus \{0\}$  and such that*

$$\lim_{\substack{x \rightarrow 0 \\ \frac{x}{|x|} \rightarrow \sigma \in S_+^{N-1}}} |x|^{\beta_*} u_k(x) = k\psi_*(\sigma). \quad (1.13)$$

Furthermore  $\lim_{k \rightarrow \infty} u_k = u_\infty$  and

$$\lim_{\substack{x \rightarrow 0 \\ \frac{x}{|x|} \rightarrow \sigma \in S_+^{N-1}}} |x|^{\beta q} u_\infty(x) = \psi_*(\sigma). \quad (1.14)$$

When  $p = N$ , then  $q_* = N - \frac{1}{2}$ ; in such a range of values we use the conformal invariance of  $\Delta_N$  and prove that the previous result holds if  $\Omega$  is any  $C^2$  domain. Finally, the isolated singularities of positive solutions of (1.1) are completely described by the two types of singular solutions obtained in the previous theorem and we prove:

**Theorem E** *Let  $\Omega$  be a bounded domain such that  $0 \in \partial\Omega$ . Assume there exists  $\delta > 0$  such that  $\Omega_\delta = B_\delta^+$  and  $0 < p - 1 < q < q_* < p \leq N$ . If  $u \in C^1(\overline{\Omega} \setminus \{0\})$  is a positive solution of (1.1) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ , then*

(i) *either there exists  $k \geq 0$  such that*

$$\lim_{\substack{x \rightarrow 0 \\ \frac{x}{|x|} \rightarrow \sigma \in S_+^{N-1}}} |x|^{\beta_*} u(x) = k\psi_*(\sigma); \quad (1.15)$$

(ii) *or*

$$\lim_{\substack{x \rightarrow 0 \\ \frac{x}{|x|} \rightarrow \sigma \in S_+^{N-1}}} |x|^{\beta q} u(x) = \psi_*(\sigma). \quad (1.16)$$

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## 2 A priori estimates

### 2.1 The gradient estimates and its applications

We recall the following estimate and its consequences which are proved in [3].

**Proposition 2.1.** *Assume  $q > p - 1$  and  $u$  is a  $C^1$  solution of (1.1) in a domain  $\Omega$ . Then*

$$|\nabla u(x)| \leq c_{N,p,q} (d(x))^{-\frac{1}{q+1-p}} \quad \forall x \in \Omega. \quad (2.1)$$

The first application is a pointwise upper bound for solutions with isolated singularities.

**Corollary 2.2.** *Assume  $q > p - 1 > 0$ ,  $R^* > 0$  and  $\Omega$  is a domain containing 0 such that  $d(0) \geq 2R^*$ . Then for any  $x \in B_{R^*} \setminus \{0\}$ , and  $0 < R \leq R^*$ , any  $u \in C^1(\Omega \setminus \{0\})$  solution of (1.1) in  $\Omega \setminus \{0\}$  satisfies*

$$|u(x)| \leq c_{N,p,q} \left| |x|^{\frac{q-p}{q+1-p}} - R^{\frac{q-p}{q+1-p}} \right| + \max\{|u(z)| : |z| = R\}, \quad (2.2)$$

if  $p \neq q$ , and

$$|u(x)| \leq c_{N,p} (\ln R - \ln |x|) + \max\{|u(z)| : |z| = R\}, \quad (2.3)$$

if  $p = q$ .

The second application corresponds to solutions with boundary blow-up. For  $\delta > 0$  small enough we set  $\Omega_\delta := \{z \in \Omega : d(z) < \delta\}$ .

**Corollary 2.3.** *Assume  $q > p - 1 > 0$ ,  $\Omega$  is a bounded domain with a  $C^2$  boundary. Then there exists  $\delta_1 > 0$  which depends only on  $\Omega$  such that any  $u \in C^1(\Omega)$  solution of (1.1) in  $\Omega$  satisfies*

$$|u(x)| \leq c_{N,p,q} \left| (d(x))^{\frac{q-p}{q+1-p}} - \delta_1^{\frac{q-p}{q+1-p}} \right| + \max\{|u(z)| : d(z) = \delta_1\} \quad \forall x \in \Omega_{\delta_1} \quad (2.4)$$

if  $p \neq q$ , and

$$|u(x)| \leq c_{N,p,q} (\ln \delta_1 - \ln d(x)) + \max\{|u(z)| : d(z) = \delta_1\} \quad \forall x \in \Omega_{\delta_1} \quad (2.5)$$

if  $p = q$ .

*Remark.* As a consequence of (2.4) there holds for  $p > q > p - 1$

$$u(x) \leq (c_{N,p,q} + K \max\{|u(z)| : d(z) \geq \delta_1\}) (d(x))^{\frac{q-p}{q+1-p}} \quad \forall x \in \Omega \quad (2.6)$$

where  $K = (\text{diam}(\Omega))^{\frac{p-q}{q+1-p}}$ , with the standard modification if  $p = q$ .

As a variant of Corollary 2.3 the following upper estimate of solutions in an exterior domain will be used in the sequel.

**Corollary 2.4.** *Assume  $q > p - 1 > 0$ ,  $R > 0$  and  $u \in C^1(B_{R_0}^c)$  is any solution of (1.1) in  $B_{R_0}^c$ . Then for any  $R > R_0$  there holds*

$$|u(x)| \leq c_{N,p,q} \left| (|x| - R_0)^{\frac{q-p}{q+1-p}} - (R - R_0)^{\frac{q-p}{q+1-p}} \right| + \max\{|u(z)| : |z| = R\} \quad \forall x \in B_R^c \quad (2.7)$$

if  $p \neq q$  and

$$|u(x)| \leq c_{N,p,q} (\ln(|x| - R_0) - \ln(R - R_0)) + \max\{|u(z)| : |z| = R\} \quad \forall x \in B_R^c \quad (2.8)$$

if  $p = q$ .

*Proof.* The proof is a consequence of the identity

$$u(x) = u(z) + \int_0^1 \frac{d}{dt} u(tx + (1-t)z) dt = \int_0^1 \langle \nabla u(tx + (1-t)z), x - z \rangle dt$$

where  $z = \frac{R}{|x|}x$ . Since by (2.1)

$$|\nabla u(tx + (1-t)z)| \leq C_{N,p,q} (t|x| + (1-t)R - R_0)^{-\frac{1}{q+1-p}},$$

(2.7) and (2.8) follow by integration. □

## 2.2 Boundary a priori estimates

The next result is the extension to isolated boundary singularities of a previous regularity estimate dealing with singularity in a domain proved in [3, Lemma 3.10].

**Lemma 2.5.** *Assume  $p - 1 < q < p$ ,  $\Omega$  is a bounded  $C^2$  domain such that  $0 \in \partial\Omega$ . Let  $u \in C^1(\overline{\Omega} \setminus \{0\})$  be a solution of (1.1) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies*

$$|u(x)| \leq \phi(|x|) \quad \forall x \in \Omega, \quad (2.9)$$

where  $\phi : \mathbb{R}_+^* \mapsto \mathbb{R}_+$  is continuous, nonincreasing and satisfies

$$\phi(rs) \leq \gamma \phi(r)\phi(s) \quad \text{and} \quad r^{\frac{p-q}{q+1-p}} \phi(r) \leq c, \quad (2.10)$$

for some  $\gamma, c > 0$  and any  $r, s > 0$ . There exist  $\alpha \in (0, 1)$  and  $c_1 = c_1(p, q, \Omega) > 0$  such that

$$\begin{aligned} (i) \quad & |\nabla u(x)| \leq c_1 \phi(|x|) |x|^{-1} & \forall x \in \Omega, \\ (ii) \quad & |\nabla u(x) - \nabla u(y)| \leq c_1 \phi(|x|) |x|^{-1-\alpha} |x - y|^\alpha & \forall x, y \in \Omega, |x| \leq |y|. \end{aligned} \quad (2.11)$$

Furthermore

$$u(x) \leq c_1 \phi(|x|) \frac{d(x)}{|x|} \quad \forall x \in \Omega. \quad (2.12)$$

*Proof.* For  $\ell > 0$ , we set  $\Omega^\ell := \frac{1}{\ell} \Omega$ . If  $\ell \in (0, 1]$  the curvature of  $\partial\Omega^\ell$  remains uniformly bounded. As in [5, p 622], there exists  $0 < \delta_0 \leq 1$  and an involutive diffeomorphism  $\psi$  from  $\overline{B}_{\delta_0} \cap \overline{\Omega}^{\delta_0}$  into  $\overline{B}_{\delta_0} \cap (\Omega^{\delta_0})^c$  which is the identity on  $\overline{B}_{\delta_0} \cap \partial\Omega^{\delta_0}$  and such that  $D\psi(\xi)$  is the symmetry with respect to the tangent plane  $T_\xi \partial\Omega$  for any  $\xi \in \partial\Omega \cap \overline{B}_{\delta_0}$ . We extend any function  $v$  defined in  $\overline{B}_{\delta_0} \cap \overline{\Omega}^{\delta_0}$  and vanishing on  $\overline{B}_{\delta_0} \cap \partial\Omega^{\delta_0}$  into a function  $\tilde{v}$  defined in  $\overline{B}_{\delta_0}$  by

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in \overline{B}_{\delta_0} \cap \overline{\Omega}^{\delta_0} \\ -v \circ \psi(x) & \text{if } x \in \overline{B}_{\delta_0} \cap (\Omega^{\delta_0})^c, \end{cases} \quad (2.13)$$

If  $v \in C^1(\overline{B}_{\delta_0} \cap \overline{\Omega}^{\delta_0})$  is a solution of (1.1) in  $B_{\delta_0} \cap \Omega^{\delta_0}$  which vanishes on  $\partial\Omega^{\delta_0} \cap \overline{B}_{\delta_0}$ ,  $\tilde{v}$  satisfies

$$-\sum_j \frac{\partial}{\partial x_j} \tilde{A}_j(x, \nabla \tilde{v}) + B(x, \nabla \tilde{v}) = 0 \quad \text{in } B_{\delta_0}. \quad (2.14)$$

As in [5, (2.37)] the  $A_j$  and  $B$  satisfy the following estimates

$$\begin{aligned} (i) \quad & \tilde{A}_j(x, 0) = 0 \\ (ii) \quad & \sum_{i,j} \frac{\partial}{\partial \eta_i} \tilde{A}_j(x, \eta) \xi_i \xi_j \geq C_1 |\eta|^{p-1} |\xi|^2 \\ (iii) \quad & \sum_{i,j} \left| \frac{\partial}{\partial \eta_j} \tilde{A}_j(x, \eta) \right| \leq C_2 |\eta|^{p-2}, \end{aligned} \quad (2.15)$$



and

$$|B(x, \eta)| \leq C_3(1 + |\eta|)^p, \quad (2.16)$$

where the  $C_j$  are positive constants. These estimates are the ones needed to apply Tolksdorf's result [15, Th 1,2]. There exists a constant  $C$ , such that for any ball  $\overline{B}_{3R} \subset \overline{B}_{\delta_0}$ , there holds

$$\|\nabla \tilde{v}\|_{L^\infty(B_R)} \leq C, \quad (2.17)$$

where  $C$  depends on the constants  $C_k$  ( $k = 1, 2, 3$ ),  $N$ ,  $p$  and  $\|\tilde{v}\|_{L^\infty(B_{3R})}$ . We define

$$\Phi_\ell[u](y) := u_\ell = \frac{1}{\phi(\ell)} u(\ell y) \quad \forall y \in \Omega^\ell. \quad (2.18)$$

Then

$$|u_\ell(y)| \leq \frac{\phi(\ell|y|)}{\phi(\ell)} \leq \gamma\phi(|y|) \quad \forall y \in \Omega^\ell \quad (2.19)$$

and

$$-\Delta_p u_\ell + (\ell^{\beta q} \phi(\ell))^{q+1-p} |\nabla u_\ell|^q = 0 \quad \text{in } \Omega^\ell. \quad (2.20)$$

Using formula (2.13) we extend  $u_\ell$  into a function  $\tilde{u}_\ell$  which satisfies

$$-\sum_j \frac{\partial}{\partial y_j} \tilde{A}_j(y, \nabla \tilde{u}_\ell) + (\ell^{\beta q} \phi(\ell))^{q+1-p} B(y, \nabla \tilde{u}_\ell) = 0 \quad \text{in } B_{\delta_0}. \quad (2.21)$$

For  $0 < |x| < \delta_0$  there exists  $\ell \in (0, 2)$  such that  $\frac{\delta_0 \ell}{2} \leq |x| \leq \delta_0 \ell$ . Then  $y \mapsto \tilde{u}_\ell(y)$  with  $y = \frac{x}{\ell}$  satisfies (2.21) in  $B_{\delta_0}$  and  $|\tilde{u}_\ell(y)| \leq \gamma_* \phi(|y|)$  since  $\psi$  is a diffeomorphism and  $D\psi(\xi) \in O(N)$  for any  $\xi \in \partial\Omega \cap B_{\delta_0}$ . The function  $\tilde{u}_\ell$  remains bounded on any ball  $B_{3R}(z) \subset \Gamma := \{y \in \mathbb{R}^N : \frac{\delta_0}{2} < |y| < \delta_0\}$ , therefore  $|\nabla \tilde{u}_\ell(y)| \leq c$  for any  $y \in B_R(z)$ , for some constant  $c > 0$ . This implies

$$|\nabla u(x)| \leq c\gamma_* \delta_0 \phi\left(\frac{2}{\delta_0}\right) \phi(|x|) |x|^{-1} \quad \forall x \in \Omega \cap B_{\delta_0}, \quad (2.22)$$

which is (2.11)-(i). Moreover, by standard regularity estimates [10], there exists  $\alpha \in (0, 1)$  such that  $|\nabla \tilde{u}_\ell(y) - \nabla \tilde{u}_\ell(y')| \leq c|y - y'|^\alpha$  for all  $y$  and  $y'$  belonging to  $B_R(z)$ . This implies (2.11)-(ii).

Next we prove (2.12). Let  $0 < \delta_1 \leq \delta_0$  such that at any boundary point  $z$  there exist two closed balls of radius  $\delta_1$  tangent to  $\partial\Omega$  at  $z$  and which are included in  $\Omega \cup \{z\}$  and in  $\overline{\Omega}^c \cup \{z\}$  respectively ( $\delta_1$  corresponds to the maximal radius of the interior and exterior sphere condition). Let  $x \in \Omega$  such that  $d(x) \leq \delta_1$  (this is not a loss of generality) and  $z_x$  be the projection of  $x$  on  $\partial\Omega$ . We first assume that  $x$  does not belong to the cone  $\Sigma_{\frac{\pi}{4}}$  with vertex 0, axis  $-\mathbf{n}_0$ , where  $\mathbf{n}_0$  is the normal outward unit vector at 0, and angle  $\frac{\pi}{4}$ . Consider the path  $\zeta$  from  $z_x$  to  $x$  defined by  $\zeta(t) = tx + (1-t)z_x$  with  $0 \leq t \leq 1$ . Then

$$u(x) = \int_0^1 \frac{d}{dt} u \circ \zeta(t) dt = \int_0^1 \langle \nabla u \circ \zeta(t), x - z_x \rangle dt \quad (2.23)$$

Thus, by the Cauchy-Schwarz inequality, using (2.11),

$$|u(x)| \leq c_1 d(x) \int_0^1 \frac{\phi(|\zeta(t)|)}{|\zeta(t)|} dt. \quad (2.24)$$

Since  $x \notin \Sigma_{\frac{\pi}{4}}$ ,  $\zeta(t) \notin \Sigma_{\frac{\pi}{4}}$  and there exists  $c_2 > 0$  depending on  $\Omega$  such that  $c_2^{-1}|x| \leq |\zeta(t)| \leq c_2|x|$  for all  $0 \leq t \leq 1$ . Therefore  $\phi(|\zeta(t)|) \leq \phi(c_2|x|) \leq \gamma\phi(c_2)\phi(|x|)$  by (2.10). This implies

$$|u(x)| \leq \gamma c_1 c_2 \phi(c_2) \frac{d(x)\phi(|x|)}{|x|} \quad (2.25)$$

by (2.12) whenever  $x \notin \Sigma_{\frac{\pi}{4}}$ . When  $x \in \Sigma_{\frac{\pi}{4}}$  then  $d(x) \leq |x| \leq c_3 d(x)$  where  $c_3 > 0$  depends on the curvature of  $\partial\Omega$ . Then (2.9) combined with (2.10) implies the claim.  $\square$

**Lemma 2.6.** *Assume  $p - 1 < q \leq p$ ,  $\Omega$  is a bounded  $C^2$  domain such that  $0 \in \partial\Omega$  and  $R_0 = \max\{|z| : z \in \Omega\}$ . If  $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$  is a positive solution of (1.1) which vanishes on  $\partial\Omega \setminus \{0\}$ , it satisfies*

$$u(x) \leq \begin{cases} c_2 \left( |x|^{\frac{q-p}{q+1-p}} - R_0^{\frac{q-p}{q+1-p}} \right) & \text{if } q < p \\ (p-1) \ln \left( \frac{R_0}{|x|} \right) & \text{if } q = p \end{cases} \quad (2.26)$$

for all  $x \in \Omega$ , where  $c_2 = c_2(p, q) > 0$ .

*Proof.* For  $\epsilon > 0$  we denote by  $P_\epsilon : \mathbb{R} \mapsto \mathbb{R}_+$  the function defined by

$$P_\epsilon(r) = \begin{cases} 0 & \text{if } 0 \leq r \leq \epsilon \\ -\frac{r^4}{2\epsilon^3} + \frac{3r^3}{\epsilon^2} - \frac{6r^2}{\epsilon} + 5r - \frac{3\epsilon}{2} & \text{if } \epsilon < r < 2\epsilon \\ r - \frac{3\epsilon}{2} & \text{if } r \geq 2\epsilon, \end{cases} \quad (2.27)$$

and by  $u_\epsilon$  the extension of  $P_\epsilon(u)$  by zero outside  $\Omega$ . There exists  $R_0$  such that  $\Omega \subset B_{R_0}$ . Since  $0 \leq P_\epsilon(r) \leq |r|$  and  $P_\epsilon$  is convex,  $u_\epsilon \in C(\mathbb{R}^N \setminus \{0\}) \cap W_{loc}^{1,p}(\mathbb{R}^N \setminus \{0\})$  and

$$-\Delta_p u_\epsilon + |\nabla u_\epsilon|^q \leq 0 \quad \text{in } \mathbb{R}^N.$$

Let  $R > R_0$ . If  $p - 1 < q < p$

$$U_{\epsilon,R}(|x|) = c_2 \left( (|x| - \epsilon)^{\frac{q-p}{q+1-p}} - (R - \epsilon)^{\frac{q-p}{q+1-p}} \right) \quad \text{in } B_R \setminus B_\epsilon, \quad (2.28)$$

with  $c_2 = (p-q)^{-1}(q+p-1)^{\frac{q-p}{q+1-p}}$ . Then  $-\Delta_p U_{\epsilon,R} + |\nabla U_{\epsilon,R}|^q \geq 0$ . Since  $u_\epsilon$  vanishes on  $\partial B_R$  and is finite on  $\partial B_\epsilon$ , it follows  $u_\epsilon \leq U_{\epsilon,R}$ . Letting successively  $\epsilon \rightarrow 0$  and  $R \rightarrow R_0$  yields to (2.26). If  $q = p$  we take

$$U_{\epsilon,R}(|x|) = (p-1) \ln \left( \frac{R - \epsilon}{|x| - \epsilon} \right) \quad \text{in } B_R \setminus B_\epsilon, \quad (2.29)$$

which turns out to be a supersolution of (1.1); the end of the proof is similar.  $\square$

As a consequence of Lemma 2.5 and Lemma 2.6, we obtain.

**Corollary 2.7.** *Let  $p, q$   $\Omega$  and  $u$  be as in Lemma 2.6. Then there exists a constant  $c_3 = c_3(p, q, \Omega) > 0$  such that*

$$|\nabla u(x)| \leq c_3 |x|^{-\frac{1}{q+1-p}} \quad \forall x \in \Omega \quad (2.30)$$

and

$$u(x) \leq c_3 d(x) |x|^{-\frac{1}{q+1-p}} \quad \forall x \in \overline{\Omega} \setminus \{0\}. \quad (2.31)$$

*Remark.* If  $\Omega$  is locally flat near 0, then estimates (2.30) and (2.31) are valid without any sign assumption on  $u$ . More precisely, if  $\partial\Omega \cap B_{\delta_0} = T_0\partial\Omega \cap B_{\delta_0}$  we can perform the reflection of  $u$  through the tangent plane  $T_0\partial\Omega$  to  $\partial\Omega$  at 0 and the new function  $\tilde{u}$  is a solution of (1.1) in  $B_{\delta_0} \setminus \{0\}$ . By Proposition 2.1, it satisfies

$$|\nabla\tilde{u}(x)| \leq c_{N,p,q} |x|^{-\frac{1}{q+1-p}} \quad \forall x \in B_{\frac{\delta_0}{2}} \setminus \{0\}. \quad (2.32)$$

Integrating this relation as in [3], we derive that for any  $x \in B_{\frac{\delta_0}{2}} \cap \Omega$ , there holds

$$|u(x)| \leq \begin{cases} c_{N,p,q} \left( |x|^{-\beta_q} - \left(\frac{\delta_0}{2}\right)^{-\beta_q} \right) + \max\{|u(z)| : |z| = \frac{\delta_0}{2}\} & \text{if } p \neq q \\ c_{N,p} \ln\left(\frac{\delta_0}{2|x|}\right) + \max\{|u(z)| : |z| = \frac{\delta_0}{2}\} & \text{if } p = q. \end{cases} \quad (2.33)$$

In the next result we allow the boundary singular set to be a compact set.

**Proposition 2.8.** *Let  $p-1 < q < p$  and  $\delta_1$  as above. There exist  $r^* \in (0, \delta_1]$  and  $c_4 = c_4(N, p, q) > 0$  such that for any nonempty compact set  $K \subset \partial\Omega$ ,  $K \neq \partial\Omega$  and any positive solution  $u \in C^1(\bar{\Omega} \setminus K) \cap C^1(\Omega)$  of (1.1) which vanishes on  $\partial\Omega \setminus K$ , there holds*

$$u(x) \leq c_4 d(x) (d_K(x))^{-\frac{1}{q+1-p}} \quad \forall x \in \partial\Omega \text{ s.t. } d(x) \leq r^*, \quad (2.34)$$

where  $d_K(x) = \text{dist}(x, K)$ .

*Proof. Step 1: Tangential estimates.* Let  $x \in \Omega$  such that  $d(x) \leq \delta_1$ . We denote by  $\sigma(x)$  the projection of  $x$  onto  $\partial\Omega$ , unique since  $d(x) \leq \delta_1$ . Let  $r, r', \tau > 0$  such that  $\frac{3}{4}r < r' < \frac{7}{8}r$  and  $0 < \tau \leq \frac{r'}{2}$  and put  $\omega_{\tau,x} = \sigma(x) + \tau \mathbf{n}_{\sigma(x)}$ . Since  $\partial\Omega$  is  $C^2$ , there exists  $0 < r^* \leq \delta_1$  depending on  $\Omega$  such that  $d_K(\omega_{\tau,x}) > \frac{7}{8}r$  whenever  $d(x) \leq r^*$ . Let  $a > 0$  and  $b > 0$  to be specified later on; we define  $\tilde{v}(s) = a(r' - s)^{\frac{q-p}{q+1-p}} - b$  and  $v(y) = \tilde{v}(|y - \omega_{\tau,x}|)$  in  $[0, r')$  and  $B_{r'}(\omega_{\tau,x})$  respectively. Then

$$|\tilde{v}'|^{p-2} \left( |\tilde{v}'|^{q+2-p} - (p-1)\tilde{v}'' - \frac{N-1}{s}\tilde{v}' \right) = a^{p-1} \left( \frac{p-q}{q+1-p} \right)^{p-1} (r' - s)^{-\frac{q}{q+1-p}} X(s)$$

where

$$X(s) = \left( a \frac{p-q}{q+1-p} \right)^{q+1-p} - \frac{p-1}{q+1-p} - \frac{(N-1)(r'-s)}{s}.$$

For any  $\tau \in (0, r')$  there exists  $a > 0$  such that

$$\left( a \frac{p-q}{q+1-p} \right)^{q+1-p} \geq \frac{p-1}{q+1-p} + \frac{(N-1)(r'-s)}{s} \quad \forall \tau \leq s \leq r'.$$

This implies

$$-\Delta_p v + |\nabla v|^q \geq 0 \quad \text{in } B_{r'}(\omega_{\tau,x}) \setminus B_\tau(\omega_{\tau,x}). \quad (2.35)$$

Next we take  $b = a(r' - \tau)^{\frac{q-p}{q+1-p}}$ , thus  $v = 0$  on  $\partial B_\tau(\omega_{\tau,x})$ . Clearly  $B_\tau(\omega_{\tau,x}) \subset \bar{\Omega}^c$  since  $\tau < \delta_1$ . Therefore  $v \geq 0 = u$  on  $\partial\Omega \cap B_{r'}(\omega_{\tau,x})$  and  $u \leq v = \infty$  on  $\Omega \cap \partial B_{r'}(\omega_{\tau,x})$ . By the comparison principle,  $v \geq u$  in  $\Omega \cap B_{r'}(\omega_{\tau,x})$ . In particular

$$u(x) \leq v(x) \leq a(r' - \tau - d(x))^{\frac{q-p}{q+1-p}} - a(r' - \tau)^{\frac{q-p}{q+1-p}}.$$

We take now  $\tau = \frac{r'}{2}$  and  $d(x) \leq \frac{r}{4}$  and we derive by the mean value theorem

$$u(x) \leq c'_4 r'^{-\frac{1}{q+1-p}} d(x) = c'_4 d(x) (d_K(x))^{-\frac{1}{q+1-p}}, \quad (2.36)$$

with  $c'_4 = c'_4(p, q) > 0$ . Letting  $r' \rightarrow \frac{7}{8}r$ , we get (2.12).

*Step 2: Global estimates.* If  $d(x) \geq \frac{1}{4}d_K(x)$ , there holds

$$d(x)(d_K(x))^{-\frac{1}{q+1-p}} \geq 2^{-\frac{2}{q+1-p}} (d(x))^{\frac{q-p}{q+1-p}}.$$

Combining this inequality with (2.6) and obtain (2.34).  $\square$

*Remark.* Under the assumption of Proposition 2.8, it follows from the maximum principle that  $u$  is upper bounded in the set  $\Omega'_{r^*} := \{x \in \Omega : d(x) > r^*\} = \Omega \setminus \overline{\Omega}_{r^*}$  by the solution  $w$  of

$$\begin{aligned} -\Delta_p w + |\nabla w|^q &= 0 && \text{in } \Omega_{r^*} \\ w &= c_4 d(x) (d_K(x))^{-\frac{1}{q+1-p}} && \text{in } \partial\Omega_{r^*}, \end{aligned} \quad (2.37)$$

and  $w$  itself is bounded by  $d^* = \max\{cd(x)(d_K(x))^{-\frac{1}{q+1-p}} : d(x) = r^*\}$ .

Next we prove a boundary Harnack inequality. We recall that  $\delta_1$  has been introduced at Corollary 2.3, and that the interior and exterior sphere conditions hold in the set  $\{x \in \mathbb{R}^N : d(x) \leq \delta_1\}$ .

**Theorem 2.9.** *Let  $q > p - 1$  and  $0 \in \partial\Omega$ . Then there exists  $c_5 = c_5(N, p, q, \Omega) > 0$  such that for any positive solution  $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^1(\Omega)$  of (1.1) in  $\Omega$ , vanishing on  $\partial\Omega \setminus \{0\} \cap B_{2\delta_1}$ , there holds*

$$\frac{u(y)}{c_5 d(y)} \leq \frac{u(x)}{d(x)} \leq c_5 \frac{u(y)}{d(y)} \quad (2.38)$$

for all  $x, y \in B_{\frac{2\delta_1}{3}} \cap \Omega$  such that  $\frac{1}{2}|x| \leq |y| \leq 2|x|$ .

For proving Theorem 2.9 we need some intermediate lemmas. First we recall the following result from [1].

**Lemma 2.10.** *Assume that  $a \in \partial\Omega$ ,  $0 < r < \delta_1$  and  $h > 1$  is an integer. There exists an integer  $N_0$ , depending only on  $\delta_1$ , such that for any points  $x$  and  $y$  in  $\Omega \cap B_{\frac{3r}{2}}(a)$  verifying  $\min\{d(x), d(y)\} \geq r/2^h$ , there exists a connected chain of balls  $B_1, \dots, B_j$  with  $j \leq N_0 h$  such that*

$$\begin{aligned} x \in B_1, y \in B_j, \quad B_i \cap B_{i+1} \neq \emptyset \text{ for } 1 \leq i \leq j-1 \\ \text{and } 2B_i \subset B_{2r}(Q) \cap \Omega \text{ for } 1 \leq i \leq j. \end{aligned} \quad (2.39)$$

The next result is a standard Harnack inequality.

**Lemma 2.11.** *Assume  $a \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2\delta_1}{3}}$  and  $0 < r \leq |a|/4$ . Let  $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^1(\Omega)$  be a positive solution of (1.1) vanishing on  $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$ . Then there exists a positive constant  $c_6 > 1$  depending on  $N, p, q$  and  $\delta_1$  such that*

$$u(x) \leq c_6^h u(y), \quad (2.40)$$

for every  $x, y \in B_{\frac{3r}{2}}(a) \cap \Omega$  such that  $\min\{d(x), d(y)\} \geq r/2^h$  for some  $h \in \mathbb{N}$ .

*Proof.* For  $\ell > 0$ , we define  $T_\ell[u]$  by

$$T_\ell[u](x) = \ell^{\frac{p-q}{q+1-p}} u(\ell x), \quad (2.41)$$

and we notice that if  $u$  satisfies (1.1) in  $\Omega$ , then  $T_\ell[u]$  satisfies the same equation in  $\Omega^\ell := \ell^{-1}\Omega$ . If we take in particular  $\ell = |a|$ , we can assume  $|a| = 1$ , thus the curvature of the domain  $\Omega^{|a|}$  remains bounded. By Proposition 2.8

$$u(x) \leq c'_6 \quad \forall x \in B_{2r}(a) \cap \Omega \quad (2.42)$$

where  $c'_6$  depends on  $N, q, \delta_1$ . Then we proceed as in [11], using Lemma 2.10 and internal Harnack inequality as quoted in [16, Corollary 10].  $\square$

Since the solutions are Hölder continuous, the following statement holds as in [16, Theorem 4.2]:

**Lemma 2.12.** *Let the assumptions on  $a$  and  $u$  of Lemma 2.11 be fulfilled. If  $b \in \partial\Omega \cap B_r(a)$  and  $0 < s \leq 2^{-1}r$ , there exist two positive constants  $\delta$  and  $c_7$  depending on  $N, p, q$  and  $\Omega$  such that*

$$u(x) \leq c_7 \frac{|x-b|^\delta}{s^\delta} \max\{u(z) : z \in B_r(b) \cap \Omega\} \quad (2.43)$$

for every  $x \in B_s(b) \cap \Omega$ .

As a consequence we derive the following Carleson type estimate.

**Lemma 2.13.** *Assume  $a \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2\delta_1}{3}}$  and  $0 < r \leq |a|/8$ . Let  $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^2(\Omega)$  be a positive solution of (1.1) vanishing on  $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$ . Then there exists a constant  $c_8$  depending only on  $N, p$  and  $q$  such that*

$$u(x) \leq c_8 u(a - \frac{r}{2} \mathbf{n}_a) \quad \forall x \in B_r(a) \cap \Omega. \quad (2.44)$$

*Proof.* By Lemma 2.11 it is clear that for any integer  $h$  and  $x \in B_r(a) \cap \Omega$  such that  $d(x) \geq 2^{-h}r$ , there holds

$$u(x) \leq c_6^h u(a - \frac{r}{2} \mathbf{n}_a). \quad (2.45)$$

Therefore  $u$  satisfies inequality (2.43) as any Hölder continuous function does. The proof that the constant is independent of  $r$  and  $u$  is more delicate. It is done in [1, Lemma 2.4] for linear equations, but it is based only on Lemma 2.12 and a geometric construction, thus it is also valid in our case.  $\square$

**Lemma 2.14.** *Assume  $a \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2\delta_1}{3}}$  and  $0 < r \leq |a|/8$ . Let  $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^2(\Omega)$  be a positive solution of (1.1) vanishing on  $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$ . Then there exist  $\alpha \in (0, 1/2)$  and  $c_9 > 0$  depending on  $N, p$  and  $q$  such that*

$$\frac{1}{c_9} \frac{t}{r} \leq \frac{u(b - t \mathbf{n}_b)}{u(a - \frac{r}{2} \mathbf{n}_a)} \leq c_9 \frac{t}{r} \quad (2.46)$$

for any  $b \in B_r(a) \cap \partial\Omega$  and  $0 \leq t < \frac{\alpha}{2}r$ .

*Proof.* It is similar to the one of [11, Lemma 3.15].  $\square$

*Proof of Theorem 2.9.* Assume  $x \in B_{\frac{2\delta_1}{3}} \cap \Omega$  and set  $r = \frac{|x|}{8}$ .

*Step 1: Tangential estimate:* we suppose  $d(x) < \frac{\alpha}{2}r$ . Let  $a \in \partial\Omega \setminus \{0\}$  such that  $|a| = |x|$  and  $x \in B_r(a)$ . By Lemma 2.14,

$$\frac{8}{c_9} \frac{u(a - \frac{r}{2}\mathbf{n}_a)}{|x|} \leq \frac{u(x)}{d(x)} \leq 8c_9 \frac{u(a - \frac{r}{2}\mathbf{n}_a)}{|x|}. \quad (2.47)$$

We can connect  $a - \frac{r}{2}\mathbf{n}_a$  with  $-2r\mathbf{n}_0$  by  $m_1$  (depending only on  $N$ ) connected balls  $B_i = B_{\frac{r}{4}}(x_i)$  with  $x_i \in \Omega$  and  $d(x_i) \geq \frac{r}{2}$  for every  $1 \leq i \leq m_1$ . It follows from (2.44) that

$$c_6^{-m_1} u(-2r\mathbf{n}_0) \leq u(a - \frac{r}{2}\mathbf{n}_a) \leq c_6^{m_1} u(-2r\mathbf{n}_0),$$

which, together with (2.47) leads to

$$\frac{1}{c_{10}} \frac{u(-2r\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq c_{10} \frac{u(-2r\mathbf{n}_0)}{|x|}, \quad (2.48)$$

with  $c_{10} = 8c_9c_6^{m_1}$ .

*Step 2: Internal estimate:* we suppose  $d(x) \geq \frac{\alpha}{2}r$ . We can connect  $-2r\mathbf{n}_0$  with  $x$  by  $m_2$  (depending only on  $N$ ) connected balls  $B'_i = B_{\frac{\alpha r}{4}}(x'_i)$  with  $x'_i \in \Omega$  and  $d(x'_i) \geq \frac{\alpha}{2}r$  for every  $1 \leq i \leq m_2$ . By Harnack and Carleson inequalities (2.40) and (2.44) and since  $\frac{\alpha}{4}|x| < d(x) \leq |x|$ , we get

$$\frac{\alpha}{4c_6^{m_2}} \frac{u(-2r\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq \frac{4c_6^{m_2}}{\alpha} \frac{u(-2r\mathbf{n}_0)}{|x|}. \quad (2.49)$$

*Step 3: End of proof.* Suppose  $\frac{|x|}{2} \leq s \leq 2|x|$ , we can connect  $-2r\mathbf{n}_0$  with  $-s\mathbf{n}_0$  by  $m_3$  (depending only on  $N$ ) connected balls  $B''_i = B_{\frac{r}{2}}(x''_i)$  with  $x''_i \in \Omega$  and  $d(x''_i) \geq r$  for every  $1 \leq i \leq m_3$ . This fact, jointly with (2.48) and (2.49), yields to

$$\frac{1}{c_{11}} \frac{u(-s\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq c_{11} \frac{u(-s\mathbf{n}_0)}{|x|} \quad (2.50)$$

where  $c_{11} = c_{11}(N, q, \Omega)$ . Finally, if  $y \in B_{\frac{2r_0}{3}} \cap \Omega$  satisfies  $\frac{|x|}{2} \leq |y| \leq 2|x|$ , then by applying twice (2.50) we get (2.38) with  $c_5 = c_{11}^2$ .  $\square$

The following inequality is a consequence of Theorem 2.9.

**Corollary 2.15.** *Assume  $q > p - 1$  and  $0 \in \partial\Omega$ . Then there exists  $c_{12} > 0$  depending on  $p, q$  and  $\Omega$  such that for any positive solutions  $u_1, u_2 \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^1(\Omega)$  of (1.1) in  $\Omega$ , vanishing on  $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$ , there holds*

$$\sup \left\{ \frac{u_1(y)}{u_2(y)} : y \in B_r \setminus B_{\frac{r}{2}} \right\} \leq c_{12} \inf \left\{ \frac{u_1(y)}{u_2(y)} : y \in B_r \setminus B_{\frac{r}{2}} \right\}. \quad (2.51)$$

### 3 Boundary singularities

#### 3.1 Strongly singular solutions

In this section we consider the equation (1.1) in  $\mathbb{R}_+^N$ . We denote by  $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$  the spherical coordinates in  $\mathbb{R}^N$  and

$$S_+^{N-1} = \left\{ (\sin \phi \sigma', \cos \phi) : \sigma' \in S^{N-2}, \phi \in [0, \frac{\pi}{2}] \right\}.$$

If  $v(x) = r^{-\beta} \omega(\sigma)$  satisfies (1.1) in  $\mathbb{R}_+^N$  and vanishes on  $\partial \mathbb{R}_+^N \setminus \{0\}$ , then  $\beta = \beta_q$  and  $\omega$  is a solution of

$$\begin{aligned} -\operatorname{div}' \left( (\beta_q^2 \omega^2 + |\nabla' \omega|^2)^{\frac{p-2}{2}} \nabla' \omega \right) - \beta_q \Lambda_{\beta_q} (\beta_q^2 \omega^2 + |\nabla' \omega|^2)^{\frac{p-2}{2}} \omega \\ + (\beta_q^2 \omega^2 + |\nabla' \omega|^2)^{\frac{q}{2}} = 0 \quad \text{in } S_+^{N-1} \\ \omega = 0 \quad \text{on } \partial S_+^{N-1}. \end{aligned} \quad (3.1)$$

where  $\beta_q$  and  $\Lambda_{\beta_q}$  have been defined in (1.10). We denote by  $(\beta_*, \psi_*) \in \mathbb{R}_+^* \times C^2(\overline{S_+^{N-1}})$  the unique couple such  $\max \psi_* = 1$  with the property that the function  $(r, \sigma) \mapsto r^{-\beta_*} \psi_*(\sigma)$  is positive,  $p$ -harmonic in  $\mathbb{R}_+^N$  and vanishes on  $\partial \mathbb{R}_+^N \setminus \{0\}$ . Then  $\psi_* = \psi$  satisfies

$$\begin{aligned} -\operatorname{div}' \left( (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \nabla' \psi \right) - \beta_* \Lambda_{\beta_*} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \psi = 0 \quad \text{in } S_+^{N-1} \\ \psi = 0 \quad \text{on } \partial S_+^{N-1}. \end{aligned} \quad (3.2)$$

Since the function  $\psi_*$  is unique it depends only on the azimuthal variable  $\theta_{N-1} = \cos^{-1}(\frac{x_N}{|x|})$  (see Appendix II). Our first result is the following

**Theorem 3.1.** *If  $q \geq q_*$ , or equivalently  $\beta_q \leq \beta_*$ , there exists no positive solution to problem (3.1).*

*Proof.* Suppose such a solution  $\omega$  exists and put  $\theta = \beta_q / \beta_*$ , then  $0 < \theta \leq 1$ . Set  $\eta = \psi^\theta$ , where  $\psi$  is a positive solution of (3.2), and define the operator  $\mathcal{T}$  by

$$\begin{aligned} \mathcal{T}(\eta) = -\operatorname{div}' \left( (\beta_q^2 \eta^2 + |\nabla' \eta|^2)^{\frac{p-2}{2}} \nabla' \eta \right) - \beta_q \Lambda_{\beta_q} (\beta_q^2 \eta^2 + |\nabla' \eta|^2)^{\frac{p-2}{2}} \eta \\ + (\beta_q^2 \eta^2 + |\nabla' \eta|^2)^{\frac{q}{2}}. \end{aligned} \quad (3.3)$$

Since  $\nabla \eta = \theta \psi^{\theta-1} \nabla \psi$ ,

$$\begin{aligned} (\beta_q^2 \eta^2 + |\nabla' \eta|^2)^{\frac{p-2}{2}} &= \theta^{p-2} \psi^{(\theta-1)(p-2)} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}}, \\ (\beta_q^2 \eta^2 + |\nabla' \eta|^2)^{\frac{p-2}{2}} \nabla' \eta &= \theta^{p-1} \psi^{(\theta-1)(p-1)} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \nabla' \psi, \end{aligned}$$

therefore

$$\begin{aligned} \mathcal{T}(\eta) &= -\theta^{p-1} \psi^{(\theta-1)(p-1)} \operatorname{div}' \left( (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \nabla' \psi \right) \\ &\quad - \theta^{p-1} (\theta-1)(p-1) \psi^{(\theta-1)(p-1)-1} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} |\nabla' \psi|^2 \\ &\quad - \beta_q \Lambda_{\beta_q} \theta^{p-2} \psi^{(\theta-1)(p-1)} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \psi + \theta^q \psi^{(\theta-1)q} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{q}{2}}. \end{aligned}$$

But  $\beta_q \Lambda_{\beta_q} \theta^{p-2} = \beta_* \Lambda_{\beta_*} \theta^{p-1} \leq \beta_* \Lambda_{\beta_*} \theta^{p-1}$  since  $\beta_q \leq \beta_*$ . Using (3.2), we see that  $\mathcal{T}(\eta) \geq 0$ . Because Hopf Lemma is valid, there holds  $\partial_{\mathbf{n}} \psi < 0$  on  $\partial S_+^{N-1}$ . Since  $\omega$  is  $C^1$  in  $\overline{S_+^{N-1}}$  and  $\psi$  is defined up to an homothety, there exists a smallest function  $\psi$  such that  $\eta \geq \omega$ , and the graphs of  $\eta$  and  $\omega$  over  $\overline{S_+^{N-1}}$  are tangent, either at some  $\alpha \in S_+^{N-1}$ , or only at a point  $\alpha \in \partial S_+^{N-1}$ . We put  $w = \eta - \omega$ . Then

$$\mathcal{T}(\eta) = \mathcal{T}(\eta) - \mathcal{T}(\omega) = \Phi(1) - \Phi(0), \quad (3.4)$$

where  $\Phi(t) = \mathcal{T}(\omega_t)$  with  $\omega_t = \omega + tw$ .

We use local coordinates  $(\sigma_1, \dots, \sigma_{N-1})$  on  $S^{N-1}$  near  $\alpha$ . We denote by  $g = (g_{ij})$  the metric tensor on  $S^{N-1}$  and by  $g^{jk}$  its contravariant components. Then, for any  $\varphi \in C^1(S^{N-1})$ ,

$$|\nabla \varphi|^2 = \sum_{j,k} g^{jk} \frac{\partial \varphi}{\partial \sigma_j} \frac{\partial \varphi}{\partial \sigma_k} = \langle \nabla \varphi, \nabla \varphi \rangle_g.$$

If  $X = (X^1, \dots, X^d) \in C^1(TS^{N-1})$  is a vector field, we lower indices by setting  $X^\ell = \sum_i g^{\ell i} X_i$  and define the divergence of  $X$  by

$$\operatorname{div}'_g X = \frac{1}{\sqrt{|g|}} \sum_\ell \frac{\partial}{\partial \sigma_\ell} \left( \sqrt{|g|} X^\ell \right) = \frac{1}{\sqrt{|g|}} \sum_{\ell,i} \frac{\partial}{\partial \sigma_\ell} \left( \sqrt{|g|} g^{\ell i} X_i \right).$$

We write  $\Phi(t) = \Phi_1(t) + \Phi_2(t) + \Phi_3(t)$  where

$$\Phi_1(t) = -\beta_q \Lambda_{\beta_q} \left( \beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2 \right)^{\frac{p-2}{2}} \omega_t, \quad \Phi_2(t) = \left( \beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2 \right)^{\frac{q}{2}}$$

and

$$\Phi_3(t) = -\operatorname{div}' \left( \left( \beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2 \right)^{\frac{p-2}{2}} \nabla' \omega_t \right).$$

Then

$$\Phi_1(1) - \Phi_1(0) = -\sum_j a_j \frac{\partial w}{\partial \sigma_j} - bw \quad \text{and} \quad \Phi_2(1) - \Phi_2(0) = \sum_j c_j \frac{\partial w}{\partial \sigma_j} + dw,$$

where

$$b = \beta_q \Lambda_{\beta_q} \left( \beta_q^2 \omega_t^2 + |\nabla \omega_t|^2 \right)^{\frac{p}{2}-2} \left( (p-1) \beta_q^2 \omega_t^2 + |\nabla \omega_t|^2 \right),$$

$$a_j = (p-2) \beta_q \Lambda_{\beta_q} \left( \beta_q^2 \omega_t^2 + |\nabla \omega_t|^2 \right)^{\frac{p}{2}-2} \omega_t \sum_k g^{jk} \frac{\partial \omega_t}{\partial \sigma_k},$$

$$d = q \beta_q^2 \left( \beta_q^2 \omega_t^2 + |\nabla \omega_t|^2 \right)^{\frac{q}{2}-1} \omega_t,$$

and

$$c_j = q \left( \beta_q^2 \omega_t^2 + |\nabla \omega_t|^2 \right)^{\frac{q}{2}-1} \sum_k g^{jk} \frac{\partial \omega_t}{\partial \sigma_k}.$$



Furthermore

$$\begin{aligned} \Phi_3(1) - \Phi_3(0) &= -(p-2)\operatorname{div}' \left( (\beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2)^{\frac{p-4}{2}} (\beta_q^2 \omega_t w + \langle \nabla' \omega_t, \nabla' w \rangle_g) \nabla' \omega_t \right) \\ &\quad - \operatorname{div}' \left( (\beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2)^{\frac{p-2}{2}} \nabla' w \right). \end{aligned}$$

Therefore we can write  $\Phi(1) - \Phi(0)$  under the form

$$\Phi(1) - \Phi(0) = -\operatorname{div}'(A\nabla' w) + \langle B, \nabla' w \rangle_g + Cw := \mathcal{L}w \quad (3.5)$$

where

$$\begin{aligned} \langle AX, X \rangle_g &= (\beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2)^{\frac{p-4}{2}} (p-2) \langle \nabla' \omega_t, X \rangle_g^2 + |\nabla' \omega_t|^2 |X|^2 \\ &\geq (\beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2)^{\frac{p-4}{2}} \min\{1, p-1\} |\nabla' \omega_t|^2 |X|^2. \end{aligned} \quad (3.6)$$

and  $B$  and  $C$  can be computed from the previous expressions. It is important to notice that  $\beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2$  is bounded between two positive constants  $m_1$  and  $m_2$  in  $S_+^{N-1}$ . Thus the operator  $\mathcal{L}$  is uniformly elliptic with bounded coefficients. Since  $w$  is nonnegative and either at some point  $\alpha$ ,  $\nabla' w(\alpha) = 0$  and  $w(\alpha) > 0$ , or at some boundary point  $\alpha$  where  $w(\alpha) = 0$  and  $\partial_n w(\alpha) < 0$ , it follows from the strong maximum principle or Hopf Lemma (see [7]) that  $w = 0$ , contradiction.  $\square$

**Theorem 3.2.** *Assume  $q < q_*$  or equivalently  $\beta_q > \beta_*$ . There exists a unique positive solution  $\omega_*$  to problem (3.1).*

*Proof. Existence.* It will follow from [4]. Indeed problem (3.1) can be written under the form

$$\begin{aligned} \mathbf{A}(\omega) &:= -\operatorname{div}' \mathbf{a}(\omega, \nabla' \omega) = \mathbf{B}(\omega, \nabla' \omega) && \text{in } S_+^{N-1} \\ \omega &= 0 && \text{on } \partial S_+^{N-1}, \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} \mathbf{a}(r, \xi) &= (\beta_q^2 r^2 + |\xi|^2)^{\frac{p-2}{2}} \xi, \\ \mathbf{B}(r, \xi) &= \beta_q \Lambda_{\beta_q} (\beta_q^2 r^2 + |\xi|^2)^{\frac{p-2}{2}} r - (\beta_q^2 r^2 + |\xi|^2)^{\frac{q}{2}}. \end{aligned} \quad (3.8)$$

The operator  $\mathbf{A}$  is a Leray-Lions operator which satisfies the assumptions (1.6)-(1.8) of [4, Theorem 2.1], and the term  $\mathbf{B}$  satisfies (1.9),(1.10) in the same article. Therefore the existence of a positive solution  $\omega \in W_0^{1,p}(S_+^{N-1}) \cap L^\infty(S_+^{N-1})$  is ensured whenever we can find a supersolution  $\bar{\omega} \in W^{1,p}(S_+^{N-1}) \cap L^\infty(S_+^{N-1})$  and a nontrivial subsolution  $\underline{\omega} \in W^{1,p}(S_+^{N-1})$  of (3.7) such that

$$0 \leq \underline{\omega} \leq \bar{\omega} \quad \text{in } S_+^{N-1}. \quad (3.9)$$

First we note that  $\eta = \eta_0$  is a supersolution if the positive constant  $\eta_0$  is large enough. In order to find a subsolution, we set again  $\eta = \psi^\theta$  with  $\theta = \beta_q/\beta_*$  and  $\psi$  as in (3.2). Now  $\theta > 1$ , thus  $\eta \in W_0^{1,p}(S_+^{N-1})$ . As above we have

$$\begin{aligned} \mathcal{T}(\eta) &= -\theta^{p-1} \psi^{(\theta-1)(p-1)} \operatorname{div}' \left( (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \nabla' \psi \right) \\ &\quad - \theta^{p-1} (\theta-1)(p-1) \psi^{(\theta-1)(p-1)-1} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} |\nabla' \psi|^2 \\ &\quad - \beta_q \Lambda_{\beta_q} \theta^{p-2} \psi^{(\theta-1)(p-1)} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \psi + \theta^q \psi^{(\theta-1)q} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{q}{2}}. \end{aligned}$$

Now  $\beta_q \Lambda_{\beta_q} \theta^{p-2} = \beta_* \Lambda_{\beta_q} \theta^{p-1} = \beta_* (\Lambda_{\beta_q} - \Lambda_{\beta_*}) \theta^{p-1} + \beta_* \Lambda_{\beta_*} \theta^{p-1}$  and  $\Lambda_{\beta_q} - \Lambda_{\beta_*} = (\beta_q - \beta_*) (p-1) = \beta_* (p-1)(\theta-1)$ , hence

$$\begin{aligned} \mathcal{T}(\eta) &= -\theta^{p-1} \psi^{(\theta-1)(p-1)} \operatorname{div}' \left( (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \nabla' \psi \right) \\ &\quad - \theta^{p-1} (\theta-1)(p-1) \psi^{(\theta-1)(p-1)-1} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} |\nabla' \psi|^2 \\ &\quad - \beta_* (\Lambda_{\beta_q} - \Lambda_{\beta_*}) \theta^{p-1} \psi^{(\theta-1)(p-1)} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \psi \\ &\quad - \beta_* \Lambda_{\beta_*} \theta^{p-1} \psi^{(\theta-1)(p-1)} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \psi + \theta^q \psi^{(\theta-1)q} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{q}{2}}. \end{aligned}$$

Using the equation satisfied by  $\psi$  yields to the relation

$$\begin{aligned} \mathcal{T}(\eta) &= -\theta^{p-1} (\theta-1)(p-1) \psi^{(\theta-1)(p-1)-1} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} |\nabla' \psi|^2 \\ &\quad - \beta_*^2 (p-1)(\theta-1) \theta^{p-1} \psi^{(\theta-1)(p-1)-1} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \psi^2 \\ &\quad + \theta^q \psi^{(\theta-1)q} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{q}{2}} \\ &= -\theta^{p-1} (\theta-1)(p-1) \psi^{(\theta-1)(p-1)-1} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p}{2}} \\ &\quad + \theta^q \psi^{(\theta-1)q} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{q}{2}}. \end{aligned}$$

If we replace  $\eta := \eta_1 = \psi^\theta$  by  $\eta := \eta_m = (m\psi)^\theta$  in the above computation, the inequality  $\mathcal{T}\eta_m \leq 0$  will be true provided

$$m^{\theta(q+1-p)} \psi^{(\theta-1)(q+1-p)+1} \leq \theta^{p-1-q} (\theta-1)(p-1) (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-q}{2}},$$

which is satisfied if we choose  $m$  small enough so that  $(m\psi)^\theta \leq \eta_0$  and satisfying

$$m^{\theta(q+1-p)} \leq \beta_*^{(\theta-1)(q+1-p)+1} \theta^{p-1-q} (\theta-1)(p-1) \frac{\min_{x \in S_+^{N-1}} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-q}{2}}}{\max_{x \in S_+^{N-1}} \psi^{(\theta-1)(q+1-p)+1}}.$$

Therefore  $0 < \eta_m \leq \eta_0$  and standard regularity implies that the solution  $\omega$  is  $C^1$  in  $\bar{S}_+^{N-1}$ . Actually  $\omega$  is  $C^\infty$  since the operator is not degenerate.

*Uniqueness.* We use the tangency method developed in the proof of Theorem 3.1. Assume  $\omega_1$  and  $\omega_2$  are two positive solutions of (3.2), then they are positive in  $S_+^{N-1}$  and  $\partial_n \omega_i < 0$  on  $\partial S_+^{N-1}$ . Either the  $\omega_i$  are ordered and  $\omega_1 \leq \omega_2$ , or their graphs intersect. In any case we can define

$$\tau = \inf \{s > 1 : s\omega_1 \geq \omega_2\}.$$

We set  $\omega^* = \tau\omega_1$ . Then either the graphs of  $\omega_2$  and  $\omega^*$  are tangent at some interior point  $\alpha$ , or they are not tangent in  $S_+^{N-1}$ ,  $\partial_n \omega^* \leq \partial_n \omega_2 < 0$  on  $\partial S_+^{N-1}$  and there exists  $\alpha \in \partial S_+^{N-1}$  such that  $\partial_n \omega^*(\alpha) = \partial_n \omega_2(\alpha) < 0$ . Furthermore  $\mathcal{T}(\omega^*) \geq 0$ . If we set  $w = \omega^* - \omega_2$ , then, as in Theorem 3.1,

$$-\operatorname{div}'(\tilde{A}\nabla'w) + \langle \tilde{B}, \nabla'w \rangle_g + \tilde{C}w = \tilde{\mathcal{L}}w \geq 0$$

where

$$\begin{aligned} \langle \tilde{A}X, X \rangle_g &= (\beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2)^{\frac{p-4}{2}} (p-2) \langle \nabla' \omega_t, X \rangle_g^2 + |\nabla' \omega_t|^2 |X|^2 \\ &\geq (\beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2)^{\frac{p-4}{2}} \min\{1, p-1\} |\nabla' \omega_t|^2 |X|^2, \end{aligned} \quad (3.10)$$

in which  $\omega_t = \omega_2 + t(\omega^* - \omega_2)$  and  $t \in (0, 1)$  is obtained by applying the mean value theorem and  $\tilde{B}$  and  $\tilde{C}$  are defined accordingly. Since  $\tilde{\mathcal{L}}$  is uniformly elliptic and has bounded coefficients, it follows from the strong maximum principle that  $w = 0$ . Thus  $\omega^* = \tau \omega_1 = \omega_2$  and  $\tau = 1$  from the equation. This ends the proof.  $\square$

### 3.2 Removable boundary singularities

The following is the basic result for removability of isolated singularities. It is valid in the general case, but with a local geometric constraint.

**Theorem 3.3.** *Assume  $q^* \leq q < p \leq N$ ,  $\Omega$  is a  $C^2$  bounded domain with  $0 \in \partial\Omega$ , such that  $\Omega \cap B_\delta = B_\delta^+$  for some  $\delta > 0$ . If  $u \in C^1(\bar{\Omega} \setminus \{0\})$  is a nonnegative solution of (1.1) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ , then it is identically 0.*

*Proof. Step 1: Assume  $\Omega \subset \mathbb{R}_+^N$ . For  $\epsilon > 0$ , we set  $\Omega'_\epsilon = \Omega \cap \bar{B}_\epsilon^c$  and  $H_\epsilon = \mathbb{R}_+^N \cap \bar{B}_\epsilon^c$ . For  $k, n \in \mathbb{N}_*$ ,  $n \geq \text{diam}(\Omega)$ , we denote by  $v_{k,n,\epsilon}$  ( $n \in \mathbb{N}_*$ ) the solution of the problem*

$$\begin{aligned} -\Delta_p v + |\nabla v|^q &= 0 && \text{in } H_\epsilon \cap B_n \\ v &= k \chi_{\mathbb{R}_+^N \cap \partial B_\epsilon} && \text{on } \partial(H_\epsilon \cap B_n). \end{aligned} \quad (3.11)$$

If  $k > c_2 \epsilon^{\frac{q-p}{q+1-p}}$  for a suitable  $c_2 = c_2(p, q) > 0$  (see Lemma 2.6), then  $v_{k,n,\epsilon} \geq u$  in  $\Omega'_\epsilon$ . Moreover there holds  $v_{k,n,\epsilon} \leq v_{k',n',\epsilon}$  for  $n \leq n'$  and  $k \leq k'$ . Furthermore the function

$$U_{\epsilon,n}(x) = c_2 \left( (|x| - \epsilon)^{\frac{q-p}{q+1-p}} - (n - \epsilon)^{\frac{q-p}{q+1-p}} \right)$$

is a supersolution in  $B_n \setminus B_\epsilon$ , and there holds  $v_{k,n,\epsilon} \leq U_{\epsilon,n}$ . By monotonicity and standard a priori estimate, we obtain that  $v_{k,n,\epsilon} \rightarrow v_\epsilon$  when  $n, k \rightarrow \infty$  and that the function  $v = v_\epsilon$  is solution of

$$\begin{aligned} -\Delta_p v + |\nabla v|^q &= 0 && \text{in } H_\epsilon \\ \lim_{|x| \rightarrow \epsilon} v(x) &= \infty && \\ v &= 0 && \text{on } \partial\mathbb{R}_+^N \cap \bar{B}_\epsilon^c. \end{aligned} \quad (3.12)$$

Furthermore

$$u(x) \leq v_\epsilon(x) \leq c_2 (|x| - \epsilon)^{\frac{q-p}{q+1-p}} \quad \text{in } \Omega'_\epsilon. \quad (3.13)$$

The function  $v_\epsilon$  may not be unique, however it is the minimal solution of the above problem since the  $v_{k,n,\epsilon}$  is unique, and monotonicity in  $n$  and  $k$  holds. Actually,  $v_\epsilon \leq v_{\epsilon'}$  if  $0 \leq \epsilon \leq \epsilon'$ . For  $\ell > 0$ , we recall that the transformation  $v \mapsto T_\ell[v]$  defined by (2.41) leaves equation (1.1) invariant. As a consequence of the uniqueness of the approximations we have  $T_\ell[v_{k,n,\epsilon}] = v_{\frac{p-q}{\ell^{q+1-p}} k, \ell^{-1} n, \ell^{-1} \epsilon}$ , which implies

$$T_\ell[v_\epsilon] = v_{\ell^{-1} \epsilon}. \quad (3.14)$$

Letting  $\epsilon \rightarrow 0$ , we derive from the monotonicity with respect to  $\epsilon$  and standard  $C^{1,\alpha}$  estimates, that the following identity holds:

$$T_\ell[v_0] = v_0 \quad \forall \ell > 0. \quad (3.15)$$

The function  $v_0$  is a positive and separable solution of (1.1) in  $\mathbb{R}_+^N$  which vanishes on  $\partial\Omega \setminus \{0\}$ . It follows from Theorem 3.1 that  $v_0 = 0$ , and so is  $u$ .

*Step 2: The general case.* We assume that  $\Omega \cap B_\delta \subset \mathbb{R}_+^N$  and we denote by  $M$  the maximum of  $u$  on  $\partial B_\delta \cap \Omega$ . Then the function  $(u-M)_+$  is a subsolution of (1.1) in  $\Omega \cap B_\delta$  which vanishes on  $\partial\Omega \cap B_\delta \setminus \{0\}$ . By Step 1, it is dominated by  $v_0$ , which ends the proof.  $\square$

*Remark.* The previous result is valid if  $u$  is a subsolution with the same regularity. If  $u$  is no longer assumed to be nonnegative, only  $u^+$  vanishes. Furthermore, the regularity of the boundary has not been used, but only the fact that  $\Omega$  is locally contained into a half space to the boundary of which 0 belongs.

*Remark.* If no geometric assumption is made on  $\partial\Omega$ , we can prove that  $u(x) = o(|x|^{-\beta_q})$  near 0. The next result shows that the removability holds if  $q > q_*$ .

**Theorem 3.4.** *Assume  $q^* < q < p \leq N$  and  $\Omega$  is a  $C^2$  bounded domain with  $0 \in \partial\Omega$ . If  $u$  is a nonnegative solution of (1.1) in  $\Omega$  which belongs to  $C^1(\bar{\Omega} \setminus \{0\})$  and vanishes on  $\partial\Omega \setminus \{0\}$ , it is identically 0.*

*Proof.* As it is proved in [12], for any smooth subdomain  $S \subset S^{N-1}$ , there exists a unique  $\beta_{*s} > 0$  and  $\psi_s^* > 0$ , unique up to an homothety, such that  $x \mapsto |x|^{-\beta_{*s}} \psi_s^*(|x|^{-1}x)$  is  $p$  harmonic in the cone  $C_S = \{x \in \mathbb{R}^N \setminus \{0\} : |x|^{-1}x \in S\}$  and  $\psi_s^*$  satisfies

$$\begin{aligned} -\operatorname{div}' \left( (\beta_{*s}^2 \psi_s^{*2} + |\nabla' \psi_s^*|^2)^{\frac{p-2}{2}} \nabla' \psi_s^* \right) - \beta_{*s} \Lambda_{\beta_{*s}} (\beta_{*s}^2 \psi_s^{*2} + |\nabla' \psi_s^*|^2)^{\frac{p-2}{2}} \psi_s^* &= 0 \quad \text{in } S \\ \psi_s^* &= 0 \quad \text{on } \partial S, \end{aligned} \quad (3.16)$$

Furthermore  $S \subset \tilde{S} \subset S^{N-1}$  implies  $\beta_{*\tilde{s}} \leq \beta_{*s}$ . Using the system of spherical coordinates defined in (6.5) in Appendix II, for  $\epsilon > 0$  we denote by  $S := S_\epsilon$  the spherical shell with vertex the north pole  $N$  and latitude angle  $\theta_{N-1} \in [0, \frac{\pi}{2} + \epsilon]$ . Because of uniqueness of  $\beta_{*s}$ ,  $\beta_{*s_\epsilon} \uparrow \beta_*$  as  $\epsilon \rightarrow 0$ . Therefore, if  $q > q_*$ , or equivalently  $\beta_q < \beta_*$ , there exists  $\delta, \epsilon > 0$  such that  $\Omega \cap B_\delta \subset C_{S_\epsilon} \cap B_\delta$  and  $\beta_q < \beta_{*s_\epsilon}$ . Since Theorem 3.1 is valid if  $S_+^{N-1}$  is replaced by  $S_\epsilon$  and  $\beta_q < \beta_{*s_\epsilon}$  it follows that  $u = 0$  as in the proof of Theorem 3.3, Steps 1 and 2.  $\square$

The next result, valid in the case  $p = N$ , is based upon the conformal invariance of the N-Laplacian. In this case the exponent  $\beta_*$  corresponding to the first spherical N-harmonic eigenvalue is equal to 1 and the corresponding spherical N-harmonic eigenfunction in  $S_+^{N-1}$  is  $x_N/|x|^2$ .

**Theorem 3.5.** *Assume  $N - \frac{1}{2} \leq q < N$ ,  $\Omega$  is a bounded domain and  $0 \in \partial\Omega$  is such that there exists a ball  $B \subset \Omega^\circ$  to the boundary of which 0 belongs. If  $u$  is a nonnegative solution of*

$$-\Delta_N u + |\nabla u|^q = 0 \quad \text{in } \Omega, \quad (3.17)$$

*which belongs to  $C(\bar{\Omega} \setminus \{0\}) \cap W_0^{1,N}(\Omega \setminus \bar{B}_\epsilon(0))$  for any  $\epsilon > 0$ , it is identically 0.*

*Proof.* We assume that the inward normal unit vector to  $\partial\Omega$  at 0 is  $\mathbf{e}_N = (0, 0, \dots, 1)$  and that the ball  $B = B_{\frac{1}{2}}(a)$  of center  $a = -\frac{1}{2}\mathbf{e}_N$  and radius  $\frac{1}{2}$  touches  $\partial\Omega$  at 0 and is exterior to  $\Omega$  (this can be assumed up to a rotation and a dilation). This is the consequence of the exterior sphere condition at the point 0. It is always valid if  $\partial\Omega$  is  $C^2$ . We denote by  $\mathcal{I}_\omega$  the inversion of center  $\omega = -\mathbf{e}_N$  and power 1, i.e.  $\mathcal{I}_\omega(x) = \omega + \frac{x-\omega}{|x-\omega|^2}$ . Under this transformation, the complement of the ball  $B_{\frac{1}{2}}(a)$ , which contains  $\Omega$ , is transformed into the half space  $\mathbb{R}_-^N$  which contains the image  $\tilde{\Omega}$  of  $\Omega$ . Since  $u$  satisfies (3.17),  $\tilde{u} = u \circ \mathcal{I}_\omega$  satisfies

$$-\Delta_N \tilde{u} + |x - \omega|^{2(q-N)} |\nabla \tilde{u}|^q = 0 \quad \text{in } \tilde{\Omega}. \quad (3.18)$$

Furthermore since  $0 = \mathcal{I}_\omega(0)$  and  $\mathcal{I}_\omega$  is a diffeomorphism,  $\tilde{u} \in C(\overline{\tilde{\Omega}} \setminus \{0\}) \cap C^1(\tilde{\Omega})$  and it vanishes on  $\partial\tilde{\Omega} \setminus \{0\}$ . Since  $|x - \omega| \leq 1$  and  $q < N$ ,  $\tilde{u}$  is a subsolution for (3.17) in  $\tilde{G}$ . By Theorem 3.4,  $\tilde{u} = 0$ .  $\square$

### 3.3 Weakly singular solutions

The main result of this section is the following existence and uniqueness result concerning solutions of (1.1) with a boundary weak singularity. We recall that  $\psi_*$  is unique positive solution of (1.11) such that  $\sup \psi_* = 1$ . Our first result is valid for any  $1 < p \leq N$  but it needs a geometric constraint on  $\Omega$ .

**Theorem 3.6.** *Let  $p - 1 < q < q_* < p \leq N$  and  $\Omega \subset \mathbb{R}_+^N$  be a bounded  $C^2$  domain such that  $0 \in \partial\Omega$ . Assume that there exists  $\delta > 0$  such that  $\Omega_\delta := \Omega \cap B_\delta = B_\delta^+$ . Then for any  $k > 0$  there exists a unique positive solution  $u := u_k$  of (1.1) in  $\Omega$ , which belongs to  $C^1(\overline{\Omega} \setminus \{0\})$ , vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies*

$$\lim_{x \rightarrow 0} \frac{u_k(x)}{\Psi_*(x)} = k \quad (3.19)$$

in the  $C^1$ -topology of  $S_+^{N-1}$ , where

$$\Psi_*(x) = |x|^{-\beta_*} \psi_*(|x|^{-1}x).$$

The proof of this theorem is long and difficult and requires a certain number of intermediate results.

**Lemma 3.7.** *Let the assumptions on  $p$ ,  $q$  and  $\Omega$  of Theorem 3.6 be satisfied. There exists a unique positive  $p$ -harmonic function  $\Phi_*$  in  $\Omega$ , which is continuous in  $\overline{\Omega} \setminus \{0\}$ , vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies*

$$\lim_{x \rightarrow 0} \frac{\Phi_*(x)}{\Psi_*(x)} = 1. \quad (3.20)$$

*Proof.* For  $0 < \epsilon < \delta$  let  $v_\epsilon$  be the unique nonnegative  $p$ -harmonic function in  $\Omega \setminus \overline{B_\epsilon^+}$  which is continuous in  $\overline{\Omega} \setminus B_\epsilon^+$ , vanishes on  $\partial\Omega \setminus B_\epsilon$  and achieves the value  $\Psi_*$  on  $\partial B_\epsilon \cap \Omega$ . Since  $\Omega \subset \mathbb{R}_+^N$ ,  $v_\epsilon \leq \Psi_*$  in  $\Omega \setminus B_\epsilon^+$ . Hence inequalities  $0 < \epsilon < \epsilon' \leq \delta$  imply  $v_\epsilon \leq v_{\epsilon'}$  in  $\Omega \setminus \overline{B_{\epsilon'}^+}$ . Because  $\Psi_* \leq \delta^{-\beta_*}$ , there holds

$$v_\epsilon + \delta^{-\beta_*} \geq \Psi_*, \quad (3.21)$$

in  $\Omega \setminus B_\delta^+$ . Since  $v_\epsilon$  and  $\Psi_*$  coincide on  $\partial B_\epsilon^+$  and vanish on  $\partial\mathbb{R}_+^N \cap (B_\delta^+ \setminus B_\epsilon^+)$ , (3.21) holds also in  $B_\delta^+ \setminus B_\epsilon^+$ . Because  $v_\epsilon \geq 0$  there holds

$$(\Psi_* - \delta^{-\beta_*})_+ \leq v_\epsilon \leq \Psi_* \quad \text{in } \Omega \setminus B_\epsilon^+. \quad (3.22)$$

By a standard regularity result  $v_\epsilon$  converges to a function  $\Phi_*$  continuous in  $\overline{\Omega} \setminus \{0\}$ ,  $p$ -harmonic in  $\Omega$  such that

$$(\Psi_* - \delta^{-\beta_*})_+ \leq \Phi_* \leq \Psi_*$$

in  $\Omega$ . Therefore (3.20) holds provided  $\frac{x}{|x|}$  remains in a compact subset of  $S_+^{N-1}$ . Let us define a function  $\tilde{\phi}_*$  by  $\tilde{\phi}_*(x) = |x|^{\beta_*} \Phi_*(x)$ , then  $\tilde{\phi}_*(r, \sigma) \leq \psi_*(\sigma)$  where  $r = |x|$  and  $\sigma = \frac{x}{|x|} \in S_+^{N-1}$ . By standard  $C^{1,\alpha}$  estimates,  $\tilde{\phi}_*(r, \cdot)$  is relatively compact in the  $C(\overline{S_+^{N-1}})$ -topology. Therefore the convergence of  $\frac{\Phi_*(x)}{\Psi_*(x)}$  to 1 when  $x$  to 0 holds not only when  $\frac{x}{|x|}$  remains in a compact subset of  $S_+^{N-1}$ , but uniformly on  $S_+^{N-1}$ , which implies (3.20). Uniqueness follows classically by (3.20) and the maximum principle.  $\square$

**Lemma 3.8.** *Let the assumptions on  $p$ ,  $q$  and  $\Omega$  of Theorem 3.6 be satisfied. If for some  $k > 0$  there exists a solution  $u_k$  of (1.1) in  $\Omega$ , which belongs to  $C^1(\overline{\Omega} \setminus \{0\})$ , vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies (3.19), then for any  $k > 0$  there exists such a solution.*

*Proof.* We notice that for any  $c < 1$  (resp  $c > 1$ ),  $cu_k$  is a subsolution (resp. supersolution) of (1.1) in  $\Omega$ . Let  $\Phi_*$  be as in Lemma 3.7. If  $c < 1$ , the function  $ck\Phi_*$  is a supersolution of (1.1) which vanishes on  $\partial\Omega \setminus \{0\}$ . Furthermore

$$\lim_{x \rightarrow 0} \frac{cu_k(x)}{\Psi_*(x)} = ck = \lim_{x \rightarrow 0} \frac{ck\Phi_*(x)}{\Psi_*(x)}.$$

Then there exists a solution  $u_{ck}$  of (1.1) in  $\Omega$  which satisfies  $cu_k \leq u_{ck} \leq ck\Phi_*$ . If  $c > 1$ , we set  $u^* := T_{c^\theta}[u_k]$ , which means  $u^*(x) = c^{\beta_q \theta} u_k(c^\theta x)$  with  $\theta = (\beta_q - \beta_*)^{-1}$ . Then  $u^*$  is a solution of (1.1) in  $\Omega^{c^\theta} = \frac{1}{c^\theta} \Omega$ . In particular,  $u^*$  satisfies the equation in  $B_{\frac{\delta}{c^\theta}}^+(0)$ . Since  $c^\theta > 1$ ,  $B_{\frac{\delta}{c^\theta}}^+(0) \subset B_\delta^+(0)$ . Put  $m = \max\{u^* : x \in \partial B_{\frac{\delta}{c^\theta}}^+(0)\}$ . The function  $(u^* - m)_+$ , extended by 0 outside  $B_{\frac{\delta}{c^\theta}}^+(0)$ , is a subsolution of (1.1) in  $\Omega$ . Furthermore it satisfies

$$\lim_{x \rightarrow 0} \frac{(u^* - m)_+(x)}{\Psi_*(x)} = ck,$$

uniformly on any compact subset of  $S_+^{N-1}$ . Therefore there exists a solution  $u_{ck}$  of (1.1) in  $\Omega$  which satisfies  $(u^* - m)_+ \leq u_{ck} \leq ck\Phi_*$ , and in particular it vanishes on  $\partial\Omega \setminus \{0\}$  and belongs to  $C^1(\overline{\Omega} \setminus \{0\})$ . By [13],  $u_{ck}$  is positive in  $\Omega$ . Thus  $u_{ck}$  belongs to  $C^{1,\alpha}(B_\delta^+(0) \setminus \{0\})$  and satisfies

$$|x|^{\beta_*} |u_{ck}(x)| + |x|^{1+\beta_*} |\nabla u_{ck}(x)| + |x|^{1+\beta_*+\alpha} \sup_{\substack{|y| \leq |x| \\ x \neq y}} \frac{|\nabla u_{ck}(x) - \nabla u_{ck}(y)|}{|x - y|^\alpha} \leq M$$

by (2.11). Therefore the set of functions  $\{r^{\beta_*+1} \nabla u_{ck}(r, \cdot)\}_{r>0}$  is uniformly relatively compact in the topology of uniform convergence on  $\overline{S_+^{N-1}}$ . Since it converges to  $ck \nabla' \psi_*$  uniformly on compact subsets of  $S_+^{N-1}$  as  $r \rightarrow 0$ , this convergence holds in  $C(\overline{S_+^{N-1}})$ . This implies

$$\lim_{x \rightarrow 0} \frac{u_{ck}(x)}{\Psi_*(x)} = ck. \quad (3.23)$$

□

The next Lemma is the keystone of our construction. Its proof is very delicate and needs several intermediate steps.

**Lemma 3.9.** *Under the assumptions of Theorem 3.6 there exists a real number  $R_0$  such that  $0 < R_0 \leq \delta$  and a positive subsolution  $\tilde{u}$  of (1.1) in  $B_{R_0}^+$  which is Lipschitz continuous in  $\overline{B_{R_0}^+} \setminus \{0\}$ , vanishes on  $\overline{B_{R_0}^+} \cap \partial\mathbb{R}_+^N \setminus \{0\}$ , is smaller than  $\Psi_*$  and satisfies*

$$\lim_{x \rightarrow 0} \frac{\tilde{u}(x)}{\Psi_*(x)} = 1. \quad (3.24)$$

*Proof.* The construction of the function  $\tilde{u}$ . We look for a subsolution under the form  $\tilde{u} = \Psi_* - w$  for a suitable nonnegative function  $w$ .

*Step 1: reduction of the problem.* We use spherical coordinates for a  $C^1$  function  $u : x \mapsto u(x) = u(r, \sigma)$ ,  $r = |x|$ ,  $\sigma = \frac{x}{|x|}$ . Then  $\nabla u = u_r \mathbf{e} + r^{-1} \nabla' u$  where  $\mathbf{e} = |x|^{-1} x$ ,  $|\nabla u|^2 = u_r^2 + r^{-2} |\nabla' u|^2$  and  $|\nabla u|^q = \left(u_r^2 + r^{-2} |\nabla' u|^2\right)^{\frac{q}{2}}$ . The expression of the p-Laplacian in spherical coordinates is

$$\begin{aligned} -\Delta_p u = & - \left( \left( u_r^2 + r^{-2} |\nabla' u|^2 \right)^{\frac{p-2}{2}} u_r \right)_r - \frac{N-1}{r} \left( u_r^2 + r^{-2} |\nabla' u|^2 \right)^{\frac{p-2}{2}} u_r \\ & - \frac{1}{r^2} \operatorname{div}' \left( \left( u_r^2 + r^{-2} |\nabla' u|^2 \right)^{\frac{p-2}{2}} \nabla' u \right). \end{aligned}$$

Put  $v(t, \sigma) = r^{\beta_*} u(r, \sigma)$  with  $t = \ln r \in (-\infty, \ln \delta]$ , then  $v$  satisfies

$$\begin{aligned} \mathcal{Q}[v] := & - \left( \left( (v_t - \beta_* v)^2 + |\nabla' v|^2 \right)^{\frac{p-2}{2}} (v_t - \beta_* v) \right)_t - \operatorname{div}' \left( \left( (v_t - \beta_* v)^2 + |\nabla' v|^2 \right)^{\frac{p-2}{2}} \nabla' v \right) \\ & + \Lambda_{\beta_*} \left( (v_t - \beta_* v)^2 + |\nabla' v|^2 \right)^{\frac{p-2}{2}} (v_t - \beta_* v) + e^{\nu t} \left( (v_t - \beta_* v)^2 + |\nabla' v|^2 \right)^{\frac{q}{2}} = 0 \end{aligned} \quad (3.25)$$

in  $(-\infty, \ln \delta) \times S_+^{N-1}$  where  $\nu = 1 - (q+1-p)(\beta_*+1) = 1 - \frac{\beta_*+1}{\beta_q+1} > 0$  and  $\Lambda_{\beta_*} = \beta_*(p-1) + p - N$ . Notice that  $\psi_*$  satisfies

$$-\operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \nabla' \psi_* \right) - \beta_* \Lambda_{\beta_*} \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \psi_* = 0, \quad (3.26)$$

hence it is a supersolution for (3.25). We look for a subsolution under the form

$$V(t, \sigma) = \psi_* - a(t)g(\psi_*)$$

where  $g$  is a continuous increasing function defined on  $\mathbb{R}_+$ , vanishing at 0 and smooth on  $\mathbb{R}_+^*$  and  $a(t) = e^{\gamma t}$  with  $\gamma > 0$  to be chosen. Thus  $a' = \gamma a$ ,  $a'' = \gamma^2 a$ ,  $V_t = -\gamma a g(\psi_*)$ ,  $V_t - \beta_* V =$

$-\beta_*\psi_* + a(\beta_* - \gamma)g(\psi_*)$ ,  $\nabla'V = (1 - ag'(\psi_*))\nabla'\psi_*$  and

$$\begin{aligned} (V_t - \beta_*V)^2 + |\nabla'V|^2 &= (-\beta_*\psi_* + a(\beta_* - \gamma)g(\psi_*))^2 + (1 - ag'(\psi_*))^2 |\nabla'\psi_*|^2 \\ &= (\beta_*^2\psi_*^2 + 2a\beta_*(\gamma - \beta_*)g(\psi_*)\psi_*) + (1 - 2ag'(\psi_*)) |\nabla'\psi_*|^2 + O(a^2 \|g(\psi)\|_{C^1}) \\ &= \beta_*^2\psi_*^2 + |\nabla'\psi_*|^2 + 2a \left( \beta_*(\gamma - \beta_*)\psi_*g(\psi_*) - g'(\psi_*) |\nabla'\psi_*|^2 \right) + O(a^2 \|g(\psi_*)\|_{C^1}). \end{aligned}$$

Therefore

$$\begin{aligned} &\left( (V_t - \beta_*V)^2 + |\nabla'V|^2 \right)^{\frac{p-2}{2}} \\ &= \left( \beta_*^2\psi_*^2 + |\nabla'\psi_*|^2 \right)^{\frac{p-2}{2}} \left[ 1 + (p-2)a \frac{\beta_*(\gamma - \beta_*)\psi_*g(\psi_*) - g'(\psi_*) |\nabla'\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2} \right] \\ &\quad + O(a^2 \|g(\psi)\|_{C^1}), \end{aligned}$$

and

$$\begin{aligned} &e^{\nu t} \left( (V_t - \beta_*V)^2 + |\nabla'V|^2 \right)^{\frac{q}{2}} \\ &= e^{\nu t} \left( \beta_*^2\psi_*^2 + |\nabla'\psi_*|^2 \right)^{\frac{q}{2}} \left[ 1 + qa \frac{\beta_*(\gamma - \beta_*)\psi_*g(\psi_*) - g'(\psi_*) |\nabla'\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2} \right] \\ &\quad + O(e^{\nu t} a^2 \|g(\psi_*)\|_{C^1}), \end{aligned}$$

thus

$$\begin{aligned} &\left( (V_t - \beta_*V)^2 + |\nabla'V|^2 \right)^{\frac{p-2}{2}} (V_t - \beta_*V) \\ &= -\beta_* \left( \beta_*^2\psi_*^2 + |\nabla'\psi_*|^2 \right)^{\frac{p-2}{2}} \psi_* + a(\beta_* - \gamma) \left( \beta_*^2\psi_*^2 + |\nabla'\psi_*|^2 \right)^{\frac{p-2}{2}} g(\psi_*) \\ &\quad - a\beta_*(p-2) \frac{\beta_*(\gamma - \beta_*)\psi_*g(\psi_*) - g'(\psi_*) |\nabla'\psi_*|^2}{(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2)^{\frac{4-p}{2}}} \psi_* + O(a^2 \|g(\psi_*)\|_{C^1}). \end{aligned}$$

Finally,

$$\begin{aligned} &- \left( \left( (V_t - \beta_*V)^2 + |\nabla'V|^2 \right)^{\frac{p-2}{2}} (V_t - \beta_*V) \right)_t \\ &= a \left[ (\gamma^2 - \beta_*\gamma) \left( \beta_*^2\psi_*^2 + |\nabla'\psi_*|^2 \right)^{\frac{p-2}{2}} g(\psi_*) \right. \\ &\quad \left. + \beta_*(p-2) \frac{\beta_*(\gamma^2 - \beta_*\gamma)\psi_*g(\psi_*) - \gamma g'(\psi_*) |\nabla'\psi_*|^2}{(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2)^{\frac{4-p}{2}}} \psi_* \right] + O(a^2 \|g(\psi_*)\|_{C^2}). \end{aligned} \tag{3.27}$$



Since

$$\begin{aligned}
& \left( (V_t - \beta_* V)^2 + |\nabla' V|^2 \right)^{\frac{p-2}{2}} \nabla' V = \\
& \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} (1 - a g'(\psi_*)) \left[ 1 + a(p-2) \frac{\beta_*(\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] \nabla' \psi_* \\
& \qquad \qquad \qquad + O(a^2 \|g(\psi_*)\|_{C^1}) \\
& = \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \nabla' \psi_* \\
& + a \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \left[ (p-2) \frac{\beta_*(\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} - g'(\psi_*) \right] \nabla' \psi_* \\
& \qquad \qquad \qquad + O(a^2 \|g(\psi_*)\|_{C^1}),
\end{aligned}$$

we get similarly

$$\begin{aligned}
& -\operatorname{div}' \left( \left( (V_t - \beta_* V)^2 + |\nabla' V|^2 \right)^{\frac{p-2}{2}} \nabla' V \right) = -\operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \nabla' \psi_* \right) \\
& -a \operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \left[ (p-2) \frac{\beta_*(\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} - g'(\psi_*) \right] \nabla' \psi_* \right) \\
& \qquad \qquad \qquad + O(a^2 \|g(\psi_*)\|_{C^2}).
\end{aligned} \tag{3.28}$$

Noting that

$$-\operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \nabla' \psi_* \right) \psi_* = \beta_* \Lambda_{\beta_*} \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \psi_*, \tag{3.29}$$

we obtain

$$e^{-\gamma t} \mathcal{Q}[V]$$

$$\begin{aligned}
& = \left[ (\gamma^2 - \beta_* \gamma) \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} g(\psi_*) + \beta_*(p-2) \frac{\beta_*(\gamma^2 - \beta_* \gamma) \psi_* g(\psi_*) - \gamma g'(\psi_*) |\nabla \psi_*|^2}{(\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2)^{\frac{4-p}{2}}} \psi_* \right] \\
& - \operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \left[ (p-2) \frac{\beta_*(\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} - g'(\psi_*) \right] \nabla' \psi_* \right) \\
& - \Lambda_{\beta_*} \left( (\gamma - \beta_*) \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} g(\psi_*) + \beta_*(p-2) \frac{\beta_*(\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{(\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2)^{\frac{4-p}{2}}} \psi_* \right) \\
& + e^{(\nu-\gamma)t} \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{q}{2}} \left[ 1 + qa \frac{\beta_*(\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] + O(a \|g(\psi_*)\|_{C^2}).
\end{aligned} \tag{3.30}$$

In this expression we have in particular

$$\begin{aligned}
& -\operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \left[ (p-2) \frac{\beta_*(\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} - g'(\psi_*) \right] \nabla' \psi_* \right) \\
&= (p-1) \operatorname{div}' \left[ g'(\psi_*) \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \nabla \psi_* \right] \\
&- \beta_* \operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-4}{2}} [(p-2) \beta_* \psi_* g'(\psi_*) + (p-2)(\gamma - \beta_*) g(\psi_*)] \psi_* \right) \\
&= (p-1) g''(\psi_*) \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} |\nabla \psi_*|^2 \\
&+ (p-1) g'(\psi_*) \operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \nabla \psi_* \right) \\
&- (p-2) \beta_* \operatorname{div}' \left[ \frac{((\gamma - \beta_*) g(\psi_*) \psi_* + \beta_* g'(\psi_*) \psi_*^2)}{\left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{4-p}{2}}} \nabla' \psi_* \right].
\end{aligned} \tag{3.31}$$

Using the equation (3.26) satisfied by  $\psi_*$ , it infers that

$$\begin{aligned}
& -\operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \left[ (p-2) \frac{\beta_*(\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} - g'(\psi_*) \right] \nabla' \psi_* \right) \\
&= (p-1) \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \left( g''(\psi_*) |\nabla' \psi_*|^2 - \beta_* \Lambda_{\beta_*} g'(\psi_*) \psi_* \right) \\
&- (p-2) \beta_* \operatorname{div}' \left[ \frac{((\gamma - \beta_*) g(\psi_*) \psi_* + \beta_* g'(\psi_*) \psi_*^2)}{\left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{4-p}{2}}} \nabla' \psi_* \right].
\end{aligned} \tag{3.32}$$

Plugging this identity into the expression (3.30), we obtain after some simplifications

$$e^{-\gamma t} \mathcal{Q}[V] = \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} g(\psi_*) \mathcal{Q}_1[V] + e^{(\nu-\gamma)t} R[V] + O(a \|g(\psi_*)\|_{C^2}), \tag{3.33}$$

where

$$R[V] = e^{\nu t} \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{q}{2}} \left[ 1 + q \frac{\beta_*(a' - \beta_* a) \psi_* g(\psi_*) - a g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right], \tag{3.34}$$

and

$$\begin{aligned}
\mathcal{Q}_1[V] &= (\gamma - \Lambda_{\beta_*})(\gamma - \beta_*) \left[ 1 + (p-2) \frac{\beta_*^2 \psi_*^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] - (p-1) \beta_* \Lambda_{\beta_*} \frac{\psi_* g'(\psi_*)}{g(\psi_*)} \\
&\quad + [(p-4) \beta_* \Lambda_{\beta_*} \psi_* - 2 \Delta' \psi_*] \left( \gamma - \beta_* \left( 1 - \frac{\psi_* g'(\psi_*)}{g(\psi_*)} \right) \right) \frac{\beta_* \psi_*}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \\
&\quad - (p-2) \left[ \frac{\psi_* g'(\psi_*)}{g(\psi_*)} ((\beta_* + 1) \gamma - \beta_* \Lambda_{\beta_*} + \beta_*) + \gamma - \beta_* + \beta_* \frac{\psi_*^2 g''(\psi_*)}{g(\psi_*)} \right] \frac{|\nabla' \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \\
&\quad + (p-1) \frac{g''(\psi_*)}{g(\psi_*)} |\nabla' \psi_*|^2.
\end{aligned} \tag{3.35}$$

In this expression the difficult term to deal with is  $[(p-4) \beta_* \Lambda_{\beta_*} \psi_* - 2 \Delta' \psi_*]$  since it has not a prescribed sign. However  $\Delta' \psi_* = O(\psi_*)$  by (6.19) in Appendix II.

*Step 2: The perturbation method and the computation with  $g(\psi_*) = \psi_*$ .* With such a choice of function  $g$

$$\begin{aligned}
\mathcal{Q}_1[V] &= (\gamma - \Lambda_{\beta_*})(\gamma - \beta_*) \left[ 1 + (p-2) \frac{\beta_*^2 \psi_*^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] - (p-1) \beta_* \Lambda_{\beta_*} \\
&\quad - (p-2) [(\gamma - \Lambda_{\beta_*}) \beta_* + 2\gamma] \frac{|\nabla' \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} + \gamma O(\psi_*^2).
\end{aligned} \tag{3.36}$$

Equivalently

$$\begin{aligned}
\mathcal{Q}_1[V] &= \left[ 1 + (p-2) \frac{\beta_*^2 \psi_*^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] (\gamma^2 - (\Lambda_{\beta_*} + \beta_*) \gamma) \\
&\quad - \gamma \left[ (p-2)(\beta_* + 2) \frac{|\nabla' \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} + O(\psi_*^2) \right]
\end{aligned}$$

and finally

$$\mathcal{Q}_1[V] = \left[ 1 + (p-2) \frac{\beta_*^2 \psi_*^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] \gamma [\gamma - (\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2)) + O(\psi_*^2)]. \tag{3.37}$$

Using the fact that  $\beta_* > \frac{N-1}{p-1}$  if  $1 < p < 2$  and  $1 < \beta_* < \frac{N-1}{p-1}$  if  $2 < p < N$  (see Theorem 6.1 in Appendix II), we have

$$\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2) \geq \begin{cases} \Lambda_{\beta_*} + \beta_*(p-1) & \text{if } p \geq 2 \\ N + 3(p-2) > N-3 & \text{if } 1 < p < 2. \end{cases} \tag{3.38}$$

When  $N = 2$ , we have explicitly  $\beta_* = \frac{1+2\sqrt{p^2-3p+3}}{3(p-1)}$  (see [9, Th 3.3]). Therefore for all  $N \geq 2$  and  $p > 1$ , there holds

$$\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2) > 0. \tag{3.39}$$

We fix  $\epsilon_0 > 0$  such that, whenever  $\psi_* \leq \epsilon_0$ , there holds

$$\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2) + O(\psi_*^2) > \frac{1}{2} (\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2)). \tag{3.40}$$

If we fix  $\gamma_0 > 0$  such that

$$\gamma_0 < \min \left\{ \frac{1}{2} (\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2)), \nu, \beta_* \right\}, \quad (3.41)$$

we obtain

$$\mathcal{Q}_1[V] \leq -\min\{1, p-1\} \gamma m^2 \quad \forall 0 < \gamma \leq \gamma_0, \quad (3.42)$$

whenever  $\psi_* \leq \epsilon_0$ , for some  $m$  depending only on  $p, q$  and  $N$  (through  $\psi_*$  and  $\nu$ ), which, in the same range of value of  $\psi_*$ , yields to

$$\left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} g(\psi_*) \mathcal{Q}_1[V] \leq -c_{17} \psi_* \quad \forall 0 < \gamma \leq \gamma_0, \quad (3.43)$$

for some  $c_{17} > 0$  depending on  $N, p, q$ . This estimate is valid whatever is  $p > 1$ , but only in a neighborhood of  $\psi_* = 0$ . If we replace  $g(\psi_*) = \psi_*$  by  $g_k(\psi_*) = \psi_* e^{-k\psi_*}$  for  $0 < k < 1$ , and denote by  $\mathcal{Q}_{1,k}[V]$  the corresponding expression of  $\mathcal{Q}_1[V]$  which becomes now  $\mathcal{Q}_{1,0}[V]$ . We define similarly  $\mathcal{Q}_k[V]$ , and  $\mathcal{Q}[V]$  becomes  $\mathcal{Q}_0[V]$ . Since  $g'_k(\psi_*) = e^{-k\psi_*} - k g_k(\psi_*)$  and  $g''_k = -2k e^{-k\psi_*} + k^2 g_k(\psi_*)$ , we obtain

$$\begin{aligned} \mathcal{Q}_{1,k}[V] &= \mathcal{Q}_{1,0}[V] + k(p-1)\beta_* \Lambda_{\beta_*} \psi_* + (p-1) \left( -\frac{2k}{\psi_*} + k^2 \right) |\nabla' \psi_*|^2 \\ &\quad + (2-p)\beta_* (-2k + k^2) \psi_* + O(\psi_*^2) \end{aligned} \quad (3.44)$$

Notice that  $\nabla' \psi_*$  vanishes only at the North pole  $e_N$ , thus there exists  $k_0 \in (0, 1]$  such that

$$k(1-p)\beta_* \Lambda_{\beta_*} \psi_* + (p-1) \left( \frac{2k}{\psi_*} - k^2 \right) |\nabla' \psi_*|^2 \geq \frac{1}{2} (2-p)_+ \beta_* (-2k + k^2) \psi_* \quad \forall k \leq k_0$$

whenever  $\psi_* \leq \epsilon_0$  which yields to

$$\left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} g_k(\psi_*) \mathcal{Q}_{1,k}[V] \leq -c_{18} k \quad \forall k \leq k_0 \quad (3.45)$$

for some  $c_{13} = c_{13}(N, p, q, \epsilon_0)$ . There exists  $c_{14} = c_{14}(N, p, q) > 0$  such that

$$\left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{q}{2}} \left[ 1 + q e^{\gamma t} \frac{\beta_* (\gamma - \beta_*) \psi_* g_k(\psi_*) - g'_k(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] \leq c_{14} \quad (3.46)$$

in  $S_+^{N-1} \times (-\infty, \ln \delta]$ . Moreover

$$O(a \|g(\psi_*)\|_{C^2}) \leq e^{\gamma t} \tilde{c}_k \quad (3.47)$$

for some  $\tilde{c}_k = \tilde{c}_k(N, p, q) > 0$ . We derive from (3.45)-(3.47)

$$e^{-\gamma t} \mathcal{Q}_k[V] \leq -c_{13} k + c_{14} e^{(\nu-\gamma)t} + e^{\gamma t} \tilde{c}_k \quad \forall k \leq k_0 \quad (3.48)$$

Thus there exists  $T_k \leq \ln \delta$  such that  $\mathcal{Q}_k[V] \leq 0$ , for all  $t \leq T_k$  and provided  $\psi_* \leq \epsilon_0$ . This local estimate will be used in the construction of the subsolution when  $p \geq 2$ .

*Step 3: The case  $1 < p < 2$ .* Since the function  $\psi^*$  depends only on the azimuthal angle  $\theta \in (0; \frac{\pi}{2}]$  we will write  $\psi_*(\sigma) = \psi_*(\theta)$  and  $\nabla' \psi_*(\sigma) = \psi_{*\theta}(\theta) \mathbf{n}$  where  $\mathbf{n}$  is the downward unit vector tangent to  $S^{N-1}$  in the hyperplane going through  $\sigma$  and the poles. From (6.8),

$$(p-4)\beta_*\Lambda_{\beta_*}\psi_* - 2\Delta'\psi_* = (p-2) \left( \beta_*\Lambda_{\beta_*}\psi_* + 2\frac{\beta_*^2\psi_* + \psi_{*\theta\theta}}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \right), \quad (3.49)$$

since  $\psi_{*\theta}^2 = |\nabla' \psi_*|^2$  and thus

$$\begin{aligned} & ((p-4)\beta_*\Lambda_{\beta_*}\psi_* - 2\Delta'\psi_*) \frac{\beta_*\gamma\psi_*}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \\ &= (p-2)\gamma \left( \Lambda_{\beta_*} \frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} + 2\beta_* \frac{\beta_*^2\psi_*^2 + \psi_{*\theta\theta}\psi_*}{(\beta_*^2\psi_*^2 + \psi_{*\theta}^2)^2} \right). \end{aligned} \quad (3.50)$$

From Theorem 6.1-Step 4 in Appendix II, we know that  $\beta_*^2\psi_* + \psi_{*\theta\theta} \geq 0$ , thus the contribution of this term to  $\mathcal{Q}_1[V]$  is nonpositive. We replace this expression in  $\mathcal{Q}_1[V]$  with  $g(\psi_*) = \psi_*$  and obtain

$$\begin{aligned} \mathcal{Q}_1[V] &= (\gamma - \Lambda_{\beta_*})(\gamma - \beta_*) \left( 1 + (p-2) \frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \right) - \Lambda_{\beta_*}\beta_*(p-1) \\ &\quad + (p-2)\gamma\Lambda_{\beta_*} \frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} - (p-2) \left( (\beta_* + 2)\gamma - \Lambda_{\beta_*}\beta_* \right) \frac{\psi_{*\theta}^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \\ &\quad + 2\beta_*(p-2) \frac{\beta_*^2\psi_*^2 + \psi_{*\theta\theta}\psi_*}{(\beta_*^2\psi_*^2 + \psi_{*\theta}^2)^2} \gamma \\ &\leq \gamma \left( 1 + (p-2) \frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \right) (\gamma - \Lambda_{\beta_*} - \beta_*) - (p-2)\gamma \frac{(\beta_* + 2)\psi_{*\theta}^2 - \Lambda_{\beta_*}\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \\ &\leq \gamma \left( 1 + (p-2) \frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \right) \left( \gamma - \left( \Lambda_{\beta_*} + \beta_* + (p-2) \frac{(\beta_* + 2)\psi_{*\theta}^2 - \Lambda_{\beta_*}\beta_*^2\psi_*^2}{(p-1)\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \right) \right). \end{aligned} \quad (3.51)$$

We can write

$$\begin{aligned} & \Lambda_{\beta_*} + \beta_* + (p-2) \frac{(\beta_* + 2)\psi_{*\theta}^2 - \Lambda_{\beta_*}\beta_*^2\psi_*^2}{(p-1)\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \\ &= \frac{(\Lambda_{\beta_*} + (p-1)\beta_*)\beta_*^2\psi_*^2 + (\Lambda_{\beta_*} + \beta_*(p-1) + 2(p-2))\psi_{*\theta}^2}{(p-1)\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \\ &\geq c_{15} (\Lambda_{\beta_*} + \beta_*(p-1) + 2(p-2)) \end{aligned} \quad (3.52)$$

for some positive constant  $c_{15}$ . This expression  $\Lambda_{\beta_*} + \beta_*(p-1) + 2(p-2)$  is always positive: obviously if  $N \geq 3$  and by using the explicit expression of  $\beta_*$  if  $N = 2$ . Thus there exists  $\gamma_0$  and  $c_{16} > 0$  such that  $\mathcal{Q}_1[V] < -c_{16}$  for  $0 < \gamma \leq \gamma_0$ . The perturbation method of Step 2, is valid in the whole range of values of  $\psi_*$  and we derive from (3.42)-(3.43) that (3.48) holds for all  $k \leq k_0$  and  $t \leq T_k$ . Therefore  $\mathcal{Q}_k[V] \leq 0$ .

*Step 4: The case  $p \geq 2$ .* For  $c > 0$  to be fixed and  $\psi_* \geq \epsilon_0$ ,  $\gamma \in (0, \gamma_0]$ , we take  $g(\psi_*) = c\psi_*^{1-\frac{\gamma}{\beta_*}}$ . Then

we derive from (3.35):

$$\begin{aligned}
\mathcal{Q}_1[V] &= (\gamma - \Lambda_{\beta_*})(\gamma - \beta_*) \frac{(p-1)\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} - (p-1)\beta_* \Lambda_{\beta_*} \left(1 - \frac{\gamma}{\beta_*}\right) \\
&\quad - (p-1) \frac{\gamma(\beta_* - \gamma)}{\beta_*^2} \psi_*^{-1-\frac{\gamma}{\beta_*}} |\nabla' \psi_*|^2 - (p-2)(\beta_* - \gamma)(\gamma - \Lambda_{\beta_*}) \frac{|\nabla' \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \\
&= (1-p) \left[ \gamma(\beta_* - \gamma) + \frac{\gamma(\beta_* - \gamma)}{\beta_*^2} \psi_*^{-1-\frac{\gamma}{\beta_*}} |\nabla' \psi_*|^2 \right].
\end{aligned} \tag{3.53}$$

For  $k \leq k_0$  we fix  $c$  such that  $c\epsilon_0^{1-\frac{\gamma}{\beta_*}} = \epsilon_0 e^{-k\epsilon_0} \iff c = \epsilon_0^{\frac{\gamma}{\beta_*}} e^{-k\epsilon_0}$  and we define  $g$  by

$$g(\psi_*) = \min \left\{ \psi_* e^{-k\psi_*}, \epsilon_0^{\frac{\gamma}{\beta_*}} e^{-k\epsilon_0} \psi_*^{1-\frac{\gamma}{\beta_*}} \right\} = \begin{cases} \psi_* e^{-k\psi_*} & \text{if } 0 \leq \psi_* \leq \epsilon_0 \\ \epsilon_0^{\frac{\gamma}{\beta_*}} e^{-k\epsilon_0} \psi_*^{1-\frac{\gamma}{\beta_*}} & \text{if } \epsilon_0 \leq \psi_* \leq 1, \end{cases} \tag{3.54}$$

and we set  $V(t, \sigma) = \psi^*(\sigma) - a(t)g(\psi_*(\sigma))$  with  $(t, \sigma) \in (-\infty, T_k] \times S_+^{N-1}$  and define  $\tilde{u}(r, \sigma) = r^{-\beta_*}(\psi^*(\sigma) - a(\ln r)g(\psi_*(\sigma)))$  accordingly for  $(r, \sigma) \in (-\infty, e^{T_k}] \times S_+^{N-1}$ . Since  $\psi_*$  is a decreasing function the coincidence set  $\{\sigma \in S_+^{N-1} : \psi_*(\sigma) = \epsilon_0\}$  is a circular cone  $\Sigma_{\theta_0}$  with vertex 0, axis  $\mathbf{e}_N$  and angle  $\theta_0$ . We set  $R_0 = e^{T_k}$

$$\begin{aligned}
\Gamma_1 &= \left\{ x = (r, \theta) \in B_{R_0}^+ : \theta_0 < \theta < \frac{\pi}{2} \right\} = \left\{ (r, \sigma) \in [0, R_0) \times S_+^{N-1} : 0 < \psi_*(\sigma) < \epsilon_0 \right\}, \\
\Gamma_2 &= \left\{ x = (r, \theta) \in B_{R_0}^+ : 0 < \theta < \theta_0 \right\} = \left\{ (r, \sigma) \in [0, R_0) \times S_+^{N-1} : \epsilon_0 < \psi_*(\sigma) < 1 \right\},
\end{aligned}$$

and define

$$\begin{aligned}
\tilde{u}(r, \sigma) &= r^{-\beta_*} (\psi_*(\sigma) - r^\gamma g(\psi_*(\sigma))) \\
&= \begin{cases} u_1(r, \sigma) = r^{-\beta_*} (1 - r^\gamma e^{-k\psi_*(\sigma)}) \psi_*(\sigma) & \text{if } (r, \theta) \in \Gamma_1 \\ u_2(r, \sigma) = r^{-\beta_*} \left(1 - r^\gamma \epsilon_0^{\frac{\gamma}{\beta_*}} e^{-k\epsilon_0} (\psi_*(\sigma))^{1-\frac{\gamma}{\beta_*}}\right) \psi_*(\sigma) & \text{if } (r, \theta) \in \Gamma_2. \end{cases}
\end{aligned}$$

The function  $\tilde{u}$  is a subsolution separately on  $\Gamma_1$  and  $\Gamma_2$  and is Lipschitz continuous in  $\bar{\Omega} \setminus \{0\}$ . If we denote by  $g_1$  and  $g_2$  the restriction of  $g$  to  $\Gamma_1$  and  $\Gamma_2$  respectively, that is to  $\Omega_1$  and  $\Omega_2$ , then  $g_1'(\epsilon_0) > g_2'(\epsilon_0) > 0$ . Let  $\zeta \in C_c^1(B_{R_0}^+)$  which vanishes in neighborhoods of 0 and  $\partial B_{R_0}^+$ ,  $\zeta \geq 0$ , then

$$\int_{\Gamma_i} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla \zeta dx + \int_{\Omega_i} |\nabla \tilde{u}|^q \zeta dx \leq \int_{\Sigma_{\theta_0}} |\nabla u_i|^{p-2} \partial_{\mathbf{n}_i} u_i \zeta dS, \tag{3.55}$$

where  $\mathbf{n}_i$  is the normal unit vector on  $\Sigma_{\theta_0}$  outward from  $\Gamma_i$ . Actually,  $\mathbf{n}_2 = -\mathbf{n}_1 = \mathbf{n}$  thus

$$\nabla \tilde{u} = \tilde{u}_r \mathbf{e} + r^{-\beta_*-1} (1 - r^\gamma g'(\psi_*)) \nabla' \psi_* = \tilde{u}_r \mathbf{e} + r^{-\beta_*-1} (1 - r^\gamma g'(\psi_*)) \psi_{*\theta} \mathbf{n}.$$

and on  $\Sigma_{\theta_0}$ ,

$$\nabla \tilde{u} = \begin{cases} \tilde{u}_r \mathbf{e} - r^{-\beta_*-1} (1 - r^\gamma g_1'(\epsilon_0)) \psi_{*\theta} \mathbf{n} & \text{in } \Gamma_1 \\ \tilde{u}_r \mathbf{e} + r^{-\beta_*-1} (1 - r^\gamma g_2'(\epsilon_0)) \psi_{*\theta} \mathbf{n} & \text{in } \Gamma_2 \end{cases}$$

Therefore

$$\begin{aligned} & |\nabla u_1|^{p-2} \partial_{\mathbf{n}_1} u_1 \\ &= -r^{-\beta_*-1} (1 - r^\gamma g'_1(\epsilon_0)) (\tilde{u}_r^2 + r^{-2\beta_*-2} (1 - r^\gamma g'_1(\epsilon_0))^2 \psi_{*\theta}^2)^{\frac{p-2}{2}} \psi_{*\theta} \quad \text{in } \Gamma_1 \end{aligned}$$

and

$$\begin{aligned} & |\nabla u_2|^{p-2} \partial_{\mathbf{n}_2} u_2 \\ &= r^{-\beta_*-1} (1 - r^\gamma g'_2(\epsilon_0)) (\tilde{u}_r^2 + r^{-2\beta_*-2} (1 - r^\gamma g'_2(\epsilon_0))^2 \psi_{*\theta}^2)^{\frac{p-2}{2}} \psi_{*\theta} \quad \text{in } \Gamma_2. \end{aligned}$$

By adding the two inequalities (3.55)

$$\int_{\Omega} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla \zeta dx + \int_{\Omega} |\nabla \tilde{u}|^q \zeta dx \leq \int_{\Sigma_{\theta_0}} \left( |\nabla u_1|^{p-2} \partial_{\mathbf{n}_1} u_1 + |\nabla u_2|^{p-2} \partial_{\mathbf{n}_2} u_2 \right) \zeta dS. \quad (3.56)$$

By monotonicity of the function  $X \mapsto (\tilde{u}_r^2 + X^2)^{\frac{p}{2}}$  and since

$$r^{-\beta_*-1} (1 - r^\gamma g'_2(\epsilon_0)) \geq r^{-\beta_*-1} (1 - r^\gamma g'_1(\epsilon_0)) \geq 0,$$

we derive

$$\begin{aligned} & r^{-\beta_*-1} (1 - r^\gamma g'_2(\epsilon_0)) (\tilde{u}_r^2 + r^{-2\beta_*-2} (1 - r^\gamma g'_2(\epsilon_0))^2 \psi_{*\theta}^2)^{\frac{p-2}{2}} \\ & \geq r^{-\beta_*-1} (1 - r^\gamma g'_1(\epsilon_0)) (\tilde{u}_r^2 + r^{-2\beta_*-2} (1 - r^\gamma g'_1(\epsilon_0))^2 \psi_{*\theta}^2)^{\frac{p-2}{2}} \end{aligned}$$

We derive that the right-hand side of (3.56) is nonpositive because  $\psi_{*\theta} \leq 0$ , and therefore  $\tilde{u}$  is a positive subsolution of (1.1) in  $B_{R_0}^+$  dominated by  $\Psi_*$  and satisfying (3.24).  $\square$

*Proof of Theorem 3.6.* Let  $M = \max\{\Psi_*(x) : x \in \partial B_{R_0}^+\}$ , then  $M = R_0^{-\beta_*}$ . The function  $u^*$  defined by

$$u^*(x) = \begin{cases} (\tilde{u}(x) - M)_+ & \text{if } x \in B_{R_0}^+ \\ 0 & \text{if } x \in \Omega \setminus B_{R_0}^+, \end{cases}$$

is indeed a subsolution of (1.1) in whole  $\Omega$  where it satisfies  $u^* \leq \Psi_*$  and it vanishes on  $\partial\Omega \setminus \{0\}$ . Since  $\Phi_*$  is a positive  $p$ -harmonic function in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies (3.20), it is supersolution of (1.1) and therefore it dominates  $u^*$ . Therefore there exists a solution  $u$  of (1.1) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies  $u^* \leq u \leq \Phi_*$ . This implies that (3.19) holds with  $k = 1$  and we conclude with Lemma 3.8. This ends the proof of Lemma 3.9.  $\square$

When  $p = N$  the statement of Theorem 3.6 holds without the flatness assumption on  $\partial\Omega$ . The proof of the next theorem is an easy adaptation to the one of Theorem 3.6, provided Lemma 3.7, Lemma 3.8 and Lemma 3.9 are modified accordingly.

**Theorem 3.10.** *Assume  $N - 1 < q < N - \frac{1}{2}$  and  $\Omega$  be a bounded  $C^2$  domain such that  $0 \in \partial\Omega$ . Then for any  $k > 0$  there exists a unique positive solution  $u := u_k$  of (3.17) in  $\Omega$ , which belongs to  $C^1(\bar{\Omega} \setminus \{0\})$ , vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies uniformly with respect to  $\sigma \in S_+^{N-1}$*

$$\lim_{\substack{x \rightarrow 0 \\ x/|x| \rightarrow \sigma}} |x| u_k(x) = k\psi_*(\sigma). \quad (3.57)$$

Since  $p = N$ , then  $\beta_* = 1$  and  $\psi_*(\sigma) = \frac{x_N}{|x|} = \cos \theta_{N-1}$  with the identification of  $\sigma$  and  $\theta_{N-1} := \theta$ . In a more intrinsic manner (3.57) can be written under the form

$$\lim_{\substack{x \rightarrow 0 \\ x \in \Omega}} |x|^2 \frac{u_k(x)}{d(x)} = k. \quad (3.58)$$

We recall that if  $\omega \in \mathbb{R}^N$  and  $\mathcal{I}_\omega$  denotes the inversion of center  $\omega$  and power 1, i.e.  $\mathcal{I}_\omega(x) = \omega + \frac{x-\omega}{|x-\omega|^2}$ , then  $\tilde{u} = u \circ \mathcal{I}_\omega$  satisfies (3.18).

**Lemma 3.11.** *Assume  $\Omega$  be a bounded  $C^2$  domain such that  $0 \in \partial\Omega$ . Then there exists a unique  $N$ -harmonic function  $\Phi_*$  in  $\Omega$ , which vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies*

$$\lim_{\substack{x \rightarrow 0 \\ x/|x| \rightarrow \sigma}} |x| \Phi_*(x) = \psi_*(\sigma), \quad (3.59)$$

uniformly with respect to  $\sigma \in S_+^{N-1}$ .

*Proof.* Uniqueness is standard. Let  $\omega = -\mathbf{e}_N \in \overline{\Omega}^c$ , with the notations of the proof of Theorem 3.5,  $\omega' = -\omega$ ,  $a = -\frac{1}{2}\mathbf{e}_N$  and  $a' = -a$ . We can assume that the balls  $B_{\frac{1}{2}}(a)$  and  $B_{\frac{1}{2}}(a')$  are tangent to  $\partial\Omega$  at 0 and respectively subset of  $\Omega^c$  and  $\Omega$ . The function  $x \mapsto \Psi(x) = -\frac{x_N}{|x|^2}$  which is  $N$ -harmonic in  $\mathbb{R}^N$  and vanishes on  $\partial\mathbb{R}^N \setminus \{0\}$  is transformed by the inversion  $\mathcal{I}_{\omega'}$  of center  $\omega'$  and power 1 into the function  $\Psi_{\omega'} = \Psi \circ \mathcal{I}_{\omega'}$  which is positive and  $N$ -harmonic in  $B_{\frac{1}{2}}(a')$  and vanishes on  $\partial B_{\frac{1}{2}}(a') \setminus \{0\}$ . The function  $\hat{\Psi} = -\Psi$  which is  $N$ -harmonic in  $\mathbb{R}_+^N$  and vanishes on  $\partial\mathbb{R}_+^N \setminus \{0\}$  is transformed by the inversion  $\mathcal{I}_\omega$  of center  $\omega$  and power 1 into the function  $\Psi_\omega = \hat{\Psi} \circ \mathcal{I}_\omega$  which is positive and  $N$ -harmonic in  $B_{\frac{1}{2}}^c(a)$  and vanishes on  $\partial B_{\frac{1}{2}}(a) \setminus \{0\}$ . For  $\epsilon > 0$  we denote by  $\Phi_\epsilon$  the solution of

$$\begin{aligned} -\Delta_N \Phi_\epsilon &= 0 && \text{in } \Omega \cap B_\epsilon^c \\ \Phi_\epsilon &= 0 && \text{in } (B_{\frac{1}{2}}^c(a') \cap \partial B_\epsilon) \cup (\partial\Omega \cap B_\epsilon^c) \\ \Phi_\epsilon &= \Psi_{\omega'} && \text{in } B_{\frac{1}{2}}(a') \cap \partial B_\epsilon. \end{aligned} \quad (3.60)$$

If  $0 < \epsilon' < \epsilon$ ,  $\Phi_{\epsilon'} \geq \Psi_{\omega'}$  in  $B_{\frac{1}{2}}(a') \cap \partial B_{\epsilon'}$ , thus  $\Phi_{\epsilon'} \geq \Phi_\epsilon$  in  $\Omega \cap B_{\epsilon'}^c$ . We also denote by  $\hat{U}_\epsilon$  the solution of

$$\begin{aligned} -\Delta_N \hat{\Phi}_\epsilon &= 0 && \text{in } \Omega \cap B_\epsilon^c \\ \hat{\Phi}_\epsilon &= 0 && \text{in } \partial\Omega \cap B_\epsilon^c \\ \hat{\Phi}_\epsilon &= \Psi_\omega && \text{in } \Omega \cap \partial B_\epsilon^c. \end{aligned} \quad (3.61)$$

In the same way as above

$$0 < \epsilon' < \epsilon \implies \hat{\Phi}_{\epsilon'} \leq \hat{\Phi}_\epsilon \quad \text{in } \Omega \cap \partial B_{\epsilon'}^c$$

Using the explicit form of  $\Psi$ ,  $\mathcal{I}_\omega : x \mapsto \omega + \frac{x-\omega}{|x-\omega|^2}$  and  $\mathcal{I}_{\omega'} : x \mapsto \omega' + \frac{x-\omega'}{|x-\omega'|^2}$  we see that

$$\Psi_{\omega'} \lfloor_{B_{\frac{1}{2}}(a') \cap \partial B_\epsilon} \leq \frac{1+\epsilon}{1-\epsilon} \Psi_\omega \lfloor_{B_{\frac{1}{2}}(a') \cap \partial B_\epsilon},$$

thus

$$\Phi_\epsilon \leq \frac{1+\epsilon}{1-\epsilon} \hat{\Phi}_\epsilon \quad \text{in } \Omega \cap B_\epsilon^c.$$



Letting  $\epsilon \rightarrow 0$  we conclude that  $\Phi_\epsilon$  converges uniformly in  $\bar{\Omega} \setminus \{0\}$  to  $\Phi_*$  which vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies (3.59).  $\square$

The proof of the next statement is similar to the one of Lemma 3.8 up to some minor modifications, so we omit it.

**Lemma 3.12.** *Let the assumptions on  $q$  and  $\Omega$  of Theorem 3.10 be satisfied. If for some  $k > 0$  there exists a solution  $u_k$  of (3.17) in  $\Omega$ , which belongs to  $C^1(\bar{\Omega} \setminus \{0\})$ , vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies (3.57), then for any  $k > 0$  there exists such a solution.*

**Lemma 3.13.** *Under the assumptions of Theorem 3.10 there exists a Lipschitz continuous nonnegative subsolution  $\tilde{u}$  of (3.17) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ , is smaller than  $\Phi_*$  and satisfies*

$$\lim_{\substack{x \rightarrow 0 \\ x/|x| \rightarrow \sigma}} |x| \tilde{u}(x) = \sigma, \quad (3.62)$$

uniformly with respect to  $\sigma \in S_+^{N-1}$ .

*Proof.* Let  $\tau > 0$  to be fixed and let  $w$  be the solution of

$$-\Delta_N w + |\nabla w|^q = 0 \quad \text{in } B_2^- \quad (3.63)$$

which vanishes on  $\partial B_2^- \setminus \{0\}$  and satisfies

$$\lim_{\substack{x \rightarrow 0 \\ x/|x| \rightarrow \sigma}} |x| w(x) = \sigma \quad (3.64)$$

in the  $C^1$ -topology of  $S_+^{N-1}$ . Its existence follows from Theorem 3.6 and this function is dominated by the N-harmonic function  $\Phi_*$  corresponding to this domain, obtained in Lemma 3.11. By  $\mathcal{I}_{\omega'}$ , the half-ball  $B_2^-$  is transform into the lunule  $G = B_{\frac{1}{2}}(a') \setminus B_{\frac{2}{3}}(\frac{4}{3}\omega')$  and  $\tilde{w} = w \circ \mathcal{I}_{\omega'}$  satisfies

$$-\Delta_N \tilde{w} + |x - \omega'|^{2(q-N)} |\nabla \tilde{w}|^q = 0 \quad \text{in } G. \quad (3.65)$$

Since  $|x - \omega'| \leq 1$  in  $G$ ,  $-\Delta_N \tilde{w} + |\nabla \tilde{w}|^q \leq 0$  in  $G$ . We extend  $\tilde{w}$  by 0 in  $\Omega \setminus G$  and the resulting function  $\tilde{u}$  is a subsolution of (3.17) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ , is smaller than the N-harmonic function  $\Phi_*$  obtained in Lemma 3.11, and satisfies (3.62).  $\square$

## 4 Classification of boundary singularities

We assume that  $\Omega \subset \mathbb{R}^N$  is a  $C^2$  domain and  $0 \in \partial\Omega$ . Furthermore, in order to avoid extremely technical computations, we shall assume either that  $\partial\Omega$  is flat near 0 or  $p = N$ . We suppose that the tangent plane to  $\partial\Omega$  at 0 is  $\partial\mathbb{R}_+^N = \{x = (x', 0)\}$  and the normal inward unit vector at 0 is  $\mathbf{e}_N$ , therefore  $\mathbf{n} = -\mathbf{e}_N$  in the sequel. We denote by  $\omega_{s_+^{N-1}}$  the unique positive solution of (3.1) in  $S_+^{N-1}$  and by  $U_{s_+^{N-1}}$  the corresponding singular solution of (1.1) in  $\mathbb{R}_+^N$  defined by

$$U_{s_+^{N-1}}(x) = |x|^{-\beta q} \omega_{s_+^{N-1}}\left(\frac{x}{|x|}\right). \quad (4.66)$$

We recall that  $\psi_*$  is the unique positive solution of (3.2) with maximum 1 and  $\Psi_*$  the corresponding  $p$ -harmonic function

$$\Psi_*(x) = |x|^{-\beta_*} \psi_*\left(\frac{x}{|x|}\right). \quad (4.67)$$

#### 4.1 The case $1 < p < N$

The first statement points out the link between weak and strong singularities.

**Proposition 4.1.** *Under the assumptions of Theorem 3.6 there exists  $\lim_{k \rightarrow \infty} u_k = u_\infty$  which is the unique element of  $C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$  which vanishes on  $\partial\Omega \setminus \{0\}$ , satisfies (1.1) in  $\Omega$  and*

$$\lim_{x \rightarrow 0} \frac{u_\infty(x)}{U_{s_+^{N-1}}(x)} = 1. \quad (4.68)$$

*Proof.* Uniqueness follows from (4.68) and the maximum principle. For existence, since the mapping  $k \mapsto u_k$  is increasing and  $u_k \leq U_{s_+^{N-1}}$ , there exists  $\lim_{k \rightarrow \infty} u_k := u_\infty \leq U_{s_+^{N-1}}$  and  $u_\infty \in C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$ . It vanishes on  $\partial B_\delta^+ \setminus \{0\}$  and satisfies (1.1) in  $B_\delta^+$ . In order to take into account the domain  $B_\delta^+$  in the notations, we set  $u_k = u_{k,\delta}$ . Since the mapping  $\delta \mapsto u_{k,\delta}$  is also increasing and  $u_{k,\delta} \leq k\Psi_*$ , there also exists  $\lim_{\delta \rightarrow \infty} u_{k,\delta} := u_{k,\infty} \leq k\Psi_*$ . Then, for all  $\ell > 0$ ,

$$T_\ell[u_{k,\delta}](x) = \ell^{\beta_q} u_{k,\delta}(\ell x) = u_{k\ell^{\beta_q}, \ell^{-1}\delta}(x). \quad (4.69)$$

Letting  $k \rightarrow \infty$ , we obtain

$$T_\ell[u_{\infty,\delta}](x) = \ell^{\beta_q} u_{\infty,\delta}(\ell x) = u_{\infty, \ell^{-1}\delta}(x), \quad (4.70)$$

and letting  $\delta \rightarrow \infty$ , we obtain

$$T_\ell[u_{\infty,\infty}](x) = \ell^{\beta_q} u_{\infty,\infty}(\ell x) = u_{\infty,\infty}(x). \quad (4.71)$$

This implies that

$$u_{\infty,\infty}(r, \sigma) = r^{-\beta_q} \omega'(r, \sigma), \quad (4.72)$$

and  $\omega'$  is a positive solution of problem (3.1). Therefore  $\omega' = \omega_{s_+^{N-1}}$  by Theorem 3.2. If we let  $\ell \rightarrow 0$  in (4.69) and take  $|x| = 1$ ,  $x = \sigma$ , we derive

$$\lim_{\ell \rightarrow 0} \ell^{\beta_q} u_{\infty,\delta}(\ell, \sigma) = \lim_{\ell \rightarrow 0} u_{\infty, \ell^{-1}\delta}(1, \sigma) = u_{\infty,\infty}(1, \sigma) = \omega_{s_+^{N-1}}(\sigma). \quad (4.73)$$

This convergence holds in  $C^1(\overline{S_+^{N-1}})$  because of Lemma 2.5. This implies (4.68).  $\square$

The main classification result is as follows.

**Theorem 4.2.** *Assume  $1 < p < N$ ,  $p - 1 < q < q^*$  and  $\partial\Omega \cap B_\delta = \{x = (x', 0) : |x'| < \delta\}$ , for some  $\delta > 0$ . If  $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$  is a positive solution of (1.1) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ , then we have the following alternative:*

(i) either there exists  $k \geq 0$  such that

$$\lim_{x \rightarrow 0} \frac{u(x)}{\Psi_*(x)} = k, \quad (4.74)$$

(ii) or

$$\lim_{x \rightarrow 0} \frac{u(x)}{U_{s_+^{N-1}}(x)} = 1. \quad (4.75)$$

*Proof. Step 1.* Assume

$$\liminf_{x \rightarrow 0} \frac{u(x)}{\Psi_*(x)} < \infty, \quad (4.76)$$

then we claim that (4.74) holds. We first note that if (4.76) holds, there also holds

$$\liminf_{x \rightarrow 0} \frac{u(x)}{u_1(x)} < \infty, \quad (4.77)$$

where  $u_1$  is the solution of (1.1) obtained in Theorem 3.6 with  $k = 1$ . If  $\{x_n\}$  is converging to 0 and such that for some  $k > 0$

$$\liminf_{x \rightarrow 0} \frac{u(x)}{u_1(x)} = k = \lim_{n \rightarrow \infty} \frac{u(x_n)}{u_1(x_n)},$$

there also holds by the boundary Harnack inequality (2.38) applied to both  $u$  and  $u_1$ ,

$$\frac{u(x_n)}{u_1(x_n)} = \frac{u(x_n)}{d(x_n)} \frac{d(x_n)}{u_1(x_n)} \geq c_5^{-2} \frac{u(x)}{u_1(x)} \quad \forall x \text{ s.t. } |x| = |x_n|.$$

This implies in particular

$$u(x) \leq c_5^2 (k + \epsilon_n) u_1(x) \quad \forall x \text{ s.t. } |x| = |x_n|$$

where  $\{\epsilon_n\}$  is converging to  $0_+$ , and by the comparison principle

$$u(x) \leq K u_1(x) \quad \forall x \in \mathbb{R}_+^N \text{ s.t. } |x_n| \leq |x| \leq \frac{\delta}{2},$$

for some  $K > 0$  and all  $n \in \mathbb{N}_*$ . Therefore

$$\limsup_{x \rightarrow 0} \frac{u(x)}{u_1(x)} < \infty. \quad (4.78)$$

We can assume that  $k \neq 0$ , otherwise (4.74) holds with  $k = 0$  and actually  $u$  remains bounded near 0. As a consequence of the Hopf Lemma and  $C^1$  regularity, there exists  $K > 0$  such that

$$u(x) \leq K \Psi_*(x) \quad \forall x \in B_{\frac{\delta}{2}}^+. \quad (4.79)$$

Let  $m = \max\{u(x) : |x| = \delta\}$ . For  $0 < \tau < \delta$  we denote by  $k_\tau$  the minimum of the  $\kappa > 0$  such that  $u(x) \leq \kappa \Psi_*(x) + m$  for  $\tau \leq |x| \leq \delta$ . Then  $u(x) \leq k_\tau \Psi_*(x) + m$ , and either the graphs of the mappings  $u(\cdot)$  and  $k_\tau \Psi_*(\cdot) + m$  are tangent at some  $x_\tau \in B_\delta^+ \setminus \overline{B}_\tau^+$ , or they are tangent on the boundary of the domain, and the only possibility is that they are tangent on  $|x| = \tau$ . Since

$$|\nabla \Psi_*(x)|^2 = |x|^{-2(\beta_*+1)} (\beta_*^2 \psi_*^2 + |\nabla \psi_*|^2),$$

it never vanishes. If we set  $w = u - (k_\tau \Psi_*(x) + m)$ , then

$$-\mathcal{L}w + |\nabla w|^q = 0 \quad (4.80)$$

where the operator

$$\mathcal{L} = \sum_{i,j} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial}{\partial x_j} \right)$$

is uniformly elliptic in a neighborhood of  $x_\tau$  (see [6, Lemma 1.3]). Furthermore  $w \leq 0$  and  $w(x_\tau) = 0$  by the strong maximum principle  $\nabla u(x_\tau)$  must vanish, which contradicts the fact that  $\nabla u(x_\tau) = \nabla w(x_\tau)$  by the tangency condition, and  $\nabla w(x_\tau) \neq 0$ . Therefore  $|x_\tau| = \tau$  and  $x_\tau \notin \partial \mathbb{R}_+^N$ . If  $\tau' < \tau$ ,  $k_\tau \leq k_{\tau'}$ , and we set  $k = \lim_{\tau \rightarrow 0} k_\tau$ , which is finite because of (4.79). There exists  $\{\tau_n\}$  such that  $\sigma_n := \tau^{-1} x_{\tau_n} \rightarrow \sigma_0$ . Furthermore

$$r^{\beta_*} u(r, \sigma) \leq k_\tau \psi_*(\sigma) + m r^{\beta_*} \quad \text{if } \tau \leq r \leq \delta \quad \text{and} \quad \tau^{\beta_*} u(\tau, \sigma_\tau) = k_\tau \psi_*(\sigma_\tau) + m \tau^{\beta_*}. \quad (4.81)$$

Put

$$u_\tau(x) = \tau^{\beta_*} u(\tau x) \quad (4.82)$$

Then

$$-\Delta_p u_\tau + \tau^{p-q-\beta_*(p+1-q)} |\nabla u_\tau|^q = 0 \quad \text{in } B_{\frac{\delta}{\tau}}^+ \setminus \{0\}$$

and, by (4.79),

$$0 \leq u_\tau(x) \leq K |x|^{-\beta_*} \quad \text{in } B_{\frac{\delta}{2\tau}}^+ \setminus \{0\}.$$

By Lemma 2.5, the set of functions  $\{u_\tau(\cdot)\}$  is relatively compact in the  $C_{loc}^1$  topology of  $\overline{\mathbb{R}_+^N} \setminus \{0\}$ . Therefore, as  $q < q^*$ , there exist a sequence  $\{\tau'_n\} \subset \{\tau_n\}$  converging to 0, and a positive  $p$ -harmonic function  $v$  in  $\mathbb{R}_+^N$ , continuous in  $\overline{\mathbb{R}_+^N} \setminus \{0\}$  and vanishing on  $\partial \mathbb{R}_+^N \setminus \{0\}$ , such that  $u_{\tau'_n} \rightarrow v$ , and  $v$  satisfies (4.79) in  $\overline{\mathbb{R}_+^N} \setminus \{0\}$ . By Theorem 5.1 in Appendix I, there exists  $k^*$  such that  $v = k^* \Psi_*$ . In particular,

$$\lim_{\tau'_n \rightarrow 0} u_{\tau'_n}(1, \sigma) = k^* \psi_*(\sigma) \quad (4.83)$$

in the  $C^1(\overline{S_+^{N-1}})$  topology. Combining (4.81), (4.82) and (4.83) we conclude that  $k^* = k$  and

$$\lim_{\tau'_n \rightarrow 0} \tau_n'^{\beta_*} u_{\tau'_n}(1, \sigma) = k \psi_*(\sigma) \quad (4.84)$$

Using Theorem 3.6, it is equivalent to

$$\lim_{\tau'_n \rightarrow 0} \frac{u(\tau'_n, \sigma)}{u_k(\tau'_n, \sigma)} = 1 \quad (4.85)$$

uniformly on  $S_+^{N-1}$ . For any  $\epsilon > 0$ , there exists  $n_\epsilon > 0$  such that  $n \geq n_\epsilon$  implies

$$u_{k-\epsilon}(\tau'_n, \sigma) \leq u(\tau'_n, \sigma) \leq u_{k+\epsilon}(\tau'_n, \sigma)$$

By comparison principle,

$$u_{k-\epsilon} \leq u \leq u_{k+\epsilon} + m \quad \text{in } B_\delta^+ \setminus B_{\tau'_n}^+, \quad (4.86)$$

and finally

$$u_{k-\epsilon} \leq u \leq u_{k+\epsilon} + m \quad \text{in } B_\delta^+, \quad (4.87)$$

Since  $\epsilon$  is arbitrary and using again Theorem 3.6, it implies

$$\lim_{r \rightarrow 0} \frac{u(r, \sigma)}{\Psi_*(r, \sigma)} = k, \quad (4.88)$$

locally uniformly on  $S^{N-1}$ . But since the convergence holds in  $C^1(\overline{S_+^{N-1}})$ , (4.74) follows.

*Step 2.* Assume

$$\lim_{x \rightarrow 0} \frac{u(x)}{\Psi_*(x)} = \infty. \quad (4.89)$$

For any  $0 < \epsilon < \delta$  and  $k > 0$ , there holds

$$u_k(x) \leq u(x) \leq v_\epsilon(x) \quad \text{in } B_\delta^+ \setminus B_\epsilon^+ \quad (4.90)$$

where  $v_\epsilon$  has been defined in (3.12) and  $u_k$  is given by Theorem 3.6. Letting  $\epsilon \rightarrow 0$ ,  $k \rightarrow \infty$ , and using Proposition 4.1, we derive

$$u_\infty(x) \leq u(x) \leq v_0(x) \quad \text{in } B_\delta^+ \setminus \{0\}. \quad (4.91)$$

We have seen in Theorem 3.3 that  $v_0$  is a separable solution of (1.1) in  $\mathbb{R}_+^N$  which vanishes on  $\partial\mathbb{R}_+^N \setminus \{0\}$ , therefore  $v_0(x) = U_{s_+^{N-1}}(x)$ . This implies

$$u_\infty(x) \leq u(x) \leq |x|^{-\beta q} \omega_{s_+^{N-1}}\left(\frac{x}{|x|}\right) \quad \text{in } B_\delta^+ \setminus \{0\}. \quad (4.92)$$

We conclude using Proposition 4.1.  $\square$

## 4.2 The case $p = N$

When  $p = N$ , the assumption that  $\partial\Omega$  is an hyperplane near 0 can be removed. The proof of the next results is based upon Theorem 3.10. The following result is the extension to the case  $p = N$  of Proposition 4.1.

**Proposition 4.3.** *Under the assumptions of Theorem 3.10 there exists  $\lim_{k \rightarrow \infty} u_k = u_\infty$  which is the unique element of  $C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$  which satisfies (3.17) in  $\Omega$ , vanishes on  $\partial\Omega \setminus \{0\}$  and such that*

$$\lim_{x \rightarrow 0} \frac{u_\infty(x)}{U_{s_+^{N-1}}(x)} = 1. \quad (4.93)$$

*Proof.* We denote by  $u_k^\Omega$  the unique positive solution of (3.17) satisfying (3.57) obtained in Theorem 3.6. Then

$$T_\ell[u_k^\Omega] = u_{\ell^{\beta q - \beta_* k}}^{\Omega_\ell}, \quad (4.94)$$

because of uniqueness. We denote by  $B := B_{\frac{1}{2}}(a)$  and  $B' := B_{\frac{1}{2}}(a')$  the two balls tangent to  $\partial\Omega$  at 0 respectively interior and exterior to  $\Omega$  introduced in the proof of Lemma 3.11. Estimate (3.58) implies

$$u_k^{B'/c} \leq u_k^\Omega \leq u_k^B \quad (4.95)$$

the left-hand side inequality holding in  $\Omega$  and the right-hand side one in  $B$ . Therefore

$$T_\ell[u_k^{B'c}] := u_{\ell^{\beta_q - \beta_*} k}^{B'c\ell} \leq T_\ell[u_k^\Omega] \leq T_\ell[u_k^B] := u_{\ell^{\beta_q - \beta_*} k}^{B\ell}, \quad (4.96)$$

the domains of validity of these inequalities being modified accordingly. Using again (3.58) we obtain

$$T_{\ell'}[u_{k'}^{B'c}] \leq T_\ell[u_k^{B'c}] \quad \text{in } B'^{c\ell'}, \quad (4.97)$$

for any  $0 < \ell' \leq \ell$  and  $\ell'^{\beta_q - \beta_*} k' \leq \ell^{\beta_q - \beta_*} k$ . In the same way

$$T_{\ell'}[u_{k'}^B] \geq T_\ell[u_k^B] \quad \text{in } B^\ell, \quad (4.98)$$

for any  $0 < \ell' \leq \ell$  and  $\ell'^{\beta_q - \beta_*} k' \geq \ell^{\beta_q - \beta_*} k$ . Since  $u_k^\Omega$ ,  $u_k^B$ ,  $u_k^{B'c}$  are increasing with respect to  $k$ , they converge respectively to  $u_\infty^\Omega$ ,  $u_\infty^B$ ,  $u_\infty^{B'c}$  and there holds for any  $\ell > 0$

$$T_\ell[u_\infty^{B'c}] \leq T_\ell[u_\infty^\Omega] \leq T_\ell[u_\infty^B], \quad (4.99)$$

from (4.96) and

$$\begin{aligned} (i) \quad & T_{\ell'}[u_\infty^{B'c}] \leq T_\ell[u_\infty^{B'c}] && \text{in } B'^{c\ell'} \\ (ii) \quad & T_{\ell'}[u_\infty^B] \geq T_\ell[u_\infty^B] && \text{in } B^\ell \end{aligned} \quad (4.100)$$

for any  $0 < \ell' \leq \ell$ . Notice that, replacing  $\ell$  by  $\ell\ell'$  we can rewrite (4.99) as follows

$$T_{\ell'}[T_\ell[u_\infty^{B'c}]] \leq T_{\ell'}[T_\ell[u_\infty^\Omega]] \leq T_{\ell'}[T_\ell[u_\infty^B]]. \quad (4.101)$$

Because of the monotonicity with respect to  $\ell$  the following limits exist

$$U^{B'c} = \lim_{\ell \rightarrow 0} T_\ell[u_\infty^{B'c}] \quad \text{and} \quad U^B = \lim_{\ell \rightarrow 0} T_\ell[u_\infty^B]. \quad (4.102)$$

By Lemma 2.5 applied with  $\phi(|x|) = |x|^{-\beta_q}$  and since there holds  $u_\infty^B(x) \leq c|x|^{-\beta_q}$  and  $u_\infty^{B'c}(x) \leq c|x|^{-\beta_q}$ , we derive

$$\begin{aligned} (i) \quad & |\nabla T_\ell[u_\infty^B](x)| \leq c_2|x|^{-\beta_q-1} && \forall x \in B^\ell \\ (ii) \quad & |\nabla T_\ell[u_\infty^B](x) - \nabla T_\ell[u_\infty^B](y)| \leq c_2|x|^{-\beta_q-1-\alpha}|x-y|^\alpha && \forall x, y \in B^\ell, |x| \leq |y| \\ (iii) \quad & T_\ell[u_\infty^B](x) \leq c_2|x|^{-\beta_q-1}(\text{dist}(x, \partial B^\ell))^\alpha && \forall x \in B^\ell, \end{aligned} \quad (4.103)$$

and

$$\begin{aligned} (i) \quad & |\nabla T_\ell[u_\infty^{B'c}](x)| \leq c_2|x|^{-\beta_q-1} && \forall x \in B'^{c\ell} \\ (ii) \quad & |\nabla T_\ell[u_\infty^{B'c}](x) - \nabla T_\ell[u_\infty^{B'c}](y)| \leq c_2|x|^{-\beta_q-1-\alpha}|x-y|^\alpha && \forall x, y \in B'^{c\ell}, |x| \leq |y| \\ (iii) \quad & T_\ell[u_\infty^{B'c}](x) \leq c_2|x|^{-\beta_q-1}(\text{dist}(x, \partial B'^{c\ell}))^\alpha && \forall x \in B'^{c\ell}. \end{aligned} \quad (4.104)$$

Thus the sets of functions  $\{T_\ell[u_\infty^B]\}$  and  $\{T_\ell[u_\infty^{B'c}]\}$  are equicontinuous in the  $C^1$ -loc topology and by uniqueness, the limit in (4.102) below holds in this topology. Hence  $U^{B'c}$  and  $U^B$  are positive solutions of (3.17) in  $\mathbb{R}_+^N$  which vanish on  $\partial\mathbb{R}_+^N \setminus \{0\}$ . Furthermore  $U^{B'c} \leq U^B$ . Since for any  $\ell, \ell' > 0$ ,

$T_{\ell'}[T_{\ell}[u_{\infty}^{B'c}]] = T_{\ell\ell'}[u_{\infty}^{B'c}]$ , it follows  $T_{\ell'}[U^{B'c}] = U^{B'c}$  and in the same way  $T_{\ell'}[U^B] = U^B$ . This means that  $U^B$  and  $U^{B'c}$  are self-similar solutions of (3.17) in  $\mathbb{R}_+^N$  and they vanish on  $\partial\mathbb{R}_+^N \setminus \{0\}$ . Hence

$$U^B = U^{B'c} = U_{S_+^{N-1}}. \quad (4.105)$$

Applying again Lemma 2.5 to  $u_{\infty}^{\Omega}$  with  $\phi(|x|) = |x|^{-\beta_q}$  we have

$$\begin{aligned} (i) \quad & |\nabla T_{\ell}[u_{\infty}^{\Omega}](x)| \leq c_2|x|^{-\beta_q-1} & \forall x \in \Omega^{\ell} \\ (ii) \quad & |\nabla T_{\ell}[u_{\infty}^{\Omega}](x) - \nabla T_{\ell}[u_k^{\Omega}](y)| \leq c_2|x|^{-\beta_q-1-\alpha}|x-y|^{\alpha} & \forall x, y \in \Omega^{\ell}, |x| \leq |y| \\ (iii) \quad & T_{\ell}[u_{\infty}^{\Omega}](x) \leq c_2|x|^{-\beta_q-1}(\text{dist}(x, \partial\Omega^{\ell}))^{\alpha} & \forall x \in \Omega^{\ell}. \end{aligned} \quad (4.106)$$

This implies that the set of functions  $\{T_{\ell}[u_{\infty}^{\Omega}]\}_{\ell}$  is equicontinuous in the  $C^1$ -loc topology of  $\mathbb{R}_+^N$  and there exists a sequence  $\{\ell_n\} \rightarrow 0$  and a function  $U$  such that  $T_{\ell_n}[u_{\infty}^{\Omega}] \rightarrow U^{\Omega}$  in this topology of  $\mathbb{R}_+^N$ , and  $U$  is a positive solution of (3.17) in  $\mathbb{R}_+^N$  which vanishes on  $\partial\mathbb{R}_+^N \setminus \{0\}$ . From (4.99) and (4.105) there holds  $U^{\Omega} = U_{S_+^{N-1}}$  and therefore

$$\lim_{\ell \rightarrow 0} T_{\ell}[u_{\infty}^{\Omega}] = U_{S_+^{N-1}}. \quad (4.107)$$

This implies (4.93) and

$$\lim_{r \rightarrow 0} r^{\beta_q} u_{\infty}^{\Omega}(r, \sigma) = \omega_{S_+^{N-1}}(\sigma) \quad (4.108)$$

uniformly on compact subsets of  $S_+^{N-1}$ .  $\square$

Up to minor modifications the proof of the next classification theorem is similar to the one of Theorem 4.2.

**Theorem 4.4.** *Assume  $N-1 < q < N - \frac{1}{2}$ . If  $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$  is a positive solution of (3.17) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ , then we have the following alternative:*

- (i) *either there exists  $k \geq 0$  such that (4.74) holds,*
- (ii) *or (4.75) holds.*

## 5 Appendix I: Positive $p$ -harmonic functions in a half space

In this section we prove the following rigidity result.

**Theorem 5.1.** *Assume  $1 < p \leq N$  and  $u \in C^1(\mathbb{R}_+^N) \cap C(\overline{\mathbb{R}_+^N} \setminus \{0\})$  is a positive  $p$ -harmonic function which vanishes on  $\partial\mathbb{R}_+^N \setminus \{0\}$  and such that  $|x|^{\beta_*} u(x)$  is bounded. Then there exists  $k \geq 0$  such that*

$$u(x) = k\Psi_*(x) \quad \forall x \in \mathbb{R}_+^N. \quad (5.1)$$

*Proof.* Since  $|x|^{\beta_*} u(x)$  is bounded,  $|x|^{\beta_*+1} \nabla u(x)$  is also bounded and there exists  $m > 0$  such that  $u(x) \leq m\Psi_*(x)$  in  $B_{\delta}^+$ . We denote by  $k$  the infimum of the  $c > 0$  such that  $u(x) \leq c\Psi_*(x)$ . Then

$$0 \leq u(x) \leq k\Psi_*(x) \quad \forall x \in \mathbb{R}_+^N \setminus \{0\} \quad (5.2)$$

and we assume that  $k > 0$  otherwise  $u = 0$ . Assume that the graphs over  $\mathbb{R}_+^N$  of the functions  $x \mapsto u(x)$  and  $x \mapsto k\Psi_*(x)$  are tangent at some point  $x_0 \in \mathbb{R}_+^N$  or  $x_0 \in \partial\mathbb{R}_+^N \setminus \{0\}$ . Since  $\nabla\Psi_*$  never vanishes in  $\overline{\mathbb{R}_+^N} \setminus \{0\}$  it follows from the strong maximum principle or Hopf Lemma that  $u = k\Psi_*$ . If the two graphs are not tangent in  $\overline{\mathbb{R}_+^N} \setminus \{0\}$ , either they are asymptotically tangent at 0, or at  $\infty$ .

(i) In the first case there exists two sequences  $\{k_n\}$  increasing to  $k$  and  $\{x_n\} \subset \mathbb{R}_+^N$  converging to zero such that  $\frac{u(x_n)}{\Psi_*(x_n)} = k_n$ . We set  $r_n = |x_n|$  and  $u_{r_n}(x) = r_n^{\beta_*} u(r_n x)$ . Then  $u_{r_n}$  is  $p$ -harmonic and positive and  $0 < u_{r_n}(x) \leq k|x|^{-\beta_*} \psi_*(\frac{x}{|x|})$ ; therefore

$$|\nabla u_{r_n}(x)| \leq C|x|^{-\beta_*-1} \quad \text{and} \quad |\nabla u_{r_n}(x) - \nabla u_{r_n}(x')| \leq C|x|^{-\beta_*-1-\alpha} |x - x'|^\alpha \quad (5.3)$$

for  $0 < |x| \leq |x'|$  and some constants  $C > 0$  and  $\alpha \in (0, 1)$ . Up to a subsequence, we can assume that  $u_{r_n}$  converges to some  $U$  in the  $C_{loc}^1$  topology of  $\overline{\mathbb{R}_+^N} \setminus \{0\}$  and  $\frac{x_n}{r_n} \rightarrow \xi \in S_+^{N-1}$ . The function  $U$  is  $p$ -harmonic and positive in  $\mathbb{R}_+^N$  and satisfies  $0 \leq U \leq k\Psi_*$  in  $\mathbb{R}_+^N$  and  $U(\xi) = k\Psi_*(\xi)$  if  $\xi \in S_+^{N-1}$  or  $U_{x_N}(\xi) = k\Psi_{*x_N}(\xi)$  if  $\xi \in \partial S_+^{N-1}$ . It follows from the strong maximum principle or Hopf Lemma that  $U = k\Psi_*$ . Therefore  $u_{r_n} \rightarrow k\Psi_*$  and in particular

$$\lim_{r_n \rightarrow 0} \frac{r_n^{\beta_*} u(r_n, \sigma)}{\psi_*(\sigma)} = k \quad \text{uniformly on } S_+^{N-1}. \quad (5.4)$$

For any  $\epsilon > 0$ , there exists  $n_\epsilon \in \mathbb{N}_*$  such that for  $n \geq n_\epsilon$ ,  $(k - \epsilon)\Psi_*(x) \leq u(x) \leq (k + \epsilon)\Psi_*(x)$  if  $|x| = r_n$ . This implies  $(k - \epsilon)\Psi_*(x) \leq u(x) \leq (k + \epsilon)\Psi_*$  for  $|x| \geq r_n$  and therefore in  $\mathbb{R}^N$ . Since  $\epsilon$  is arbitrary, we deduce that  $u = k\Psi_*$ .

(ii) if the two graphs are tangent at infinity, there exist two sequences  $\{k_n\}$  increasing to  $k$  and  $\{x_n\}$  such that  $r_n = |x_n| \rightarrow \infty$  with  $u(x_n) = k_n\Psi_*(x_n)$  and

$$\lim_{r_n \rightarrow \infty} \frac{r_n^{\beta_*} u(r_n, \sigma)}{\psi_*(\sigma)} = k \quad \text{uniformly on } S_+^{N-1}. \quad (5.5)$$

Therefore we look at the supremum of the  $c > 0$  such that  $u \geq c\Psi_*$ . If the set of such  $c$  is empty, it would mean that

$$\inf_{x \in \mathbb{R}_+^N} \frac{u(x)}{\Psi_*(x)} = 0.$$

Clearly, if this infimum is achieved at some point, the strong maximum principle or Hopf Lemma imply  $u \equiv 0$ , contradicting (5.5), and this relation prevents also this infimum be achieved at infinity. We are left with the case where there exists a sequence  $\{z_n\} \subset \mathbb{R}_+^N$ , converging to 0, such that

$$\lim_{n \rightarrow \infty} \frac{u(z_n)}{\Psi_*(z_n)} = 0. \quad (5.6)$$

By boundary Harnack inequality [2, th 2.11], there exists  $c > 0$  such that

$$c^{-1} \frac{u(z)}{\Psi_*(z)} \leq \frac{u(z_n)}{\Psi_*(z_n)} \leq c \frac{u(z)}{\Psi_*(z)} \quad \forall z \in \mathbb{R}_+^N \text{ s.t. } |z| = |z_n| \quad (5.7)$$



Combining (5.6) and (5.7), we derive that

$$\lim_{n \rightarrow \infty} \sup_{|z|=|z_n|} \frac{u(z)}{\Psi_*(z)} = 0, \quad (5.8)$$

Denoting by  $\epsilon_n$  the supremum in the above relation, we obtain that  $u \leq \epsilon_n \Psi_*$  in  $\mathbb{R}_+^N \setminus B_{\epsilon_n}$  and finally  $u = 0$ , contradiction. Thus we are left with the case where there exists  $k' \in (0, k]$  which is the supremum of the  $c > 0$  such that  $u \geq c\Psi_*$ . In particular  $u \geq k'\Psi_*$ . Remembering that  $u \leq k\Psi_*$  we get  $k = k'$ , which implies  $u = k\Psi_*$ .

Next we assume that  $k' < k$ . Clearly the graphs of  $u$  and  $k'\Psi_*$  cannot be tangent in  $\overline{\mathbb{R}_+^N}$ , because of strong maximum principle or Hopf Lemma. They cannot be tangent at infinity because of (5.5). Therefore there exist two sequences  $\{k'_n\}$  increasing to  $k'$  and  $\{x'_n\} \subset \mathbb{R}_+^N$  converging to 0 such that  $\frac{u(x'_n)}{\Psi_*(x'_n)} = k'_n$ . As in case (i) we obtain that

$$\lim_{r'_n \rightarrow 0} \frac{r_n'^{\beta_*} u(r'_n, \sigma)}{\psi_*(\sigma)} = k' \quad \text{uniformly on } S_+^{N-1}, \quad (5.9)$$

where  $r'_n = |x'_n|$ , and finally derive that  $u = k'\Psi_*$ , a contradiction with (5.5). Therefore  $k = k'$ , which ends the proof.  $\square$

*Remark.* In the case  $p = N$  the result holds under the weaker assumption  $\lim_{|x| \rightarrow \infty} u(x) = 0$ . This is due to the fact that this condition implies by regularity

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{\omega_{s_+^{N-1}}\left(\frac{x}{|x|}\right)} = 0$$

and therefore

$$u(x) \leq m\Psi_*(x) \quad \forall x \text{ s.t. } |x| \geq 1,$$

where  $m = \max_{|x|=1} \frac{u(x)}{\omega_{s_+^{N-1}}\left(\frac{x}{|x|}\right)}$ . Using the inversion  $x \mapsto \frac{x}{|x|^2}$ , we obtain that the estimate  $u \leq m\Psi_*$

holds  $\mathbb{R}^N$ , and we conclude by Theorem 5.1.

*Remark.* We conjecture that the rigidity result holds under the mere condition

$$\lim_{|x| \rightarrow \infty} |x|^{-\tilde{\beta}} u(x) = 0, \quad (5.10)$$

where  $\tilde{\beta}$  is the (positive) exponent corresponding to the regular spherical  $p$ -harmonic function under the form

$$\tilde{\Psi} = |x|^{\tilde{\beta}} \tilde{\psi}\left(\frac{x}{|x|}\right), \quad (5.11)$$

see [14], [12]. Note that  $\tilde{\beta} = 1$  when  $p = N$ .

## 6 Appendix II: Estimates on $\beta_*$

When  $N = 2$  and  $1 < p \leq 2$ , it is proved in [9] that

$$\beta_* = \frac{3 - p + 2\sqrt{p^2 - 5p + 7}}{3(p - 1)}. \quad (6.1)$$

Up to now no estimate is known when  $N > 2$  except in the cases  $p = 2$  where  $\beta_* = N - 1$  and  $p = N$  where  $\beta_* = 1$ , besides the classical one

$$\beta_* > \frac{N - p}{p - 1}, \quad (6.2)$$

valid when  $p < N$ . In this section we prove the following result

**Theorem 6.1.** *Assume  $1 < p < N$ . Then the following estimates hold:*

$$1 < p < 2 \implies \beta_* > \frac{N - 1}{p - 1}, \quad (6.3)$$

$$2 < p < N \implies \max \left\{ 1, \frac{N - p}{p - 1} \right\} < \beta_* < \frac{N - 1}{p - 1}. \quad (6.4)$$

*Remark.* It is worth noticing that when  $p = 2$  or  $p = N$ , there holds  $\beta_* = \frac{N-1}{p-1}$ .

*Proof of Theorem 6.1.* We consider the following set of spherical coordinates in  $\mathbb{R}_+^N$  with  $x = (x_1, \dots, x_N)$

$$\begin{aligned} x_1 &= r \sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_2 \sin \theta_1 \\ x_2 &= r \sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_2 \cos \theta_1 \\ &\vdots \\ x_{N-1} &= r \sin \theta_{N-1} \cos \theta_{N-2} \\ x_N &= r \cos \theta_{N-1} \end{aligned} \quad (6.5)$$

with  $\theta_1 \in [0, 2\pi]$  and  $\theta_k \in [0, \pi]$  for  $k = 2, \dots, N - 2$  and  $\theta_{N-1} \in [0, \frac{\pi}{2}]$ . Under this representation, a solution  $\omega$  of (3.2) verifies

$$\begin{aligned} & -\frac{1}{\sin^{N-2} \theta_{N-1}} \left[ \sin^{N-2} \theta_{N-1} \left( \beta_*^2 \omega^2 + \omega_{\theta_{N-1}}^2 + \frac{1}{\sin^2 \theta_{N-1}} |\nabla_{\theta'} \omega|^2 \right)^{\frac{p-2}{2}} \omega_{\theta_{N-1}} \right]_{\theta_{N-1}} \\ & -\frac{1}{\sin^2 \theta_{N-1}} \operatorname{div}'_{\theta'} \left[ \sin^{N-2} \theta_{N-1} \left( \beta_*^2 \omega^2 + \omega_{\theta_{N-1}}^2 + \frac{1}{\sin^2 \theta_{N-1}} |\nabla_{\theta'} \omega|^2 \right)^{\frac{p-2}{2}} \nabla_{\theta'} \omega \right] \\ & = \beta_* \Lambda_{\beta_*} \left[ \sin^{N-2} \theta_{N-1} \left( \beta_*^2 \omega^2 + \omega_{\theta_{N-1}}^2 + \frac{1}{\sin^2 \theta_{N-1}} |\nabla_{\theta'} \omega|^2 \right)^{\frac{p-2}{2}} \omega \right] \end{aligned} \quad (6.6)$$

where  $\nabla_{\theta'}$  and  $\operatorname{div}'_{\theta'}$  denotes respectively the spherical gradient the divergence in variables  $\theta' = (\theta_1, \dots, \theta_{N-2})$  parameterizing  $S^{N-2}$  and  $\Lambda_{\beta_*}$  is defined in Introduction. If  $\omega$  is the unique positive solution of (3.2)

(up to homothety), it depends only on  $\theta_{N-1}$  and is  $C^\infty$ . For simplicity we set  $\theta_{N-1} = \theta \in [0, \frac{\pi}{2}]$  and  $\omega = \omega(\theta)$  satisfies

$$-\frac{1}{\sin^{N-2}\theta} \left[ \sin^{N-2}\theta (\beta_*^2\omega^2 + \omega_\theta^2)^{\frac{p-2}{2}} \omega_\theta \right]_\theta = \beta_* \Lambda_{\beta_*} \left[ \sin^{N-2}\theta (\beta_*^2\omega^2 + \omega_\theta^2)^{\frac{p-2}{2}} \omega \right] \quad \text{in } (0, \frac{\pi}{2}) \quad (6.7)$$

$$\omega(\frac{\pi}{2}) = 0, \quad \omega_\theta(0) = 0.$$

*Step 1: The eigenvalue identity.* Equation (6.7) can also be written under the form

$$-\omega_{\theta\theta} - (N-2) \cot\theta \omega_\theta - (p-2) \frac{\beta_*^2\omega + \omega_{\theta\theta}}{\beta_*^2\omega^2 + \omega_\theta^2} \omega_\theta^2 = \beta_* \Lambda_{\beta_*} \omega. \quad (6.8)$$

By multiplying (6.8) by  $\cos\theta \sin^{N-2}\theta$  and then integrating over  $(0, \frac{\pi}{2})$  we obtain

$$-\int_0^{\frac{\pi}{2}} (\omega_{\theta\theta} + (N-2) \cot\theta \omega_\theta) \cos\theta \sin^{N-2}\theta d\theta = (N-1) \int_0^{\frac{\pi}{2}} \omega \cos\theta \sin^{N-2}\theta d\theta.$$

Noticing that

$$\beta_* \Lambda_{\beta_*} + 1 - N = (p-1) \left( \beta_* - \frac{N-1}{p-1} \right) (\beta_* + 1)$$

we derive

$$\begin{aligned} (2-p) \int_0^{\frac{\pi}{2}} \frac{\beta_*^2\omega + \omega_{\theta\theta}}{\beta_*^2\omega^2 + \omega_\theta^2} \omega_\theta^2 \omega \cos\theta \sin^{N-2}\theta d\theta \\ = (p-1) \left( \beta_* - \frac{N-1}{p-1} \right) (\beta_* + 1) \int_0^{\frac{\pi}{2}} \omega \cos\theta \sin^{N-2}\theta d\theta. \end{aligned} \quad (6.9)$$

*Step 2: Elliptic coordinates and reduction.* Writing  $\omega(\theta) = \omega(0) + a\theta^2 + o(\theta^2)$ ,  $\omega_\theta(\theta) = 2a\theta + o(\theta)$  and  $\omega_{\theta\theta}(\theta) = 2a + o(1)$ , then  $-Na = \beta_* \Lambda_{\beta_*}$ . This implies that  $\omega$  is decreasing near 0. It is immediate that it cannot have a local minimum in  $(0, \frac{\pi}{2})$ , therefore it remains decreasing in the whole interval. We parameterize the ellipse

$$E_r = \{(x, y) : x > 0, y < 0, x^2 + \beta_*^{-2}y^2 = r^2\}$$

by setting

$$\omega = r \cos\phi \quad \text{and} \quad -\omega_\theta = \beta_* r \sin\phi \quad \text{with} \quad \phi = \phi(\theta) \quad \text{and} \quad r = r(\theta).$$

The functions  $r$  and  $\phi$  are  $C^2$ . Hence  $r_\theta \cos\phi - r \sin\phi \phi_\theta = -\beta_* r \sin\phi$ , then  $r_\theta \cos\phi = (\phi_\theta - \beta_*) r \sin\phi$  and  $r_\theta = (\phi_\theta - \beta_*) r \tan\phi$ . Plugging this into (6.8), we derive

$$-\left( (p-1) \frac{r_\theta}{r} + \phi_\theta \cot\phi + (N-2) \cot\theta \right) + \Lambda_{\beta_*} \cot\phi = 0, \quad (6.10)$$

and finally

$$(p-1)(\phi_\theta - \beta_*) \tan\phi + (\phi_\theta - \Lambda_{\beta_*}) \cot\phi = (2-N) \cot\theta. \quad (6.11)$$

*Step 3: Estimates on  $\phi_\theta$ .* We can write (6.11) under the equivalent form

$$(p-1)(\phi_\theta - \beta_*) \tan^2 \phi + \phi_\theta - \Lambda_{\beta_*} = (2-N) \frac{\cos \theta \sin \phi}{\cos \phi \sin \theta}. \quad (6.12)$$

Since

$$\lim_{\theta \rightarrow 0} \frac{\sin \phi}{\sin \theta} = \lim_{\theta \rightarrow 0} \frac{\cos \phi}{\cos \theta} \phi_\theta = \phi_\theta(0),$$

we derive  $\phi_\theta(0) - \Lambda_{\beta_*} = (2-N)\phi_\theta(0)$  and thus  $\phi_\theta(0) = \frac{\Lambda_{\beta_*}}{N-1}$ . Similarly, the expansion of  $\phi(\theta)$  near  $\theta = \frac{\pi}{2}$  yields to  $\phi_\theta(\frac{\pi}{2}) = \beta_*$ . Since  $p < N$ ,  $\Lambda_{\beta_*}/(N-1) < \beta_*$ . We claim now that

$$\phi_\theta(\theta) \leq \beta_* \quad \forall \theta \in (0, \frac{\pi}{2}). \quad (6.13)$$

If  $\Lambda_{\beta_*} \leq \beta_*$ , then

$$(2-N) \cot \theta = (p-1)(\phi_\theta - \beta_*) \tan \phi + (\phi_\theta - \Lambda_{\beta_*}) \cot \phi \geq ((p-1) \tan \phi + \cot \phi)(\phi_\theta - \beta_*)$$

thus (6.13) holds.

Next we assume  $\beta_* < \Lambda_{\beta_*}$ . It means  $0 < (p-2)\beta_* - (N-p)$  and thus  $p > 2$ . We claim that

$$\beta_* \leq \frac{N-2}{p-2}. \quad (6.14)$$

We proceed by contradiction and assume

$$\beta_* > \frac{N-2}{p-2}. \quad (6.15)$$

Then

$$(p-2) \left( \beta_*^2 - \frac{N-p}{p-2} \beta_* - \frac{N-2}{p-2} \right) = (p-2) (\beta_* + 1) \left( \beta_* - \frac{N-2}{p-2} \right) > 0.$$

Equivalently

$$\beta_*(\Lambda_{\beta_*} - \beta_*) > N-2.$$

Since

$$\lim_{\theta \rightarrow \frac{\pi}{2}} \cot \theta \tan \phi = \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\cos \theta}{\cos \phi} = \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\sin \theta}{\phi_\theta \sin \phi} = \frac{1}{\beta_*}$$

and

$$\begin{aligned} (p-1)(\phi_\theta(\theta) - \beta_*) \tan^2 \phi &= \Lambda_{\beta_*} - \phi_\theta(\theta) + (2-N) \frac{\cos \theta \sin \phi}{\cos \phi \sin \theta} \\ &= \frac{1}{\beta_*} (\beta_*(\Lambda_{\beta_*} - \beta_*) + 2 - N) + o(1), \end{aligned} \quad (6.16)$$

thus, if (6.15) holds there exists  $\epsilon > 0$  such that  $\phi_\theta(\theta) > \beta_*$  for any  $\theta \in [\frac{\pi}{2} - \epsilon, \frac{\pi}{2}]$ . Since  $\phi_\theta(0) < \beta_*$ , there exists  $\bar{\theta} \in (0, \frac{\pi}{2})$  such that  $\phi_\theta(\bar{\theta}) = \beta_*$  and  $\phi_{\theta\theta}(\bar{\theta}) \geq 0$ . We compute  $\phi_{\theta\theta}$  and get

$$(p-1)\phi_\theta(\phi_\theta - \beta_*) \sec^2 \phi + ((p-1) \tan \phi + \cot \phi) \phi_{\theta\theta} - \phi_\theta(\phi_\theta - \Lambda_{\beta_*}) \csc^2 \phi = (N-2) \csc^2 \theta$$

Hence, at  $\theta = \bar{\theta}$

$$\phi_{\theta\theta}(\bar{\theta}) \left( (p-1) \tan \phi(\bar{\theta}) + \cot \phi(\bar{\theta}) \right) = \beta_*(\beta_* - \Lambda_{\beta_*}) \csc^2 \phi(\bar{\theta}) + (N-2) \csc^2 \bar{\theta}$$

From (6.11),

$$\cot \phi(\bar{\theta}) = \frac{N-2}{\Lambda_{\beta_*} - \beta_*} \cot \bar{\theta}$$

Therefore

$$\begin{aligned} A(\bar{\theta}) &:= \phi_{\theta\theta}(\bar{\theta}) \left( (p-1) \tan \phi(\bar{\theta}) + \cot \phi(\bar{\theta}) \right) \\ &= \left( 1 + \left( \frac{N-2}{\Lambda_{\beta_*} - \beta_*} \right)^2 \cot^2 \bar{\theta} \right) \beta_*(\beta_* - \Lambda_{\beta_*}) + (N-2)(1 + \cot^2 \bar{\theta}) \\ &= \beta_*(\beta_* - \Lambda_{\beta_*}) + N-2 - \left( \frac{(N-2)^2}{\Lambda_{\beta_*} - \beta_*} + 2 - N \right) \cot^2 \bar{\theta} \\ &= -(p-2)(\beta_* + 1) \left( \beta_* - \frac{N-2}{p-2} \right) - \frac{N-2}{\Lambda_{\beta_*} - \beta_*} (\beta_*(N-1) - \Lambda_{\beta_*}) \cot^2 \bar{\theta} \\ &< 0, \end{aligned} \tag{6.17}$$

using (6.15) and the fact that  $N > p$ . This is a contradiction, thus (6.14) holds.

Next, if  $\beta_* < \frac{N-2}{p-2}$ , it follows from (6.16) that there exists  $\epsilon > 0$  such that  $\phi_\theta < \beta_*$  in  $[\frac{\pi}{2} - \epsilon, \frac{\pi}{2})$ . If (6.13) is not true, there exist  $0 < \theta_1 < \theta_2 < \frac{\pi}{2} - \epsilon$  such that  $\phi_\theta(\theta_1) = \phi_\theta(\theta_2) = \beta_*$ ,  $\phi_{\theta\theta}(\theta_1) \geq 0$ ,  $\phi_{\theta\theta}(\theta_2) \leq 0$ . Using the equation satisfied by  $\phi_{\theta\theta}$ , we obtain for  $i = 1, 2$ ,

$$A(\theta_i) = (2-p)(\beta_* + 1) \left( \beta_* - \frac{N-2}{p-2} \right) - \frac{N-2}{\Lambda_{\beta_*} - \beta_*} (\beta_*(N-1) - \Lambda_{\beta_*}) \cot^2 \theta_i. \tag{6.18}$$

On one hand  $A(\theta_2) \leq 0 \leq A(\theta_1)$ , and on the other

$$A(\theta_2) - A(\theta_1) = \frac{N-2}{\Lambda_{\beta_*} - \beta_*} (\beta_*(N-1) - \Lambda_{\beta_*}) (\cot^2 \theta_1 - \cot^2 \theta_2) > 0,$$

since  $\cot$  is decreasing in  $(0, \frac{\pi}{2})$ ,  $\cot^2 \theta_1 > \cot^2 \theta_2$ , a contradiction. Therefore  $\phi_\theta \leq \beta_*$  in  $(0, \frac{\pi}{2})$ .

Finally, if  $\beta_* = \frac{N-2}{p-2}$  and the maximum of  $\phi_\theta$  on  $[0, \frac{\pi}{2})$  is larger than  $\beta_*$  and achieved at some  $\bar{\theta} < \frac{\pi}{2}$  then there exists  $\theta_1 < \bar{\theta}$  such that  $\phi_\theta(\theta_1) = \beta_*$  and  $\phi_{\theta\theta}(\theta_1) \geq 0$ . In that case

$$0 \leq A(\theta_1) = -\frac{N-2}{\Lambda_{\beta_*} - \beta_*} (\beta_*(N-1) - \Lambda_{\beta_*}) \cot^2 \theta_1 < 0$$

which is again a contradiction.

*Step 4: End of the proof.* Since  $r^2 = \beta_*^2 \omega^2 + \omega_\theta^2$ ,  $r_\theta = r(\phi_\theta - \beta_*) \tan \phi$ , we have

$$r r_\theta = (\beta_*^2 \omega + \omega_{\theta\theta}) \omega_\theta = r(\phi_\theta - \beta_*) \tan \phi.$$

Since  $\omega_\theta < 0$  on  $(0, \frac{\pi}{2})$ , it follows from Step 3 that  $\beta_*^2 \omega + \omega_{\theta\theta} \geq 0$  and thus

$$\int_0^{\frac{\pi}{2}} \frac{\beta_*^2 \omega + \omega_{\theta\theta}}{\beta_*^2 \omega^2 + \omega_\theta^2} \omega_\theta^2 \omega \cos \theta \sin^{N-2} \theta d\theta > 0,$$

since the integrand cannot be identically 0. The conclusion follows from (6.9).  $\square$

*Remark.* Since  $\omega_\theta(\frac{\pi}{2}) = -c^2 < 0$ , it follows  $\omega(\theta) = -\omega_\theta(\theta) \cot \theta + O(\frac{\pi}{2} - \theta)$  as  $\theta \rightarrow \frac{\pi}{2}$ , and from the eigenfunction equation (6.8)

$$\frac{\beta_*^2 \omega + \omega_{\theta\theta}}{\beta_*^2 \omega^2 + \omega_\theta^2} \omega_\theta^2 = (\beta_*^2 \omega + \omega_{\theta\theta})(1 + o(1)).$$

Therefore

$$-(p-1)\omega_{\theta\theta} = (\beta_* \Lambda_{\beta_*} + (p-2)\beta_*^2 + 2 - N)\omega(1 + o(1)) \quad \text{as } \theta \rightarrow \frac{\pi}{2}$$

and since  $\Delta' \omega := \omega_{\theta\theta} + (N-2) \cot \theta \omega_\theta$

$$-\Delta' \omega = \frac{\beta_*(\beta_*(2p-3) + p - N) + (p-2)(N-2)}{p-1} \omega(1 + o(1)) \quad \text{as } \theta \rightarrow \frac{\pi}{2}.$$

Because  $\omega$  is  $C^\infty$  we obtain finally

$$|\Delta' \omega| \leq c\omega, \tag{6.19}$$

for some  $c > 0$ .

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