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# Boundary singularities of positive solutions of quasilinear Hamilton-Jacobi equations

**Marie-Françoise Bidaut-Véron**
**Marta Garcia-Huidobro**
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**Abstract** We study the boundary behaviour of the solutions of (E) $-\Delta_p u + |\nabla u|^q = 0$ in a domain $\Omega \subset \mathbb{R}^N$, when $N \geq p > q > p-1$. We show the existence of a critical exponent $q_* < p$ such that if $p-1 < q < q_*$ there exist positive solutions of (E) with an isolated singularity on $\Omega$ and that these solutions belong to two different classes of singular solutions. If $q_* \leq q < p$ no such solution exists and actually any boundary isolated singularity of a positive solution of (E) is removable. We prove that all the singular positive solutions are classified according the two types of singular solutions that we have constructed.

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*Key words.* p-Laplace operator; singularities; spherical p-harmonic equations; Leray-Lions operators; Schauder fixed point theorem.

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# 1 Introduction

Let $N \geq p > 1$, $q > p-1$ and $\Omega \subset \mathbb{R}^N$ $(N > 1)$ be a $C^2$ bounded domain such that $0 \in \partial\Omega$. In this article we study boundary singularities of functions $u \in C(\overline{\Omega} \setminus \{0\})$ which satisfy

$$-\Delta_p u + |\nabla u|^q = 0 \qquad \text{in } \Omega \tag{1.1}$$

where $\Delta_p u := \mathrm{div}\left(\nabla \mathrm{u}|^{\mathrm{p}-2}|\nabla \mathrm{u}|\right)$. The two main questions we consider are as follows:

Q-1- Existence of positive solutions of (1.1).

Q-2- Description of positive solutions with an isolated boundary singularity at 0.

When $p = 2$ a fairly complete description of positive solutions of

$$-\Delta u + |\nabla u|^q = 0 \tag{1.2}$$

in $\Omega$ is provided by Nguyen-Phuoc and Véron [22]. In particular they prove the following series of results in the range of values $1 < q < 2$.

*1-* Any signed solution of (1.3) verifies

$$|\nabla u(x)| \leq c_{N,q}(\mathrm{dist}\,(x)^{-\frac{1}{q-1}} \qquad \forall x \in \Omega, \tag{1.3}$$

where $d(x) = \mathrm{dist}\,(x, \partial\Omega)$. As a consequence, if $u \in C(\overline{\Omega} \setminus \{0\})$ is a solution which vanishes on $\partial\Omega \setminus \{0\}$ it satisfies

$$|u(x)| \leq c_{q,\Omega} d(x)|x|^{-\frac{1}{q-1}} \qquad \forall x \in \Omega. \tag{1.4}$$

*2-* If $\frac{N+1}{N} \leq q < 2$ any positive solution of (1.3) in $\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$ is identically 0. A point is a removable singularities.

*3-* If $1 < q < \frac{N+1}{N}$ and $k > 0$ there exists a unique positive solution $u := u_k$ of (1.3) in $\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies $u(x) \sim c_N k P^\Omega(x, 0)$ where $P^\Omega$ is the Poisson kernel in $\Omega \times \partial\Omega$.

*4-* If $1 < q < \frac{N+1}{N}$ there exists a unique positive solution $u$ of (1.3) in $\mathbb{R}^N_+$ under the form $u(x) = |x|^{-\frac{1}{q-1}}\omega(|x|^{-1}x)$ which vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$. The function $\omega$ is the unique positive solution of

$$\begin{aligned} -\Delta' u + ((q-1)^2\omega^2 + |\nabla'\omega|^2)^{\frac{q}{2}} - \lambda_{N,q}\omega = 0 \qquad & \text{in } S^{N-1}_+ \\ \omega = 0 \qquad & \text{in } \partial S^{N-1}_+ \end{aligned} \tag{1.5}$$

where $\Delta'$ is the Laplace-Beltrami operator on $S^{N-1}$ and $\lambda_{N,q} > 0$ is an explicit constant.

*5-* If $1 < q < \frac{N+1}{N}$ and $u$ is a positive solution of (1.3) in $\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$ the following dichotomy holds:

*(i)* either $u(x) \sim |x|^{-\frac{1}{q-1}}\omega(|x|^{-1}x)$ as $x \to 0$,

*(ii)* or $u(x) \sim k c_N P^\Omega(x, 0)$ as $x \to 0$ for some $k \geq 0$.

In this article we extend to the quasilinear case $1 < p \leq N$ the above mentioned results. The following pointwise gradient estimate valid for any signed solution $u$ of (1.1) is proved in [7]: if $0 < p-1 < q$ there exists a constant $c_{N,p,q} > 0$ such that

$$|\nabla u(x)| \leq c_{N,p,q}(\mathrm{dist}\,(x))^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega. \tag{1.6}$$

As a consequence, any solution $u \in C^1(\overline{\Omega} \setminus \{0\}$ satisfies

$$|u(x)| \leq c_{p,q,\Omega} d(x)\,|x|^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega. \tag{1.7}$$

Concerning boundary singularities, the situation is more complicated than in the case $p = 2$ and the threshold of critical exponent less explicit. We first consider the problem in the half space $\mathbb{R}^N_+ := \{x = (x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > 0\}$. Assuming $p-1 < q \leq p$, separable solutions of (1.1) in $\mathbb{R}^N_+$ and vanishing on $\mathbb{R}^N_+ \setminus \{0\}$ can be looked for in spherical coordinates $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$ under the form

$$u(x) = u(r,\sigma) = r^{-\frac{p-q}{q+1-p}}\omega(\sigma), \quad r > 0\,,\ \sigma \in S^{N-1}_+ := \{S^{N-1} \cap \mathbb{R}^N_+\}. \tag{1.8}$$

Then $\omega$ is solution of the following problem

$$\begin{aligned} -div\left(\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla'\omega\right) - \beta_q\Lambda_{\beta_q}\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}}\omega \\ + \left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{q}{2}} = 0 \quad \text{in } S^{N-1}_+ \\ \omega = 0 \quad \text{on } \partial S^{N-1}_+, \end{aligned} \tag{1.9}$$

where $\beta_q = \frac{p-q}{q+1-p}$ and $\Lambda_{\beta_q} = \beta_q(p-1) + p - N$. The existence of positive solution to this problem is conditioned to the *existence of spherical p-harmonic functions* which have been studied by Krol [18], Tolksdorf [29], Kichenassamy and Véron [19] and more recently by Porretta and Véron [25]. One of the main point is that there exists a separable positive p-harmonic function $\psi$ under the form $\psi(r,\sigma) = r^{-\beta}\psi(\sigma)$ which is positive on $S^{N-1}_+$ and negative on $S^{N-1}_- := -S^{N-1}_+$ if and only if $\psi$ satisfies *the spherical p-harmonic eigenvalue problem*

$$\begin{aligned} -div\left(\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi\right) - \beta_*\Lambda_{\beta_*}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi = 0 \qquad \text{in } S^{N-1}_+ \\ \psi = 0 \qquad \text{on } \partial S^{N-1}_+, \end{aligned} \tag{1.10}$$

There exists a unique couple $(\psi_*, \beta_*)$ for which (1.10) has a solution with $\beta_* > 0$ and $\psi_* > 0$ (up to an homothety on $\psi_*$). The exponent $\beta_*$ which is larger than $\max\{1, \frac{N-p}{p-1}\}$, is not known explicitely except in the cases $p = 2$, and it has the value $N-1$, $p = N$ and it is 1, and $N = 2$ (see [18], [19]) and it is the root of an algebraic equation of degree 2. In Appendix II we prove the following new estimate:

**Theorem B** *Let* $1 < p \leq N$.

*(i) If $2 \le p \le N$, then $\beta_* \le \frac{N-1}{p-1}$ with equality only if $p = 2$ or $N$.*
*(ii) If $1 \le p < 2$, then $\beta_* > \frac{N-1}{p-1}$.*

To this exponent $\beta_*$ is associated the critical value $q_*$ of $q$ defined by $\beta_* = \beta_q$, or equivalently

$$q_* := \frac{\beta_*(p-1)+p}{\beta_*+1} = p - \frac{\beta_*}{\beta_*+1}. \tag{1.11}$$

The following result characterizes strong singularities.

**Theorem C** *Let $0 < p-1 \le N$, then*
*(i) If $p-1 < q < q_*$ problem (1.9) admits a unique positive solution $\omega_*$.*
*(ii) If $q_* \le q < p$ problem (1.9) admits no positive solution.*

This critical exponent corresponds to the threshold of criticality for boundary singularities.

**Theorem D** *Assume $q_* \le q < p \le N$. If $u \in C(\overline{\Omega} \setminus \{0\})$ is a nonnegative solution of (1.1) in $\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$, it is identical zero.*

As in the case $p = 2$ there exist positive solutions (1.1) in $\Omega$ with weak boundary singularities which are characterized by their blow-up near the singularity; their existence, based upon the delicate construction of sub and super solutions is much more difficult than in the case $p = 2$ (see [22]). Furthermore it is done only if $\Omega$ is locally an hyperplane near 0. In the sequel we denote by $B_R(a)$ the open ball of center $a$ and radius $R > 0$ and $B_R = B_R(0)$. We also set $B_R^+(a) := \mathbb{R}^N_+ \cap B_R(a)$, $B_R^+ := \mathbb{R}^N_+ \cap B_R$, $B_R^-(a) := \mathbb{R}^N_- \cap B_R(a)$ and $B_R^- := \mathbb{R}^N_- \cap B_R$.

**Theorem E** *Assume $0 < p-1 < q < q_* < p \le N$, $R > 0$ and $\Omega_R := \Omega \cap B_R = B_R^+$. Then for any $k > 0$ there exists a unique $u := u_k \in C(\overline{\Omega}_R \setminus \{0\})$, solution of (1.1) in $\Omega_R$, vanishing on $\partial\Omega_R \setminus \{0\}$ and such that*

$$\lim_{x\to 0} \frac{u_k(x)}{\psi_*(x)} = k. \tag{1.12}$$

*Furthermore $\lim_{k\to\infty} u_k = u_\infty$ and*

$$\lim_{x\to 0} |x|^{\beta_q} u_\infty(x) = \omega_*(|x|^{-1}x). \tag{1.13}$$

When $p = N$ and $N-1 < q < N - \frac{1}{2}$ we prove that the previous result holds if $\Omega$ is any $C^2$ by using the conformal invariance of $\Delta_N$. Finally, the isolated singularities of positive solutions of (1.1) are completely described by the two types of singular solutions obtained in the previous theorem and we prove:

**Theorem F** *Assume $0 < p-1 < q < q_* < p \le N$, $R > 0$ and $\Omega_R = B_R^+$. If $u \in C(\overline{\Omega} \setminus \{0\})$ is a positive solution of (1.1) in $\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$, then*
*(i) either there exists $k \ge 0$ such that*

$$\lim_{x\to 0} \frac{u(x)}{\psi_*(x)} = k. \tag{1.14}$$

*(ii) or*

$$\lim_{x\to 0} |x|^{\beta_q} u(x) = \omega_*(|x|^{-1}x). \tag{1.15}$$

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# 2 A priori estimates

## 2.1 The gradient estimates and its applications

We denote by $d(x)$ the distance from $x \in \overline{\Omega}$ to $\partial\Omega$ and we recall the following estimate proved in [7].

**Proposition 2.1** *Assume $q > p-1$ and $u$ is a $C^1$ solution of* (1.1) *in a domain $\Omega$. Then*

$$|\nabla u(x)| \le c_{N,p,q}(d(x))^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega. \tag{2.1}$$

The first application is a pointwise upper bound for solutions with isolated singularities.

**Corollary 2.2** *Assume $q > p-1 > 0$, $R^* > 0$ and $\Omega$ is a domain containing $0$ such that $d(0) \ge 2R^*$. Then for any $x \in B_{R^*} \setminus \{0\}$, and $0 < R \le R^*$, any $u \in C^1(\Omega \setminus \{0\})$ solution of* (1.1) *in $\Omega \setminus \{0\}$) satisfies*

$$|u(x)| \le c_{N,p,q} \left| |x|^{\frac{q-p}{q+1-p}} - R^{\frac{q-p}{q+1-p}} \right| + \max\{|u(z)| : |z| = R\}, \tag{2.2}$$

*if $p \ne q$, and*

$$|u(x)| \le c_{N,p} \left(\ln R - \ln |x|\right) + \max\{|u(z)| : |z| = R\}, \tag{2.3}$$

*if $p = q$.*

The second estimate corresponds to solutions with boundary blow-up.

**Corollary 2.3** *Assume $q > p-1 > 0$, $\Omega$ is a bounded domain with a $C^2$ boundary. Then there exists $\delta_1 > 0$ such that if we denote $\Omega_{\delta_1} := \{z \in \Omega : d(z) \le \delta_1\}$, any $u \in C^1(\Omega)$ solution of* (1.1) *in $\Omega$ satisfies*

$$|u(x)| \le c_{N,p,q} \left| (d(x))^{\frac{q-p}{q+1-p}} - \delta_1^{\frac{q-p}{q+1-p}} \right| + \max\{|u(z)| : d(z) = \delta_1\} \quad \forall x \in \Omega_{\delta_1} \tag{2.4}$$

*if $p \ne q$, and*

$$|u(x)| \le c_{N,p,q} \left(\ln \delta_1 - \ln d(x)\right) + \max\{|u(z)| : d(z) = \delta_1\} \quad \forall x \in \Omega_{\delta_1} \tag{2.5}$$

*if $p = q$.*

*Remark.* As a consequence of (2.5) there holds for $p > q > p-1$

$$u(x) \le \left(c_{N,p,q} + K \max\{|u(z)| : d(z) \ge \delta_1\}\right)(d(x))^{\frac{q-p}{q+1-p}} \quad \forall x \in \Omega \tag{2.6}$$

where $K = (\mathrm{diam}(\Omega))^{\frac{p-q}{q+1-p}}$, with the standard modification if $p = q$.

As a variant of Corollary 2.3 we have an upper estimate of solutions in an exterior domain.

**Corollary 2.4** *Assume $q > p-1 > 0$, $R > 0$ and $u \in C^1(B^c_{R_0})$ is any solution of (1.1) in $B^c_{R_0}$. Then for any $R > R_0$ there holds*

$$|u(x)| \le c_{N,p,q}\left|(|x|-R_0)^{\frac{q-p}{q+1-p}} - (R-R_0)^{\frac{q-p}{q+1-p}}\right| + \max\{|u(z)| : |z| = R\} \quad \forall x \in B^c_R \tag{2.7}$$

*if $p \ne q$ and*

$$|u(x)| \le c_{N,p,q}\left(\ln(|x|-R_0) - \ln(R-R_0)\right) + \max\{|u(z)| : |z| = R\} \quad \forall x \in B^c_R \tag{2.8}$$

*if $p = q$.*

*Proof.* The proof is a consequence of the identity

$$u(x) = u(z) + \int_0^1 \frac{d}{dt} u(tx + (1-t)z)dt = \int_0^1 \langle \nabla u(tx + (1-t)z), x - z\rangle dt$$

where $z = \frac{R}{|x|}x$. Since

$$|\nabla u(tx + (1-t)z)| \le C_{N,p,q}(t\,|x| + (1-t)R - R_0)^{-\frac{1}{q+1-p}}$$

by estimate (2.1), the result follows by integration. □

## 2.2 Boundary a priori estimates

The next result is the extension of a regularity estimate dealing with singularity in a domain proved in [7].

**Lemma 2.5** *Assume $p-1 < q < p$, $\Omega$ is a bounded $C^2$ domain such that $0 \in \partial\Omega$. Let $u \in C^1(\overline{\Omega} \setminus \{0\})$ be a solutions of (1.1) in $\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies*

$$|u(x)| \le \phi(|x|) \qquad \forall x \in \Omega, \tag{2.9}$$

*where $\phi : \mathbb{R}^*_+ \mapsto \mathbb{R}_+$ is continuous, nonincreasing and satisfies $\phi(rs) \le \gamma\phi(r)\phi(s)$ and $r^{\frac{p-q}{q+1-p}}\phi(r) \le c$ for some $\gamma, c > 0$ and any $r, s > 0$. Then there exist $\alpha \in (0,1)$ and $c_2 = c_2(p,q,\Omega) > 0$ such that*

$$\begin{array}{lll} (i) & |\nabla u(x)| \le c_2\phi(|x|)\,|x|^{-1} & \forall x \in \Omega, \\ (ii) & |\nabla u(x) - \nabla u(y)| \le c_2\phi(|x|)\,|x|^{-1-\alpha}\,|x-y|^{\alpha} & \forall x, y \in \Omega,\ |x| \le |y|\,. \end{array} \tag{2.10}$$

*Furthermore*

$$u(x) \le c_2\phi(|x|)\frac{d(x)}{|x|} \qquad \forall x \in \Omega. \tag{2.11}$$

*Proof.* For $\ell > 0$, we set $\Omega^\ell := \frac{1}{\ell}\Omega$. If $\ell \in (0,1]$ is bounded the curvature of $\partial\Omega^\ell$ remains uniformly bounded. As in [12, p 622], there exists $0 < \delta_0 \leq 1$ and an involutive diffeomorphism $\psi$ from $\overline{B}_{\delta_0} \cap \overline{\Omega}^{\delta_0}$ into $\overline{B}_{\delta_0} \cap (\Omega^{\delta_0})^c$ which is the identity on $\overline{B}_{\delta_0} \cap \partial\Omega^{\delta_0}$ and such that $D\psi(\xi)$ is the symmetry with respect to the tangent plane $T_\xi\partial\Omega$ for any $\xi \in \partial\Omega \cap \overline{B}_{\delta_0}$. We extend any function $v$ defined in $\overline{B}_{\delta_0} \cap \overline{\Omega}^{\delta_0}$ and vanishing on $\overline{B}_{\delta_0} \cap \partial\Omega^{\delta_0}$ into a function $\tilde{v}$ defined in $\overline{B}_{\delta_0}$ by

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in \overline{B}_{\delta_0} \cap \overline{\Omega}^{\delta_0} \\ -v \circ \psi(x) & \text{if } x \in \overline{B}_{\delta_0} \cap (\Omega^{\delta_0})^c, \end{cases} \tag{2.12}$$

If $v \in C^1(\overline{B}_{\delta_0} \cap \overline{\Omega}^{\delta_0})$ is a solution of (1.1) in $B_{\delta_0} \cap \Omega^{\delta_0}$ which vanishes on $\partial\Omega^{\delta_0} \cap \overline{B}_{\delta_0}$, $\tilde{v}$ satisfies

$$-\sum_j \frac{\partial}{\partial x_j}\tilde{A}_j(x, \nabla\tilde{v}) + B(x, \nabla\tilde{v}) = 0 \quad \text{in } B_{\delta_0}. \tag{2.13}$$

As in [12, (2.37)] the $A_j$ satisfy the following estimates

$$\begin{aligned} &(i) && \tilde{A}_j(x, 0) = 0 \\ &(ii) && \sum_{i,j} \frac{\partial}{\partial \eta_i}\tilde{A}_j(x,\eta)\xi_i\xi_j \geq \Gamma_1 |\eta|^{p-1} |\xi|^2 \\ &(iii) && \sum_{i,j} \left|\frac{\partial}{\partial \eta_j}\tilde{A}_j(x,\eta)\right| \leq \Gamma_2 |\eta|^{p-2}, \end{aligned} \tag{2.14}$$

as for $B$ there holds

$$|B(x,\eta)| \leq \Gamma_3(1 + |\eta|)^p. \tag{2.15}$$

These estimates are the ones needed to apply Tolksdorf's result [30, Th 1,2]. There exists a constant $C$, such that for any ball $\overline{B}_{3R} \subset \overline{B}_{\delta_0}$, there holds

$$\|\nabla\tilde{v}\|_{L^\infty(B_R)} \leq C \tag{2.16}$$

where $C$ depends on the constants $\Gamma_k$ $(k = 1, 2, 3)$, $N$, $p$ and $\|\tilde{v}\|_{L^\infty(B_{3R})}$. We define

$$\Phi_\ell[u](y) = \frac{1}{\phi(\ell)}u(\ell y) \qquad \forall y \in \Omega^\ell. \tag{2.17}$$

Then $\Phi_\ell[u] := u_\ell$ satisfies

$$|u_\ell(y)| \leq \frac{\phi(\ell\,|y|)}{\phi(\ell)} \leq \gamma\phi(|y|) \qquad \forall y \in \Omega^\ell \tag{2.18}$$

and

$$-\Delta_p u_\ell + (\ell^{\beta_q}\phi(\ell))^{q+1-p}\,|\nabla u_\ell|^q = 0 \qquad \text{in } \Omega^\ell. \tag{2.19}$$

We extend $u_\ell$ by the formula (2.12) into a function $\tilde{u}_\ell$ which satisfies

$$-\sum_j \frac{\partial}{\partial y_j}\tilde{A}_j(y, \nabla\tilde{u}_\ell) + (\ell^{\beta_q}\phi(\ell))^{q+1-p}B(y, \nabla\tilde{u}_\ell) = 0 \quad \text{in } B_{\delta_0}. \tag{2.20}$$

For $0 < |x| < \delta_0$ there exists $\ell \in (0,2)$ such that $\frac{\delta_0 \ell}{2} \leq |x| \leq \delta_0 \ell$. Then $y \mapsto \tilde{u}_\ell(y)$ with $y = \frac{x}{\ell}$ satisfies (2.20) in $B_{\delta_0}$ and $|\tilde{u}_\ell(y)| \leq \gamma_* \phi(|y|)$ since $\psi$ is a diffeomorphism and $D\psi(\xi) \in O(N)$ for any $\xi \in \partial\Omega \cap B_{\delta_0}$. The function $\tilde{u}_\ell$ remains bounded on any ball $B_{3R}(z) \subset \Gamma := \{y \in \mathbb{R}^N : \frac{\delta_0}{2} < |y| < \delta_0\}$, therefore $|\nabla \tilde{u}_\ell(y)| \leq c$ for any $y \in B_R(z)$, for some constant $c > 0$. This implies

$$|\nabla u(x)| \leq c\gamma_* \delta_0 \phi(\tfrac{2}{\delta_0}) \phi(|x|) |x|^{-1} \quad \forall x \in \Omega \cap B_{\delta_0}. \tag{2.21}$$

For proving 2.10)-(ii) there exists $\alpha \in (0,1)$ such that $|\nabla \tilde{u}_\ell(y) - \nabla \tilde{u}_\ell(y')| \leq c\,|y - y'|^\alpha$ for all $y$ and $y'$ belonging to $B_R(z)$. This implies the claim.

Let $0 < \delta_1 \leq \delta_0$ such that at any boundary point $z$ there exist two closed balls of radius $\delta_1$ tangent to $\partial\Omega$ at $z$ and included in $\Omega \cup \{z\}$ and in $\overline{\Omega}^c \cup \{z\}$ respectively ($\delta_1$ corresponds to the maximal radius of the interior and exterior sphere condition). Let $a \in \partial\Omega$ and $x \in \Omega$ such that $|x| = |a| = r \leq \delta_1$ and let $b = -r\mathbf{n}_0$ where $\mathbf{n}_0$ is the normal outward unit vector to $\partial\Omega$ at 0. Let $\theta \in (0, \frac{\pi}{2})$ be the angle between $\overrightarrow{0a}$ and $\overrightarrow{0x}$. Consider the path $\gamma$ from $a$ to $x$ defined by $\gamma(t) = \cos(t\theta)a + \sin(t\theta)b$ with $0 \leq t \leq 1$. Then $x = \cos\theta a + \sin\theta b$ and

$$u(x) = \int_0^1 \frac{d}{dt} u \circ \gamma(t) dt = \int_0^1 \theta \langle \nabla u \circ \gamma(t), \cos(t\theta)b - \sin(t\theta)a \rangle dt$$

Thus, by Cauchy-Schwarz and using (2.9),

$$u(x) \leq \theta\, |\nabla u \circ \gamma(t)| \leq c_2' \theta\, |x|^{-\frac{1}{q+1-p}},$$

where $c_2' = c_2'(p, q, \Omega)$. But there exists $c_2'' > 0$ depending only on $\delta_1$ and the curvature of $\partial\Omega$ such that $c^{-1}\theta \leq d(x) \leq c\theta$. This implies (2.11). □

**Lemma 2.6** *Assume $p - 1 < q \leq p$, $\Omega$ is a bounded $C^2$ domain such that $0 \in \partial\Omega$ and $R_0 = \max\{|z| : z \in \Omega\}$. If $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$ is a positive solution of* (1.1) *which vanishes on $\partial\Omega \setminus \{0\}$, it satisfies*

$$u(x) \leq \begin{cases} c_1 \left( |x|^{\frac{q-p}{q+1-p}} - R_0^{\frac{q-p}{q+1-p}} \right) & \text{if } q < p \\ (p-1) \ln\left(\frac{R_0}{|x|}\right) & \text{if } q = p \end{cases} \tag{2.22}$$

*for all $x \in \Omega$, where $c_1 = c_1(p, q) > 0$.*

*Proof.* For $\epsilon > 0$ we denote by $P_\epsilon : \mathbb{R} \mapsto \mathbb{R}_+$ the function defined by

$$P_\epsilon(r) = \begin{cases} 0 & \text{if } 0 \leq r \leq \epsilon \\ -\frac{r^4}{2\epsilon^3} + \frac{3r^3}{\epsilon^2} - \frac{6r^2}{\epsilon} + 5r - \frac{3\epsilon}{2} & \text{if } \epsilon < r < 2\epsilon \\ r - \frac{3\epsilon}{2} & \text{if } r \geq 2\epsilon, \end{cases} \tag{2.23}$$

and by $u_\epsilon$ the extension of $P_\epsilon(u)$ by zero outside $\Omega$. There exists $R_0$ such that $\Omega \subset B_{R_0}$ . Since $0 \leq P_\epsilon(r) \leq |r|$ and $P_\epsilon$ is convex, $u_\epsilon \in C(\mathbb{R}^N \setminus \{0\}) \cap W^{1,p}_{loc}(\mathbb{R}^N \setminus \{0\})$ and

$$-\Delta_p u_\epsilon + |\nabla u_\epsilon|^q \leq 0 \qquad \text{in } \mathbb{R}^N.$$

Let $R > R_0$. If $p-1 < q < p$

$$U_{\epsilon,R}(|x|) = c_1 \left( (|x|-\epsilon)^{\frac{q-p}{q+1-p}} - (R-\epsilon)^{\frac{q-p}{q+1-p}} \right) \qquad \text{in } B_R \setminus B_\epsilon, \tag{2.24}$$

with $c_1 = (p-q)^{-1}(q+p-1)^{\frac{q-p}{q+1-p}}$. Then $-\Delta_p U_\epsilon + |\nabla U_\epsilon|^q \geq 0$. Since $u_\epsilon$ vanishes on $\partial B_R$ and is finite on $\partial B_\epsilon$, it follows $u_\epsilon \leq U_\epsilon$. Letting successively $\epsilon \to 0$ and $R \to \infty$ yields to (2.22). If $q = p$ we take

$$U_{\epsilon,R}(|x|) = (p-1)\ln\left(\frac{R-\epsilon}{|x|-\epsilon}\right) \qquad \text{in } B_R \setminus B_\epsilon, \tag{2.25}$$

which turns out to be a super solution of (1.1); the end of the proof is similar. □

As a consequence of Lemma 2.5 and Lemma 2.6, we have.

**Corollary 2.7** *Let $p, q$ $\Omega$ and $u$ be as in Lemma 2.6. Then there exists a constant $c_3 = c_3(p,q,\Omega) > 0$ such that*

$$|\nabla u(x)| \leq c_3 \, |x|^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega \tag{2.26}$$

*and*

$$u(x) \leq c_3 d(x) \, |x|^{-\frac{1}{q+1-p}} \qquad \forall x \in \overline{\Omega} \setminus \{0\}. \tag{2.27}$$

*Remark.* If $\Omega$ is locally flat near 0 estimates (2.26) and (2.26) are valid without any sign assumption on $u$. More precisely, if $\partial\Omega \cap B_{\delta_0} = T_0\partial\Omega \cap B_{\delta_0}$ we can perform the reflexion of $u$ thrue the tangent plane $T_0\partial\Omega$ to $\partial\Omega$ at 0 and the new function $\tilde{u}$ is a solution of (1.1) in $B_{\delta_0} \setminus \{0\}$. By Proposition 2.1, it satisfies

$$|\nabla \tilde{u}(x)| \leq c_{N,p,q} \, |x|^{-\frac{1}{q+1-p}} \qquad \forall x \in B_{\frac{\delta_0}{2}} \setminus \{0\}. \tag{2.28}$$

Integrating this relation as in Corollary 2.2, we derive that for any $x \in B_{\frac{\delta_0}{2}} \cap \Omega$, there holds

$$|u(x)| \leq \begin{cases} c_{N,p,q}\left(|x|^{-\beta_q} - (\frac{\delta_0}{2})^{-\beta_q}\right) + \max\{|u(z)| : |z| = \frac{\delta_0}{2}\} & \text{if } p \neq q \\ c_{N,p} \ln\left(\frac{\delta_0}{2|x|}\right) + \max\{|u(z)| : |z| = \frac{\delta_0}{2}\} & \text{if } p = q. \end{cases} \tag{2.29}$$

In the next result we authorize the boundary singular set be a compact set.

**Proposition 2.8** *Let $p-1 < q < p$. There exist $0 \leq r^* \leq \delta_1$ and $c_4 = c_4(N,p,q) > 0$ such that for any nonempty compact set $K \subset \partial\Omega$, $K \neq \partial\Omega$ and any positive solution $u \in C(\overline{\Omega} \setminus K) \cap C^1(\Omega)$ of (1.1) which vanishes on $\partial\Omega \setminus K$, there holds*

$$u(x) \leq c_4 d(x)(d_K(x))^{-\frac{1}{q+1-p}} \qquad \forall x \in \partial\Omega \text{ s.t. } d(x) \leq r^*, \tag{2.30}$$

*where $d_K(x) = \operatorname{dist}(x, K)$.*

*Proof. Step 1: Tangential estimates.* Let $x \in \Omega$ such that $d(x) \le \delta_1$. We denote by $\sigma(x)$ the projection of $x$ onto $\partial\Omega$, unique since $d(x) \le \delta_1$. Let $r\,, r', \tau > 0$ such that $\frac{3}{4}r < r' < \frac{7}{8}r$ and $0 < \tau \le \frac{r'}{2}$ and put $\omega_{\tau,x} = \sigma(x) + \tau \mathbf{n}_{\sigma(x)}$. Since $\partial\Omega$ is $C^2$, there exists $0 < r^* \le \delta_1$ depending on $\Omega$ such that $d_K(\omega_{\tau,x}) > \frac{7}{8}r$ whenever $d(x) \le r^*$. Let $A > 0$ and $B > 0$ to be specified later on; we define $\tilde{v}(s) = A(r'-s)^{\frac{q-p}{q+1-p}} - B$ and $v(y) = \tilde{v}(|y - \omega_{\tau,x}|)$ in $[0, r')$ and $B_{r'}(\omega_{\tau,x})$. Then

$$|\tilde{v}'|^{p-2}\left(|\tilde{v}'|^{q+2-p} - (p-1)\tilde{v}'' - \frac{N-1}{s}\tilde{v}'\right) = A^{p-1}\left(\frac{p-q}{q+1-p}\right)^{p-1}(r'-s)^{-\frac{q}{q+1-p}}X(s)$$

where

$$X(s) = \left(A\frac{p-q}{q+1-p}\right)^{q+1-p} - \frac{p-1}{q+1-p} - \frac{(N-1)(r'-s))}{s}.$$

For any $\tau \in (0, r')$ there exists $A > 0$ such that

$$\left(A\frac{p-q}{q+1-p}\right)^{q+1-p} \ge \frac{p-1}{q+1-p} + \frac{(N-1)(r'-s))}{s} \qquad \forall \tau \le s \le r'.$$

This implies

$$-\Delta_p v + |\nabla v|^q \ge 0 \qquad \text{in } B_{r'}(\omega_{\tau,x}) \setminus B_\tau(\omega_{\tau,x}). \tag{2.31}$$

Next we take $B = A(r'-\tau)^{\frac{q-p}{q+1-p}}$, thus $v = 0$ on $\partial B_\tau(\omega_{\tau,x})$. Clearly $B_\tau(\omega_{\tau,x}) \subset \overline{\Omega}^c$ since $\tau < \delta_1$. Therefore $v \ge 0 = u$ on $\partial\Omega \cap B_{r'}(\omega_{\tau,x})$ and $u \le v = \infty$ on $\Omega \cap \partial B_{r'}(\omega_{\tau,x})$. By the comparison principle, $v \ge u$ in $\Omega \cap B_{r'}(\omega_{\tau,x})$. In particular

$$u(x) \le v(x) \le A(r' - \tau - d(x))^{\frac{q-p}{q+1-p}} - A(r'-\tau)^{\frac{q-p}{q+1-p}}.$$

We take now $\tau = \frac{r'}{2}$ and $d(x) \le \frac{r}{4}$ and we get by the mean value theorem

$$u(x) \le c_4' r'^{-\frac{1}{q+1-p}} d(x) = c_4' d(x)(d_K'(x))^{-\frac{1}{q+1-p}}, \tag{2.32}$$

with $c_4' = c_4'(p,q) > 0$ Letting $r' \to \frac{7}{8}r$, we get (2.11).
*Step 2: Global estimates.* If $d(x) \ge \frac{1}{4}d_K(x)$, there holds

$$d(x)(d_K(x))^{-\frac{1}{q+1-p}} \ge 2^{\frac{2}{q+1-p}}(d(x))^{\frac{q-p}{q+1-p}}.$$

We combine (2.11) with (2.6) and obtain (2.11). □

*Remark.* Under the assumption of Proposition 2.8, it follows from the maximum principle that $u$ is upper bounded in the set $\Omega'_{r^*} := \{x \in \Omega : d(x) > r^*\} = \Omega \setminus \overline{\Omega}_{r^*}$ by the solution $w$ of

$$\begin{aligned} -\Delta_p w + |\nabla w|^q &= 0 && \text{in } \Omega_{r^*} \\ w &= c_4 d(x)(d_K(x))^{-\frac{1}{q+1-p}} && \text{in } \partial\Omega_{r^*}, \end{aligned} \tag{2.33}$$

and $w$ itself is bounded by $d^* = \max\{cd(x)(d_K(x))^{-\frac{1}{q+1-p}} : d(x) = r^*\}$.

We prove here the boundary Harnack inequality. We recall that $\delta_1$ has already been defined, and that the interior and exterior sphere conditions holds in the set $\{x \in \mathbb{R}^N : \text{dist}\,(x\partial\Omega) \le \delta_1\}$.

**Proposition 2.9** *Let $q > p-1$ and $0 \in \partial\Omega$. Then there exists $c_5 = c_5(N, p, q, \Omega) > 0$ such that for any positive solution $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}) \cap C^1(\Omega)$ of (1.1) in $\Omega$, vanishing on $\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$, there holds*

$$\frac{u(y)}{c_5 d(y)} \leq \frac{u(x)}{d(x)} \leq c_5 \frac{u(y)}{d(y)} \tag{2.34}$$

*for all $x, y \in B_{\frac{2\delta_1}{3}} \cap \Omega$ such that $\frac{1}{2}|x| \leq |y| \leq 2|x|$.*

We also recall the following result [3].

**Lemma 2.10** *Assume that $a \in \partial\Omega$, $0 < r < \delta_1$ and $h > 1$ is an integer. There exists an integer $N_0$ depending only on $\delta_1$ such that for any points $x$ and $y$ in $\Omega \cap B_{\frac{3r}{2}}(a)$ verifying $\min\{d(x), d(y)\} \geq r/2^h$, there exists a connected chain of balls $B_1, ..., B_j$ with $j \leq N_0 h$ such that*

$$\begin{array}{l} x \in B_1, y \in B_j, \quad B_i \cap B_{i+1} \neq \emptyset \text{ for } 1 \leq i \leq j-1 \\ \text{and } 2B_i \subset B_{2r}(Q) \cap \Omega \text{ for } 1 \leq i \leq j. \end{array} \tag{2.35}$$

The next result is a standard Harnack inequality.

**Lemma 2.11** *Assume $a \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2\delta^*}{3}}$ and $0 < r \leq |a|/4$. Let $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^1(\Omega)$ be a positive solution of (1.1) vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$. Then there exists a positive constant $c_6 > 1$ depending on $N$, $p$, $q$ and $\delta_1$ such that*

$$u(x) \leq c_6^h u(y), \tag{2.36}$$

*for every $x, y \in B_{\frac{3r}{2}}(a) \cap \Omega$ such that $\min\{d(x), d(y)\} \geq r/2^h$ for some $h \in \mathbb{N}$.*

*Proof.* We first notice that for any $\ell > 0$, $T_\ell[u]$ satisfies (1.1) in $\Omega^\ell := \ell^{-1}\Omega$ where $T_\ell$ is defined by

$$T_\ell[u](x) = \ell^{\frac{p-q}{q+1-p}} u(\ell x). \tag{2.37}$$

If we take in particular $\ell = |a|$, we can assume $|a| = 1$ thus the curvature of the domain $\Omega^{|a|}$ remains bounded. By Proposition 2.8

$$u(x) \leq c_6' \quad \forall x \in B_{2r}(a) \cap \Omega \tag{2.38}$$

where $c_6'$ depends on $N$, $q$, $\delta_1$. Then we proceed as in [22], using Lemma 2.10 and Harnack inequality as quoted in [31, Corollary 10]. □

Since the solutions are Hölder continuous, there holds as in [31, Theorem 4.2]

**Lemma 2.12** *Assume the assumptions on $a$ and $u$ of Lemma 2.11 are fulfilled. If $b \in \partial\Omega \cap B_r(a)$ and $0 < s \leq 2^{-1}r$, there exist two positive constants $\delta$ and $c_7$ depending on $N$, $p$, $q$ and $\Omega$ such that*

$$u(x) \leq c_7 \frac{|x-b|^\delta}{s^\delta} \max\{u(z) : z \in B_r(b) \cap \Omega\} \tag{2.39}$$

*for every $x \in B_s(b) \cap \Omega$.*

As a consequence we derive the following Carleson type estimate.

**Lemma 2.13** *Assume* $a \in (\partial\Omega\backslash\{0\})\cap B_{\frac{2\delta_1}{3}}$ *and* $0 < r \leq |a|/8$. *Let* $u \in C(\Omega\cup((\partial\Omega\backslash\{0\})\cap B_{2\delta_1}))\cap C^2(\Omega)$ *be a positive solution of* (1.1) *vanishing on* $(\partial\Omega\setminus\{0\})\cap B_{2\delta_1}$. *Then there exists a constant* $c_8$ *depending only on* $N$, $p$, $q$ *and* $\delta_1$ *such that*

$$u(x) \leq c_8 u(a - \tfrac{r}{2}\mathbf{n}_a) \quad \forall x \in B_r(a)\cap\Omega. \tag{2.40}$$

*Proof.* By Lemma 2.11 it is clear that for any integer $h$ and $x \in B_r(a)\cap\Omega$ such that $d(x) \geq 2^{-h}r$, there holds

$$u(x) \leq c_6^h u(a - \tfrac{r}{2}\mathbf{n}_a). \tag{2.41}$$

Therefore $u$ satisfies inequality (2.39) as any Hölder continuous function does it. The proof that the constant is independent of $r$ and $u$ is more delicate. It is done in [3, Lemma 2.4] for linear equations, but it based only on Lemma 2.12 and a geometric construction, thus it is also valid in our case. □

**Lemma 2.14** *Assume* $a \in (\partial\Omega\backslash\{0\})\cap B_{\frac{2\delta_1}{3}}$ *and* $0 < r \leq |a|/8$. *Let* $u \in C(\Omega\cup((\partial\Omega\backslash\{0\})\cap B_{2\delta_1}))\cap C^2(\Omega)$ *be a positive solution of* (1.1) *vanishing on* $(\partial\Omega\setminus\{0\})\cap B_{2\delta_1}$. *Then there exist* $\alpha \in (0, 1/2)$ *and* $c_9 > 0$ *depending on* $N$, $p$, $q$ *and* $\delta_1$ *such that*

$$\frac{1}{c_9}\frac{t}{r} \leq \frac{u(b - t\mathbf{n}_b)}{u(a - \frac{r}{2}\mathbf{n}_a)} \leq c_9\frac{t}{r} \tag{2.42}$$

*for any* $b \in B_r(a)\cap\partial\Omega$ *and* $0 \leq t < \frac{\alpha}{2}r$.

*Proof.* It is similar to the one of [22, Lemma 3.15]. □

*Proof of Proposition 2.9.* Assume $x \in B_{\frac{2\delta_1}{3}}\cap\Omega$ and set $r = \frac{|x|}{8}$.

*Step 1: Tangential estimate: we suppose* $d(x) < \frac{\alpha}{2}r$. Let $a \in \partial\Omega\setminus\{0\}$ such that $|a| = |x|$ and $x \in B_r(a)$. By Lemma 2.14,

$$\frac{8}{c_9}\frac{u(a - \frac{r}{2}\mathbf{n}_a)}{|x|} \leq \frac{u(x)}{d(x)} \leq 8c_9\frac{u(a - \frac{r}{2}\mathbf{n}_a)}{|x|}. \tag{2.43}$$

We can connect $a - \frac{r}{2}\mathbf{n}_a$ with $-2r\mathbf{n}_0$ by $m_1$ (depending only on $N$) connected balls $B_i = B_{\frac{r}{4}}(x_i)$ with $x_i \in \Omega$ and $d(x_i) \geq \frac{r}{2}$ for every $1 \leq i \leq m_1$. It follows from (2.40) that

$$c_6^{-m_1}u(-2r\mathbf{n}_0) \leq u(a - \tfrac{r}{2}\mathbf{n}_a) \leq c_6^{m_1}u(-2r\mathbf{n}_0),$$

which, together with (2.43) leads to

$$\frac{1}{c_9'}\frac{u(-2r\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq c_9'\frac{u(-2r\mathbf{n}_0)}{|x|}, \tag{2.44}$$

with $c_9' = 8c_9c_6^{m_1}$.

*Step 2: Internal estimate: we suppose* $d(x) \geq \frac{\alpha}{2}r$. We can connect $-2r\mathbf{n}_0$ with $x$ by $m_2$ (depending only on $N$) connected balls $B'_i = B_{\frac{\alpha r}{4}}(x'_i)$ with $x'_i \in \Omega$ and $d(x'_i) \geq \frac{\alpha}{2}r$ for every $1 \leq i \leq m_2$. By Harnack and Carleson inequalities (2.36) and (2.40) and since $\frac{\alpha}{4}|x| < d(x) \leq |x|$, we get

$$\frac{\alpha}{4c_6'^{m_2}}\frac{u(-2r\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq \frac{4c_6'^{m_2}}{\alpha}\frac{u(-2r\mathbf{n}_0)}{|x|}. \tag{2.45}$$

*Step 3: End of proof.* Suppose $\frac{|x|}{2} \leq s \leq 2|x|$, we can connect $-2r\mathbf{n}_Q$ with $-s\mathbf{n}_Q$ by $m_3$ (depending only on $N$) connected balls $B''_i = B_{\frac{r}{2}}(x''_i)$ with $x''_i \in \Omega$ and $d(x''_i) \geq r$ for every $1 \leq i \leq m_3$. This fact, jointly with (2.44) and (2.45), yields

$$\frac{1}{c_{10}}\frac{u(-s\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq c_{10}\frac{u(-s\mathbf{n}_0)}{|x|} \tag{2.46}$$

where $c_{10} = c_{10}(N, q, \Omega)$. Finally, if $y \in B_{\frac{2r_0}{3}} \cap \Omega$ satisfies $\frac{|x|}{2} \leq |y| \leq 2|x|$, then by applying twice (2.46) we get (2.34) with $c_5 = c_{10}^2$. □

As a consequence of Proposition 2.9, we have

**Corollary 2.15** *Assume* $q > p-1$ *and* $0 \in \partial\Omega$. *Then there exists* $c_{11} = c_{11}(N, p, q, \Omega) > 0$ *such that for any positive solutions* $u_1$, $u_2 \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}) \cap C^1(\Omega)$ *of* (1.1) *in* $\Omega$, *vanishing on* $\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$, *there holds*

$$\sup\left\{\frac{u_1(y)}{u_2(y)} : y \in B_r \setminus B_{\frac{r}{2}}\right\} \leq c_{11} \inf\left\{\frac{u_1(y)}{u_2(y)} : y \in B_r \setminus B_{\frac{r}{2}}\right\}. \tag{2.47}$$

# 3 Boundary singularities

## 3.1 Strongly singular solutions

We next consider the equation (1.1) in $\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}^*_+\}$. We denote by $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$ the spherical coordinates in $\mathbb{R}^N$ and

$$S^{N-1}_+ = \left\{(\sin\phi\sigma', \cos\phi) : \sigma' \in S^{N-2}, \phi \in [0, \frac{\pi}{2})\right\}.$$

Set

$$\beta_q := \frac{p-q}{q+1-p} \quad \text{and} \quad \Lambda_{\beta_q} = \beta_q(p-1) + p - N$$

If $v(x) = r^{-\beta}\omega(\sigma)$ satisfies (1.1) in $\mathbb{R}^N_+$ and vanishes on $\overline{\mathbb{R}^N_+} \setminus \{0\}$, then $\beta = \beta_q$ and $\omega$ is a solution of

$$\begin{aligned} -div\left(\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla'\omega\right) - \beta_q\Lambda_{\beta_q}\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}}\omega \\ + \left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{q}{2}} = 0 \quad \text{in } S^{N-1}_+ \\ \omega = 0 \quad \text{on } \partial S^{N-1}_+. \end{aligned} \tag{3.1}$$

where $\nabla'$ denotes the covariant derivative on $S^{N-1}$ identified with the tangential derivative. There exists a unique $\beta_* = \beta_*(N,p) > 0$ such that the function $(r,\sigma) \mapsto r^{-\beta_*}\psi(\sigma)$ is positive, p-harmonic in $\mathbb{R}^N_+$ and vanishes on $\overline{\mathbb{R}^N_+} \setminus \{0\}$ (see [30], [25]). The function $\psi$ is defined up to a multiplicative constant and it satisfies

$$\begin{aligned} -div\left(\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi\right) - \beta_*\Lambda_{\beta_*}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi = 0 \quad \text{in } S^{N-1}_+ \\ \psi = 0 \quad \text{on } \partial S^{N-1}_+. \end{aligned} \tag{3.2}$$

We denote by $\psi_*$ the solution which maximal value 1. The coefficient $\beta_*$ is unknown, except when $N = p$, and it is 1, or $p = 2$ and it is $N-1$. Because of uniqueness $\psi$ depends only on the azimuthal variable $\theta_{N-1} = \cos^{-1}(\frac{x_N}{|x|})$ (see Appendix II). Our first result is the following

**Theorem 3.1** *If $\beta_q \le \beta_*$ there exists no positive solution to problem* (3.1).

*Proof.* Suppose such a solution $\omega$ exists and put $\theta = \beta_q/\beta_*$. Then $0 < \theta \le 1$. Set $\eta = \psi\theta$, where $\psi$ is a positive solution of (3.2) and define the operator $\mathcal{T}$ by

$$\begin{aligned} \mathcal{T}(\eta) = -div\left(\left(\beta_q^2\eta^2 + |\nabla'\eta|^2\right)^{\frac{p-2}{2}}\nabla'\eta\right) - \beta_q\Lambda_{\beta_q}\left(\beta_q^2\eta^2 + |\nabla'\eta|^2\right)^{\frac{p-2}{2}}\eta \\ + \left(\beta_q^2\eta^2 + |\nabla'\eta|^2\right)^{\frac{q}{2}}. \end{aligned} \tag{3.3}$$

Since $\nabla\eta = \theta\psi^{\theta-1}\nabla\psi$, $\left(\beta_q^2\eta^2 + |\nabla'\eta|^2\right)^{\frac{p-2}{2}} = \theta^{p-2}\psi^{(\theta-1)(p-2)}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}$, $\left(\beta_q^2\eta^2 + |\nabla'\eta|^2\right)^{\frac{p-2}{2}}\nabla'\eta = \theta^{p-1}\psi^{(\theta-1)(p-1)}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi$, therefore

$$\begin{aligned} \mathcal{T}(\eta) = &-\theta^{p-1}\psi^{(\theta-1)(p-1)}\mathrm{div}\left(\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi\right) \\ &- \theta^{p-1}(\theta-1)(p-1)\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}|\nabla'\psi|^2 \\ &- \beta_q\Lambda_{\beta_q}\theta^{p-2}\psi^{(\theta-1)(p-1)}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi + \theta^q\psi^{(\theta-1)q}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{q}{2}} \end{aligned}$$

But $\beta_q\Lambda_{\beta_q}\theta^{p-2} = \beta_*\Lambda_{\beta_q}\theta^{p-1} \le \beta_*\Lambda_{\beta_*}\theta^{p-1}$ since $\beta_q \le \beta_*$. Using (3.2), we see that $\mathcal{T}(\eta) \ge 0$. Because Hopf lemma is valid, $\partial_{\mathbf{n}}\psi < 0$ on $\partial S^{N-1}_+$. Since $\omega$ is $C^1$ in $\overline{S^{N-1}_+}$ and $\psi$ is defined up to an homothety, there exists a minimal $\psi$ such that $\eta \ge \omega$, and the graphs of $\eta$ and $\omega$ over $\overline{S^{N-1}_+}$ are tangent, either at some $\alpha \in S^{N-1}_+$, or only at point $\alpha \in \partial S^{N-1}_+$. We put $w = \eta - \omega$. Then

$$\mathcal{T}(\eta) - \mathcal{T}(\omega) = \Phi(1) - \Phi(0), \tag{3.4}$$

where $\Phi(t) = \mathcal{T}(\omega_t)$ with $\omega_t = \omega + tw$.

We use local coordinates $(\sigma_1, ..., \sigma_{N-1})$ on $S^{N-1}$ near $\alpha$. We denote by $g = (g_{ij})$ the metric tensor on $S^{N-1}$ and by $g^{jk}$ its contravariant components. Then, for any $\varphi \in C^1(S^{N-1})$,

$$|\nabla\varphi|^2 = \sum_{j,k} g^{jk}\frac{\partial\varphi}{\partial\sigma_j}\frac{\partial\varphi}{\partial\sigma_k} = \langle\nabla\varphi, \nabla\varphi\rangle_g.$$

If $X = (X^1, ...X^d) \in C^1(TS^{N-1})$ is a vector field, we lower indices by setting $X^\ell = \sum_i g^{\ell i} X_i$ and define the divergence of $X$ by

$$div_g X = \frac{1}{\sqrt{|g|}} \sum_\ell \frac{\partial}{\partial \sigma_\ell} \left( \sqrt{|g|} X^\ell \right) = \frac{1}{\sqrt{|g|}} \sum_{\ell,i} \frac{\partial}{\partial \sigma_\ell} \left( \sqrt{|g|} g^{\ell i} X_i \right).$$

We write $\Phi(t) = \Phi_1(t) + \Phi_2(t) + \Phi_3(t)$ where

$$\Phi_1(t) = -\beta_q \Lambda_{\beta_q} \left( \beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2 \right)^{\frac{p-2}{2}} \omega_t,$$

$$\Phi_2(t) = \left( \beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2 \right)^{\frac{q}{2}}$$

and

$$\Phi_3(t) = -\mathrm{div} \left( \left( \beta_\mathrm{q}^2 \omega_\mathrm{t}^2 + |\nabla' \omega_\mathrm{t}|^2 \right)^{\frac{\mathrm{p}-2}{2}} \nabla' \omega_\mathrm{t} \right).$$

Then

$$\Phi_1(1) - \Phi_1(0) = -\sum_j a_j \frac{\partial w}{\partial \sigma_j} - bw$$

where

$$b = \beta_q \Lambda_{\beta_q} \left( \beta_q^2 {\omega_t}^2 + |\nabla \omega_t|^2 \right)^{\frac{p}{2}-2} \left( (p-1) \beta_q^2 \omega_t^2 + |\nabla \omega_t|^2 \right)$$

and

$$a_j = (p-2) \beta_q \Lambda_{\beta_q} \left( \beta_q^2 {\omega_t}^2 + |\nabla \omega_t|^2 \right)^{\frac{p}{2}-2} \omega_t \sum_k g^{jk} \frac{\partial \omega_t}{\partial \sigma_k};$$

$$\Phi_2(1) - \Phi_2(0) = \sum_j c_j \frac{\partial w}{\partial \sigma_j} + dw$$

where

$$d = q \beta_q^2 \left( \beta^2 {\omega_t}^2 + |\nabla \omega_t|^2 \right)^{\frac{q}{2}-1} \omega_t$$

$$c_j = q \left( \beta_q^2 {\omega_t}^2 + |\nabla \omega_t|^2 \right)^{\frac{q}{2}-1} \sum_k g^{jk} \frac{\partial \omega_t}{\partial \sigma_k};$$

$$\begin{aligned} \Phi_3(1) - \Phi_3(0) = &-(p-2) \mathrm{div} \left( \left( \beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2 \right)^{\frac{p-4}{2}} \left( \beta_q^2 \omega_t w + \langle \nabla' \omega_t, \nabla' w \rangle_g \right) \nabla' \omega_t \right) \\ &- \mathrm{div} \left( \left( \beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2 \right)^{\frac{p-2}{2}} \nabla' w \right). \end{aligned}$$

Therefore we have

$$\Phi(1) - \Phi(0) = -\mathrm{div}(A \nabla' w) + \langle B, \nabla' w \rangle_g + Cw := \mathcal{L} w \tag{3.5}$$

where

$$\begin{aligned}\langle AX, X\rangle_g &= \left(\beta_q^2\omega_t^2 + |\nabla'\omega_t|^2\right)^{\frac{p-4}{2}} \left(p-2)\langle\nabla'\omega_t, X\rangle_g^2 + |\nabla'\omega_t|^2|X|^2\right) \\ &\geq \left(\beta_q^2\omega_t^2 + |\nabla'\omega_t|^2\right)^{\frac{p-4}{2}} \min\{1, p-1\}|\nabla'\omega_t|^2|X|^2.\end{aligned} \tag{3.6}$$

and $B$ and $C$ can be computed from the previous expressions. It is important to notice that $\beta_q^2\omega_t^2 + |\nabla'\omega_t|^2$ is bounded between two positive constants $c_1$ and $c_2$ in $\overline{S_+^{N-1}}$. Thus the operator $\mathcal{L}$ is uniformly elliptic with bounded coefficients. Since $w$ is nonnegative and vanishes at some point $\alpha$ where $\nabla' w(\alpha) = 0$ and $w(\alpha) > 0$ or at some boundary point $\alpha$ where $w(\alpha) = 0$ and $\partial_{\mathbf{n}} w(\alpha) < 0$ it follows from the strong maximum principle or Hopf boundary lemma (see [17]) that $w = 0$, contradiction. □

**Theorem 3.2** *Assume $\beta_q > \beta_*$. There exists a unique positive solution $\omega_*$ to problem* (3.1).

*Proof. Existence.* It will follow from [10]. Indeed problem (3.1) can be written

$$\begin{aligned}\mathbf{A}(\omega) := -div\,\mathbf{a}(\omega, \nabla'\omega) = B(\omega, \nabla'\omega) \qquad & \text{in } S_+^{N-1} \\ \omega = 0 \qquad & \text{on } \partial S_+^{N-1}\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}\mathbf{a}(r,\xi) &= \left(\beta_q^2 r^2 + |\xi|^2\right)^{\frac{p-2}{2}} \xi \\ B(r,\xi) &= \beta_q\Lambda_{\beta_q}\left(\beta_q^2 r^2 + |\xi|^2\right)^{\frac{p-2}{2}} r - \left(\beta_q^2 r^2 + |\xi|^2\right)^{\frac{q}{2}}\end{aligned} \tag{3.8}$$

The operator $\mathbf{A}$ is a Leray-Lions operator which satisfies the assumptions (1.6)-(1.8) of [10, Theorem 2.1], and the term $B$ satisfies (1.9),(1.10) in the same article. Therefore existence of a positive solution $\omega \in W_0^{1,p}(S_+^{N-1}) \cap L^\infty(S_+^{N-1})$ is ensured whenever we can find a super solution $\overline{\omega} \in W^{1,p}(S_+^{N-1}) \cap L^\infty(S_+^{N-1})$ and a nontrivial subsolution $\underline{\omega} \in W^{1,p}(S_+^{N-1})$ of (3.7) such that

$$0 \leq \underline{\omega} \leq \overline{\omega} \qquad \text{in } S_+^{N-1}. \tag{3.9}$$

First we note that $\eta = \eta_0$ is a supersolution if the positive constant $\eta_0$ is large enough. In order to find a subsolution, we set $\eta = \psi^\theta$ with $\theta = \beta_q/\beta_*$ and $\psi$ as in (3.2). Then $\theta > 1$, thus $\eta \in W_0^{1,p}(S_+^{N-1})$. As above we have

$$\begin{aligned}\mathcal{T}(\eta) = &-\theta^{p-1}\psi^{(\theta-1)(p-1)}\mathrm{div}\left(\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi\right) \\ &- \theta^{p-1}(\theta-1)(p-1)\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}|\nabla'\psi|^2 \\ &- \beta_q\Lambda_{\beta_q}\theta^{p-2}\psi^{(\theta-1)(p-1)}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi + \theta^q\psi^{(\theta-1)q}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{q}{2}}.\end{aligned}$$

Now $\beta_q\Lambda_{\beta_q}\theta^{p-2} = \beta_*\Lambda_{\beta_q}\theta^{p-1} = \beta_*(\Lambda_{\beta_q} - \Lambda_{\beta_*})\theta^{p-1} + \beta_*\Lambda_{\beta_*}\theta^{p-1}$ and $\Lambda_{\beta_q} - \Lambda_{\beta_*} = (\beta_q - \beta_*)(p-1) = \beta_*(p-1)(\theta-1)$, hence

$$\begin{aligned}\mathcal{T}(\eta) = &-\theta^{p-1}\psi^{(\theta-1)(p-1)}\mathrm{div}\left(\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi\right)\\ &- \theta^{p-1}(\theta-1)(p-1)\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}|\nabla'\psi|^2\\ &- \beta_*(\Lambda_{\beta_q} - \Lambda_{\beta_*})\theta^{p-1}\psi^{(\theta-1)(p-1)}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi\\ &- \beta_*\Lambda_{\beta_*}\theta^{p-1}\psi^{(\theta-1)(p-1)}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi + \theta^q\psi^{(\theta-1)q}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{q}{2}}.\end{aligned}$$

Using the equation satisfied by $\psi$ yields

$$\begin{aligned}\mathcal{T}(\eta) &= -\theta^{p-1}(\theta-1)(p-1)\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}|\nabla'\psi|^2\\ &\quad - \beta_*^2(p-1)(\theta-1)\theta^{p-1}\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi^2\\ &\quad + \theta^q\psi^{(\theta-1)q}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{q}{2}}\\ &= -\theta^{p-1}(\theta-1)(p-1)\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p}{2}}\\ &\quad + \theta^q\psi^{(\theta-1)q}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{q}{2}}.\end{aligned}$$

We will see that replacing $\psi$ with $m\psi$ in the above computation, the right choice of $m$ will give $\mathcal{T}(\eta) \le 0$: Indeed, we need

$$m^{\theta(q+1-p)}\psi^{(\theta-1)(q+1-p)+1} \le \theta^{p-1-q}(\theta-1)(p-1)\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-q}{2}}$$

which will hold true if we choose $m$ to satisfy

$$m^{\theta(q+1-p)} \le \beta_*^{(\theta-1)(q+1-p)+1}\theta^{p-1-q}(\theta-1)(p-1)\frac{\min_{x\in S_+^{N-1}}\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-q}{2}}}{\max_{x\in S_+^{N-1}}\psi^{(\theta-1)(q+1-p)+1}}.$$

Therefore $0 < \eta \le \eta_0$ and standard regularity implies that the solution $\omega$ is $C^1$ in $\overline{S}_+^{N-1}$. Actulally since the operator is not degenerate, $\omega$ is $C^\infty$.

*Uniqueness.* We use the tangency method developed in the proof of Theorem 3.1. Assume $\omega_1$ and $\omega_2$ are two positive solutions of (3.2), then they are positive in $S_+^{N-1}$ and $\partial_{\mathbf{n}}\omega_i < 0$ on $\partial S_+^{N-1}$. Either the $\omega_i$ are ordered and $\omega_1 \le \omega_2$ or their graphs intersect. In any case we can define

$$\tau = \inf\{s > 1 : s\omega_1 \ge \omega_2\}.$$

We set $\omega^* = \tau\omega_1$. Then either the graphs of $\omega_2$ and $\omega^*$ are tangent at some interior point $\alpha$, or they are not tangent in $S_+^{N-1}$, $\partial_{\mathbf{n}}\omega^* \le \partial_{\mathbf{n}}\omega_2 < 0$ on $\partial S_+^{N-1}$ and there exists $\alpha \in \partial S_+^{N-1}$ such that $\partial_{\mathbf{n}}\omega^*(\alpha) = \partial_{\mathbf{n}}\omega_2(\alpha) < 0$. Furthermore $\mathcal{T}(\omega^*) \ge 0$. If we set $w = \omega^* - \omega_2$, then, as in Theorem 3.1,

$$-\mathrm{div}(A^*\nabla' w) + \langle B^*, \nabla' w\rangle_g + C^* w = \mathcal{L}^* w \ge 0$$

where

$$\begin{aligned}\langle A^*X, X\rangle_g &= \left(\beta_q^2\omega_t^2 + |\nabla'\omega_t|^2\right)^{\frac{p-4}{2}}\left(p-2)\langle\nabla'\omega_t, X\rangle_g^2 + |\nabla'\omega_t|^2|X|^2\right)\\ &\geq \left(\beta_q^2\omega_t^2 + |\nabla'\omega_t|^2\right)^{\frac{p-4}{2}}\min\{1, p-1\}|\nabla'\omega_t|^2|X|^2,\end{aligned} \tag{3.10}$$

in which $\omega_t = \omega_2 + t(\omega^* - \omega_2)$ and $t \in (0,1)$ is obtained by applying the mean value theorem and $B^*$ and $C^*$ are defined accordingly. Since $\mathcal{L}^*$ is uniformly elliptic and has bounded coefficient, it follows from the strong maximum principle that $w = 0$. Thus $\omega^* = \tau\omega_1 = \omega_2$ and $\tau = 1$ from the equation. This ends the proof. □

## 3.2 Removable boundary singularities

The following is the basic result for removability of isolated singularities. It is valid in the general case, but with a local geometric constraint.

**Theorem 3.3** *Assume $q^* \leq q < p \leq N$, $\Omega$ is a $C^2$ bounded domain with $0 \in \partial\Omega$, such that $\Omega \cap B_\delta \subset \mathbb{R}^N_+$ for some $\delta > 0$. If $u \in C^1(\overline{\Omega} \setminus \{0\})$ is a nonnegative solution of* (1.1) *in $\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$, it is identically* 0.

*Proof. Step 1: Assume* $\Omega \subset \mathbb{R}^N_+$. For $\epsilon > 0$, we set $\Omega'_\epsilon = \Omega \cap \overline{B^c_\epsilon}$ and $H_\epsilon = \mathbb{R}^N_+ \cap \overline{B^c_\epsilon}$. For $k, n \in \mathbb{N}_*$, $n \geq \operatorname{diam}(\Omega)$, we denotes by $v_{k,n,\epsilon}$ ($n \in \mathbb{N}_*$) the solution of the problem

$$\begin{aligned}-\Delta_p v + |\nabla v|^q &= 0 && \text{in } H_\epsilon \cap B_n\\ v &= k\chi_{\mathbb{R}^N_+ \cap \partial B_\epsilon} && \text{on } \partial(H_\epsilon \cap B_n).\end{aligned} \tag{3.11}$$

If $k > c_2\epsilon^{\frac{q-p}{q+1-p}}$ then $v_{k,n,\epsilon} \geq u$ in $\Omega'_\epsilon$. Moreover there holds $v_{k,n,\epsilon} \leq v_{k',n',\epsilon}$ for $n \leq n'$ and $k \leq k'$. Furthermore the function

$$U_{\epsilon,n}(x) = c_1\left((|x| - \epsilon)^{\frac{q-p}{q+1-p}} - (n-\epsilon)^{\frac{q-p}{q+1-p}}\right)$$

is a super solution in $B_n \setminus B_\epsilon$ for a suitable $c_1 = c_1(N, p, q) > 0$ (see Lemma 2.6), and there holds $v_{k,n,\epsilon} \leq U_{\epsilon,n}$. By monotonicity and standard a priori estimate, we obtain that $v_{k,n,\epsilon} \to v_\epsilon$ when $n, k \to \infty$ and that the function $v = v_\epsilon$ is solution of

$$\begin{aligned}-\Delta_p v + |\nabla v|^q &= 0 && \text{in } H_\epsilon\\ \lim_{|x|\to\epsilon} v(x) &= \infty\\ v &= 0 && \text{on } \partial\mathbb{R}^N_+ \cap \overline{B^c_\epsilon}.\end{aligned} \tag{3.12}$$

Furthermore

$$u(x) \leq v_\epsilon(x) \leq c_1(|x| - \epsilon)^{\frac{q-p}{q+1-p}} \quad \text{in } \Omega'_\epsilon. \tag{3.13}$$

The function $v_\epsilon$ may not be unique, however it is the minimal solution of the above problem since the $v_{k,n,\epsilon}$ is unique, and monotonicity in $n$ and $k$ holds. Actually, $v_\epsilon \leq v_{\epsilon'}$ if $0 \leq \epsilon \leq \epsilon'$. For $\ell > 0$, we recall that the transformation $T_\ell$ defined by

$$T_\ell[v](x) = \ell^{\frac{p-q}{q+1-p}} v(\ell x), \tag{3.14}$$

leaves equation (1.1) invariant. As a consequence of the uniqueness of the approximations we have $T_\ell[v_{k,n,\epsilon}] = v_{\ell^{\frac{p-q}{q+1-p}}k,\ell^{-1}n,\ell^{-1}\epsilon}$, which yields

$$T_\ell[v_\epsilon](x) = v_{\ell^{-1}\epsilon}. \tag{3.15}$$

Letting $\epsilon \to 0$, we derive from the monotonicity with respect to $\epsilon$ and standard $C^{1,\alpha}$ estimates,

$$T_\ell[v_0] = v_0 \qquad \forall \ell > 0. \tag{3.16}$$

The function $v_0$ is a positive and separable solution of (1.1) in $\mathbb{R}^N_+$ which vanishes on $\partial\Omega \setminus \{0\}$. It follows from Theorem 3.1 that $v_0 = 0$, and so is $u$.

*Step 2: The general case.* We assume that $\Omega \cap B_\delta \subset \mathbb{R}^N_+$ and we denote by $M$ the maximum of $u$ on $\partial B_\delta \cap \Omega$. Then the function $(u - M)_+$ is a subsolution of (1.1) in $\Omega \cap B_\delta$ which vanishes on $\partial\Omega \cap B_\delta \setminus \{0\}$. By Step 1, it is dominated by $v_0$, which ends the proof. □

*Remark.* The previous result is valid if $u$ is a subsolution with the same regularity as $u$. If $u$ is no longer assumed to be nonnegative, only $u^+$ vanishes. Furthermore, the regularity of the boundary has not been used, but only the fact that $\Omega$ is locally contained into a half space to the boundary of which 0 belongs.

*Remark.* If no geometric assumption is made on $\partial\Omega$, we can prove that $u(x) = \circ(|x|^{-\beta_q})$ near 0. The next result shows that the removability holds if $q > q_*$.

**Theorem 3.4** *Assume $q^* < q < p \leq N$ and $\Omega$ is a $C^2$ bounded domain with $0 \in \partial\Omega$. If $u$ is a nonnegative solution of* (1.1) *in $\Omega$ which belongs to $C^1(\overline{\Omega} \setminus \{0\})$ which vanishes on $\partial\Omega \setminus \{0\}$, it is identically* 0.

*Proof.* In [25] is proved that for any smooth subdomain $S \subset S^{N-1}$, there exists a unique $\beta_s > 0$ and $\psi_s > 0$, up to an homothety, such that $x \mapsto |x|^{-\beta_s}\, \psi_s(|x|^{-1}\, x)$ is $p$ harmonic in the cone $C_S = \{x \in \mathbb{R}^N \setminus \{0\} : |x|^{-1}\, x \in S\}$ and $\psi_s$ satisfies

$$\begin{aligned} -div\left(\left(\beta_s^2\psi_s^2 + |\nabla'\psi_s|^2\right)^{\frac{p-2}{2}} \nabla'\psi_s\right) - \beta_s\Lambda_{\beta_s}\left(\beta_s^2\psi_s^2 + |\nabla'\psi_s|^2\right)^{\frac{p-2}{2}} \psi_s = 0 \qquad &\text{in } S \\ \psi_s = 0 \qquad &\text{on } \partial S, \end{aligned} \tag{3.17}$$

By construction $S \mapsto \beta_s$ is decreasing. If, for $\epsilon > 0$, we denote by $S := S_\epsilon$ the spherical shell with vertex the north pole $N$ and latitude angle $\theta_{N-1} \in [0, \frac{\pi}{2} + \epsilon]$- we use here the spherical coordinates system defined in (6.5 ). Because of uniqueness of $\beta_s$, $\beta_{s_\epsilon} \uparrow \beta_*$ as $\epsilon \to 0$. Therefore, if $q > q_*$, or equivalently $\beta_q < \beta_*$, there exists $\delta, \epsilon > 0$ such that $\Omega \cap B_\delta \subset C_{S_\epsilon} \cap B_\delta$ and $\beta_q < \beta_{s_\epsilon}$. Since Theorem 3.1 is valid if $S^{N-1}_+$ is replaced by $S_\epsilon$ and $\beta_q < \beta_{s_\epsilon}$ it follows that $u = 0$ as in the proof of Theorem 3.3, Step 1 and 2. □

The next result, valid in the case $p = N$, is based upon the conformal invariance of the N-Laplacian. In this case the exponent $\beta_*$ corresponding to the first spherical N-harmonic eigenvalue is equal to 1 and the corresponding spherical N-harmonic eigenfunction in $S^{N-1}_+ = S^{N-1} \cap \{x = (x', x_N), x_N > 0\}$ is $x_N/\,|x|^2$.

**Theorem 3.5** *Assume $N-\frac{1}{2}\leq q<N$, $\Omega$ is a bounded domain and $0\in\partial\Omega$ is such that there exists a ball in $\Omega^c$ to the boundary of which a belongs. If $u$ is a nonnegative solution of*

$$-\Delta_N u+|\nabla u|^q=0 \qquad \text{in } \Omega, \tag{3.18}$$

*which belongs to $C(\overline{\Omega}\setminus\{0\})\cap W_0^{1,N}(\Omega\setminus\overline{B}_\epsilon(a))$ for any $\epsilon>0$, it is identically 0.*

*Proof.* We assume that the inward normal unit vector to $\partial\Omega$ at 0 is $\mathbf{e}_N=(0,0,...,1)$ and that the ball $B=B_{\frac{1}{2}}(a)$ of center $a=-\frac{1}{2}\mathbf{e}_N$ and radius $\frac{1}{2}$ touches $\partial\Omega$ at 0 and is exterior to $\Omega$ (this can be assumed up to a rotation and a dilation). This is the consequence of the exterior sphere condition at the point 0. It is always valid if $\partial\Omega$ is $C^2$. We denote by $\mathcal{I}_\omega$ the inversion of center $\omega$ and power 1, i.e. $\mathcal{I}_\omega(x)=\omega+\frac{x-\omega}{|x-\omega|^2}$. Under this transformation, the complement of the ball $B$, which contains $\Omega$ is transformed into the half space $\{x=(x',x_N):x_N<0\}$ which contains the image $\tilde{\Omega}$ of $\Omega$. Since $u$ satisfies (3.18), $\tilde{u}=u\circ\mathcal{I}_\omega$ satisfies

$$-\Delta_N\tilde{u}+|x-\omega|^{2(q-N)}\,|\nabla\tilde{u}|^q=0 \qquad \text{in } \tilde{\Omega}. \tag{3.19}$$

Furthermore since $0=\mathcal{I}_\omega(0)$ and $\mathcal{I}_\omega$ is a diffeomorphism, $\tilde{u}\in C(\overline{\tilde{\Omega}}\setminus\{0\})\cap C^1(\tilde{\Omega})$ and it vanishes on $\partial\tilde{\Omega}\setminus\{0\}$. Since $|x-\omega|\leq 1$ and $q<N$, $\tilde{u}$ is a subsolution for (3.18) in $\tilde{G}$. By Theorem 3.5, $\tilde{u}=0$. □

## 3.3 Weakly singular solutions

The main result of this section is the following existence and uniqueness result concerning solutions of (1.1) with a boundary weak singularity. We recall that $\psi_*$ is unique positive solution of (1.10) such that $\sup\psi_*=1$. Our first result is valid for any $1<p\leq N$ but it needs a geometric constraint on $\Omega$.

**Theorem 3.6** *Let $p-1<q<q_*<p\leq N$ and $\Omega$ be a bounded $C^2$ domain such that $0\in\partial\Omega$. Assume that there exists $\delta>0$ such that $\Omega\cap B_\delta=B_\delta^+$. Then for any $k>0$ there exists a unique positive solution $u:=u_k$ of (1.1) in $\Omega$, which belongs to $C^1(\overline{\Omega}\setminus\{0\})$, vanishes on $\partial\Omega\setminus\{0\}$ and satisfies*

$$\lim_{x\to 0}\frac{u_k(x)}{\Psi_*(x)}=k \tag{3.20}$$

*in the $C^1$-topology of $S_+^{N-1}$, where $\Psi_*(x)=|x|^{-\beta_*}\,\psi_*(|x|^{-1}x)$.*

**Lemma 3.7** *Let the assumptions on $p$, $q$ and $\Omega$ of Theorem 3.6 be satisfied. There exists a unique positive $p$-harmonic function $\Phi_*$ in $\Omega$, which is continuous in $\overline{\Omega}\setminus\{0\}$, vanishes on $\partial\Omega\setminus\{0\}$ and satisfies*

$$\lim_{x\to 0}\frac{\Phi_*(x)}{\Psi_*(x)}=1. \tag{3.21}$$

*Proof.* For $0 < \epsilon < \delta$ let $v_\epsilon$ be the unique p-harmonic function in $\Omega \setminus \overline{B_\epsilon^+}$ which is continuous in $\overline{\Omega} \setminus B_\epsilon^+$, vanishes on $\partial\Omega \setminus B_\epsilon$ and achieves the value $\Psi_*$ on $\partial B_\epsilon \cap \Omega$. By the maximum principle, and since $\Psi_*(x) \le D^{-\beta_*}$ where $D = \max\{|z| : z \in \Omega\}$, there holds

$$(\Psi_* - D^{-\beta_*})_+ \le v_\epsilon \le \Psi_* \quad \text{in } \Omega \setminus \overline{B_\epsilon^+}. \tag{3.22}$$

If $\epsilon < \epsilon' < \delta$, then $v_\epsilon \le v_{\epsilon'}$ in $\Omega \setminus \overline{B_{\epsilon'}^+}$. By standard regularity result $v_\epsilon$ converges to a function $\Phi_*$ continuous in $\overline{\Omega} \setminus \{0\}$, p-harmonic in $\Omega$ and satisfies $(\Psi_* - \delta^{-\beta_*})_+ \le \Phi_* \le \Psi_*$ in $\Omega$. Therefore (3.21) holds provided $\frac{x}{|x|}$ remains in a compact subset of $S_+^{N-1}$. If we define the function $\tilde{\phi}_*$ by $\tilde{\phi}_*(x) = |x|^{\beta_*}\, \Phi_*(x)$, then $\tilde{\phi}_*(r,\sigma) \le \psi_*(\sigma)$ where $r = |x|$ and $\sigma = \frac{x}{|x|} \in S_+^{N-1}$. By standard $C^{1,\alpha}$ estimates, $\tilde{\phi}_*(r,.)$ is relatively compact in the $C(\overline{S_+^{N-1}})$-topology. Therefore the convergence of $\frac{\Phi_*(x)}{\Psi_*(x)}$ to 1 when $x$ to 0 holds not only when $\frac{x}{|x|}$ remains in a compact subset of $S_+^{N-1}$, but uniformly on $S_+^{N-1}$, which implies (3.22). Uniqueness follows classically by (3.22) and the maximum principle. □

**Lemma 3.8** *Let the assumptions on $p$, $q$ and $\Omega$ of Theorem 3.6 be satisfied. If for some $k > 0$ there exists a solution $u_k$ of* (1.1) *in $\Omega$, which belongs to $C^1(\overline{\Omega} \setminus \{0\})$, vanishes on $\partial\Omega \setminus \{0\}$ and satisfies* (3.20)*, then for any $k > 0$ there exists such a solution.*

*Proof.* We notice that for any $c < 1$ (resp $c > 1$), $cu_k$ is a subsolution (resp. supersolution) of (1.1) in $\Omega$. If $c < 1$, the function $ck\Phi_*$ is a supersolution of (1.1) which vanishes on $\partial\Omega \setminus \{0\}$. Furthermore

$$\lim_{x\to 0} \frac{cu_k(x)}{\Psi_*(x)} = \lim_{x\to 0} \frac{ck\Phi_*(x)}{\Psi_*(x)} = ck.$$

Then there exists a solution $u_{ck}$ of (1.1) in $\Omega$ which satisfies $cu_k \le u_{ck} \le ck\Phi_*$. If $c > 1$, we consider $u^* := T_{c^\theta}[u_k] : x \mapsto c^{\beta_q\theta} u_k(c^\theta\, x)$ with $\theta = (\beta_q - \beta_*)^{-1}$. Then $u^*$ is a solution of (1.1) in $\Omega^{c^\theta} = \frac{1}{c^\theta}\Omega$. In particular, $u^*$ satisfies the equation in $B^+_{\frac{\delta}{c^\theta}}(0)$. Since $c^\theta > 1$, $B^+_{\frac{\delta}{c^\theta}}(0) \subset B^+_\delta(0)$. Put $m = \max\{u^* : x \in \partial B^+_{\frac{\delta}{c^\theta}}(0)\}$. The function $(u^* - m)_+$, extended by 0 outside $B^+_{\frac{\delta}{c^\theta}}(0)$, is a subsolution of (1.1) in $\Omega$. Furthermore it satisfies

$$\lim_{x\to 0} \frac{(u^* - m)_+(x)}{\Psi_*(x)} = ck,$$

uniformly on any compact subset of $S_+^{N-1}$. Therefore there exists a solution $u_{ck}$ of (1.1) in $\Omega$ which satisfies $(u^* - m)_+ \le u_{ck} \le ck\Phi_*$, and in particular it vanishes on $\partial\Omega \setminus \{0\}$ and belongs to $C^1(\overline{\Omega} \setminus \{0\})$. By [27], $u_{ck}$ is positive in $\Omega$. Since $u_{ck}$ belongs to $C^{1,\alpha}(\overline{B^+_\delta}(0) \setminus \{0\})$ and satisfies

$$|x|^{\beta_*}\, |u_{ck}(x)| + |x|^{1+\beta_*}\, |\nabla u_{ck}(x)| + |x|^{1+\beta_*+\alpha} \sup_{\substack{|y| \le |x| \\ x \ne y}} \frac{|\nabla u_{ck}(x) - \nabla u_{ck}(y)|}{|x-y|^\alpha} \le M$$

by (2.10), the set of functions $\{r^{\beta_*+1}\nabla u_{ck}(r,.)\}_{r>0}$ is uniformly relatively compact in the topology of uniform convergence on $\overline{S}_+^{N-1}$. Since it converges to $ck\nabla'\psi_*$ uniformly on compact subsets of $S_+^{N-1}$, this convergence holds in $C(\overline{S}_+^{N-1})$. This implies

$$\lim_{x\to 0}\frac{u_{ck}(x)}{\Psi_*(x)}=ck. \tag{3.23}$$

□

**Lemma 3.9** *Under the assumptions of Theorem 3.6 there exists a real number $R_0$ such that $0<R_0\leq\delta$ and a positive subsolution $\tilde{u}$ of (1.1) in $B_{R_0}^+$ which is Lipschitz continuous in $\overline{B_{R_0}^+}\setminus\{0\}$, vanishes on $\overline{B}_{R_0}^+\cap\partial\mathbb{R}_+^N\setminus\{0\}$, is smaller than $\Psi_*$ and satisfies*

$$\lim_{x\to 0}\frac{\tilde{u}(x)}{\Psi_*(x)}=1. \tag{3.24}$$

*Proof.* The construction of the function $\tilde{u}$ is delicate and needs several intermediate steps. We look for a solution under the form $\tilde{u}=\Psi_*-w$ for a suitable nonnegative function $w$.

*Step 1: reduction of the problem.* We use spherical coordinates for a $C^1$ function $u:x\mapsto u(x)=u(r,\sigma)$, $r=|x|$, $\sigma=\frac{x}{|x|}$. Then $\nabla u=u_r\mathbf{e}+r^{-1}\nabla' u$ where $\mathbf{e}=|x|^{-1}x$, $|\nabla u|^2=u_r^2+r^{-2}|\nabla' u|^2$ and $|\nabla u|^q=\left(u_r^2+r^{-2}|\nabla' u|^2\right)^{\frac{q}{2}}$. The expression of the p-Laplacian in spherical coordinates is

$$\begin{aligned}-\Delta_p u=&-\left(\left(u_r^2+r^{-2}|\nabla' u|^2\right)^{\frac{p-2}{2}}u_r\right)_r-\frac{N-1}{r}\left(u_r^2+r^{-2}|\nabla' u|^2\right)^{\frac{p-2}{2}}u_r\\&-\frac{1}{r^2}div'\left(\left(u_r^2+r^{-2}|\nabla' u|^2\right)^{\frac{p-2}{2}}\nabla' u\right).\end{aligned}$$

Put $v(t,\sigma)=r^{\beta_*}u(r,\sigma)$ with $t=\ln r\in(-\infty,\ln\delta]$, then $v$ satisfies

$$\begin{aligned}&\mathcal{Q}[v]:\\&=-\left(\left((v_t-\beta_*v)^2+|\nabla' v|^2\right)^{\frac{p-2}{2}}(v_t-\beta_*v)\right)_t-div'\left(\left((v_t-\beta_*v)^2+|\nabla' v|^2\right)^{\frac{p-2}{2}}\nabla' v\right)\\&\quad+\Lambda_{\beta_*}\left((v_t-\beta_*v)^2+|\nabla' v|^2\right)^{\frac{p-2}{2}}(v_t-\beta_*v)+e^{\nu t}\left((v_t-\beta_*v)^2+|\nabla' v|^2\right)^{\frac{q}{2}}=0\end{aligned} \tag{3.25}$$

in $(-\infty,\ln\delta)\times S_+^{N-1}$ where $\nu=1-(q+1-p)(\beta_*+1)=1-\frac{\beta_*+1}{\beta_q+1}>0$ and $\Lambda_{\beta_*}=\beta_*(p-1)+p-N$. Notice that since $\psi_*$ satisfies

$$-div'\left(\left(\beta_*^2\psi_*^2+\left|\nabla'\psi_*\right|^2\right)^{\frac{p-2}{2}}\nabla'\psi_*\right)-\beta_*\Lambda_{\beta_*}\left(\beta_*^2\psi_*^2+\left|\nabla'\psi_*\right|^2\right)^{\frac{p-2}{2}}\psi_*=0, \tag{3.26}$$

it is a supersolution for (3.25). We look for a subsolution under the form

$$V(t,\sigma) = \psi_* - a(t)g(\psi_*)$$

where $g$ is a a continuous increasing function defined on $\mathbb{R}_+$, vanishing at 0 and smooth on $\mathbb{R}_+^*$ and $a(t) = e^{\gamma t}$ with $\gamma > 0$ is to be chosen. Thus $a' = \gamma a$, $a'' = \gamma^2 a$, $V_t = -\gamma a g(\psi_*)$, $V_t - \beta_* V = -\beta_*\psi_* + a(\beta_* - \gamma)g(\psi_*)$, $\nabla' V = (1 - ag'(\psi_*))\nabla'\psi_*$ and

$$\begin{aligned}
(V_t - \beta_* V)^2 + |\nabla' V|^2 &= (-\beta_*\psi_* + a(\beta_* - \gamma)g(\psi_*))^2 + (1 - ag'(\psi_*))^2 |\nabla'\psi_*|^2 \\
&= \left(\beta_*^2\psi_*^2 + 2a\beta_*(\gamma - \beta_*)g(\psi_*)\psi_*\right) + (1 - 2ag'(\psi_*)) |\nabla'\psi_*|^2 + O(a^2 \|g(\psi)\|_{C^1}) \\
&= \beta_*^2\psi_*^2 + |\nabla'\psi_*|^2 + 2a\left(\beta_*(\gamma - \beta_*)\psi_* g(\psi_*) - g'(\psi_*) |\nabla\psi_*|^2\right) + O(a^2 \|g(\psi_*)\|_{C^1}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left(V_t - \beta_* V)^2 + |\nabla' V|^2\right)^{\frac{p-2}{2}} \\
&= \left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} \left[1 + (p-2)a\frac{\beta_*(\gamma - \beta_*)\psi_* g(\psi_*) - g'(\psi_*) |\nabla\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}\right] \\
&\qquad\qquad + O(a^2 \|g(\psi)\|_{C^1}),
\end{aligned}$$

and

$$\begin{aligned}
&e^{\nu t}\left(V_t - \beta_* V)^2 + |\nabla' V|^2\right)^{\frac{q}{2}} \\
&= e^{\nu t}\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{q}{2}} \left[1 + qa\frac{\beta_*(\gamma - \beta_*)\psi_* g(\psi_*) - g'(\psi_*) |\nabla\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}\right] \\
&\qquad\qquad + O(e^{\nu t} a^2 \|g(\psi_*)\|_{C^1}),
\end{aligned}$$

thus

$$\begin{aligned}
&\left((V_t - \beta_* V)^2 + |\nabla' V|^2\right)^{\frac{p-2}{2}} (V_t - \beta_* V) \\
&\qquad = -\beta_*\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} \psi_* + a(\beta_* - \gamma)\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} g(\psi_*) \\
&\qquad\quad - a\beta_*(p-2)\frac{\beta_*(\gamma - \beta_*)\psi_* g(\psi_*) - g'(\psi_*) |\nabla\psi_*|^2}{(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2)^{\frac{4-p}{2}}}\psi_* + O(a^2 \|g(\psi_*)\|_{C^1}).
\end{aligned}$$

Finally,

$$\begin{aligned}
&-\left(\left((V_t - \beta_* V)^2 + |\nabla' V|^2\right)^{\frac{p-2}{2}} (V_t - \beta_* V)\right)_t \\
&= a\left[(\gamma^2 - \beta_*\gamma)\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} g(\psi_*) + \beta_*(p-2)\frac{\beta_*(\gamma^2 - \beta_*\gamma)\psi_* g(\psi_*) - \gamma g'(\psi_*) |\nabla\psi_*|^2}{(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2)^{\frac{4-p}{2}}}\psi_*\right] \\
&\qquad\qquad + O(a^2 \|g(\psi_*)\|_{C^2}).
\end{aligned} \tag{3.27}$$

Since

$$\left((V_t-\beta_* V)^2+|\nabla' V|^2\right)^{\frac{p-2}{2}}\nabla' V=$$

$$\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}(1-ag'(\psi_*))\left[1+a(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_* g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}\right]\nabla'\psi_*$$

$$+O(a^2\,\|g(\psi_*)\|_{C^1})$$

$$=\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla'\psi_*$$

$$+a\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left[(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_* g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-g'(\psi_*)\right]\nabla'\psi_*$$

$$+O(a^2\,\|g(\psi_*)\|_{C^1})$$

we get similarly

$$-div'\left(\left((V_t-\beta_* V)^2+|\nabla' V|^2\right)^{\frac{p-2}{2}}\nabla' V\right)=-div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla'\psi_*\right)$$

$$-a\,div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left[(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_* g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-g'(\psi_*)\right]\nabla'\psi_*\right)$$

$$+O(a^2\,\|g(\psi_*)\|_{C^2}). \tag{3.28}$$

Noting that

$$-div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla'\psi_*\right)\psi_*=\beta_*\Lambda_{\beta_*}\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\psi_*, \tag{3.29}$$

we obtain

$$e^{-\gamma t}\mathcal{Q}[V]$$

$$=\left[(\gamma^2-\beta_*\gamma)\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}g(\psi_*)+\beta_*(p-2)\frac{\beta_*(\gamma^2-\beta_*\gamma)\psi_* g(\psi_*)-\gamma g'(\psi_*)\,|\nabla\psi_*|^2}{(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2)^{\frac{4-p}{2}}}\psi_*\right]$$

$$-div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left[(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_* g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-g'(\psi_*)\right]\nabla'\psi_*\right)$$

$$-\Lambda_{\beta_*}\left((\gamma-\beta_*)\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}g(\psi_*)+\beta_*(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_* g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2)^{\frac{4-p}{2}}}\psi_*\right)$$

$$+e^{(\nu-\gamma)t}\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{q}{2}}\left[1+qa\frac{\beta_*(\gamma-\beta_*)\psi_* g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}\right]+O(a\,\|g(\psi_*)\|_{C^2}). \tag{3.30}$$

In this expression we have in particular

$$
\begin{aligned}
-div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left[(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_* g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-g'(\psi_*)\right]\nabla'\psi_*\right)\\
=(p-1)div'\left[g'(\psi_*)\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla\psi_*\right]\\
-\beta_* div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-4}{2}}\left[(p-2)\beta_*\psi_* g'(\psi_*)+(p-2)(\gamma-\beta_*)g(\psi_*)\right]\psi_*\right)\\
=(p-1)g''(\psi_*)\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}|\nabla\psi_*|^2\\
+(p-1)g'(\psi_*)div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla\psi_*\right)\\
-(p-2)\beta_* div'\left[\frac{\left((\gamma-\beta_*)g(\psi_*)\psi_*+\beta_* g'(\psi_*)\psi_*^2\right)}{\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{4-p}{2}}}\nabla'\psi_*\right].
\end{aligned}
\tag{3.31}
$$

Using equation (3.26) satisfied by $\psi_*$, it follows

$$
\begin{aligned}
-div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left[(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_* g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-g'(\psi_*)\right]\nabla'\psi_*\right)\\
=(p-1)\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left(g''(\psi_*)|\nabla'\psi_*|^2-\beta_*\Lambda_{\beta_*}g'(\psi_*)\psi_*\right)\\
-(p-2)\beta_* div'\left[\frac{\left((\gamma-\beta_*)g(\psi_*)\psi_*+\beta_* g'(\psi_*)\psi_*^2\right)}{\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{4-p}{2}}}\nabla'\psi_*\right].
\end{aligned}
\tag{3.32}
$$

Plugging this identity into the expression (3.30), we obtain after some simplifications

$$
e^{-\gamma t}\mathcal{Q}[V]=\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}g(\psi_*)\mathcal{Q}_1[V]+e^{(\nu-\gamma)t}R[V]+O(a\,\|g(\psi_*)\|_{C^2}),
\tag{3.33}
$$

where

$$
R[V]=e^{\nu t}\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{q}{2}}\left[1+q\frac{\beta_*(a'-\beta_* a)\psi_* g(\psi_*)-a g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}\right],
\tag{3.34}
$$

and

$$\mathcal{Q}_1[V] = (\gamma - \Lambda_{\beta_*})(\gamma - \beta_*)\left[1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}\right] - (p-1)\beta_*\Lambda_{\beta_*}\frac{\psi_* g'(\psi_*)}{g(\psi_*)}$$
$$+ [(p-4)\beta_*\Lambda_{\beta_*}\psi_* - 2\Delta'\psi_*]\left(\gamma - \beta_*\left(1 - \frac{\psi_* g'(\psi_*)}{g(\psi_*)}\right)\right)\frac{\beta_*\psi_*}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2} + (p-1)\frac{g''(\psi_*)}{g(\psi_*)}|\nabla'\psi_*|^2$$
$$- (p-2)\left[\frac{\psi_* g'(\psi_*)}{g(\psi_*)}((\beta_*+1)\gamma - \beta_*\Lambda_{\beta_*} + \beta_*) + \gamma - \beta_* + \beta_*\frac{\psi_*^2 g''(\psi_*)}{g(\psi_*)}\right]\frac{|\nabla'\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}. \tag{3.35}$$

In this expression the difficult term to treat is $[(p-4)\beta_*\Lambda_{\beta_*}\psi_* - 2\Delta'\psi_*]$ because of its lack of sign. However $\Delta'\psi_* = O(\psi_*)$ by (6.20).

*Step 2: The perturbation method and computation with $g(\psi_*) = \psi_*$.* With such a choice of function $g$

$$\begin{aligned}\mathcal{Q}_1[V] = (\gamma - \Lambda_{\beta_*})(\gamma - \beta_*)&\left[1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}\right]\\ &- (p-2)\left[(\gamma - \Lambda_{\beta_*})\beta_* + 2\gamma\right]\frac{|\nabla'\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}.\end{aligned} \tag{3.36}$$

Equivalently

$$\begin{aligned}\mathcal{Q}_1[V] = &\left[1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}\right]\left(\gamma^2 - (\Lambda_{\beta_*} + \beta_*)\gamma\right)\\ &- \left[(p-2)(\beta_*+2)\frac{|\nabla'\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2} + 0(\psi_*^2)\right]\end{aligned}$$

and finally

$$\mathcal{Q}_1[V] = \left[1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}\right]\gamma\left[\gamma - (\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_*+2)) + 0(\psi_*^2)\right]. \tag{3.37}$$

Using the fact that $\beta > \frac{N-1}{p-1}$ if $1 < p < 2$ and $1 < \beta < \frac{N-1}{p-1}$ if $2 < p < N$ (see Appendix II), we have

$$\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_*+2) \geq \begin{cases} \Lambda_{\beta_*} + \beta_*(p-1) & \text{if } p \geq 2 \\ N + 3(p-2) > N-3 & \text{if } 1 < p < 2. \end{cases} \tag{3.38}$$

When $N = 2$, we have explicitly $\beta_* = \frac{1+2\sqrt{p^2-3p+3}}{3(p-1)}$ (see [19, Th 3.3]). Therefore for all $N \geq 2$ and $p > 1$, there holds

$$\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_*+2) > 0. \tag{3.39}$$

We fix $\epsilon_0 > 0$ such that, whenever $\psi_* \leq \epsilon_0$, there holds

$$\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_*+2) + 0(\psi_*^2) > \frac{1}{2}\left(\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_*+2)\right). \tag{3.40}$$

If we fix $\gamma_0 > 0$ such that

$$\gamma_0 < \min\left\{\frac{1}{2}\left(\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_*+2)\right), \nu, \beta_*\right\}, \tag{3.41}$$

we obtain

$$\mathcal{Q}_1[V] \leq -\min\{1, p-1\}\gamma B^2 \qquad \forall\, 0 < \gamma \leq \gamma_0, \tag{3.42}$$

whenever $\psi_* \leq \epsilon_0$, for some $B$ depending only on $p$, $q$ and $N$ (through $\psi_*$ and $\nu$), which yields, in the same range of value of $\psi_*$,

$$\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} g(\psi_*)\mathcal{Q}_1[V] \leq -c_0\psi_* \qquad \forall\, 0 < \gamma \leq \gamma_0, \tag{3.43}$$

for some $c_0 > 0$ depending on $N, p, q$. This estimate is valid whatever is $p$, but only in a neighborhood of $\psi_* = 0$. If we replace $g(\psi_*) = \psi_*$ by $g_k(\psi_*) = \psi_* e^{-k\psi_*}$ for $0 < k < 1$, and denote by $\mathcal{Q}_{1,k}[V]$ the corresponding expression of $\mathcal{Q}_1[V]$ which becomes now $\mathcal{Q}_{1,0}[V]$. We define similarly $\mathcal{Q}_k[V]$, and $\mathcal{Q}[V]$ becomes $\mathcal{Q}_0[V]$. Since $g_k'(\psi_*) = e^{-k\psi_*} - kg_k(\psi_*)$ and $g_k'' = -2ke^{-k\psi_*} + k^2 g_k(\psi_*)$, we obtain

$$\begin{aligned}\mathcal{Q}_{1,k}[V] = \mathcal{Q}_{1,0}[V] + k(p-1)\beta_*\Lambda_{\beta_*}\psi_* + (p-1)\left(-\frac{2k}{\psi_*} + k^2\right)|\nabla'\psi_*|^2 \\ + (2-p)\beta_*\left(-2k + k^2\right)\psi_* + O(\psi_*^2)\end{aligned} \tag{3.44}$$

Because $\nabla'\psi_*$ vanishes only at the North pole $\mathbf{e}_N$, there exists $k_0 \in (0,1]$ such that

$$k(1-p)\beta_*\Lambda_{\beta_*}\psi_* + (p-1)\left(\frac{2k}{\psi_*} - k^2\right)\left|\nabla'\psi_*\right|^2 \geq \frac{1}{2}(2-p)_+\beta_*\left(-2k + k^2\right)\psi_* \qquad \forall k \leq k_0$$

whenever $\psi_* \leq \epsilon_0$ which yields

$$\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} g_k(\psi_*)\mathcal{Q}_{1,k}[V] \leq -c_1 k \qquad \forall k \leq k_0 \tag{3.45}$$

for some $c_1 = c_1(N, p, q, \epsilon_0)$. There exists $c_2 = c_2(N, p, q) > 0$ such that

$$\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{q}{2}}\left[1 + qe^{\gamma t}\frac{\beta_*(\gamma - \beta_*)\psi_* g_k(\psi_*) - g_k'(\psi_*)\left|\nabla\psi_*\right|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}\right] \leq c_2 \tag{3.46}$$

in $S_+^{N-1} \times (-\infty, \ln\delta]$. Moreover

$$O(a\left\|g(\psi_*)\right\|_{C^2}) \leq e^{\gamma t}c_k \tag{3.47}$$

for some $c_k = c_k(N, p, q) > 0$. We derive from (3.45)-(3.48)

$$e^{-\gamma t}\mathcal{Q}_k[V] \leq -c_1 k + c_2 e^{(\nu-\gamma)t} + e^{\gamma t}c_k \qquad \forall k \leq k_0 \tag{3.48}$$

Thus there exists $T_k \leq \ln\delta$ such that $\mathcal{Q}_k[V] \leq 0$, for all $t \leq T_k$ and provided $\psi_* \leq \epsilon_0$. This local estimate will be used in the construction of the sub-solution when $p \geq 2$.

*Step 3: The case* $1 < p < 2$. Since the function $\psi^*$ depends only on the azimuthal angle $\theta \in (0; \frac{\pi}{2}]$ we will write $\psi_*(\theta) = \psi_*(\theta)$ and $\nabla'\psi_*(\sigma) = \psi_{*\theta}(\theta)\mathbf{n}$ where $\mathbf{n}$ is the downward unit vector tangent to $S^{N-1}$ in the hyperplane going through $\sigma$ and the poles. From (6.8),

$$(p-4)\beta_*\Lambda_{\beta_*}\psi_- 2\Delta'\psi_* = (p-2)\left(\beta_*\Lambda_{\beta_*}\psi_* + 2\frac{\beta_*^2\psi_* + \psi_{*\theta\theta}}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\right), \tag{3.49}$$

since $\psi_{*\theta}^2 = |\nabla'\psi_*|^2$ and thus

$$\begin{aligned}((p-4)\beta_*\Lambda_{\beta_*}\psi_* - 2\Delta'\psi_*)\,\frac{\beta_*\gamma\psi_*}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \\ = (p-2)\gamma\left(\Lambda_{\beta_*}\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} + 2\beta_*\frac{\beta_*^2\psi_*^2 + \psi_{*\theta\theta}\psi_*}{(\beta_*^2\psi_*^2 + \psi_{*\theta}^2)^2}\psi_{*\theta}^2\right).\end{aligned} \tag{3.50}$$

From Theorem 6.1-Step 4 we know that $\beta_*^2\psi_* + \psi_{*\theta\theta} \geq 0$, thus the contribution of this term to $\mathcal{Q}_1[V]$ is nonpositive. We replace this expression in $\mathcal{Q}_1[V]$ with $g(\psi_*) = \psi_*$, then

$$\begin{aligned}\mathcal{Q}_1[V] &= (\gamma - \Lambda_{\beta_*})(\gamma - \beta_*)\left(1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\right) - \Lambda_{\beta_*}\beta_*(p-1) \\ &\quad + (p-2)\gamma\Lambda_{\beta_*}\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} - (p-2)\left((\beta_*+2)\gamma - \Lambda_{\beta_*}\beta_*\right)\frac{\psi_{*\theta}^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \\ &\quad + 2\beta_*(p-2)\frac{\beta_*^2\psi_*^2 + \psi_{*\theta\theta}\psi_*}{(\beta_*^2\psi_*^2 + \psi_{*\theta}^2)^2}\psi_{*\theta}^2 \\ &\leq \gamma\left(1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\right)(\gamma - \Lambda_{\beta_*} - \beta_*) - (p-2)\gamma\frac{(\beta_*+2))\psi_{*\theta}^2 - \Lambda_{\beta_*}\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \\ &\leq \gamma\left(1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\right)\left(\gamma - \left(\Lambda_{\beta_*} + \beta_* + (p-2)\frac{(\beta_*+2)\psi_{*\theta}^2 - \Lambda_{\beta_*}\beta_*^2\psi_*^2}{(p-1)\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\right)\right).\end{aligned} \tag{3.51}$$

We can write

$$\begin{aligned}\Lambda_{\beta_*} + \beta_* + (p-2)\frac{(\beta_*+2)\psi_{*\theta}^2 - \Lambda_{\beta_*}\beta_*^2\psi_*^2}{(p-1)\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \\ = \frac{\left(\Lambda_{\beta_*} + (p-1)\beta_*\right)\beta_*^2\psi_*^2 + \left(\Lambda_{\beta_*} + \beta_*(p-1) + 2(p-2)\right)\psi_{*\theta}^2}{(p-1)\beta_*^2\psi_*^2 + \psi_{*\theta}^2} \\ \geq c_0\left(\Lambda_{\beta_*} + \beta_*(p-1) + 2(p-2)\right).\end{aligned} \tag{3.52}$$

This expression $\Lambda_{\beta_*} + \beta_*(p-1) + 2(p-2)$ is always positive: obviously if $N \geq 3$ and by using the explicit expression of $\beta_*$ if $N = 2$. Thus there exists $\gamma_0$ and $c_1 > 0$ such that $\mathcal{Q}_1[V] < -c_1$ for $0 < \gamma \leq \gamma_0$. The pertubation method of Step 2, is valid in the whole range of values of $\psi_*$ and we derive from (3.42)-(3.43) that (3.48) holds for all $k \leq k_0$ and $t \leq T_k$. Therefore $\mathcal{Q}_k[V] \leq 0$.

*Step 4: The case $p \geq 2$.* For $c > 0$ to be fixed and $\psi_* \geq \epsilon_0$, $\gamma \in (0, \gamma_0]$, we take $g(\psi_*) = c\psi_*^{1-\frac{\gamma}{\beta_*}}$. Then we derive from (3.35):

$$\begin{aligned}\mathcal{Q}_1[V] &= (\gamma - \Lambda_{\beta_*})(\gamma - \beta_*)\frac{(p-1)\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2} - (p-1)\beta_*\Lambda_{\beta_*}\left(1 - \frac{\gamma}{\beta_*}\right) \\ &\quad - (p-1)\frac{\gamma(\beta_* - \gamma)}{\beta_*^2}\psi_*^{-1-\frac{\gamma}{\beta_*}}|\nabla'\psi_*|^2 - (p-2)(\beta_* - \gamma)(\gamma - \Lambda_{\beta_*})\frac{|\nabla'\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2} \\ &= (1-p)\left[\gamma(\beta_* - \gamma) + \frac{\gamma(\beta_* - \gamma)}{\beta_*^2}\psi_*^{-1-\frac{\gamma}{\beta_*}}|\nabla'\psi_*|^2\right].\end{aligned} \tag{3.53}$$

For $k \leq k_0$ we fix $c$ such that $c\epsilon_0^{1-\frac{\gamma}{\beta_*}} = \epsilon_0 e^{-k\epsilon_0} \Longleftrightarrow c = \epsilon_0^{\frac{\gamma}{\beta_*}} e^{-k\epsilon_0}$ and we define $g$ by

$$g(\psi_*) = \min\left\{\psi_* e^{-k\psi_*}, \epsilon_0^{\frac{\gamma}{\beta_*}} e^{-k\epsilon_0}\psi_*^{1-\frac{\gamma}{\beta_*}}\right\} = \begin{cases} \psi_* e^{-k\psi_*} & \text{if } 0 \leq \psi_* \leq \epsilon_0 \\ \epsilon_0^{\frac{\gamma}{\beta_*}} e^{-k\epsilon_0}\psi_*^{1-\frac{\gamma}{\beta_*}} & \text{if } \epsilon_0 \leq \psi_* \leq 1, \end{cases} \tag{3.54}$$

and we set $V(t,\sigma) = \psi^*(\sigma) - a(t)g(\psi_*(\sigma))$ with $(t,\sigma) \in (-\infty, T_k] \times S_+^{N-1}$ and define $\tilde{u}(r,\sigma) = r^{-\beta_*}(\psi^*(\sigma) - a(e^t)g(\psi_*(\sigma)))$ accordingly for $(r,\sigma) \in (-\infty, e^{T_k}] \times S_+^{N-1}$. Since $\psi_*$ is a decreasing function the coincidence set $\{\sigma \in S_+^{N-1} : \psi_*(\sigma) = \epsilon_0\}$ is a circular cone $\Sigma_{\theta_0}$ with vertex 0, axis $\mathbf{e}_N$ and angle $\theta_0$. We set $R_0 = e^{T_k}$

$$\begin{aligned}\Gamma_1 &= \left\{x = (r,\theta) \in B_{R_0}^+ : \theta_0 < \theta < \tfrac{\pi}{2}\right\} = \left\{(r,\sigma) \in [0, R_0) \times S_+^{N-1} : 0 < \psi_*(\sigma) < \epsilon_0\right\}, \\ \Gamma_2 &= \left\{x = (r,\theta) \in B_{R_0}^+ : 0 < \theta < \theta_0\right\} = \left\{(r,\sigma) \in [0, R_0) \times S_+^{N-1} : \epsilon_0 < \psi_*(\sigma) < 1\right\},\end{aligned}$$

and define

$$\begin{aligned}\tilde{u}(r,\sigma) &= r^{-\beta_*}\left(\psi_*(\sigma) - r^\gamma g(\psi_*(\sigma))\right) \\ &= \begin{cases} u_1(r,\sigma) = r^{-\beta_*}(1 - r^\gamma e^{-k\psi_*(\sigma)})\psi_*(\sigma) & \text{if } (r,\theta) \in \Gamma_1 \\ u_2(r,\sigma) = r^{-\beta_*}\left(1 - r^\gamma \epsilon_0^{\frac{\gamma}{\beta_*}} e^{-k\epsilon_0}(\psi_*(\sigma))^{1-\frac{\gamma}{\beta_*}}\right)\psi_*(\sigma) & \text{if } (r,\theta) \in \Gamma_2. \end{cases}\end{aligned}$$

The function $\tilde{u}$ is a subsolution separately on $\Gamma_1$ and $\Gamma_2$ and is Lipschitz continuous in $\overline{\Omega} \setminus \{0\}$. If we denote by $g_1$ and $g_2$ the restriction of $g$ to $\Gamma_1$ and $\Gamma_2$ respectively, that is to $\Omega_1$ and $\Omega_2$, then $g_1'(\epsilon_0) > g_2'(\epsilon_0) > 0$. Let $\zeta \in C_c^1(B_{R_0}^+)$ which vanishes in neighborhoods of 0 and $\partial B_{R_0}^+$, $\zeta \geq 0$, then

$$\int_{\Gamma_i} |\nabla\tilde{u}|^{p-2}\nabla\tilde{u}.\nabla\zeta dx + \int_{\Omega_i} |\nabla\tilde{u}|^q \zeta dx \leq \int_{\Sigma_{\theta_0}} |\nabla u_i|^{p-2}\partial_{\mathbf{n}_i} u_i \zeta dS, \tag{3.55}$$

where $\mathbf{n}_i$ is the normal unit vector on $\Sigma_{\theta_0}$ outward from $\Gamma_i$. Actually, $\mathbf{n}_2 = -\mathbf{n}_1 = \mathbf{n}$ thus

$$\nabla\tilde{u} = \tilde{u}_r\mathbf{e} + r^{-\beta_*-1}(1 - r^\gamma g'(\psi_*))\nabla'\psi_* = \tilde{u}_r\mathbf{e} + r^{-\beta_*-1}(1 - r^\gamma g'(\psi_*))\psi_{*\theta}\,\mathbf{n}.$$

and on $\Sigma_{\theta_0}$,

$$\nabla\tilde{u} = \begin{cases} \tilde{u}_r\mathbf{e} - r^{-\beta_*-1}(1-r^\gamma g_1'(\epsilon_0))\psi_{*\theta}\,\mathbf{n} & \text{in } \Gamma_1 \\ \tilde{u}_r\mathbf{e} + r^{-\beta_*-1}(1-r^\gamma g_2'(\epsilon_0))\psi_{*\theta}\,\mathbf{n} & \text{in } \Gamma_2 \end{cases}$$

Therefore

$$|\nabla u_1|^{p-2}\,\partial_{\mathbf{n}_1}u_1 = -r^{-\beta_*-1}(1-r^\gamma g_1'(\epsilon_0))\left(\tilde{u}_r^2 + r^{-2\beta_*-2}(1-r^\gamma g_1'(\epsilon_0))^2\psi_{*\theta}^2\right)^{\frac{p-2}{2}}\psi_{*\theta} \quad \text{in } \Gamma_1$$

and

$$|\nabla u_2|^{p-2}\,\partial_{\mathbf{n}_2}u_2 = r^{-\beta_*-1}(1-r^\gamma g_2'(\epsilon_0))\left(\tilde{u}_r^2 + r^{-2\beta_*-2}(1-r^\gamma g_2'(\epsilon_0))^2\psi_{*\theta}^2\right)^{\frac{p-2}{2}}\psi_{*\theta} \quad \text{in } \Gamma_2.$$

By adding the two inequalities (3.55)

$$\int_\Omega |\nabla\tilde{u}|^{p-2}\,\nabla\tilde{u}.\nabla\zeta dx + \int_\Omega |\nabla\tilde{u}|^q\,\zeta dx \le \int_{\Sigma_{\theta_0}}\left(|\nabla u_1|^{p-2}\,\partial_{\mathbf{n_1}}u_1 + |\nabla u_2|^{p-2}\,\partial_{\mathbf{n_2}}u_2\right)\zeta dS. \tag{3.56}$$

By monotonicity of the function $X \mapsto \left(\tilde{u}_r^2 + X^2\right)^{\frac{p}{2}}$ and since

$$r^{-\beta_*-1}(1-r^\gamma g_2'(\epsilon_0)) \ge r^{-\beta_*-1}(1-r^\gamma g_1'(\epsilon_0)) \ge 0,$$

we derive

$$\begin{aligned} &r^{-\beta_*-1}(1-r^\gamma g_2'(\epsilon_0))\left(\tilde{u}_r^2 + r^{-2\beta_*-2}(1-r^\gamma g_2'(\epsilon_0))^2\psi_{*\theta}^2\right)^{\frac{p-2}{2}} \\ &\qquad \ge r^{-\beta_*-1}(1-r^\gamma g_1'(\epsilon_0))\left(\tilde{u}_r^2 + r^{-2\beta_*-2}(1-r^\gamma g_1'(\epsilon_0))^2\psi_{*\theta}^2\right)^{\frac{p-2}{2}} \end{aligned}$$

We derive that the right-hand side of (3.56) is nonpositive because $\psi_{*\theta} \le 0$, and therefore $\tilde{u}$ is a positive subsolution of (1.1) in $B_{R_0}^+$ dominated by $\Psi_*$ and satisfying (3.24). □

*Proof of Theorem 3.6.* Let $M = \max\{\Psi_*(x) : x \in \partial B_{R_0}^+\}$, then $M = R_0^{-\beta_*}$. The function $u^*$ defined by

$$u^*(x) = \begin{cases} (\tilde{u}(x) - M)_+ & \text{if } x \in B_{R_0}^+ \\ 0 & \text{if } x \in \Omega \setminus B_{R_0}^+, \end{cases}$$

is indeed a subsolution of (1.1) in whole $\Omega$ where it satisfies $u^* \le \Psi_*$ and it vanishes on $\partial\Omega \setminus \{0\}$. Since $\Phi_*$ is a positive p-harmonic function in $\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies (3.21), it is supersolution of (1.1) and therefore it dominates $u^*$. Therefore there exists a solution $u$ of (1.1) in $\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies $u^* \le u \le \Phi_*$. This implies that (3.20) holds with $k = 1$ and we conclude with Lemma 3.8. □

When $p = N$ the statement of Theorem 3.6 holds without the assumption on $\partial\Omega$.

**Theorem 3.10** *Assume $N-1<q<N+\frac{N}{2}$ and $\Omega$ be a bounded $C^2$ domain such that $0\in\partial\Omega$. Then for any $k>0$ there exists a unique positive solution $u:=u_k$ of* (3.18) *in $\Omega$, which belongs to $C^1(\overline{\Omega}\setminus\{0\})$, vanishes on $\partial\Omega\setminus\{0\}$ and satisfies uniformly with respect to $\sigma\in S^{N-1}_+$*

$$\lim_{\substack{x\to 0\\ x/|x|\to\sigma}} |x|\,u_k(x)=\sigma. \tag{3.57}$$

Since $p=N$, $\beta_*=1$ and $\psi_*(\sigma)=\frac{x_N}{|x|}=\cos\theta_{N-1}$ with the identification of $\sigma$ and $\theta_{N-1}:=\theta$. We recall that if $\omega\in\mathbb{R}^N$ and $\mathcal{I}_\omega$ denotes the inversion of center $\omega$ and power 1, i.e. $\mathcal{I}_\omega(x)=\omega+\frac{x-\omega}{|x-\omega|^2}$, then $\tilde{u}=u\circ\mathcal{I}_\omega$ satisfies (3.19).

**Lemma 3.11** *Assume $\Omega$ be a bounded $C^2$ domain such that $0\in\partial\Omega$. Then there exists a unique $N$-harmonic function $\Phi_*$ in $\Omega$, which vanishes on $\partial\Omega\setminus\{0\}$ and satisfies*

$$\lim_{\substack{x\to 0\\ x/|x|\to\sigma}} |x|\,\Phi_*(x)=\sigma, \tag{3.58}$$

*uniformly with respect to $\sigma\in S^{N-1}_+$.*

*Proof.* Uniqueness is standard. Let $\omega=-\mathbf{e}_N\in\overline{\Omega}^c$, with the notations of the proof of Theorem 3.5, $\omega'=-\omega$, $a=-\frac{1}{2}\mathbf{e}_N$ and $a'=-a$. We can assume that the balls $B_{\frac{1}{2}}(a)$ and $B_{\frac{1}{2}}(a')$ are tangent to $\partial\Omega$ at 0 and respectively subset of $\Omega^c$ and $\Omega$. The function $x\mapsto\Psi(x)=-\frac{x_N}{|x|^2}$ which is $N$-harmonic in $\mathbb{R}^N_-$ and vanishes on $\partial\mathbb{R}^{N-1}_-\setminus\{0\}$ is transformed by the inversion $\mathcal{I}_{\omega'}$ of center $\omega'$ and power 1 into the function $\Psi_{\omega'}=\Psi\circ\mathcal{I}_\omega$ which is positive and $N$-harmonic in $B_{\frac{1}{2}}(a')$ and vanishes on $\partial B_{\frac{1}{2}}(a')\setminus\{0\}$. The function $\hat{\Psi}=-\Psi$ which is $N$-harmonic in $\mathbb{R}^N_+$ and vanishes on $\partial\mathbb{R}^{N-1}_+\setminus\{0\}$ is transformed by the inversion $\mathcal{I}_\omega$ of center $\omega$ and power 1 into the function $\Psi_\omega=\hat{\Psi}\circ\mathcal{I}_\omega$ which is positive and $N$-harmonic in $B^c_{\frac{1}{2}}(a)$ and vanishes on $\partial B_{\frac{1}{2}}(a)\setminus\{0\}$. For $\epsilon>0$ we denote by $\Phi_\epsilon$ the solution of

$$\begin{aligned} -\Delta_N\Phi_\epsilon&=0 && \text{in } \Omega\cap B^c_\epsilon\\ \Phi_\epsilon&=0 && \text{in } (B^c_{\frac{1}{2}}(a')\cap\partial B_\epsilon)\cup(\partial\Omega\cap B^c_\epsilon)\\ \Phi_\epsilon&=\Psi_{\omega'} && \text{in } B_{\frac{1}{2}}(a')\cap\partial B_\epsilon. \end{aligned} \tag{3.59}$$

If $0<\epsilon'<\epsilon$, $\Phi_{\epsilon'}\geq\Psi_{\omega'}$ in $B_{\frac{1}{2}}(a')\cap\partial B_\epsilon$, thus $\Phi_{\epsilon'}\geq\Phi_{\epsilon'}$ in $\Omega\cap B^c_\epsilon$. We also denote by $\hat{U}_\epsilon$ the solution of

$$\begin{aligned} -\Delta_N\hat{\Phi}_\epsilon&=0 && \text{in } \Omega\cap B^c_\epsilon\\ \hat{\Phi}_\epsilon&=0 && \text{in } \partial\Omega\cap B^c_\epsilon\\ \hat{\Phi}_\epsilon&=\Psi_\omega && \text{in } \Omega\cap\partial B^c_\epsilon. \end{aligned} \tag{3.60}$$

In the same way as above

$$0 < \epsilon' < \epsilon \Longrightarrow \hat{\Phi}_{\epsilon'} \leq \hat{\Phi}_{\epsilon} \quad \text{in } \Omega \cap \partial B_{\epsilon}^{c}$$

Using the explicit form of $\Psi$, $\mathcal{I}_{\omega} : x \mapsto \omega + \frac{x-\omega}{|x-\omega|^2}$ and $\mathcal{I}_{\omega'} : x \mapsto \omega' + \frac{x-\omega'}{|x-\omega'|^2}$ we see that

$$\Psi_{\omega'} \lfloor_{B_{\frac{1}{2}}(a') \cap \partial B_{\epsilon}} \leq \frac{1+\epsilon}{1-\epsilon} \Psi_{\omega} \lfloor_{B_{\frac{1}{2}}(a') \cap \partial B_{\epsilon}},$$

thus

$$\Phi_{\epsilon} \leq \frac{1+\epsilon}{1-\epsilon} \hat{\Phi}_{\epsilon} \quad \text{in } \Omega \cap B_{\epsilon}^{c}.$$

Letting $\epsilon \to 0$ we conclude that $\Phi_{\epsilon}$ converges uniformly in $\overline{\Omega} \setminus \{0\}$ to $\Phi_*$ which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies (3.58). □

The proof of the next statement is similar to the one of Lemma 3.8 up to some minor modifications.

**Lemma 3.12** *Let the assumptions on $q$ and $\Omega$ of Theorem 3.10 be satisfied. If for some $k > 0$ there exists a solution $u_k$ of* (3.18) *in $\Omega$, which belongs to $C^1(\overline{\Omega} \setminus \{0\})$, vanishes on $\partial\Omega \setminus \{0\}$ and satisfies* (3.57)*, then for any $k > 0$ there exists such a solution.*

**Lemma 3.13** *Under the assumptions of Theorem 3.10 there exists a Lipschitz continuous nonnegative subsolution $\tilde{u}$ of* (3.18) *in $\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$), is smaller than $\Phi_*$ and satisfies*

$$\lim_{\substack{x \to 0 \\ x/|x| \to \sigma}} |x|\, \tilde{u}(x) = \sigma, \tag{3.61}$$

*uniformly with respect to $\sigma \in S_{+}^{N-1}$.*

*Proof.* Let $\tau > 0$ to be fixed and let $w$ be the solution of

$$-\Delta_N w + |\nabla w|^q = 0 \qquad \text{in } B_2^- \tag{3.62}$$

which vanishes on $\partial B_2^- \setminus \{0\}$ and satisfies

$$\lim_{\substack{x \to 0 \\ x/|x| \to \sigma}} |x| w(x) = \sigma \tag{3.63}$$

in the $C^1$-topology of $S_{-}^{N-1}$. Its existence follows from Theorem 3.6 and this function is dominated by the N-harmonic function $\Phi_*$ corresponding to this domain obtain in Lemma 3.11. By $\mathcal{I}_{\omega'}$, the half-ball $B_2^-$ is transform into the lunule $G = B_{\frac{1}{2}}(a') \setminus B_{\frac{2}{3}}(\frac{4}{3}\omega')$ and $\tilde{w} = w \circ \mathcal{I}_{\omega'}$ satisfies

$$-\Delta_N \tilde{w} + |x - \omega'|^{2(q-N)} |\nabla \tilde{w}|^q = 0 \qquad \text{in } G. \tag{3.64}$$

Since $|x-\omega'|\leq 1$ in $G$, $-\Delta_N\tilde{w}+|\nabla\tilde{w}|^q\leq 0$ in $G$. We extend $\tilde{w}$ by 0 in $\Omega\setminus G$ and the resulting function $\tilde{u}$ is a subsolution of (3.18) in $\Omega$ which vanishes on $\partial\Omega\setminus\{0\}$), is smaller than the N-harmonic function $\Phi_*$ obtained in Lemma 3.11, and satisfies (3.61). $\square$

The proof of Theorem 3.10 is similar to the one of Theorem 3.6.

# 4 Classification of boundary singularities

We assume that $\Omega\subset\mathbb{R}^N$ is a $C^2$ domain and $0\in\partial\Omega$. Furthermore, in order to avoid extremely technical computations, we shall assume either that $\partial\Omega$ is flat near 0 or $p=N$. We denote $\mathbb{R}^N_+=\{x=(x',x_N),x_N>0\}$ and suppose that the tangent plane to $\partial\Omega$ at 0 is $\partial\mathbb{R}^N_+=\{x=(x',0)$ and the normal inward unit vector at 0 is $\mathbf{e}_N$, therefore $\mathbf{n}=-\mathbf{e}_N$ in the sequel. We denote by $\omega_s$ the unique positive solution of (3.1) in $S^{N-1}_+$ and by $U_s$ the corresponding singular solution of (3.1) in $\mathbb{R}^N_+$ defined by

$$U_s(x)=|x|^{-\beta_q}\,\omega_s(\frac{x}{|x|}). \tag{4.65}$$

We denote by $\psi_*$ a the unique positive solution of (3.2) with maximum 1 and by $\Psi_*$ the corresponding p-harmonic function

$$\Psi_*(x)=|x|^{-\beta_*}\,\psi_*(\frac{x}{|x|}). \tag{4.66}$$

**Proposition 4.1** *Assume $N-1<q<N-\frac{1}{2}$ and $\Omega$ is a bounded $C^2$ domain with $0\in\partial\Omega$. Then for any $k>0$ there exists a unique $u\in C(\overline{\Omega}\setminus\{0\})\cap C^1(\Omega)$, $u>0$ solution of* (3.18) *in $\Omega$ which vanishes on $\partial\Omega\setminus\{0\}$ and satisfies*

$$\lim_{\substack{x\to 0\\ x\in\Omega}} d^2(x)\frac{u_k(x)}{x}=k. \tag{4.67}$$

Next we can study the link between weak and strong singularities.

**Proposition 4.2** *Under the assumptions of Proposition* **??** *there exists $\lim_{k\to\infty}u_k=u_\infty$ which is the unique element of $C(\overline{\Omega}\setminus\{0\})\cap C^1(\Omega)$ vanishes on $\partial B^+_\delta\setminus\{0\}$, satisfies* (1.1) *in $B^+_\delta$ and*

$$\lim_{x\to 0}\frac{u_\infty(x)}{U_s(x)}=1. \tag{4.68}$$

*Proof.* Uniqueness follows from (4.68) and the maximum principle. For existence, since the mapping $k\mapsto u_k$ is increasing and $u_k\leq U_s$, there exists $\lim_{k\to\infty}u_k:=u_\infty\leq U_s$ and $u_\infty\in C(\overline{\Omega}\setminus\{0\})\cap C^1(\Omega)$. It vanishes on $\partial B^+_\delta\setminus\{0\}$ and satisfies (1.1) in $B^+_\delta$. In order to take into account the domain $B^+_\delta$ in the notations, we set

$u_k = u_{k,\delta}$. Since the mapping $\delta \mapsto u_{k,\delta}$ is also increasing and $u_{k,\delta} \leq k\Psi_*$, there also exists $\lim_{\delta\to\infty} u_{k,\delta} := u_{k,\infty} \leq k\Psi_*$ Then, for all $\ell > 0$,

$$T_\ell[u_{k,\delta}](x) = \ell^{\beta_q} u_{k,\delta}(\ell x) = u_{k\ell^{\beta_q},\ell^{-1}\delta}(x). \tag{4.69}$$

Letting $k \to \infty$, we obtain

$$T_\ell[u_{\infty,\delta}](x) = \ell^{\beta_q} u_{\infty,\delta}(\ell x) = u_{\infty,\ell^{-1}\delta}(x), \tag{4.70}$$

and letting $\delta \to \infty$, we obtain

$$T_\ell[u_{\infty,\infty}](x) = \ell^{\beta_q} u_{\infty,\infty}(\ell x) = u_{\infty,\infty}(x). \tag{4.71}$$

Thins implies that

$$u_{\infty,\infty}(r,\sigma) = r^{-\beta_q}\omega'_s(\sigma), \tag{4.72}$$

and $\omega'_s$ is a positive solution of problem (3.1). Therefore $\omega'_s = \omega_s$ by Theorem 3.2. If we let $\ell \to 0$ in (4.69) and take $|x| = 1$, $x = \sigma$, we derive

$$\lim_{\ell\to 0} \ell^{\beta_q} u_{\infty,\delta}(\ell,\sigma) = \lim_{\ell\to 0} u_{\infty,\ell^{-1}\delta}(1,\sigma) = u_{\infty,\infty}(1,\sigma) = \omega_s(\sigma). \tag{4.73}$$

This convergence holds in $C^1(\overline{S_+^{N-1}})$ because of Lemma **??**. This implies (4.68). $\square$

Similarly we have

**Proposition 4.3** *Under the assumptions of Proposition 4.1 there exists* $\lim_{k\to\infty} u_k = u_\infty$ *which is the unique element of* $C(\overline{\Omega}\setminus\{0\})\cap C^1(\Omega)$ *which vanishes on* $\partial B_\delta^+ \setminus \{0\}$, *satisfies* (3.18) *in* $\Omega$ *and*

$$\lim_{x\to 0} \frac{u_\infty(x)}{U_s(x)} = 1. \tag{4.74}$$

**Theorem 4.4** *Assume* $1 < p < N$, $\beta_q > \beta_*$ *and* $\partial\Omega \cap B_\delta = \{x = (x',0) : |x|' < \delta\}$, *for some* $\delta > 0$. *If* $u \in C(\overline{\Omega}\setminus\{0\}) \cap C^1(\Omega)$ *is a positive solution of* (1.1) *in* $\Omega$ *which vanishes on* $\partial\Omega \setminus \{0\}$, *then we have the following alternative*

*(i) either there exists* $k \geq 0$ *such that*

$$\lim_{x\to 0} \frac{u(x)}{\Psi_*(x)} = k, \tag{4.75}$$

*(ii) or*

$$\lim_{x\to 0} \frac{u(x)}{U_s(x)} = 1. \tag{4.76}$$

*Proof. Step 1.* Assume

$$\liminf_{x\to 0} \frac{u(x)}{\Psi_*(x)} < \infty, \tag{4.77}$$

then we claim that (**??**) holds. We first note that if 4.77 holds, there also holds

$$\liminf_{x\to 0} \frac{u(x)}{u_1(x)} < \infty, \tag{4.78}$$

where $u_1$ is the solution of (1.1) obtained in Proposition **??** with $k = 1$. If $\{x_n\}$ is converging to 0 and such that

$$\liminf_{x\to 0} \frac{u(x)}{u_1(x)} = k = \lim_{n\to 0} \frac{u(x_n)}{u_1(x_n)},$$

there also holds by the boundary Harnack inequality (2.34)

$$\frac{u(x_n)}{u_1(x_n)} = \frac{u(x_n)}{dx_n)} \frac{d(x_n)}{u_1(x_n)} \geq c_5^{-2} \frac{u(x)}{u_1(x)} \quad \forall x \text{ s.t. } |x| = |x_n|\,.$$

This implies in particular

$$u(x) \leq c_5^2 (k + \epsilon_n) u_1(x) \qquad \forall x \text{ s.t. } |x| = |x_n|$$

where $\{\epsilon_n\}$ is converging to $0_+$, and by comparison principle

$$u(x) \leq K u_1(x) \qquad \forall x \in \mathbb{R}^N_+ \text{ s.t. } |x_n| \leq |x| \leq \frac{\delta}{2},$$

for some $K > 0$ and all $n \in \mathbb{N}_*$. Therefore

$$\limsup_{x\to 0} \frac{u(x)}{u_1(x)} < \infty. \tag{4.79}$$

We can assume that $k \neq 0$, otherwhile (4.75) holds with $k = 0$ and actually $u$ remains bounded near 0. As a consequence, there exists $K > 0$ such that

$$u(x) \leq K \Psi_*(x) \qquad \forall x \in B^+_{\frac{\delta}{2}}. \tag{4.80}$$

Let $m = \max\{u(x) : |x| = \delta\}$. For $0 < \tau < \delta$ we denote by $k_\tau$ the minimum of the $\kappa > 0$ such that $u(x) \leq \kappa \Psi_*(x) + m$ for $\tau \leq |x| \leq \delta$. Then $u(x) \leq k_\tau \Psi_*(x) + m$, and either the graphs of the mappings $u(.)$ and $k_\tau \Psi_*(.) + m$ are tangent at some $x_\tau \in B^+_\delta \setminus \overline{B}^+_\tau$, or they are tangent on the boundary of the domain, and the only possibility is that they are tangent on $|x| = \tau$. Since

$$|\nabla \Psi_*(x)|^2 = |x|^{-2(\tau+1)} \left(\tau^2 \psi^2 + |\nabla \eta|^2\right),$$

it never vanishes. If we set $w = u - (k_\tau \Psi_*(x) + m)$, then

$$-\mathcal{L} w + |\nabla u|^q = 0 \tag{4.81}$$

where the operator

$$\mathcal{L} = \sum_{i,j} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial x}{\partial x_j} \right)$$

is uniformly elliptic in a neighborhood of $x_\tau$ (see [16, Lemma 1.3]). Furthermore $w \leq 0$ and $w(x_\tau) = 0$ by the strong maximum principle $\nabla u(x_\tau)$ must vanishes, which contradicts the fact that $\nabla u(x_\tau) = \nabla w(x_\tau)$ by the tangency condition, and $\nabla w(x_\tau) \neq 0$. Therefore $|x_\tau| = \tau$ and $x_\tau \notin \partial\mathbb{R}^N_+$. If $\tau' < \tau$, $k_\tau \leq k_{\tau'}$, and we set $k = \lim_{\tau\to 0} k_\tau$, which is finite because of (4.80). There exists $\{\tau_n\}$ such that $\sigma_n := \tau^{-1}x_{\tau_n} \to \sigma_0$. Furthermore

$$r^{\beta_*}u(r,\sigma) \leq k_\tau\psi_*(\sigma) + mr^{\beta_*} \quad \text{if } \tau \leq r \leq \delta \quad \text{and } \tau^{\beta_*}u(\tau,\sigma_\tau) = k_\tau\psi_*(\sigma_\tau) + mr^{\beta_*}. \tag{4.82}$$

Put

$$u_\tau(x) = \tau^{\beta_*}u(\tau x) \tag{4.83}$$

Then

$$-\Delta u_\tau + \tau^{p-q-\beta_*(q+1-q)}\,|\nabla u_\tau|^q = 0 \qquad \text{in } B^+_{\frac{\delta}{\tau}} \setminus \{0\}$$

and, by (4.80),

$$0 \leq u_\tau(x) \leq K\,|x|^{-\beta_*} \qquad \text{in } B^+_{\frac{\delta}{2\tau}} \setminus \{0\}.$$

By Lemma **??**, there holds the set of functions $\{u_\tau(.)\}$ is relatively compact in the $C^1_{loc}$ topology of $\overline{\mathbb{R}^N_+} \setminus \{0\}$. Therefore, there exists a sequence $\{\tau'_n\} \subset \{\tau_n\}$ converging to 0, and a positive p-harmonic function $v$ in $\mathbb{R}^N_+$, continuous in $\overline{\mathbb{R}^N_+} \setminus \{0\}$ and vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$, such that $u_{\tau'_n} \to v$, and $v$ satisfies (4.80) in $\overline{\mathbb{R}^N_+} \setminus \{0\}$. By Theorem 5.1 $v$ is separable, therefore there exists $k^*$ such that $v = k^*\Psi_*$. In particular,

$$\lim_{\tau'_n\to 0} u_{\tau'_n}(1,\sigma) = k^*\psi_*(\sigma) \tag{4.84}$$

in the $C^1(\overline{S^{N-1}_+})$ topology. Combining (4.82), (4.83)and (4.84) we conclude that $k^* = k$ and

$$\lim_{\tau'_n\to 0} \tau_n'^{\beta_*}u_{\tau'_n}(1,\sigma) = k\psi_*(\sigma) \tag{4.85}$$

Using Proposition **??**, it is equivalent to

$$\lim_{\tau'_n\to 0} \frac{u(\tau'_n,\sigma)}{u_k(\tau'_n,\sigma)} = 1 \tag{4.86}$$

uniformly on $S^{N-1}_+$. For any $\epsilon > 0$, there exists $n_\epsilon > 0$ such that $n \geq n_\epsilon$ implies

$$u_{k-\epsilon}(\tau'_n,\sigma) \leq u(\tau'_n,\sigma) \leq u_{k+\epsilon}(\tau'_n,\sigma)$$

By comparison principle,

$$u_{k-\epsilon} \leq u \leq u_{k+\epsilon} + m \qquad \text{in } B^+_\delta \setminus B^+_{\tau'_n}, \tag{4.87}$$

and finally

$$u_{k-\epsilon} \leq u \leq u_{k+\epsilon} + m \qquad \text{in } B^+_\delta, \tag{4.88}$$

Since $\epsilon$ is arbitrary and using again Proposition **??**, it implies

$$\lim_{r\to 0}\frac{u(r,\sigma)}{\Psi_*(r,\sigma)}=k, \tag{4.89}$$

locally uniformly on $S^{N-1}$. But since the convergence holds in $C^1(\overline{S^{N-1}_+})$, (4.75) follows.

*Step 2.* Assume

$$\lim_{x\to 0}\frac{u(x)}{\Psi_*(x)}=\infty. \tag{4.90}$$

For any $0<\epsilon<\delta$ and $k>0$, there holds

$$u_k(x)\le u(x)\le v_\epsilon(x) \qquad \text{in } B^+_\delta\setminus B^+_\epsilon \tag{4.91}$$

where $v_\epsilon$ has been defined in (3.12), and letting $\epsilon$ to 0 a,d $k\to\infty$, we derive

$$u_\infty(x)\le u(x)\le v_0(x) \qquad \text{in } B^+_\delta\setminus\{0\}. \tag{4.92}$$

We have seen in Theorem 3.3 that $v_0$ is a separable solution of (1.1) in $\mathbb{R}^N_+$ which vanishes on $\partial\mathbb{R}^N_+\setminus\{0\}$, therefore $v_0(x)=U_s(x)$. This implies

$$u_\infty(x)\le u(x)\le |x|^{-\beta_q}\,\omega_s(\frac{x}{|x|}) \qquad \text{in } B^+_\delta\setminus\{0\}. \tag{4.93}$$

We conclude using Proposition 4.2. □

When $p=N$, the assumption that $\partial\Omega$ is an hyperplane near 0 can be removed and the following result holds. Its proof is similar and based upon the use of Theorem 3.10.

**Theorem 4.5** *Assume $N-1<q<N-\frac{1}{2}$ If $u\in C(\overline{\Omega}\setminus\{0\})\cap C^1(\Omega)$ is a positive solution of* (3.18) *in $\Omega$ which vanishes on $\partial\Omega\setminus\{0\}$, then we have the following alternative*

*(i) either there exists $k\ge 0$ such that* (4.75) *holds,*

*(ii) or* (4.76) *holds.*

# 5 Appendix I: Positive p-harmonic functions in a half space

We recall that $\mathbb{R}^N_+=\{x=(x',x_N):x_N>0\}$. In this section we prove the following rigidity result.

**Theorem 5.1** *Assume $1<p\le N$ and $u\in C^1(\mathbb{R}^N_+)\cap C(\overline{\mathbb{R}^N_+}\setminus\{0\})$ is a positive p-harmonic function which vanishes on $\partial\mathbb{R}^N_+\setminus\{0\}$ and such that $|x|^{\beta_*}\,u(x)$ is bounded. Then there exists $k\ge 0$ such that*

$$u(x)=k\Psi_*(x) \qquad \forall x\in\mathbb{R}^N_+. \tag{5.1}$$

*Proof.* Since $|x|_*^\beta u(x)$ is bounded, $|x|^{\beta_*+1}\nabla u(x)$ is also bounded and there exists $m>0$ such that $u(x)\le m\Psi_*(x)$ in $B_\delta^+$. We denote by $k$ the infimum of the $c>0$ such that $u(x)\le c\Psi_S(x)$. Then

$$0\le u(x)\le k\Psi_*(x)\qquad \forall x\in\mathbb{R}_+^N\setminus\{0\} \tag{5.2}$$

and we assume that $k>0$ otherwhile $u=0$. Assume that the graphs over $\mathbb{R}_+^N$) of the functions $x\mapsto u(x)$ and $x\mapsto k\Psi_S(x)$ are tangent at some point $x_0\in\mathbb{R}_+^N$ or $x_0\in\partial\mathbb{R}_+^N\setminus\{0\}$. Since $\nabla\Psi_S$ never vanishes in $\overline{\mathbb{R}}_+^N\setminus\{0\}$ it follows from the strong maximum principle or Hopf boundary lemma that $u=k\Psi_*$. If the two graphs are not tangent in $\overline{\mathbb{R}}_+^N\setminus\{0\}$, either they are asymptotically tangent at 0, or at $\infty$.
(i) In the first case there exists two sequences $\{k_n\}$ increasing to $k$ and $\{x_n\}\subset\mathbb{R}_+^N$ converging to zero such that $\frac{u(x_n)}{\Psi_*(x_n)}=k_n$. We set $r_n=|x_n|$ and $u_{r_n}(x)=r_n^{\beta_*}u(r_nx)$. Then $u_{r_n}$ is p-harmonic and positive and $0<u_{r_n}(x)\le k\,|x|^{-\beta_*}\,\psi_*(\frac{x}{|x|})$; therefore

$$|\nabla u_{r_n}(x)|\le C\,|x|^{-\beta_*-1}\ \text{ and }\ \left|\nabla u_{r_n}(x)-\nabla u_{r_n}(x')\right|\le C\,|x|^{-\beta_*-1-\alpha}\left|x-x'\right|^\alpha \tag{5.3}$$

for $0<|x|\le|x|'$ and some constants $C>0$ and $\alpha\in(0,1)$. Up to a subsequence, we can assume that $u_{r_n}\to U$ in the $C^1_{loc}$ topology of $\overline{\mathbb{R}}_+^N\setminus\{0\}$ and $\frac{x_n}{r_n}\to\xi\in S_+^{N-1}$. The function $U$ is p-harmonic and positive in $\mathbb{R}_+^N$ and satisfies $0\le U\le k\Psi_*$ in $\mathbb{R}_+^N$ and $U(\xi)=k\Psi_*(\xi)$ if $\xi\in S_+^{N-1}$ or $U_{x_N}(\xi)=k\Psi_{s\,x_N}(\xi)$ if $\xi\in\partial S_+^{N-1}$. It follows from the strong maximum principle or Hopf boundary lemma that $U=k\Psi_*$. Therefore $u_{r_n}\to k\Psi_*$ and in particular

$$\lim_{r_n\to0}\frac{r_n^{\beta_*}u(r_n,\sigma)}{\psi_*(\sigma)}=k\quad\text{uniformly on }\ S_+^{N-1} \tag{5.4}$$

For any $\epsilon>0$, there exists $n_\epsilon\in\mathbb{N}_*$ such that for $n\ge n_\epsilon$, $(k-\epsilon)\Psi_*(x)\le u(x)\le(k+\epsilon)\Psi_*(x)$ if $|x|=r_n$. Which implies $(k-\epsilon)\Psi_*(x)\le u(x)\le(k+\epsilon)\Psi_*$ for $|x|\ge r_n$ and therefore in $\mathbb{R}^N$ Since $\epsilon$ is arbitrary, we deduce that $u=k\Psi_*$.
(ii) if the two graphs are tangent at infinity, there exist two sequences $\{k_n\}$ increasing to $k$ and $\{x_n\}$ such that $r_n=|x_n|\to\infty$ with $u(x_n)=k_n\Psi_*(x_n)$ and

$$\lim_{r_n\to\infty}\frac{r_n^{\beta_*}u(r_n,\sigma)}{\psi_*(\sigma)}=k\quad\text{uniformly on }\ S_+^{N-1}. \tag{5.5}$$

Therefore we look at the supremum of the $c>0$ such that $u\ge c\Psi_*$. If the set of such $c$ is empty, it would means that

$$\inf_{x\in\mathbb{R}_+^N}\frac{u(x)}{\Psi_*(x)}=0.$$

Clearly, if this infimum is achieved at some point, the strong maximum principle or Hopf boundary lemma imply $u\equiv0$, contradicting (5.5), and this relation prevents

also this infimum be achieved at infinity. We are left with the case where there exists a sequence $\{z_n\} \subset \mathbb{R}^N_+$, converging to 0, such that

$$\lim_{n\to\infty} \frac{u(z_n)}{\Psi_*(z_n)} = 0. \tag{5.6}$$

By boundary Harnack inequality [5, th 2.11], there exists $c > 0$ such that

$$c^{-1}\frac{u(z)}{\Psi_*(z)} \leq \frac{u(z_n)}{\Psi_*(z_n)} \leq c\frac{u(z)}{\Psi_*(z)} \quad \forall z \in \mathbb{R}^N_+ \text{ s.t. } |z| = |z_n| \tag{5.7}$$

Combining (5.6) and (5.7), we derive that

$$\lim_{n\to\infty} \sup_{|z|=|z_n|} \frac{u(z)}{\Psi_*(z)} = 0, \tag{5.8}$$

Denoting by $\epsilon_n$ the infimum in the above relation, we obtain that $u \leq \epsilon_n \Psi_*$ in $\mathbb{R}^N_+ \setminus B_{\epsilon_n}$ and finally $u = 0$, contradiction. Thus we are left with the case where there exists $k' \in (0, k]$ which is the supremum of the $c > 0$ such that $u \geq c\Psi_*$. In particular $u \geq k'\Psi_*$. Remembering that $u \leq k\Psi_*$we obtain that $k = k'$ implies $u = k\Psi_*$.

Next we assume that $k' < k$. Clearly the graphs of $u$ and $k'\Psi_*$ cannot be tangent in $\overline{\mathbb{R}}^N_+$, because of strong maximum principle or Hopf boundary lemma. They cannot be tangent at infinity because of (5.5). Therefore there exists two sequences $\{k'_n\}$ increasing to $k'$ and $\{x'_n\} \subset \mathbb{R}^N_+$ converging to 0 such that $\frac{u(x'_n)}{\Psi_*(x'_n)} = k'_n$. As in Case (i) we obtain that

$$\lim_{r'_n\to 0} \frac{r'^{\beta_*}_n u(r'_n, \sigma)}{\psi_*(\sigma)} = k' \quad \text{uniformly on } S^{N-1}_+ \tag{5.9}$$

where $r'_n = |x'_n|$, and finally derive that $u = k'\Psi_*$, a contradiction with (5.5). Therefore $k = k'$, which ends the proof. □

*Remark.* In the case $p = N$ the result holds under the weaker assumption $\lim_{|x|\to\infty} u(x) = 0$. This is due to the fact that this condition implies by regularity

$$\lim_{|x|\to\infty} \frac{u(x)}{\omega_s(\frac{x}{|x|})} = 0$$

and therefore

$$u(x) \leq m\Psi_*(x) \forall x \text{ s.t. } |x| \geq 1$$

where $m = \max_{|x|=1} \dfrac{u(x)}{\omega_s(\frac{x}{|x|})}$. Using the inversion $x \mapsto \frac{x}{|x|^2}$, we obtain that the estimate $u \leq m\Psi_*$ holds $\mathbb{R}^N$, and we conclude by Theorem 5.1.

*Remark.* We conjecture that the rigidity result holds under the mere condition

$$\lim_{|x|\to\infty} |x|^{-\tilde{\beta}_*} u(x) = 0, \tag{5.10}$$

were $\tilde{\beta}_*$ is the exponent (positive) corresponding to the regular spherical p-harmonic function under the form

$$\tilde{\Psi}_* = |x|^{\tilde{\beta}_*}\, \tilde{\psi}_*(\frac{x}{|x|}), \tag{5.11}$$

see [29], [25]. When $p = N$, $\tilde{\beta}_* = 1$.

# 6 Appendix II: Estimates on $\beta_*$

When $N = 2$ and $1 < p \leq 2$, it is proved in [19] that

$$\beta_* = \frac{3 - p + 2\sqrt{p^2 - 5p + 7}}{3(p-1)}. \tag{6.1}$$

Up to now no estimate is known when $N > 2$ except in the cases $p = 2$ where $\beta_* = N - 1$ and $p = N$ where $\beta_* = 1$, besides the classical one

$$\beta_* > \frac{N-p}{p-1}, \tag{6.2}$$

valid when $p < N$. In this section we prove the following result

**Theorem 6.1** *Assume $1 < p < N$. Then the following estimates hold*

$$1 < p < 2 \Longrightarrow \beta_* > \frac{N-1}{p-1} \tag{6.3}$$

$$2 < p < N \Longrightarrow \max\left\{1, \frac{N-p}{p-1}\right\} < \beta_* < \frac{N-1}{p-1} \tag{6.4}$$

*Remark.* It is worth noticing that when $p = 2$ or $p = N$, there holds $\beta_* = \frac{N-1}{p-1}$.

*Proof of Theorem 6.1.* We consider the following set of spherical coordinates in $\mathbb{R}^N_+$ with $x = (x_1, ..., x_N)$

$$\begin{aligned} x_1 &= r \sin\theta_{N-1} \sin\theta_{N-2} ... \sin\theta_2 \sin\theta_1 \\ x_2 &= r \sin\theta_{N-1} \sin\theta_{N-2} ... \sin\theta_2 \cos\theta_1 \\ &\vdots \\ x_{N-1} &= r \sin\theta_{N-1} \cos\theta_{N-2} \\ x_N &= r \cos\theta_{N-1} \end{aligned} \tag{6.5}$$

with $\theta_1 \in [0, 2\pi]$ and $\theta_k \in [0, \pi]$ for $k = 2, ..., N-2$ and $\theta_{N-1} \in [0, \frac{\pi}{2}]$. Under this representation, a solution $\omega$ of (3.2) verifies

$$\begin{aligned} &-\frac{1}{\sin^{N-2}\theta_{N-1}} \left[ \sin^{N-2}\theta_{N-1} \left( \beta_*^2 \omega^2 + \omega_{\theta_{N-1}}^2 + \frac{1}{\sin^2\theta_{N-1}} |\nabla_{\theta'}\omega|^2 \right)^{\frac{p-2}{2}} \omega_{\theta_{N-1}} \right]_{\theta_{N-1}} \\ &-\frac{1}{\sin^2\theta_{N-1}} div_{\theta'} \left[ \sin^{N-2}\theta_{N-1} \left( \beta_*^2 \omega^2 + \omega_{\theta_{N-1}}^2 + \frac{1}{\sin^2\theta_{N-1}} |\nabla_{\theta'}\omega|^2 \right)^{\frac{p-2}{2}} \nabla_{\theta'}\omega \right] \\ &= \beta_* \Lambda_{\beta_*} \left[ \sin^{N-2}\theta_{N-1} \left( \beta_*^2 \omega^2 + \omega_{\theta_{N-1}}^2 + \frac{1}{\sin^2\theta_{N-1}} |\nabla_{\theta'}\omega|^2 \right)^{\frac{p-2}{2}} \omega \right] \end{aligned} \tag{6.6}$$

where $\nabla_{\theta'}$ and $div_{\theta'}$ denotes respectively the spherical gradient the divergence in variables $\theta' = (\theta_1, ..., \theta_{N-2})$ parametrizing $S^{N-2}$ and $\Lambda_{\beta_*} = \beta_*(p-1) + p - N$. If $\omega$ is the unique positive solution of (3.2) (up to homothety), it depends only on $\theta_{N-1}$ and is $C^\infty$. For simplicity we set $\theta_{N-1} = \theta \in [0, \frac{\pi}{2}]$ and $\omega = \omega(\theta)$ satisfies

$$-\frac{1}{\sin^{N-2}\theta}\left[\sin^{N-2}\theta\left(\beta_*^2\omega^2+\omega_\theta^2\right)^{\frac{p-2}{2}}\omega_\theta\right]_\theta = \beta_*\Lambda_{\beta_*}\left[\sin^{N-2}\theta\left(\beta_*^2\omega^2+\omega_\theta^2\right)^{\frac{p-2}{2}}\omega\right] \quad \text{in } (0,\tfrac{\pi}{2})$$
$$\omega(\tfrac{\pi}{2}) = 0\,,\ \omega_\theta(0) = 0. \tag{6.7}$$

*Step 1: The eigenvalue identity.* Equation (6.7) can also be written under the form

$$-\omega_{\theta\theta} - (N-2)\cot\theta\,\omega_\theta - (p-2)\frac{\beta_*^2\omega+\omega_{\theta\theta}}{\beta_*^2\omega^2+\omega_\theta^2}\omega_\theta^2 = \beta_*\Lambda_{\beta_*}\omega \tag{6.8}$$

Since

$$-\int_0^{\frac{\pi}{2}}(\omega_{\theta\theta}+(N-2)\cot\theta\,\omega_\theta)\cos\theta\sin^{N-2}\theta d\theta = (N-1)\int_0^{\frac{\pi}{2}}\omega\cos\theta\sin^{N-2}\theta d\theta,$$

we multiplying by $\cos\theta\sin^{N-2}\theta$ and integrate on $(0,\frac{\pi}{2})$. Noticing that

$$\beta_*\Lambda_{\beta_*} + 1 - N = (p-1)\left(\beta_* - \frac{N-1}{p-1}\right)(\beta_*+1)$$

we derive

$$\begin{aligned}(2-p)&\int_0^{\frac{\pi}{2}}\frac{\beta_*^2\omega+\omega_{\theta\theta}}{\beta_*^2\omega^2+\omega_\theta^2}\omega_\theta^2\omega\cos\theta\sin^{N-2}\theta d\theta\\ &= (p-1)\left(\beta_* - \frac{N-1}{p-1}\right)(\beta_*+1)\int_0^{\frac{\pi}{2}}\omega\cos\theta\sin^{N-2}\theta d\theta.\end{aligned} \tag{6.9}$$

*Step 2: Elliptic coordinates and reduction.* Writting $\omega(\theta) = \omega(0) + a\theta^2 + o(\theta^2)$, $\omega_\theta(\theta) = 2a\theta + o(\theta)$ and $\omega_{\theta\theta}(\theta) = 2a + o(1)$, then $-Na = \beta_*\Lambda_{\beta_*}$. This implies that $\omega$ is decreasing near 0. It is immediate that it cannot have a local minimum in $(0,\frac{\pi}{2})$, therefore it remains decreasing in the whole interval. We parametrize the ellipse

$$E_r = \{(x,y) :, x > 0,\ y < 0,\ x^2 + \beta^{-2}y^2 = r^2\}$$

and set $\omega = r\cos\phi$ and $-\omega_\theta = \beta r\sin\phi$, with $\phi = \phi(\theta)$ and $r = r(\theta)$; $r$ and $\phi$ are $C^2$ functions. Hence $r_\theta\cos\phi - r\sin\phi\phi_\theta = -\beta r\sin\phi$, then $r_\theta\cos\phi = (\phi_\theta - \phi)r\sin\phi$ and $r_\theta = (\phi_\theta - \beta)r\tan\phi$. Plugging this into (6.9), we derive

$$-\left((p-1)\frac{r_\theta}{r} + \phi_\theta\cot\phi + (N-2)\cot\theta\right) + \Lambda_{\beta_*}\cot\phi = 0, \tag{6.10}$$

and finally

$$(p-1)(\phi_\theta - \beta)\tan\phi + (\phi_\theta - \Lambda_{\beta_*})\cot\phi = (2-N)\cot\theta. \tag{6.11}$$

*Step 3: Estimates on $\phi_\theta$.* We can write (6.11) under the equivalent form

$$(p-1)(\phi_\theta-\beta)\tan^2\phi+\phi_\theta-\Lambda_{\beta_*}=(2-N)\frac{\cos\theta}{\cos\phi}\frac{\sin\phi}{\sin\theta}. \tag{6.12}$$

Since

$$\lim_{\theta\to 0}\frac{\sin\phi}{\sin\theta}=\lim_{\theta\to 0}\frac{\cos\phi}{\cos\theta}\phi_\theta=\phi_\theta(0)$$

we derive $\phi_\theta(0)-\Lambda_{\beta_*}=(2-N)\phi_\theta(0)$ and thus

$$\phi_\theta(0)=\frac{\Lambda_{\beta_*}}{N-1}. \tag{6.13}$$

Similarly, the expansion of $\phi(\theta)$ near $\theta=\frac{\pi}{2}$ yields to

$$\phi_\theta(\tfrac{\pi}{2})=\beta_*. \tag{6.14}$$

Since $p<N$, $\Lambda_{\beta_*}/(N-1)<\beta_*$. We claim now that

$$\phi_\theta(\theta)<\beta_* \qquad \forall\theta\in(0,\tfrac{\pi}{2}). \tag{6.15}$$

If $\Lambda_{\beta_*}\le\beta_*$, then

$$(2-N)\cot\theta=(p-1)(\phi_\theta-\beta)\tan\phi+(\phi_\theta-\Lambda_{\beta_*})\cot\phi\ge((p-1)\tan\phi+\cot\phi)(\phi_\theta-\beta)$$

thus (6.15) holds.

Next we assume $\beta_*<\Lambda_{\beta_*}$. It means $0<(p-2)\beta_*-(N-p)$ and thus $p>2$. We claim that

$$\beta_*\le\frac{N-2}{p-2}. \tag{6.16}$$

We proceed by contradiction and assume

$$\beta_*>\frac{N-2}{p-2}. \tag{6.17}$$

Then

$$(p-2)\left(\beta_*^2-\frac{N-p}{p-2}\beta_*-\frac{N-2}{p-2}\right)=(p-2)\left(\beta_*+1\right)\left(\beta_*-\frac{N-2}{p-2}\right)>0.$$

Equivalently

$$\beta_*(\Lambda_{\beta_*}-\beta_*)>N-2.$$

Since

$$\lim_{\theta\to\frac{\pi}{2}}\cot\theta\tan\phi=\lim_{\theta\to\frac{\pi}{2}}\frac{\cos\theta}{\cos\phi}=\lim_{\theta\to\frac{\pi}{2}}\frac{\sin\theta}{\phi_\theta\sin\phi}=\frac{1}{\beta_*}$$

and

$$\begin{aligned}(p-1)(\phi_\theta(\theta)-\beta_*)\tan^2\phi&=\Lambda_{\beta_*}-\phi_\theta(\theta)+(2-N)\frac{\cos\theta}{\cos\phi}\frac{\sin\phi}{\sin\theta}\\&=\frac{1}{\beta_*}\left(\beta_*(\Lambda_{\beta_*}-\beta_*)+2-N\right)+o(1)\end{aligned}$$

if follows that (6.17) is equivalent to the fact that there exists $\epsilon > 0$ such that $\phi_\theta(\theta) > \frac{\pi}{2}$ for any $\theta \in [\frac{\pi}{2} - \epsilon, \frac{\pi}{2})$. Since $\phi_\theta(0) < \beta_*$, there exists $\bar{\theta} \in (0, \frac{\pi}{2})$ such that $\phi_\theta(\bar{\theta}) = \beta$ and $\phi_{\theta\theta}(\bar{\theta}) \geq 0$. We compute $\phi_{\theta\theta}$ and get

$$\begin{aligned}(p-1)\phi_\theta(\theta)(\phi_\theta(\theta) - \beta_*)\sec^2\phi(\theta) + (p-1)\phi_{\theta\theta}(\theta)\tan\phi(\theta) + \phi_{\theta\theta}(\theta)\cot\phi(\theta)\\ - \phi_\theta(\theta)(\phi_\theta(\theta) - \Lambda_{\beta_*})\csc^2\phi(\theta) = (N-2)\csc^2\theta\end{aligned}$$

Hence, at $\theta = \bar{\theta}$

$$(p-1)\phi_{\theta\theta}(\bar{\theta})\left((p-1)\tan\phi(\bar{\theta}) + \cot\phi(\bar{\theta})\right) = \beta_*(\beta_* - \Lambda_{\beta_*})\csc^2\phi(\theta) + (N-2)\csc^2\bar{\theta}$$

From (6.11),

$$\cot\phi(\bar{\theta}) = \frac{N-2}{\Lambda_{\beta_*} - \beta_*}\cot\bar{\theta}$$

Therefore

$$\begin{aligned}&(p-1)\phi_{\theta\theta}(\bar{\theta})\left((p-1)\tan\phi(\bar{\theta}) + \cot\phi(\bar{\theta})\right)\\ &\quad= \left(1 + \left(\frac{N-2}{\Lambda_{\beta_*} - \beta_*}\right)^2\cot^2\bar{\theta}\right)\beta_*(\beta_* - \Lambda_{\beta_*}) + (N-2)(1+\cot^2\bar{\theta})\\ &\quad= \beta_*(\beta_* - \Lambda_{\beta_*}) + N - 2 - \left(\frac{(N-2)^2}{\Lambda_{\beta_*} - \beta_*} + 2 - N\right)\cot^2\bar{\theta}\\ &\quad= -(p-2)(\beta_*+1)\left(\beta_* - \frac{N-2}{p-2}\right) - \frac{N-2}{\Lambda_{\beta_*} - \beta_*}\left(\beta_*(N-1) - \Lambda_{\beta_*}\right)\cot^2\bar{\theta}\\ &\quad< 0,\end{aligned} \tag{6.18}$$

using (6.17) and the fact that $N > p$. This is a contradiction, thus (6.16) holds and there exists $\epsilon > 0$ such that $\phi_\theta < \beta_*$ in $[\frac{\pi}{2} - \epsilon, \frac{\pi}{2})$.

We claim now that $\phi_\theta < \beta_*$ in $[0, \frac{\pi}{2})$. If it is not true, there exist $\theta_1 < \theta_2 < \frac{\pi}{2}$ such that $\phi_\theta(\theta_1) = \phi_\theta(\theta_2) = \beta_*$, $\phi_{\theta\theta}(\theta_1) > 0$, $\phi_{\theta\theta}(\theta_1) < 0$, since Cauchy-Lipschitz theorem applies to equation (6.11) in $(0, \frac{\pi}{2})$. We put $\phi(\theta_1) = \phi_1$ and $\phi(\theta_2) = \phi_2$. Using the equation satisfied by $\phi_{\theta\theta}$, we obtain for $i = 1, 2$,

$$\begin{aligned}&\left((p-1)\tan\phi_i + \cot\phi_i\right)\phi_{\theta\theta}(\theta_i)\\ &\quad= (2-p)(\beta_*+1)\left(\beta_* - \frac{N-2}{p-2}\right) - \frac{N-2}{\Lambda_{\beta_*} - \beta_*}(\beta_*(N-1) - \Lambda_{\beta_*})\cot^2\theta_i.\end{aligned} \tag{6.19}$$

Since cot is decreasing in $(0, \frac{\pi}{2})$, $\cot^2\theta_1 > \cot^2\theta_2$, hence

$$0 < \left((p-1)\tan\phi_1 + \cot\phi_1\right)\phi_{\theta\theta}(\theta_1) < \left((p-1)\tan\phi_2 + \cot\phi_2\right)\phi_{\theta\theta}(\theta_2) < 0,$$

a contradiction. Therefore $\phi_\theta < \beta_*$ in $(0, \frac{\pi}{2})$.

*Step 4: End of the proof.* Since $r^2 = \beta_*^2\omega^2 + \omega_\theta^2$, $r_\theta = r(\phi_\theta - \beta_*)\tan\phi$, thus

$$rr_\theta = \left(\beta_*^2\omega + \omega_{\theta\theta}\right)\omega_\theta = r(\phi_\theta - \beta_*)\tan\phi.$$

Since $\omega_\theta < 0$ on $(0, \frac{\pi}{2})$, it follows from Step 3 that $\beta_*^2\omega + \omega_{\theta\theta} > 0$ and finally

$$\int_0^{\frac{\pi}{2}} \frac{\beta_*^2\omega + \omega_{\theta\theta}}{\beta_*^2\omega^2 + \omega_\theta^2}\omega_\theta^2\omega\cos\theta\sin^{N-2}\theta d\theta > 0.$$

The conclusion follows from (6.9). □

*Remark.* Since $\omega_\theta(\frac{\pi}{2}) = -c^2 < 0$, it follows $\omega(\theta) = -\omega_\theta(\theta)\cot\theta + O(\frac{\pi}{2} - \theta)$ and from the eigenfunction equation (6.8)

$$\frac{\beta_*^2\omega + \omega_{\theta\theta}}{\beta_*^2\omega^2 + \omega_\theta^2}\omega_\theta^2 = (\beta_*^2\omega + \omega_{\theta\theta})(1 + o(1)).$$

Therefore

$$-(p-1)\omega_{\theta\theta} = (\beta_*\Lambda_{\beta_*} + (p-2)\beta_*^2 + 2 - N)\omega(1 + o(1)) \quad \text{as } \theta \to \frac{\pi}{2}$$

and since $\Delta'\omega := \omega_{\theta\theta} + (N-2)\cot\theta\,\omega_\theta$

$$-\Delta'\omega = \frac{\beta_*(\beta_*(2p-3) + p - N) + (p-2)(N-2)}{p-1}\omega(1 + o(1)) \quad \text{as } \theta \to \frac{\pi}{2}.$$

Because $\omega$ is $C^\infty$ we obtain finally

$$\left|\Delta'\omega\right| \le c\omega, \tag{6.20}$$

for some $c > 0$.

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