



**HAL**  
open science

## Boundary singularities of positive solutions of quasilinear Hamilton-Jacobi equations

Marie-Françoise Bidaut-Véron, Marta Garcia-Huidobro, Laurent Véron

► **To cite this version:**

Marie-Françoise Bidaut-Véron, Marta Garcia-Huidobro, Laurent Véron. Boundary singularities of positive solutions of quasilinear Hamilton-Jacobi equations. 2014. hal-01095162v1

**HAL Id: hal-01095162**

**<https://hal.science/hal-01095162v1>**

Preprint submitted on 15 Dec 2014 (v1), last revised 9 Sep 2015 (v5)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Boundary singularities of positive solutions of quasilinear Hamilton-Jacobi equations

Marie-Françoise Bidaut-Véron  
Marta Garcia-Huidobro  
Laurent Véron

**Abstract** We study the boundary behaviour of the solutions of (E)  $-\Delta_p u + |\nabla u|^q = 0$  in a domain  $\Omega \subset \mathbb{R}^N$ , when  $N \geq p > q > p - 1$ . We show the existence of a critical exponent  $q_* < p$  such that if  $p - 1 < q < q_*$  there exist positive solutions of (E) with an isolated singularity on  $\Omega$  and that these solutions belong to two different classes of singular solutions. If  $q_* \leq q < p$  no such solution exists and actually any boundary isolated singularity of a positive solution of (E) is removable. We prove that all the singular positive solutions are classified according the two types of singular solutions that we have constructed.

*2010 Mathematics Subject Classification.* 35J62; 35J92; 35B40; 35A20 .

*Key words.* p-Laplace operator; singularities; spherical p-harmonic equations; Leray-Lions operators; Schauder fixed point theorem.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>A priori estimates</b>	<b>5</b>
2.1	The gradient estimates and its applications . . . . .	5
2.2	Boundary a priori estimates . . . . .	6
<b>3</b>	<b>Boundary singularities</b>	<b>13</b>
3.1	Strongly singular solutions . . . . .	13
3.2	Removable boundary singularities . . . . .	18
3.3	Weakly singular solutions . . . . .	20
<b>4</b>	<b>Classification of boundary singularities</b>	<b>33</b>
<b>5</b>	<b>Appendix I: Positive p-harmonic functions in a half space</b>	<b>37</b>
<b>6</b>	<b>Appendix II: Estimates on <math>\beta_*</math></b>	<b>40</b>

# 1 Introduction

Let  $N \geq p > 1$ ,  $q > p - 1$  and  $\Omega \subset \mathbb{R}^N$  ( $N > 1$ ) be a  $C^2$  bounded domain such that  $0 \in \partial\Omega$ . In this article we study boundary singularities of functions  $u \in C(\overline{\Omega} \setminus \{0\})$  which satisfy

$$-\Delta_p u + |\nabla u|^q = 0 \quad \text{in } \Omega \quad (1.1)$$

where  $\Delta_p u := \operatorname{div}(\nabla u^{p-2} |\nabla u|)$ . The two main questions we consider are as follows:

Q-1- Existence of positive solutions of (1.1).

Q-2- Description of positive solutions with an isolated boundary singularity at 0.

When  $p = 2$  a fairly complete description of positive solutions of

$$-\Delta u + |\nabla u|^q = 0 \quad (1.2)$$

in  $\Omega$  is provided by Nguyen-Phuoc and Véron [22]. In particular they prove the following series of results in the range of values  $1 < q < 2$ .

1- Any signed solution of (1.3) verifies

$$|\nabla u(x)| \leq c_{N,q} (\operatorname{dist}(x))^{-\frac{1}{q-1}} \quad \forall x \in \Omega, \quad (1.3)$$

where  $d(x) = \operatorname{dist}(x, \partial\Omega)$ . As a consequence, if  $u \in C(\overline{\Omega} \setminus \{0\})$  is a solution which vanishes on  $\partial\Omega \setminus \{0\}$  it satisfies

$$|u(x)| \leq c_{q,\Omega} d(x) |x|^{-\frac{1}{q-1}} \quad \forall x \in \Omega. \quad (1.4)$$

2- If  $\frac{N+1}{N} \leq q < 2$  any positive solution of (1.3) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$  is identically 0. A point is a removable singularities.

3- If  $1 < q < \frac{N+1}{N}$  and  $k > 0$  there exists a unique positive solution  $u := u_k$  of (1.3) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies  $u(x) \sim c_N k P^\Omega(x, 0)$  where  $P^\Omega$  is the Poisson kernel in  $\Omega \times \partial\Omega$ .

4- If  $1 < q < \frac{N+1}{N}$  there exists a unique positive solution  $u$  of (1.3) in  $\mathbb{R}_+^N$  under the form  $u(x) = |x|^{-\frac{1}{q-1}} \omega(|x|^{-1}x)$  which vanishes on  $\partial\mathbb{R}_+^N \setminus \{0\}$ . The function  $\omega$  is the unique positive solution of

$$\begin{aligned} -\Delta' u + ((q-1)^2 \omega^2 + |\nabla' \omega|^2)^{\frac{q}{2}} - \lambda_{N,q} \omega &= 0 & \text{in } S_+^{N-1} \\ \omega &= 0 & \text{in } \partial S_+^{N-1} \end{aligned} \quad (1.5)$$

where  $\Delta'$  is the Laplace-Beltrami operator on  $S^{N-1}$  and  $\lambda_{N,q} > 0$  is an explicit constant.

5- If  $1 < q < \frac{N+1}{N}$  and  $u$  is a positive solution of (1.3) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$  the following dichotomy holds:

(i) either  $u(x) \sim |x|^{-\frac{1}{q-1}} \omega(|x|^{-1}x)$  as  $x \rightarrow 0$ ,

(ii) or  $u(x) \sim k c_N P^\Omega(x, 0)$  as  $x \rightarrow 0$  for some  $k \geq 0$ .

In this article we extend to the quasilinear case  $1 < p \leq N$  the above mentioned results. The following pointwise gradient estimate valid for any signed solution  $u$  of (1.1) is proved in [7]: if  $0 < p - 1 < q$  there exists a constant  $c_{N,p,q} > 0$  such that

$$|\nabla u(x)| \leq c_{N,p,q} (\text{dist}(x))^{-\frac{1}{q+1-p}} \quad \forall x \in \Omega. \quad (1.6)$$

As a consequence, any solution  $u \in C^1(\overline{\Omega} \setminus \{0\})$  satisfies

$$|u(x)| \leq c_{p,q,\Omega} d(x) |x|^{-\frac{1}{q+1-p}} \quad \forall x \in \Omega. \quad (1.7)$$

Concerning boundary singularities, the situation is more complicated than in the case  $p = 2$  and the threshold of critical exponent less explicit. We first consider the problem in the half space  $\mathbb{R}_+^N := \{x = (x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > 0\}$ . Assuming  $p - 1 < q \leq p$ , separable solutions of (1.1) in  $\mathbb{R}_+^N$  and vanishing on  $\mathbb{R}_+^N \setminus \{0\}$  can be looked for in spherical coordinates  $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$  under the form

$$u(x) = u(r, \sigma) = r^{-\frac{p-q}{q+1-p}} \omega(\sigma), \quad r > 0, \quad \sigma \in S_+^{N-1} := \{S^{N-1} \cap \mathbb{R}_+^N\}. \quad (1.8)$$

Then  $\omega$  is solution of the following problem

$$\begin{aligned} -\text{div} \left( (\beta_q^2 \omega^2 + |\nabla' \omega|^2)^{\frac{p-2}{2}} \nabla' \omega \right) - \beta_q \Lambda_{\beta_q} (\beta_q^2 \omega^2 + |\nabla' \omega|^2)^{\frac{p-2}{2}} \omega \\ + (\beta_q^2 \omega^2 + |\nabla' \omega|^2)^{\frac{q}{2}} = 0 \quad \text{in } S_+^{N-1} \\ \omega = 0 \quad \text{on } \partial S_+^{N-1}, \end{aligned} \quad (1.9)$$

where  $\beta_q = \frac{p-q}{q+1-p}$  and  $\Lambda_{\beta_q} = \beta_q(p-1) + p - N$ . The existence of positive solution to this problem is conditioned to the *existence of spherical  $p$ -harmonic functions* which have been studied by Krol [18], Tolksdorf [29], Kichenassamy and Véron [19] and more recently by Porretta and Véron [25]. One of the main point is that there exists a separable positive  $p$ -harmonic function  $\psi$  under the form  $\psi(r, \sigma) = r^{-\beta} \psi(\sigma)$  which is positive on  $S_+^{N-1}$  and negative on  $S_-^{N-1} := -S_+^{N-1}$  if and only if  $\psi$  satisfies the *spherical  $p$ -harmonic eigenvalue problem*

$$\begin{aligned} -\text{div} \left( (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \nabla' \psi \right) - \beta_* \Lambda_{\beta_*} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \psi = 0 \quad \text{in } S_+^{N-1} \\ \psi = 0 \quad \text{on } \partial S_+^{N-1}, \end{aligned} \quad (1.10)$$

There exists a unique couple  $(\psi_*, \beta_*)$  for which (1.10) has a solution with  $\beta_* > 0$  and  $\psi_* > 0$  (up to an homothety on  $\psi_*$ ). The exponent  $\beta_*$  which is larger than  $\max\{1, \frac{N-p}{p-1}\}$ , is not known explicitly except in the cases  $p = 2$ , and it has the value  $N - 1$ ,  $p = N$  and it is 1, and  $N = 2$  (see [18], [19]) and it is the root of an algebraic equation of degree 2. In Appendix II we prove the following new estimate:

**Theorem B** *Let  $1 < p \leq N$ .*

- (i) If  $2 \leq p \leq N$ , then  $\beta_* \leq \frac{N-1}{p-1}$  with equality only if  $p = 2$  or  $N$ .  
(ii) If  $1 \leq p < 2$ , then  $\beta_* > \frac{N-1}{p-1}$ .

To this exponent  $\beta_*$  is associated the critical value  $q_*$  of  $q$  defined by  $\beta_* = \beta_q$ , or equivalently

$$q_* := \frac{\beta_*(p-1) + p}{\beta_* + 1} = p - \frac{\beta_*}{\beta_* + 1}. \quad (1.11)$$

The following result characterizes strong singularities.

**Theorem C** *Let  $0 < p - 1 \leq N$ , then*

- (i) *If  $p - 1 < q < q_*$  problem (1.9) admits a unique positive solution  $\omega_*$ .*  
(ii) *If  $q_* \leq q < p$  problem (1.9) admits no positive solution.*

This critical exponent corresponds to the threshold of criticality for boundary singularities.

**Theorem D** *Assume  $q_* \leq q < p \leq N$ . If  $u \in C(\overline{\Omega} \setminus \{0\})$  is a nonnegative solution of (1.1) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ , it is identical zero.*

As in the case  $p = 2$  there exist positive solutions (1.1) in  $\Omega$  with weak boundary singularities which are characterized by their blow-up near the singularity; their existence, based upon the delicate construction of sub and super solutions is much more difficult than in the case  $p = 2$  (see [22]). Furthermore it is done only if  $\Omega$  is locally an hyperplane near 0. In the sequel we denote by  $B_R(a)$  the open ball of center  $a$  and radius  $R > 0$  and  $B_R = B_R(0)$ . We also set  $B_R^+(a) := \mathbb{R}_+^N \cap B_R(a)$ ,  $B_R^- := \mathbb{R}_+^N \cap B_R$ ,  $B_R^-(a) := \mathbb{R}_-^N \cap B_R(a)$  and  $B_R^- := \mathbb{R}_-^N \cap B_R$ .

**Theorem E** *Assume  $0 < p - 1 < q < q_* < p \leq N$ ,  $R > 0$  and  $\Omega_R := \Omega \cap B_R = B_R^+$ . Then for any  $k > 0$  there exists a unique  $u := u_k \in C(\overline{\Omega}_R \setminus \{0\})$ , solution of (1.1) in  $\Omega_R$ , vanishing on  $\partial\Omega_R \setminus \{0\}$  and such that*

$$\lim_{x \rightarrow 0} \frac{u_k(x)}{\psi_*(x)} = k. \quad (1.12)$$

Furthermore  $\lim_{k \rightarrow \infty} u_k = u_\infty$  and

$$\lim_{x \rightarrow 0} |x|^{\beta_q} u_\infty(x) = \omega_*(|x|^{-1}x). \quad (1.13)$$

When  $p = N$  and  $N - 1 < q < N - \frac{1}{2}$  we prove that the previous result holds if  $\Omega$  is any  $C^2$  by using the conformal invariance of  $\Delta_N$ . Finally, the isolated singularities of positive solutions of (1.1) are completely described by the two types of singular solutions obtained in the previous theorem and we prove:

**Theorem F** *Assume  $0 < p - 1 < q < q_* < p \leq N$ ,  $R > 0$  and  $\Omega_R = B_R^+$ . If  $u \in C(\overline{\Omega} \setminus \{0\})$  is a positive solution of (1.1) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ , then*

- (i) *either there exists  $k \geq 0$  such that*

$$\lim_{x \rightarrow 0} \frac{u(x)}{\psi_*(x)} = k. \quad (1.14)$$

(ii) or

$$\lim_{x \rightarrow 0} |x|^{\beta q} u(x) = \omega_*(|x|^{-1} x). \quad (1.15)$$

**Acknowledgements** This article has been prepared with the support of the Math-Amsud collaboration program 13MATH-02 QUESP. The first two authors were supported by Fondecyt grant N°1110268.

## 2 A priori estimates

### 2.1 The gradient estimates and its applications

We denote by  $d(x)$  the distance from  $x \in \bar{\Omega}$  to  $\partial\Omega$  and we recall the following estimate proved in [7].

**Proposition 2.1** *Assume  $q > p - 1$  and  $u$  is a  $C^1$  solution of (1.1) in a domain  $\Omega$ . Then*

$$|\nabla u(x)| \leq c_{N,p,q} (d(x))^{-\frac{1}{q+1-p}} \quad \forall x \in \Omega. \quad (2.1)$$

The first application is a pointwise upper bound for solutions with isolated singularities.

**Corollary 2.2** *Assume  $q > p - 1 > 0$ ,  $R^* > 0$  and  $\Omega$  is a domain containing 0 such that  $d(0) \geq 2R^*$ . Then for any  $x \in B_{R^*} \setminus \{0\}$ , and  $0 < R \leq R^*$ , any  $u \in C^1(\Omega \setminus \{0\})$  solution of (1.1) in  $\Omega \setminus \{0\}$  satisfies*

$$|u(x)| \leq c_{N,p,q} \left| |x|^{\frac{q-p}{q+1-p}} - R^{\frac{q-p}{q+1-p}} \right| + \max\{|u(z)| : |z| = R\}, \quad (2.2)$$

if  $p \neq q$ , and

$$|u(x)| \leq c_{N,p} (\ln R - \ln |x|) + \max\{|u(z)| : |z| = R\}, \quad (2.3)$$

if  $p = q$ .

The second estimate corresponds to solutions with boundary blow-up.

**Corollary 2.3** *Assume  $q > p - 1 > 0$ ,  $\Omega$  is a bounded domain with a  $C^2$  boundary. Then there exists  $\delta_1 > 0$  such that if we denote  $\Omega_{\delta_1} := \{z \in \Omega : d(z) \leq \delta_1\}$ , any  $u \in C^1(\Omega)$  solution of (1.1) in  $\Omega$  satisfies*

$$|u(x)| \leq c_{N,p,q} \left| (d(x))^{\frac{q-p}{q+1-p}} - \delta_1^{\frac{q-p}{q+1-p}} \right| + \max\{|u(z)| : d(z) = \delta_1\} \quad \forall x \in \Omega_{\delta_1} \quad (2.4)$$

if  $p \neq q$ , and

$$|u(x)| \leq c_{N,p,q} (\ln \delta_1 - \ln d(x)) + \max\{|u(z)| : d(z) = \delta_1\} \quad \forall x \in \Omega_{\delta_1} \quad (2.5)$$

if  $p = q$ .

*Remark.* As a consequence of (2.5) there holds for  $p > q > p - 1$

$$u(x) \leq (c_{N,p,q} + K \max\{|u(z)| : d(z) \geq \delta_1\}) (d(x))^{\frac{q-p}{q+1-p}} \quad \forall x \in \Omega \quad (2.6)$$

where  $K = (\text{diam}(\Omega))^{\frac{p-q}{q+1-p}}$ , with the standard modification if  $p = q$ .

As a variant of Corollary 2.3 we have an upper estimate of solutions in an exterior domain.

**Corollary 2.4** *Assume  $q > p - 1 > 0$ ,  $R > 0$  and  $u \in C^1(B_{R_0}^c)$  is any solution of (1.1) in  $B_{R_0}^c$ . Then for any  $R > R_0$  there holds*

$$|u(x)| \leq c_{N,p,q} \left| (|x| - R_0)^{\frac{q-p}{q+1-p}} - (R - R_0)^{\frac{q-p}{q+1-p}} \right| + \max\{|u(z)| : |z| = R\} \quad \forall x \in B_R^c \quad (2.7)$$

if  $p \neq q$  and

$$|u(x)| \leq c_{N,p,q} (\ln(|x| - R_0) - \ln(R - R_0)) + \max\{|u(z)| : |z| = R\} \quad \forall x \in B_R^c \quad (2.8)$$

if  $p = q$ .

*Proof.* The proof is a consequence of the identity

$$u(x) = u(z) + \int_0^1 \frac{d}{dt} u(tx + (1-t)z) dt = \int_0^1 \langle \nabla u(tx + (1-t)z), x - z \rangle dt$$

where  $z = \frac{R}{|x|}x$ . Since

$$|\nabla u(tx + (1-t)z)| \leq C_{N,p,q} (t|x| + (1-t)R - R_0)^{-\frac{1}{q+1-p}}$$

by estimate (2.1), the result follows by integration.  $\square$

## 2.2 Boundary a priori estimates

The next result is the extension of a regularity estimate dealing with singularity in a domain proved in [7].

**Lemma 2.5** *Assume  $p - 1 < q < p$ ,  $\Omega$  is a bounded  $C^2$  domain such that  $0 \in \partial\Omega$ . Let  $u \in C^1(\overline{\Omega} \setminus \{0\})$  be a solutions of (1.1) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies*

$$|u(x)| \leq \phi(|x|) \quad \forall x \in \Omega, \quad (2.9)$$

where  $\phi : \mathbb{R}_+^* \mapsto \mathbb{R}_+$  is continuous, nonincreasing and satisfies  $\phi(rs) \leq \gamma\phi(r)\phi(s)$  and  $r^{\frac{p-q}{q+1-p}}\phi(r) \leq c$  for some  $\gamma, c > 0$  and any  $r, s > 0$ . Then there exist  $\alpha \in (0, 1)$  and  $c_2 = c_2(p, q, \Omega) > 0$  such that

$$\begin{aligned} (i) \quad & |\nabla u(x)| \leq c_2 \phi(|x|) |x|^{-1} & \forall x \in \Omega, \\ (ii) \quad & |\nabla u(x) - \nabla u(y)| \leq c_2 \phi(|x|) |x|^{-1-\alpha} |x - y|^\alpha & \forall x, y \in \Omega, |x| \leq |y|. \end{aligned} \quad (2.10)$$

Furthermore

$$u(x) \leq c_2 \phi(|x|) \frac{d(x)}{|x|} \quad \forall x \in \Omega. \quad (2.11)$$

*Proof.* For  $\ell > 0$ , we set  $\Omega^\ell := \frac{1}{\ell}\Omega$ . If  $\ell \in (0, 1]$  is bounded the curvature of  $\partial\Omega^\ell$  remains uniformly bounded. As in [12, p 622], there exists  $0 < \delta_0 \leq 1$  and an involutive diffeomorphism  $\psi$  from  $\overline{B_{\delta_0}} \cap \overline{\Omega^{\delta_0}}$  into  $\overline{B_{\delta_0}} \cap (\Omega^{\delta_0})^c$  which is the identity on  $\overline{B_{\delta_0}} \cap \partial\Omega^{\delta_0}$  and such that  $D\psi(\xi)$  is the symmetry with respect to the tangent plane  $T_\xi\partial\Omega$  for any  $\xi \in \partial\Omega \cap \overline{B_{\delta_0}}$ . We extend any function  $v$  defined in  $\overline{B_{\delta_0}} \cap \overline{\Omega^{\delta_0}}$  and vanishing on  $\overline{B_{\delta_0}} \cap \partial\Omega^{\delta_0}$  into a function  $\tilde{v}$  defined in  $\overline{B_{\delta_0}}$  by

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in \overline{B_{\delta_0}} \cap \overline{\Omega^{\delta_0}} \\ -v \circ \psi(x) & \text{if } x \in \overline{B_{\delta_0}} \cap (\Omega^{\delta_0})^c, \end{cases} \quad (2.12)$$

If  $v \in C^1(\overline{B_{\delta_0}} \cap \overline{\Omega^{\delta_0}})$  is a solution of (1.1) in  $B_{\delta_0} \cap \Omega^{\delta_0}$  which vanishes on  $\partial\Omega^{\delta_0} \cap \overline{B_{\delta_0}}$ ,  $\tilde{v}$  satisfies

$$-\sum_j \frac{\partial}{\partial x_j} \tilde{A}_j(x, \nabla \tilde{v}) + B(x, \nabla \tilde{v}) = 0 \quad \text{in } B_{\delta_0}. \quad (2.13)$$

As in [12, (2.37)] the  $A_j$  satisfy the following estimates

$$\begin{aligned} (i) \quad & \tilde{A}_j(x, 0) = 0 \\ (ii) \quad & \sum_{i,j} \frac{\partial}{\partial \eta_i} \tilde{A}_j(x, \eta) \xi_i \xi_j \geq \Gamma_1 |\eta|^{p-1} |\xi|^2 \\ (iii) \quad & \sum_{i,j} \left| \frac{\partial}{\partial \eta_j} \tilde{A}_j(x, \eta) \right| \leq \Gamma_2 |\eta|^{p-2}, \end{aligned} \quad (2.14)$$

as for  $B$  there holds

$$|B(x, \eta)| \leq \Gamma_3 (1 + |\eta|)^p. \quad (2.15)$$

These estimates are the ones needed to apply Tolksdorf's result [30, Th 1,2]. There exists a constant  $C$ , such that for any ball  $\overline{B_{3R}} \subset \overline{B_{\delta_0}}$ , there holds

$$\|\nabla \tilde{v}\|_{L^\infty(B_R)} \leq C \quad (2.16)$$

where  $C$  depends on the constants  $\Gamma_k$  ( $k = 1, 2, 3$ ),  $N$ ,  $p$  and  $\|\tilde{v}\|_{L^\infty(B_{3R})}$ . We define

$$\Phi_\ell[u](y) = \frac{1}{\phi(\ell)} u(\ell y) \quad \forall y \in \Omega^\ell. \quad (2.17)$$

Then  $\Phi_\ell[u] := u_\ell$  satisfies

$$|u_\ell(y)| \leq \frac{\phi(\ell |y|)}{\phi(\ell)} \leq \gamma \phi(|y|) \quad \forall y \in \Omega^\ell \quad (2.18)$$

and

$$-\Delta_p u_\ell + (\ell^{\beta q} \phi(\ell))^{q+1-p} |\nabla u_\ell|^q = 0 \quad \text{in } \Omega^\ell. \quad (2.19)$$

We extend  $u_\ell$  by the formula (2.12) into a function  $\tilde{u}_\ell$  which satisfies

$$-\sum_j \frac{\partial}{\partial y_j} \tilde{A}_j(y, \nabla \tilde{u}_\ell) + (\ell^{\beta q} \phi(\ell))^{q+1-p} B(y, \nabla \tilde{u}_\ell) = 0 \quad \text{in } B_{\delta_0}. \quad (2.20)$$



For  $0 < |x| < \delta_0$  there exists  $\ell \in (0, 2)$  such that  $\frac{\delta_0 \ell}{2} \leq |x| \leq \delta_0 \ell$ . Then  $y \mapsto \tilde{u}_\ell(y)$  with  $y = \frac{x}{\ell}$  satisfies (2.20) in  $B_{\delta_0}$  and  $|\tilde{u}_\ell(y)| \leq \gamma_* \phi(|y|)$  since  $\psi$  is a diffeomorphism and  $D\psi(\xi) \in O(N)$  for any  $\xi \in \partial\Omega \cap B_{\delta_0}$ . The function  $\tilde{u}_\ell$  remains bounded on any ball  $B_{3R}(z) \subset \Gamma := \{y \in \mathbb{R}^N : \frac{\delta_0}{2} < |y| < \delta_0\}$ , therefore  $|\nabla \tilde{u}_\ell(y)| \leq c$  for any  $y \in B_R(z)$ , for some constant  $c > 0$ . This implies

$$|\nabla u(x)| \leq c \gamma_* \delta_0 \phi\left(\frac{2}{\delta_0}\right) \phi(|x|) |x|^{-1} \quad \forall x \in \Omega \cap B_{\delta_0}. \quad (2.21)$$

For proving 2.10)-(ii) there exists  $\alpha \in (0, 1)$  such that  $|\nabla \tilde{u}_\ell(y) - \nabla \tilde{u}_\ell(y')| \leq c |y - y'|^\alpha$  for all  $y$  and  $y'$  belonging to  $B_R(z)$ . This implies the claim.

Let  $0 < \delta_1 \leq \delta_0$  such that at any boundary point  $z$  there exist two closed balls of radius  $\delta_1$  tangent to  $\partial\Omega$  at  $z$  and included in  $\Omega \cup \{z\}$  and in  $\overline{\Omega}^c \cup \{z\}$  respectively ( $\delta_1$  corresponds to the maximal radius of the interior and exterior sphere condition). Let  $a \in \partial\Omega$  and  $x \in \Omega$  such that  $|x| = |a| = r \leq \delta_1$  and let  $b = -r \mathbf{n}_0$  where  $\mathbf{n}_0$  is the normal outward unit vector to  $\partial\Omega$  at  $0$ . Let  $\theta \in (0, \frac{\pi}{2})$  be the angle between  $\overrightarrow{0a}$  and  $\overrightarrow{0x}$ . Consider the path  $\gamma$  from  $a$  to  $x$  defined by  $\gamma(t) = \cos(t\theta)a + \sin(t\theta)b$  with  $0 \leq t \leq 1$ . Then  $x = \cos \theta a + \sin \theta b$  and

$$u(x) = \int_0^1 \frac{d}{dt} u \circ \gamma(t) dt = \int_0^1 \theta \langle \nabla u \circ \gamma(t), \cos(t\theta)b - \sin(t\theta)a \rangle dt$$

Thus, by Cauchy-Schwarz and using (2.9),

$$u(x) \leq \theta |\nabla u \circ \gamma(t)| \leq c'_2 \theta |x|^{-\frac{1}{q+1-p}},$$

where  $c'_2 = c'_2(p, q, \Omega)$ . But there exists  $c''_2 > 0$  depending only on  $\delta_1$  and the curvature of  $\partial\Omega$  such that  $c^{-1}\theta \leq d(x) \leq c\theta$ . This implies (2.11).  $\square$

**Lemma 2.6** *Assume  $p - 1 < q \leq p$ ,  $\Omega$  is a bounded  $C^2$  domain such that  $0 \in \partial\Omega$  and  $R_0 = \max\{|z| : z \in \Omega\}$ . If  $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$  is a positive solution of (1.1) which vanishes on  $\partial\Omega \setminus \{0\}$ , it satisfies*

$$u(x) \leq \begin{cases} c_1 \left( |x|^{\frac{q-p}{q+1-p}} - R_0^{\frac{q-p}{q+1-p}} \right) & \text{if } q < p \\ (p-1) \ln \left( \frac{R_0}{|x|} \right) & \text{if } q = p \end{cases} \quad (2.22)$$

for all  $x \in \Omega$ , where  $c_1 = c_1(p, q) > 0$ .

*Proof.* For  $\epsilon > 0$  we denote by  $P_\epsilon : \mathbb{R} \mapsto \mathbb{R}_+$  the function defined by

$$P_\epsilon(r) = \begin{cases} 0 & \text{if } 0 \leq r \leq \epsilon \\ -\frac{r^4}{2\epsilon^3} + \frac{3r^3}{\epsilon^2} - \frac{6r^2}{\epsilon} + 5r - \frac{3\epsilon}{2} & \text{if } \epsilon < r < 2\epsilon \\ r - \frac{3\epsilon}{2} & \text{if } r \geq 2\epsilon, \end{cases} \quad (2.23)$$

and by  $u_\epsilon$  the extension of  $P_\epsilon(u)$  by zero outside  $\Omega$ . There exists  $R_0$  such that  $\Omega \subset B_{R_0}$ . Since  $0 \leq P_\epsilon(r) \leq |r|$  and  $P_\epsilon$  is convex,  $u_\epsilon \in C(\mathbb{R}^N \setminus \{0\}) \cap W_{loc}^{1,p}(\mathbb{R}^N \setminus \{0\})$  and

$$-\Delta_p u_\epsilon + |\nabla u_\epsilon|^q \leq 0 \quad \text{in } \mathbb{R}^N.$$

Let  $R > R_0$ . If  $p - 1 < q < p$

$$U_{\epsilon,R}(|x|) = c_1 \left( (|x| - \epsilon)^{\frac{q-p}{q+1-p}} - (R - \epsilon)^{\frac{q-p}{q+1-p}} \right) \quad \text{in } B_R \setminus B_\epsilon, \quad (2.24)$$

with  $c_1 = (p - q)^{-1}(q + p - 1)^{\frac{q-p}{q+1-p}}$ . Then  $-\Delta_p U_\epsilon + |\nabla U_\epsilon|^q \geq 0$ . Since  $u_\epsilon$  vanishes on  $\partial B_R$  and is finite on  $\partial B_\epsilon$ , it follows  $u_\epsilon \leq U_\epsilon$ . Letting successively  $\epsilon \rightarrow 0$  and  $R \rightarrow \infty$  yields to (2.22). If  $q = p$  we take

$$U_{\epsilon,R}(|x|) = (p - 1) \ln \left( \frac{R - \epsilon}{|x| - \epsilon} \right) \quad \text{in } B_R \setminus B_\epsilon, \quad (2.25)$$

which turns out to be a super solution of (1.1); the end of the proof is similar.  $\square$

As a consequence of Lemma 2.5 and Lemma 2.6, we have.

**Corollary 2.7** *Let  $p, q, \Omega$  and  $u$  be as in Lemma 2.6. Then there exists a constant  $c_3 = c_3(p, q, \Omega) > 0$  such that*

$$|\nabla u(x)| \leq c_3 |x|^{-\frac{1}{q+1-p}} \quad \forall x \in \Omega \quad (2.26)$$

and

$$u(x) \leq c_3 d(x) |x|^{-\frac{1}{q+1-p}} \quad \forall x \in \overline{\Omega} \setminus \{0\}. \quad (2.27)$$

*Remark.* If  $\Omega$  is locally flat near 0 estimates (2.26) and (2.27) are valid without any sign assumption on  $u$ . More precisely, if  $\partial\Omega \cap B_{\delta_0} = T_0 \partial\Omega \cap B_{\delta_0}$  we can perform the reflexion of  $u$  thru the tangent plane  $T_0 \partial\Omega$  to  $\partial\Omega$  at 0 and the new function  $\tilde{u}$  is a solution of (1.1) in  $B_{\delta_0} \setminus \{0\}$ . By Proposition 2.1, it satisfies

$$|\nabla \tilde{u}(x)| \leq c_{N,p,q} |x|^{-\frac{1}{q+1-p}} \quad \forall x \in B_{\frac{\delta_0}{2}} \setminus \{0\}. \quad (2.28)$$

Integrating this relation as in Corollary 2.2, we derive that for any  $x \in B_{\frac{\delta_0}{2}} \cap \Omega$ , there holds

$$|u(x)| \leq \begin{cases} c_{N,p,q} \left( |x|^{-\beta_q} - \left(\frac{\delta_0}{2}\right)^{-\beta_q} \right) + \max\{|u(z)| : |z| = \frac{\delta_0}{2}\} & \text{if } p \neq q \\ c_{N,p} \ln \left( \frac{\delta_0}{2|x|} \right) + \max\{|u(z)| : |z| = \frac{\delta_0}{2}\} & \text{if } p = q. \end{cases} \quad (2.29)$$

In the next result we authorize the boundary singular set be a compact set.

**Proposition 2.8** *Let  $p - 1 < q < p$ . There exist  $0 \leq r^* \leq \delta_1$  and  $c_4 = c_4(N, p, q) > 0$  such that for any nonempty compact set  $K \subset \partial\Omega$ ,  $K \neq \partial\Omega$  and any positive solution  $u \in C(\overline{\Omega} \setminus K) \cap C^1(\Omega)$  of (1.1) which vanishes on  $\partial\Omega \setminus K$ , there holds*

$$u(x) \leq c_4 d(x) (d_K(x))^{-\frac{1}{q+1-p}} \quad \forall x \in \partial\Omega \text{ s.t. } d(x) \leq r^*, \quad (2.30)$$

where  $d_K(x) = \text{dist}(x, K)$ .

*Proof. Step 1: Tangential estimates.* Let  $x \in \Omega$  such that  $d(x) \leq \delta_1$ . We denote by  $\sigma(x)$  the projection of  $x$  onto  $\partial\Omega$ , unique since  $d(x) \leq \delta_1$ . Let  $r, r', \tau > 0$  such that  $\frac{3}{4}r < r' < \frac{7}{8}r$  and  $0 < \tau \leq \frac{r'}{2}$  and put  $\omega_{\tau,x} = \sigma(x) + \tau \mathbf{n}_{\sigma(x)}$ . Since  $\partial\Omega$  is  $C^2$ , there exists  $0 < r^* \leq \delta_1$  depending on  $\Omega$  such that  $d_K(\omega_{\tau,x}) > \frac{7}{8}r$  whenever  $d(x) \leq r^*$ . Let  $A > 0$  and  $B > 0$  to be specified later on; we define  $\tilde{v}(s) = A(r' - s)^{\frac{q-p}{q+1-p}} - B$  and  $v(y) = \tilde{v}(|y - \omega_{\tau,x}|)$  in  $[0, r')$  and  $B_{r'}(\omega_{\tau,x})$ . Then

$$|\tilde{v}'|^{p-2} \left( |\tilde{v}'|^{q+2-p} - (p-1)\tilde{v}'' - \frac{N-1}{s}\tilde{v}' \right) = A^{p-1} \left( \frac{p-q}{q+1-p} \right)^{p-1} (r' - s)^{-\frac{q}{q+1-p}} X(s)$$

where

$$X(s) = \left( A \frac{p-q}{q+1-p} \right)^{q+1-p} - \frac{p-1}{q+1-p} - \frac{(N-1)(r'-s)}{s}.$$

For any  $\tau \in (0, r')$  there exists  $A > 0$  such that

$$\left( A \frac{p-q}{q+1-p} \right)^{q+1-p} \geq \frac{p-1}{q+1-p} + \frac{(N-1)(r'-s)}{s} \quad \forall \tau \leq s \leq r'.$$

This implies

$$-\Delta_p v + |\nabla v|^q \geq 0 \quad \text{in } B_{r'}(\omega_{\tau,x}) \setminus B_\tau(\omega_{\tau,x}). \quad (2.31)$$

Next we take  $B = A(r' - \tau)^{\frac{q-p}{q+1-p}}$ , thus  $v = 0$  on  $\partial B_\tau(\omega_{\tau,x})$ . Clearly  $B_\tau(\omega_{\tau,x}) \subset \overline{\Omega}^c$  since  $\tau < \delta_1$ . Therefore  $v \geq 0 = u$  on  $\partial\Omega \cap B_{r'}(\omega_{\tau,x})$  and  $u \leq v = \infty$  on  $\Omega \cap \partial B_{r'}(\omega_{\tau,x})$ . By the comparison principle,  $v \geq u$  in  $\Omega \cap B_{r'}(\omega_{\tau,x})$ . In particular

$$u(x) \leq v(x) \leq A(r' - \tau - d(x))^{\frac{q-p}{q+1-p}} - A(r' - \tau)^{\frac{q-p}{q+1-p}}.$$

We take now  $\tau = \frac{r'}{2}$  and  $d(x) \leq \frac{r}{4}$  and we get by the mean value theorem

$$u(x) \leq c'_4 r'^{-\frac{1}{q+1-p}} d(x) = c'_4 d(x) (d'_K(x))^{-\frac{1}{q+1-p}}, \quad (2.32)$$

with  $c'_4 = c'_4(p, q) > 0$ . Letting  $r' \rightarrow \frac{7}{8}r$ , we get (2.11).

*Step 2: Global estimates.* If  $d(x) \geq \frac{1}{4}d_K(x)$ , there holds

$$d(x)(d_K(x))^{-\frac{1}{q+1-p}} \geq 2^{\frac{2}{q+1-p}} (d(x))^{\frac{q-p}{q+1-p}}.$$

We combine (2.11) with (2.6) and obtain (2.11).  $\square$

*Remark.* Under the assumption of Proposition 2.8, it follows from the maximum principle that  $u$  is upper bounded in the set  $\Omega'_{r^*} := \{x \in \Omega : d(x) > r^*\} = \Omega \setminus \overline{\Omega}_{r^*}$  by the solution  $w$  of

$$\begin{aligned} -\Delta_p w + |\nabla w|^q &= 0 && \text{in } \Omega_{r^*} \\ w &= c_4 d(x) (d_K(x))^{-\frac{1}{q+1-p}} && \text{in } \partial\Omega_{r^*}, \end{aligned} \quad (2.33)$$

and  $w$  itself is bounded by  $d^* = \max\{cd(x)(d_K(x))^{-\frac{1}{q+1-p}} : d(x) = r^*\}$ .

We prove here the boundary Harnack inequality. We recall that  $\delta_1$  has already been defined, and that the interior and exterior sphere conditions holds in the set  $\{x \in \mathbb{R}^N : \text{dist}(x, \partial\Omega) \leq \delta_1\}$ .

**Proposition 2.9** *Let  $q > p-1$  and  $0 \in \partial\Omega$ . Then there exists  $c_5 = c_5(N, p, q, \Omega) > 0$  such that for any positive solution  $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^1(\Omega)$  of (1.1) in  $\Omega$ , vanishing on  $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$ , there holds*

$$\frac{u(y)}{c_5 d(y)} \leq \frac{u(x)}{d(x)} \leq c_5 \frac{u(y)}{d(y)} \quad (2.34)$$

for all  $x, y \in B_{\frac{2\delta_1}{3}} \cap \Omega$  such that  $\frac{1}{2}|x| \leq |y| \leq 2|x|$ .

We also recall the following result [3].

**Lemma 2.10** *Assume that  $a \in \partial\Omega$ ,  $0 < r < \delta_1$  and  $h > 1$  is an integer. There exists an integer  $N_0$  depending only on  $\delta_1$  such that for any points  $x$  and  $y$  in  $\Omega \cap B_{\frac{3r}{2}}(a)$  verifying  $\min\{d(x), d(y)\} \geq r/2^h$ , there exists a connected chain of balls  $B_1, \dots, B_j$  with  $j \leq N_0 h$  such that*

$$\begin{aligned} x \in B_1, y \in B_j, \quad B_i \cap B_{i+1} \neq \emptyset \text{ for } 1 \leq i \leq j-1 \\ \text{and } 2B_i \subset B_{2r}(Q) \cap \Omega \text{ for } 1 \leq i \leq j. \end{aligned} \quad (2.35)$$

The next result is a standard Harnack inequality.

**Lemma 2.11** *Assume  $a \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2\delta_1}{3}}$  and  $0 < r \leq |a|/4$ . Let  $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^1(\Omega)$  be a positive solution of (1.1) vanishing on  $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$ . Then there exists a positive constant  $c_6 > 1$  depending on  $N, p, q$  and  $\delta_1$  such that*

$$u(x) \leq c_6^h u(y), \quad (2.36)$$

for every  $x, y \in B_{\frac{3r}{2}}(a) \cap \Omega$  such that  $\min\{d(x), d(y)\} \geq r/2^h$  for some  $h \in \mathbb{N}$ .

*Proof.* We first notice that for any  $\ell > 0$ ,  $T_\ell[u]$  satisfies (1.1) in  $\Omega^\ell := \ell^{-1}\Omega$  where  $T_\ell$  is defined by

$$T_\ell[u](x) = \ell^{\frac{p-q}{q+1-p}} u(\ell x). \quad (2.37)$$

If we take in particular  $\ell = |a|$ , we can assume  $|a| = 1$  thus the curvature of the domain  $\Omega^{|a|}$  remains bounded. By Proposition 2.8

$$u(x) \leq c'_6 \quad \forall x \in B_{2r}(a) \cap \Omega \quad (2.38)$$

where  $c'_6$  depends on  $N, q, \delta_1$ . Then we proceed as in [22], using Lemma 2.10 and Harnack inequality as quoted in [31, Corollary 10].  $\square$

Since the solutions are Hölder continuous, there holds as in [31, Theorem 4.2]

**Lemma 2.12** *Assume the assumptions on  $a$  and  $u$  of Lemma 2.11 are fulfilled. If  $b \in \partial\Omega \cap B_r(a)$  and  $0 < s \leq 2^{-1}r$ , there exist two positive constants  $\delta$  and  $c_7$  depending on  $N, p, q$  and  $\Omega$  such that*

$$u(x) \leq c_7 \frac{|x-b|^\delta}{s^\delta} \max\{u(z) : z \in B_r(b) \cap \Omega\} \quad (2.39)$$

for every  $x \in B_s(b) \cap \Omega$ .

As a consequence we derive the following Carleson type estimate.

**Lemma 2.13** *Assume  $a \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2\delta_1}{3}}$  and  $0 < r \leq |a|/8$ . Let  $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^2(\Omega)$  be a positive solution of (1.1) vanishing on  $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$ . Then there exists a constant  $c_8$  depending only on  $N, p, q$  and  $\delta_1$  such that*

$$u(x) \leq c_8 u(a - \frac{r}{2} \mathbf{n}_a) \quad \forall x \in B_r(a) \cap \Omega. \quad (2.40)$$

*Proof.* By Lemma 2.11 it is clear that for any integer  $h$  and  $x \in B_r(a) \cap \Omega$  such that  $d(x) \geq 2^{-h}r$ , there holds

$$u(x) \leq c_6^h u(a - \frac{r}{2} \mathbf{n}_a). \quad (2.41)$$

Therefore  $u$  satisfies inequality (2.39) as any Hölder continuous function does it. The proof that the constant is independent of  $r$  and  $u$  is more delicate. It is done in [3, Lemma 2.4] for linear equations, but it based only on Lemma 2.12 and a geometric construction, thus it is also valid in our case.  $\square$

**Lemma 2.14** *Assume  $a \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2\delta_1}{3}}$  and  $0 < r \leq |a|/8$ . Let  $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^2(\Omega)$  be a positive solution of (1.1) vanishing on  $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$ . Then there exist  $\alpha \in (0, 1/2)$  and  $c_9 > 0$  depending on  $N, p, q$  and  $\delta_1$  such that*

$$\frac{1}{c_9} \frac{t}{r} \leq \frac{u(b - t \mathbf{n}_b)}{u(a - \frac{r}{2} \mathbf{n}_a)} \leq c_9 \frac{t}{r} \quad (2.42)$$

for any  $b \in B_r(a) \cap \partial\Omega$  and  $0 \leq t < \frac{\alpha}{2}r$ .

*Proof.* It is similar to the one of [22, Lemma 3.15].  $\square$

*Proof of Proposition 2.9.* Assume  $x \in B_{\frac{2\delta_1}{3}} \cap \Omega$  and set  $r = \frac{|x|}{8}$ .

*Step 1: Tangential estimate:* we suppose  $d(x) < \frac{\alpha}{2}r$ . Let  $a \in \partial\Omega \setminus \{0\}$  such that  $|a| = |x|$  and  $x \in B_r(a)$ . By Lemma 2.14,

$$\frac{8}{c_9} \frac{u(a - \frac{r}{2} \mathbf{n}_a)}{|x|} \leq \frac{u(x)}{d(x)} \leq 8c_9 \frac{u(a - \frac{r}{2} \mathbf{n}_a)}{|x|}. \quad (2.43)$$

We can connect  $a - \frac{r}{2} \mathbf{n}_a$  with  $-2r \mathbf{n}_0$  by  $m_1$  (depending only on  $N$ ) connected balls  $B_i = B_{\frac{r}{4}}(x_i)$  with  $x_i \in \Omega$  and  $d(x_i) \geq \frac{r}{2}$  for every  $1 \leq i \leq m_1$ . It follows from (2.40) that

$$c_6^{-m_1} u(-2r \mathbf{n}_0) \leq u(a - \frac{r}{2} \mathbf{n}_a) \leq c_6^{m_1} u(-2r \mathbf{n}_0),$$

which, together with (2.43) leads to

$$\frac{1}{c_9'} \frac{u(-2r \mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq c_9' \frac{u(-2r \mathbf{n}_0)}{|x|}, \quad (2.44)$$

with  $c_9' = 8c_9 c_6^{m_1}$ .

*Step 2: Internal estimate:* we suppose  $d(x) \geq \frac{\alpha}{2}r$ . We can connect  $-2r\mathbf{n}_0$  with  $x$  by  $m_2$  (depending only on  $N$ ) connected balls  $B'_i = B_{\frac{\alpha r}{4}}(x'_i)$  with  $x'_i \in \Omega$  and  $d(x'_i) \geq \frac{\alpha}{2}r$  for every  $1 \leq i \leq m_2$ . By Harnack and Carleson inequalities (2.36) and (2.40) and since  $\frac{\alpha}{4}|x| < d(x) \leq |x|$ , we get

$$\frac{\alpha}{4c_6^{m_2}} \frac{u(-2r\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq \frac{4c_6^{m_2}}{\alpha} \frac{u(-2r\mathbf{n}_0)}{|x|}. \quad (2.45)$$

*Step 3: End of proof.* Suppose  $\frac{|x|}{2} \leq s \leq 2|x|$ , we can connect  $-2r\mathbf{n}_Q$  with  $-s\mathbf{n}_Q$  by  $m_3$  (depending only on  $N$ ) connected balls  $B''_i = B_{\frac{r}{2}}(x''_i)$  with  $x''_i \in \Omega$  and  $d(x''_i) \geq r$  for every  $1 \leq i \leq m_3$ . This fact, jointly with (2.44) and (2.45), yields

$$\frac{1}{c_{10}} \frac{u(-s\mathbf{n}_Q)}{|x|} \leq \frac{u(x)}{d(x)} \leq c_{10} \frac{u(-s\mathbf{n}_Q)}{|x|} \quad (2.46)$$

where  $c_{10} = c_{10}(N, q, \Omega)$ . Finally, if  $y \in B_{\frac{2r_0}{3}} \cap \Omega$  satisfies  $\frac{|x|}{2} \leq |y| \leq 2|x|$ , then by applying twice (2.46) we get (2.34) with  $c_5 = c_{10}^2$ .  $\square$

As a consequence of Proposition 2.9, we have

**Corollary 2.15** *Assume  $q > p-1$  and  $0 \in \partial\Omega$ . Then there exists  $c_{11} = c_{11}(N, p, q, \Omega) > 0$  such that for any positive solutions  $u_1, u_2 \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^1(\Omega)$  of (1.1) in  $\Omega$ , vanishing on  $\partial\Omega \setminus \{0\} \cap B_{2\delta_1}$ , there holds*

$$\sup \left\{ \frac{u_1(y)}{u_2(y)} : y \in B_r \setminus B_{\frac{r}{2}} \right\} \leq c_{11} \inf \left\{ \frac{u_1(y)}{u_2(y)} : y \in B_r \setminus B_{\frac{r}{2}} \right\}. \quad (2.47)$$

## 3 Boundary singularities

### 3.1 Strongly singular solutions

We next consider the equation (1.1) in  $\mathbb{R}_+^N = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}_+^*\}$ . We denote by  $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$  the spherical coordinates in  $\mathbb{R}^N$  and

$$S_+^{N-1} = \left\{ (\sin \phi \sigma', \cos \phi) : \sigma' \in S^{N-2}, \phi \in [0, \frac{\pi}{2}] \right\}.$$

Set

$$\beta_q := \frac{p-q}{q+1-p} \quad \text{and} \quad \Lambda_{\beta_q} = \beta_q(p-1) + p - N$$

If  $v(x) = r^{-\beta}\omega(\sigma)$  satisfies (1.1) in  $\mathbb{R}_+^N$  and vanishes on  $\overline{\mathbb{R}_+^N} \setminus \{0\}$ , then  $\beta = \beta_q$  and  $\omega$  is a solution of

$$\begin{aligned} -\operatorname{div} \left( (\beta_q^2 \omega^2 + |\nabla' \omega|^2)^{\frac{p-2}{2}} \nabla' \omega \right) - \beta_q \Lambda_{\beta_q} (\beta_q^2 \omega^2 + |\nabla' \omega|^2)^{\frac{p-2}{2}} \omega \\ + (\beta_q^2 \omega^2 + |\nabla' \omega|^2)^{\frac{q}{2}} = 0 \quad \text{in } S_+^{N-1} \\ \omega = 0 \quad \text{on } \partial S_+^{N-1}. \end{aligned} \quad (3.1)$$

where  $\nabla'$  denotes the covariant derivative on  $S^{N-1}$  identified with the tangential derivative. There exists a unique  $\beta_* = \beta_*(N, p) > 0$  such that the function  $(r, \sigma) \mapsto r^{-\beta_*} \psi(\sigma)$  is positive,  $p$ -harmonic in  $\mathbb{R}_+^N$  and vanishes on  $\overline{\mathbb{R}_+^N} \setminus \{0\}$  (see [30], [25]). The function  $\psi$  is defined up to a multiplicative constant and it satisfies

$$\begin{aligned} -\operatorname{div} \left( (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \nabla' \psi \right) - \beta_* \Lambda_{\beta_*} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \psi &= 0 \quad \text{in } S_+^{N-1} \\ \psi &= 0 \quad \text{on } \partial S_+^{N-1}. \end{aligned} \quad (3.2)$$

We denote by  $\psi_*$  the solution which maximal value 1. The coefficient  $\beta_*$  is unknown, except when  $N = p$ , and it is 1, or  $p = 2$  and it is  $N - 1$ . Because of uniqueness  $\psi$  depends only on the azimuthal variable  $\theta_{N-1} = \cos^{-1}(\frac{x_N}{|x|})$  (see Appendix II). Our first result is the following

**Theorem 3.1** *If  $\beta_q \leq \beta_*$  there exists no positive solution to problem (3.1).*

*Proof.* Suppose such a solution  $\omega$  exists and put  $\theta = \beta_q/\beta_*$ . Then  $0 < \theta \leq 1$ . Set  $\eta = \psi\theta$ , where  $\psi$  is a positive solution of (3.2) and define the operator  $\mathcal{T}$  by

$$\begin{aligned} \mathcal{T}(\eta) = -\operatorname{div} \left( (\beta_q^2 \eta^2 + |\nabla' \eta|^2)^{\frac{p-2}{2}} \nabla' \eta \right) - \beta_q \Lambda_{\beta_q} (\beta_q^2 \eta^2 + |\nabla' \eta|^2)^{\frac{p-2}{2}} \eta \\ + (\beta_q^2 \eta^2 + |\nabla' \eta|^2)^{\frac{q}{2}}. \end{aligned} \quad (3.3)$$

Since  $\nabla \eta = \theta \psi^{\theta-1} \nabla \psi$ ,  $(\beta_q^2 \eta^2 + |\nabla' \eta|^2)^{\frac{p-2}{2}} = \theta^{p-2} \psi^{(\theta-1)(p-2)} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}}$ ,  $(\beta_q^2 \eta^2 + |\nabla' \eta|^2)^{\frac{p-2}{2}} \nabla' \eta = \theta^{p-1} \psi^{(\theta-1)(p-1)} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \nabla' \psi$ , therefore

$$\begin{aligned} \mathcal{T}(\eta) &= -\theta^{p-1} \psi^{(\theta-1)(p-1)} \operatorname{div} \left( (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \nabla' \psi \right) \\ &\quad - \theta^{p-1} (\theta - 1) (p - 1) \psi^{(\theta-1)(p-1)-1} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} |\nabla' \psi|^2 \\ &\quad - \beta_q \Lambda_{\beta_q} \theta^{p-2} \psi^{(\theta-1)(p-1)} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \psi + \theta^q \psi^{(\theta-1)q} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{q}{2}} \end{aligned}$$

But  $\beta_q \Lambda_{\beta_q} \theta^{p-2} = \beta_* \Lambda_{\beta_*} \theta^{p-1} \leq \beta_* \Lambda_{\beta_*} \theta^{p-1}$  since  $\beta_q \leq \beta_*$ . Using (3.2), we see that  $\mathcal{T}(\eta) \geq 0$ . Because Hopf lemma is valid,  $\partial_n \psi < 0$  on  $\partial S_+^{N-1}$ . Since  $\omega$  is  $C^1$  in  $S_+^{N-1}$  and  $\psi$  is defined up to an homothety, there exists a minimal  $\psi$  such that  $\eta \geq \omega$ , and the graphs of  $\eta$  and  $\omega$  over  $S_+^{N-1}$  are tangent, either at some  $\alpha \in S_+^{N-1}$ , or only at point  $\alpha \in \partial S_+^{N-1}$ . We put  $w = \eta - \omega$ . Then

$$\mathcal{T}(\eta) - \mathcal{T}(\omega) = \Phi(1) - \Phi(0), \quad (3.4)$$

where  $\Phi(t) = \mathcal{T}(\omega_t)$  with  $\omega_t = \omega + tw$ .

We use local coordinates  $(\sigma_1, \dots, \sigma_{N-1})$  on  $S^{N-1}$  near  $\alpha$ . We denote by  $g = (g_{ij})$  the metric tensor on  $S^{N-1}$  and by  $g^{jk}$  its contravariant components. Then, for any  $\varphi \in C^1(S^{N-1})$ ,

$$|\nabla \varphi|^2 = \sum_{j,k} g^{jk} \frac{\partial \varphi}{\partial \sigma_j} \frac{\partial \varphi}{\partial \sigma_k} = \langle \nabla \varphi, \nabla \varphi \rangle_g.$$

If  $X = (X^1, \dots, X^d) \in C^1(TS^{N-1})$  is a vector field, we lower indices by setting  $X^\ell = \sum_i g^{\ell i} X_i$  and define the divergence of  $X$  by

$$\operatorname{div}_g X = \frac{1}{\sqrt{|g|}} \sum_\ell \frac{\partial}{\partial \sigma_\ell} \left( \sqrt{|g|} X^\ell \right) = \frac{1}{\sqrt{|g|}} \sum_{\ell, i} \frac{\partial}{\partial \sigma_\ell} \left( \sqrt{|g|} g^{\ell i} X_i \right).$$

We write  $\Phi(t) = \Phi_1(t) + \Phi_2(t) + \Phi_3(t)$  where

$$\Phi_1(t) = -\beta_q \Lambda_{\beta_q} \left( \beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2 \right)^{\frac{p-2}{2}} \omega_t,$$

$$\Phi_2(t) = \left( \beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2 \right)^{\frac{q}{2}}$$

and

$$\Phi_3(t) = -\operatorname{div} \left( \left( \beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2 \right)^{\frac{p-2}{2}} \nabla' \omega_t \right).$$

Then

$$\Phi_1(1) - \Phi_1(0) = - \sum_j a_j \frac{\partial w}{\partial \sigma_j} - bw$$

where

$$b = \beta_q \Lambda_{\beta_q} \left( \beta_q^2 \omega_t^2 + |\nabla \omega_t|^2 \right)^{\frac{p}{2}-2} \left( (p-1) \beta_q^2 \omega_t^2 + |\nabla \omega_t|^2 \right)$$

and

$$a_j = (p-2) \beta_q \Lambda_{\beta_q} \left( \beta_q^2 \omega_t^2 + |\nabla \omega_t|^2 \right)^{\frac{p}{2}-2} \omega_t \sum_k g^{jk} \frac{\partial \omega_t}{\partial \sigma_k};$$

$$\Phi_2(1) - \Phi_2(0) = \sum_j c_j \frac{\partial w}{\partial \sigma_j} + dw$$

where

$$d = q \beta_q^2 \left( \beta_q^2 \omega_t^2 + |\nabla \omega_t|^2 \right)^{\frac{q}{2}-1} \omega_t$$

$$c_j = q \left( \beta_q^2 \omega_t^2 + |\nabla \omega_t|^2 \right)^{\frac{q}{2}-1} \sum_k g^{jk} \frac{\partial \omega_t}{\partial \sigma_k};$$

$$\begin{aligned} \Phi_3(1) - \Phi_3(0) &= -(p-2) \operatorname{div} \left( \left( \beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2 \right)^{\frac{p-4}{2}} \left( \beta_q^2 \omega_t w + \langle \nabla' \omega_t, \nabla' w \rangle_g \right) \nabla' \omega_t \right) \\ &\quad - \operatorname{div} \left( \left( \beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2 \right)^{\frac{p-2}{2}} \nabla' w \right). \end{aligned}$$

Therefore we have

$$\Phi(1) - \Phi(0) = -\operatorname{div}(A \nabla' w) + \langle B, \nabla' w \rangle_g + Cw := \mathcal{L}w \quad (3.5)$$



where

$$\begin{aligned} \langle AX, X \rangle_g &= (\beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2)^{\frac{p-4}{2}} (p-2) \langle \nabla' \omega_t, X \rangle_g^2 + |\nabla' \omega_t|^2 |X|^2 \\ &\geq (\beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2)^{\frac{p-4}{2}} \min\{1, p-1\} |\nabla' \omega_t|^2 |X|^2. \end{aligned} \quad (3.6)$$

and  $B$  and  $C$  can be computed from the previous expressions. It is important to notice that  $\beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2$  is bounded between two positive constants  $c_1$  and  $c_2$  in  $\overline{S_+^{N-1}}$ . Thus the operator  $\mathcal{L}$  is uniformly elliptic with bounded coefficients. Since  $w$  is nonnegative and vanishes at some point  $\alpha$  where  $\nabla' w(\alpha) = 0$  and  $w(\alpha) > 0$  or at some boundary point  $\alpha$  where  $w(\alpha) = 0$  and  $\partial_{\mathbf{n}} w(\alpha) < 0$  it follows from the strong maximum principle or Hopf boundary lemma (see [17]) that  $w = 0$ , contradiction.  $\square$

**Theorem 3.2** *Assume  $\beta_q > \beta_*$ . There exists a unique positive solution  $\omega_*$  to problem (3.1).*

*Proof. Existence.* It will follow from [10]. Indeed problem (3.1) can be written

$$\begin{aligned} \mathbf{A}(\omega) &:= -\operatorname{div} \mathbf{a}(\omega, \nabla' \omega) = B(\omega, \nabla' \omega) && \text{in } S_+^{N-1} \\ \omega &= 0 && \text{on } \partial S_+^{N-1} \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} \mathbf{a}(r, \xi) &= (\beta_q^2 r^2 + |\xi|^2)^{\frac{p-2}{2}} \xi \\ B(r, \xi) &= \beta_q \Lambda_{\beta_q} (\beta_q^2 r^2 + |\xi|^2)^{\frac{p-2}{2}} r - (\beta_q^2 r^2 + |\xi|^2)^{\frac{q}{2}} \end{aligned} \quad (3.8)$$

The operator  $\mathbf{A}$  is a Leray-Lions operator which satisfies the assumptions (1.6)-(1.8) of [10, Theorem 2.1], and the term  $B$  satisfies (1.9), (1.10) in the same article. Therefore existence of a positive solution  $\omega \in W_0^{1,p}(S_+^{N-1}) \cap L^\infty(S_+^{N-1})$  is ensured whenever we can find a super solution  $\bar{\omega} \in W^{1,p}(S_+^{N-1}) \cap L^\infty(S_+^{N-1})$  and a nontrivial subsolution  $\underline{\omega} \in W^{1,p}(S_+^{N-1})$  of (3.7) such that

$$0 \leq \underline{\omega} \leq \bar{\omega} \quad \text{in } S_+^{N-1}. \quad (3.9)$$

First we note that  $\eta = \eta_0$  is a supersolution if the positive constant  $\eta_0$  is large enough. In order to find a subsolution, we set  $\eta = \psi^\theta$  with  $\theta = \beta_q/\beta_*$  and  $\psi$  as in (3.2). Then  $\theta > 1$ , thus  $\eta \in W_0^{1,p}(S_+^{N-1})$ . As above we have

$$\begin{aligned} \mathcal{T}(\eta) &= -\theta^{p-1} \psi^{(\theta-1)(p-1)} \operatorname{div} \left( (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \nabla' \psi \right) \\ &\quad - \theta^{p-1} (\theta-1)(p-1) \psi^{(\theta-1)(p-1)-1} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} |\nabla' \psi|^2 \\ &\quad - \beta_q \Lambda_{\beta_q} \theta^{p-2} \psi^{(\theta-1)(p-1)} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \psi + \theta^q \psi^{(\theta-1)q} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{q}{2}}. \end{aligned}$$

Now  $\beta_q \Lambda_{\beta_q} \theta^{p-2} = \beta_* \Lambda_{\beta_q} \theta^{p-1} = \beta_* (\Lambda_{\beta_q} - \Lambda_{\beta_*}) \theta^{p-1} + \beta_* \Lambda_{\beta_*} \theta^{p-1}$  and  $\Lambda_{\beta_q} - \Lambda_{\beta_*} = (\beta_q - \beta_*) (p-1) = \beta_* (p-1) (\theta-1)$ , hence

$$\begin{aligned} \mathcal{T}(\eta) &= -\theta^{p-1} \psi^{(\theta-1)(p-1)} \operatorname{div} \left( (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \nabla' \psi \right) \\ &\quad - \theta^{p-1} (\theta-1) (p-1) \psi^{(\theta-1)(p-1)-1} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} |\nabla' \psi|^2 \\ &\quad - \beta_* (\Lambda_{\beta_q} - \Lambda_{\beta_*}) \theta^{p-1} \psi^{(\theta-1)(p-1)} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \psi \\ &\quad - \beta_* \Lambda_{\beta_*} \theta^{p-1} \psi^{(\theta-1)(p-1)} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \psi + \theta^q \psi^{(\theta-1)q} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{q}{2}}. \end{aligned}$$

Using the equation satisfied by  $\psi$  yields

$$\begin{aligned} \mathcal{T}(\eta) &= -\theta^{p-1} (\theta-1) (p-1) \psi^{(\theta-1)(p-1)-1} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} |\nabla' \psi|^2 \\ &\quad - \beta_*^2 (p-1) (\theta-1) \theta^{p-1} \psi^{(\theta-1)(p-1)-1} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-2}{2}} \psi^2 \\ &\quad + \theta^q \psi^{(\theta-1)q} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{q}{2}} \\ &= -\theta^{p-1} (\theta-1) (p-1) \psi^{(\theta-1)(p-1)-1} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p}{2}} \\ &\quad + \theta^q \psi^{(\theta-1)q} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{q}{2}}. \end{aligned}$$

We will see that replacing  $\psi$  with  $m\psi$  in the above computation, the right choice of  $m$  will give  $\mathcal{T}(\eta) \leq 0$ : Indeed, we need

$$m^{\theta(q+1-p)} \psi^{(\theta-1)(q+1-p)+1} \leq \theta^{p-1-q} (\theta-1) (p-1) (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-q}{2}}$$

which will hold true if we choose  $m$  to satisfy

$$m^{\theta(q+1-p)} \leq \beta_*^{(\theta-1)(q+1-p)+1} \theta^{p-1-q} (\theta-1) (p-1) \frac{\min_{x \in S_+^{N-1}} (\beta_*^2 \psi^2 + |\nabla' \psi|^2)^{\frac{p-q}{2}}}{\max_{x \in S_+^{N-1}} \psi^{(\theta-1)(q+1-p)+1}}.$$

Therefore  $0 < \eta \leq \eta_0$  and standard regularity implies that the solution  $\omega$  is  $C^1$  in  $\overline{S_+^{N-1}}$ . Actually since the operator is not degenerate,  $\omega$  is  $C^\infty$ .

*Uniqueness.* We use the tangency method developed in the proof of Theorem 3.1. Assume  $\omega_1$  and  $\omega_2$  are two positive solutions of (3.2), then they are positive in  $S_+^{N-1}$  and  $\partial_{\mathbf{n}} \omega_i < 0$  on  $\partial S_+^{N-1}$ . Either the  $\omega_i$  are ordered and  $\omega_1 \leq \omega_2$  or their graphs intersect. In any case we can define

$$\tau = \inf \{s > 1 : s\omega_1 \geq \omega_2\}.$$

We set  $\omega^* = \tau\omega_1$ . Then either the graphs of  $\omega_2$  and  $\omega^*$  are tangent at some interior point  $\alpha$ , or they are not tangent in  $S_+^{N-1}$ ,  $\partial_{\mathbf{n}} \omega^* \leq \partial_{\mathbf{n}} \omega_2 < 0$  on  $\partial S_+^{N-1}$  and there exists  $\alpha \in \partial S_+^{N-1}$  such that  $\partial_{\mathbf{n}} \omega^*(\alpha) = \partial_{\mathbf{n}} \omega_2(\alpha) < 0$ . Furthermore  $\mathcal{T}(\omega^*) \geq 0$ . If we set  $w = \omega^* - \omega_2$ , then, as in Theorem 3.1,

$$-\operatorname{div}(A^* \nabla' w) + \langle B^*, \nabla' w \rangle_g + C^* w = \mathcal{L}^* w \geq 0$$

where

$$\begin{aligned} \langle A^* X, X \rangle_g &= (\beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2)^{\frac{p-4}{2}} (p-2) \langle \nabla' \omega_t, X \rangle_g^2 + |\nabla' \omega_t|^2 |X|^2 \\ &\geq (\beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2)^{\frac{p-4}{2}} \min\{1, p-1\} |\nabla' \omega_t|^2 |X|^2, \end{aligned} \quad (3.10)$$

in which  $\omega_t = \omega_2 + t(\omega^* - \omega_2)$  and  $t \in (0, 1)$  is obtained by applying the mean value theorem and  $B^*$  and  $C^*$  are defined accordingly. Since  $\mathcal{L}^*$  is uniformly elliptic and has bounded coefficient, it follows from the strong maximum principle that  $w = 0$ . Thus  $\omega^* = \tau \omega_1 = \omega_2$  and  $\tau = 1$  from the equation. This ends the proof.  $\square$

### 3.2 Removable boundary singularities

The following is the basic result for removability of isolated singularities. It is valid in the general case, but with a local geometric constraint.

**Theorem 3.3** *Assume  $q^* \leq q < p \leq N$ ,  $\Omega$  is a  $C^2$  bounded domain with  $0 \in \partial\Omega$ , such that  $\Omega \cap B_\delta \subset \mathbb{R}_+^N$  for some  $\delta > 0$ . If  $u \in C^1(\overline{\Omega} \setminus \{0\})$  is a nonnegative solution of (1.1) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ , it is identically 0.*

*Proof. Step 1: Assume  $\Omega \subset \mathbb{R}_+^N$ . For  $\epsilon > 0$ , we set  $\Omega'_\epsilon = \Omega \cap \overline{B_\epsilon^c}$  and  $H_\epsilon = \mathbb{R}_+^N \cap \overline{B_\epsilon^c}$ . For  $k, n \in \mathbb{N}_*$ ,  $n \geq \text{diam}(\Omega)$ , we denote by  $v_{k,n,\epsilon}$  ( $n \in \mathbb{N}_*$ ) the solution of the problem*

$$\begin{aligned} -\Delta_p v + |\nabla v|^q &= 0 && \text{in } H_\epsilon \cap B_n \\ v &= k \chi_{\mathbb{R}_+^N \cap \partial B_\epsilon} && \text{on } \partial(H_\epsilon \cap B_n). \end{aligned} \quad (3.11)$$

If  $k > c_2 \epsilon^{\frac{q-p}{q+1-p}}$  then  $v_{k,n,\epsilon} \geq u$  in  $\Omega'_\epsilon$ . Moreover there holds  $v_{k,n,\epsilon} \leq v_{k',n',\epsilon}$  for  $n \leq n'$  and  $k \leq k'$ . Furthermore the function

$$U_{\epsilon,n}(x) = c_1 \left( (|x| - \epsilon)^{\frac{q-p}{q+1-p}} - (n - \epsilon)^{\frac{q-p}{q+1-p}} \right)$$

is a super solution in  $B_n \setminus B_\epsilon$  for a suitable  $c_1 = c_1(N, p, q) > 0$  (see Lemma 2.6), and there holds  $v_{k,n,\epsilon} \leq U_{\epsilon,n}$ . By monotonicity and standard a priori estimate, we obtain that  $v_{k,n,\epsilon} \rightarrow v_\epsilon$  when  $n, k \rightarrow \infty$  and that the function  $v = v_\epsilon$  is solution of

$$\begin{aligned} -\Delta_p v + |\nabla v|^q &= 0 && \text{in } H_\epsilon \\ \lim_{|x| \rightarrow \epsilon} v(x) &= \infty && \\ v &= 0 && \text{on } \partial \mathbb{R}_+^N \cap \overline{B_\epsilon^c}. \end{aligned} \quad (3.12)$$

Furthermore

$$u(x) \leq v_\epsilon(x) \leq c_1 (|x| - \epsilon)^{\frac{q-p}{q+1-p}} \quad \text{in } \Omega'_\epsilon. \quad (3.13)$$

The function  $v_\epsilon$  may not be unique, however it is the minimal solution of the above problem since the  $v_{k,n,\epsilon}$  is unique, and monotonicity in  $n$  and  $k$  holds. Actually,  $v_\epsilon \leq v_{\epsilon'}$  if  $0 \leq \epsilon \leq \epsilon'$ . For  $\ell > 0$ , we recall that the transformation  $T_\ell$  defined by

$$T_\ell[v](x) = \ell^{\frac{p-q}{q+1-p}} v(\ell x), \quad (3.14)$$

leaves equation (1.1) invariant. As a consequence of the uniqueness of the approximations we have  $T_\ell[v_{k,n,\epsilon}] = v_{\frac{p-q}{\ell^{q+1-p}k, \ell^{-1}n, \ell^{-1}\epsilon}}$ , which yields

$$T_\ell[v_\epsilon](x) = v_{\ell^{-1}\epsilon}. \quad (3.15)$$

Letting  $\epsilon \rightarrow 0$ , we derive from the monotonicity with respect to  $\epsilon$  and standard  $C^{1,\alpha}$  estimates,

$$T_\ell[v_0] = v_0 \quad \forall \ell > 0. \quad (3.16)$$

The function  $v_0$  is a positive and separable solution of (1.1) in  $\mathbb{R}_+^N$  which vanishes on  $\partial\Omega \setminus \{0\}$ . It follows from Theorem 3.1 that  $v_0 = 0$ , and so is  $u$ .

*Step 2: The general case.* We assume that  $\Omega \cap B_\delta \subset \mathbb{R}_+^N$  and we denote by  $M$  the maximum of  $u$  on  $\partial B_\delta \cap \Omega$ . Then the function  $(u - M)_+$  is a subsolution of (1.1) in  $\Omega \cap B_\delta$  which vanishes on  $\partial\Omega \cap B_\delta \setminus \{0\}$ . By Step 1, it is dominated by  $v_0$ , which ends the proof.  $\square$

*Remark.* The previous result is valid if  $u$  is a subsolution with the same regularity as  $u$ . If  $u$  is no longer assumed to be nonnegative, only  $u^+$  vanishes. Furthermore, the regularity of the boundary has not been used, but only the fact that  $\Omega$  is locally contained into a half space to the boundary of which 0 belongs.

*Remark.* If no geometric assumption is made on  $\partial\Omega$ , we can prove that  $u(x) = o(|x|^{-\beta_q})$  near 0. The next result shows that the removability holds if  $q > q_*$ .

**Theorem 3.4** *Assume  $q^* < q < p \leq N$  and  $\Omega$  is a  $C^2$  bounded domain with  $0 \in \partial\Omega$ . If  $u$  is a nonnegative solution of (1.1) in  $\Omega$  which belongs to  $C^1(\overline{\Omega} \setminus \{0\})$  which vanishes on  $\partial\Omega \setminus \{0\}$ , it is identically 0.*

*Proof.* In [25] is proved that for any smooth subdomain  $S \subset S^{N-1}$ , there exists a unique  $\beta_s > 0$  and  $\psi_s > 0$ , up to an homothety, such that  $x \mapsto |x|^{-\beta_s} \psi_s(|x|^{-1}x)$  is  $p$  harmonic in the cone  $C_S = \{x \in \mathbb{R}^N \setminus \{0\} : |x|^{-1}x \in S\}$  and  $\psi_s$  satisfies

$$\begin{aligned} -\operatorname{div} \left( (\beta_s^2 \psi_s^2 + |\nabla' \psi_s|^2)^{\frac{p-2}{2}} \nabla' \psi_s \right) - \beta_s \Lambda_{\beta_s} (\beta_s^2 \psi_s^2 + |\nabla' \psi_s|^2)^{\frac{p-2}{2}} \psi_s &= 0 \quad \text{in } S \\ \psi_s &= 0 \quad \text{on } \partial S, \end{aligned} \quad (3.17)$$

By construction  $S \mapsto \beta_s$  is decreasing. If, for  $\epsilon > 0$ , we denote by  $S := S_\epsilon$  the spherical shell with vertex the north pole  $N$  and latitude angle  $\theta_{N-1} \in [0, \frac{\pi}{2} + \epsilon]$ - we use here the spherical coordinates system defined in (6.5). Because of uniqueness of  $\beta_s$ ,  $\beta_{s_\epsilon} \uparrow \beta_*$  as  $\epsilon \rightarrow 0$ . Therefore, if  $q > q_*$ , or equivalently  $\beta_q < \beta_*$ , there exists  $\delta, \epsilon > 0$  such that  $\Omega \cap B_\delta \subset C_{S_\epsilon} \cap B_\delta$  and  $\beta_q < \beta_{s_\epsilon}$ . Since Theorem 3.1 is valid if  $S_+^{N-1}$  is replaced by  $S_\epsilon$  and  $\beta_q < \beta_{s_\epsilon}$  it follows that  $u = 0$  as in the proof of Theorem 3.3, Step 1 and 2.  $\square$

The next result, valid in the case  $p = N$ , is based upon the conformal invariance of the N-Laplacian. In this case the exponent  $\beta_*$  corresponding to the first spherical N-harmonic eigenvalue is equal to 1 and the corresponding spherical N-harmonic eigenfunction in  $S_+^{N-1} = S^{N-1} \cap \{x = (x', x_N), x_N > 0\}$  is  $x_N/|x|^2$ .

**Theorem 3.5** *Assume  $N - \frac{1}{2} \leq q < N$ ,  $\Omega$  is a bounded domain and  $0 \in \partial\Omega$  is such that there exists a ball in  $\Omega^c$  to the boundary of which  $a$  belongs. If  $u$  is a nonnegative solution of*

$$-\Delta_N u + |\nabla u|^q = 0 \quad \text{in } \Omega, \quad (3.18)$$

*which belongs to  $C(\overline{\Omega} \setminus \{0\}) \cap W_0^{1,N}(\Omega \setminus \overline{B}_\epsilon(a))$  for any  $\epsilon > 0$ , it is identically 0.*

*Proof.* We assume that the inward normal unit vector to  $\partial\Omega$  at 0 is  $\mathbf{e}_N = (0, 0, \dots, 1)$  and that the ball  $B = B_{\frac{1}{2}}(a)$  of center  $a = -\frac{1}{2}\mathbf{e}_N$  and radius  $\frac{1}{2}$  touches  $\partial\Omega$  at 0 and is exterior to  $\Omega$  (this can be assumed up to a rotation and a dilation). This is the consequence of the exterior sphere condition at the point 0. It is always valid if  $\partial\Omega$  is  $C^2$ . We denote by  $\mathcal{I}_\omega$  the inversion of center  $\omega$  and power 1, i.e.  $\mathcal{I}_\omega(x) = \omega + \frac{x-\omega}{|x-\omega|^2}$ . Under this transformation, the complement of the ball  $B$ , which contains  $\Omega$  is transformed into the half space  $\{x = (x', x_N) : x_N < 0\}$  which contains the image  $\tilde{\Omega}$  of  $\Omega$ . Since  $u$  satisfies (3.18),  $\tilde{u} = u \circ \mathcal{I}_\omega$  satisfies

$$-\Delta_N \tilde{u} + |x - \omega|^{2(q-N)} |\nabla \tilde{u}|^q = 0 \quad \text{in } \tilde{\Omega}. \quad (3.19)$$

Furthermore since  $0 = \mathcal{I}_\omega(0)$  and  $\mathcal{I}_\omega$  is a diffeomorphism,  $\tilde{u} \in C(\overline{\tilde{\Omega}} \setminus \{0\}) \cap C^1(\tilde{\Omega})$  and it vanishes on  $\partial\tilde{\Omega} \setminus \{0\}$ . Since  $|x - \omega| \leq 1$  and  $q < N$ ,  $\tilde{u}$  is a subsolution for (3.18) in  $\tilde{G}$ . By Theorem 3.5,  $\tilde{u} = 0$ .  $\square$

### 3.3 Weakly singular solutions

The main result of this section is the following existence and uniqueness result concerning solutions of (1.1) with a boundary weak singularity. We recall that  $\psi_*$  is unique positive solution of (1.10) such that  $\sup \psi_* = 1$ . Our first result is valid for any  $1 < p \leq N$  but it needs a geometric constraint on  $\Omega$ .

**Theorem 3.6** *Let  $p - 1 < q < q_* < p \leq N$  and  $\Omega$  be a bounded  $C^2$  domain such that  $0 \in \partial\Omega$ . Assume that there exists  $\delta > 0$  such that  $\Omega \cap B_\delta = B_\delta^+$ . Then for any  $k > 0$  there exists a unique positive solution  $u := u_k$  of (1.1) in  $\Omega$ , which belongs to  $C^1(\overline{\Omega} \setminus \{0\})$ , vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies*

$$\lim_{x \rightarrow 0} \frac{u_k(x)}{\Psi_*(x)} = k \quad (3.20)$$

*in the  $C^1$ -topology of  $S_+^{N-1}$ , where  $\Psi_*(x) = |x|^{-\beta_*} \psi_*(|x|^{-1}x)$ .*

**Lemma 3.7** *Let the assumptions on  $p$ ,  $q$  and  $\Omega$  of Theorem 3.6 be satisfied. There exists a unique positive  $p$ -harmonic function  $\Phi_*$  in  $\Omega$ , which is continuous in  $\overline{\Omega} \setminus \{0\}$ , vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies*

$$\lim_{x \rightarrow 0} \frac{\Phi_*(x)}{\Psi_*(x)} = 1. \quad (3.21)$$

*Proof.* For  $0 < \epsilon < \delta$  let  $v_\epsilon$  be the unique p-harmonic function in  $\Omega \setminus \overline{B_\epsilon^+}$  which is continuous in  $\overline{\Omega} \setminus B_\epsilon^+$ , vanishes on  $\partial\Omega \setminus B_\epsilon$  and achieves the value  $\Psi_*$  on  $\partial B_\epsilon \cap \Omega$ . By the maximum principle, and since  $\Psi_*(x) \leq D^{-\beta_*}$  where  $D = \max\{|z| : z \in \Omega\}$ , there holds

$$(\Psi_* - D^{-\beta_*})_+ \leq v_\epsilon \leq \Psi_* \quad \text{in } \Omega \setminus \overline{B_\epsilon^+}. \quad (3.22)$$

If  $\epsilon < \epsilon' < \delta$ , then  $v_\epsilon \leq v_{\epsilon'}$  in  $\Omega \setminus \overline{B_{\epsilon'}^+}$ . By standard regularity result  $v_\epsilon$  converges to a function  $\Phi_*$  continuous in  $\overline{\Omega} \setminus \{0\}$ , p-harmonic in  $\Omega$  and satisfies  $(\Psi_* - \delta^{-\beta_*})_+ \leq \Phi_* \leq \Psi_*$  in  $\Omega$ . Therefore (3.21) holds provided  $\frac{x}{|x|}$  remains in a compact subset of  $S_+^{N-1}$ . If we define the function  $\tilde{\phi}_*$  by  $\tilde{\phi}_*(x) = |x|^{\beta_*} \Phi_*(x)$ , then  $\tilde{\phi}_*(r, \sigma) \leq \psi_*(\sigma)$  where  $r = |x|$  and  $\sigma = \frac{x}{|x|} \in S_+^{N-1}$ . By standard  $C^{1,\alpha}$  estimates,  $\tilde{\phi}_*(r, \cdot)$  is relatively compact in the  $C(\overline{S_+^{N-1}})$ -topology. Therefore the convergence of  $\frac{\Phi_*(x)}{\Psi_*(x)}$  to 1 when  $x$  to 0 holds not only when  $\frac{x}{|x|}$  remains in a compact subset of  $S_+^{N-1}$ , but uniformly on  $S_+^{N-1}$ , which implies (3.22). Uniqueness follows classically by (3.22) and the maximum principle.  $\square$

**Lemma 3.8** *Let the assumptions on  $p, q$  and  $\Omega$  of Theorem 3.6 be satisfied. If for some  $k > 0$  there exists a solution  $u_k$  of (1.1) in  $\Omega$ , which belongs to  $C^1(\overline{\Omega} \setminus \{0\})$ , vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies (3.20), then for any  $k > 0$  there exists such a solution.*

*Proof.* We notice that for any  $c < 1$  (resp  $c > 1$ ),  $cu_k$  is a subsolution (resp. supersolution) of (1.1) in  $\Omega$ . If  $c < 1$ , the function  $ck\Phi_*$  is a supersolution of (1.1) which vanishes on  $\partial\Omega \setminus \{0\}$ . Furthermore

$$\lim_{x \rightarrow 0} \frac{cu_k(x)}{\Psi_*(x)} = \lim_{x \rightarrow 0} \frac{ck\Phi_*(x)}{\Psi_*(x)} = ck.$$

Then there exists a solution  $u_{ck}$  of (1.1) in  $\Omega$  which satisfies  $cu_k \leq u_{ck} \leq ck\Phi_*$ . If  $c > 1$ , we consider  $u^* := T_{c^\theta}[u_k] : x \mapsto c^{\beta_q \theta} u_k(c^\theta x)$  with  $\theta = (\beta_q - \beta_*)^{-1}$ . Then  $u^*$  is a solution of (1.1) in  $\Omega^{c^\theta} = \frac{1}{c^\theta} \Omega$ . In particular,  $u^*$  satisfies the equation in  $B_{\frac{\delta}{c^\theta}}^+(0)$ .

Since  $c^\theta > 1$ ,  $B_{\frac{\delta}{c^\theta}}^+(0) \subset B_\delta^+(0)$ . Put  $m = \max\{u^* : x \in \partial B_{\frac{\delta}{c^\theta}}^+(0)\}$ . The function  $(u^* - m)_+$ , extended by 0 outside  $B_{\frac{\delta}{c^\theta}}^+(0)$ , is a subsolution of (1.1) in  $\Omega$ . Furthermore it satisfies

$$\lim_{x \rightarrow 0} \frac{(u^* - m)_+(x)}{\Psi_*(x)} = ck,$$

uniformly on any compact subset of  $S_+^{N-1}$ . Therefore there exists a solution  $u_{ck}$  of (1.1) in  $\Omega$  which satisfies  $(u^* - m)_+ \leq u_{ck} \leq ck\Phi_*$ , and in particular it vanishes on  $\partial\Omega \setminus \{0\}$  and belongs to  $C^1(\overline{\Omega} \setminus \{0\})$ . By [27],  $u_{ck}$  is positive in  $\Omega$ . Since  $u_{ck}$  belongs to  $C^{1,\alpha}(\overline{B_\delta^+(0)} \setminus \{0\})$  and satisfies

$$|x|^{\beta_*} |u_{ck}(x)| + |x|^{1+\beta_*} |\nabla u_{ck}(x)| + |x|^{1+\beta_*+\alpha} \sup_{\substack{|y| \leq |x| \\ x \neq y}} \frac{|\nabla u_{ck}(x) - \nabla u_{ck}(y)|}{|x - y|^\alpha} \leq M$$

by (2.10), the set of functions  $\{r^{\beta_*+1}\nabla u_{ck}(r, \cdot)\}_{r>0}$  is uniformly relatively compact in the topology of uniform convergence on  $\overline{S}_+^{N-1}$ . Since it converges to  $ck\nabla'\psi_*$  uniformly on compact subsets of  $S_+^{N-1}$ , this convergence holds in  $C(\overline{S}_+^{N-1})$ . This implies

$$\lim_{x \rightarrow 0} \frac{u_{ck}(x)}{\Psi_*(x)} = ck. \quad (3.23)$$

□

**Lemma 3.9** *Under the assumptions of Theorem 3.6 there exists a real number  $R_0$  such that  $0 < R_0 \leq \delta$  and a positive subsolution  $\tilde{u}$  of (1.1) in  $B_{R_0}^+$  which is Lipschitz continuous in  $\overline{B_{R_0}^+} \setminus \{0\}$ , vanishes on  $\overline{B_{R_0}^+} \cap \partial\mathbb{R}_+^N \setminus \{0\}$ , is smaller than  $\Psi_*$  and satisfies*

$$\lim_{x \rightarrow 0} \frac{\tilde{u}(x)}{\Psi_*(x)} = 1. \quad (3.24)$$

*Proof.* The construction of the function  $\tilde{u}$  is delicate and needs several intermediate steps. We look for a solution under the form  $\tilde{u} = \Psi_* - w$  for a suitable nonnegative function  $w$ .

*Step 1: reduction of the problem.* We use spherical coordinates for a  $C^1$  function  $u : x \mapsto u(x) = u(r, \sigma)$ ,  $r = |x|$ ,  $\sigma = \frac{x}{|x|}$ . Then  $\nabla u = u_r \mathbf{e} + r^{-1} \nabla' u$  where  $\mathbf{e} = |x|^{-1} x$ ,  $|\nabla u|^2 = u_r^2 + r^{-2} |\nabla' u|^2$  and  $|\nabla u|^q = \left(u_r^2 + r^{-2} |\nabla' u|^2\right)^{\frac{q}{2}}$ . The expression of the p-Laplacian in spherical coordinates is

$$\begin{aligned} -\Delta_p u = & - \left( \left( u_r^2 + r^{-2} |\nabla' u|^2 \right)^{\frac{p-2}{2}} u_r \right)_r - \frac{N-1}{r} \left( u_r^2 + r^{-2} |\nabla' u|^2 \right)^{\frac{p-2}{2}} u_r \\ & - \frac{1}{r^2} \operatorname{div}' \left( \left( u_r^2 + r^{-2} |\nabla' u|^2 \right)^{\frac{p-2}{2}} \nabla' u \right). \end{aligned}$$

Put  $v(t, \sigma) = r^{\beta_*} u(r, \sigma)$  with  $t = \ln r \in (-\infty, \ln \delta]$ , then  $v$  satisfies

$\mathcal{Q}[v]$  :

$$\begin{aligned} = & - \left( \left( (v_t - \beta_* v)^2 + |\nabla' v|^2 \right)^{\frac{p-2}{2}} (v_t - \beta_* v) \right)_t - \operatorname{div}' \left( \left( (v_t - \beta_* v)^2 + |\nabla' v|^2 \right)^{\frac{p-2}{2}} \nabla' v \right) \\ & + \Lambda_{\beta_*} \left( (v_t - \beta_* v)^2 + |\nabla' v|^2 \right)^{\frac{p-2}{2}} (v_t - \beta_* v) + e^{\nu t} \left( (v_t - \beta_* v)^2 + |\nabla' v|^2 \right)^{\frac{q}{2}} = 0 \end{aligned} \quad (3.25)$$

in  $(-\infty, \ln \delta) \times S_+^{N-1}$  where  $\nu = 1 - (q+1-p)(\beta_*+1) = 1 - \frac{\beta_*+1}{\beta_q+1} > 0$  and  $\Lambda_{\beta_*} = \beta_*(p-1) + p - N$ . Notice that since  $\psi_*$  satisfies

$$-\operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \nabla' \psi_* \right) - \beta_* \Lambda_{\beta_*} \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \psi_* = 0, \quad (3.26)$$

it is a supersolution for (3.25). We look for a subsolution under the form

$$V(t, \sigma) = \psi_* - a(t)g(\psi_*)$$

where  $g$  is a continuous increasing function defined on  $\mathbb{R}_+$ , vanishing at 0 and smooth on  $\mathbb{R}_+^*$  and  $a(t) = e^{\gamma t}$  with  $\gamma > 0$  is to be chosen. Thus  $a' = \gamma a$ ,  $a'' = \gamma^2 a$ ,  $V_t = -\gamma a g(\psi_*)$ ,  $V_t - \beta_* V = -\beta_* \psi_* + a(\beta_* - \gamma)g(\psi_*)$ ,  $\nabla' V = (1 - a g'(\psi_*))\nabla' \psi_*$  and

$$\begin{aligned} (V_t - \beta_* V)^2 + |\nabla' V|^2 &= (-\beta_* \psi_* + a(\beta_* - \gamma)g(\psi_*))^2 + (1 - a g'(\psi_*))^2 |\nabla' \psi_*|^2 \\ &= (\beta_*^2 \psi_*^2 + 2a\beta_*(\gamma - \beta_*)g(\psi_*)\psi_*) + (1 - 2a g'(\psi_*)) |\nabla' \psi_*|^2 + O(a^2 \|g(\psi)\|_{C^1}) \\ &= \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 + 2a \left( \beta_*(\gamma - \beta_*)\psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2 \right) + O(a^2 \|g(\psi_*)\|_{C^1}). \end{aligned}$$

Therefore

$$\begin{aligned} &\left( (V_t - \beta_* V)^2 + |\nabla' V|^2 \right)^{\frac{p-2}{2}} \\ &= \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \left[ 1 + (p-2)a \frac{\beta_*(\gamma - \beta_*)\psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] \\ &\quad + O(a^2 \|g(\psi)\|_{C^1}), \end{aligned}$$

and

$$\begin{aligned} &e^{\nu t} \left( (V_t - \beta_* V)^2 + |\nabla' V|^2 \right)^{\frac{q}{2}} \\ &= e^{\nu t} \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{q}{2}} \left[ 1 + qa \frac{\beta_*(\gamma - \beta_*)\psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] \\ &\quad + O(e^{\nu t} a^2 \|g(\psi_*)\|_{C^1}), \end{aligned}$$

thus

$$\begin{aligned} &\left( (V_t - \beta_* V)^2 + |\nabla' V|^2 \right)^{\frac{p-2}{2}} (V_t - \beta_* V) \\ &= -\beta_* \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \psi_* + a(\beta_* - \gamma) \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} g(\psi_*) \\ &\quad - a\beta_*(p-2) \frac{\beta_*(\gamma - \beta_*)\psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{(\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2)^{\frac{4-p}{2}}} \psi_* + O(a^2 \|g(\psi_*)\|_{C^1}). \end{aligned}$$

Finally,

$$\begin{aligned} &-\left( \left( (V_t - \beta_* V)^2 + |\nabla' V|^2 \right)^{\frac{p-2}{2}} (V_t - \beta_* V) \right)_t \\ &= a \left[ (\gamma^2 - \beta_* \gamma) \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} g(\psi_*) + \beta_*(p-2) \frac{\beta_*(\gamma^2 - \beta_* \gamma)\psi_* g(\psi_*) - \gamma g'(\psi_*) |\nabla \psi_*|^2}{(\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2)^{\frac{4-p}{2}}} \psi_* \right] \\ &\quad + O(a^2 \|g(\psi_*)\|_{C^2}). \end{aligned}$$

(3.27)



Since

$$\begin{aligned}
& \left( (V_t - \beta_* V)^2 + |\nabla' V|^2 \right)^{\frac{p-2}{2}} \nabla' V = \\
& \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} (1 - a g'(\psi_*)) \left[ 1 + a(p-2) \frac{\beta_*(\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] \nabla' \psi_* \\
& \qquad \qquad \qquad + O(a^2 \|g(\psi_*)\|_{C^1}) \\
& = \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \nabla' \psi_* \\
& + a \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \left[ (p-2) \frac{\beta_*(\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} - g'(\psi_*) \right] \nabla' \psi_* \\
& \qquad \qquad \qquad + O(a^2 \|g(\psi_*)\|_{C^1})
\end{aligned}$$

we get similarly

$$\begin{aligned}
& -\operatorname{div}' \left( \left( (V_t - \beta_* V)^2 + |\nabla' V|^2 \right)^{\frac{p-2}{2}} \nabla' V \right) = -\operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \nabla' \psi_* \right) \\
& -a \operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \left[ (p-2) \frac{\beta_*(\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} - g'(\psi_*) \right] \nabla' \psi_* \right) \\
& \qquad \qquad \qquad + O(a^2 \|g(\psi_*)\|_{C^2}).
\end{aligned} \tag{3.28}$$

Noting that

$$-\operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \nabla' \psi_* \right) \psi_* = \beta_* \Lambda_{\beta_*} \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \psi_*, \tag{3.29}$$

we obtain

$$\begin{aligned}
& e^{-\gamma t} \mathcal{Q}[V] \\
& = \left[ (\gamma^2 - \beta_* \gamma) \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} g(\psi_*) + \beta_* (p-2) \frac{\beta_*(\gamma^2 - \beta_* \gamma) \psi_* g(\psi_*) - \gamma g'(\psi_*) |\nabla \psi_*|^2}{(\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2)^{\frac{4-p}{2}}} \psi_* \right] \\
& - \operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \left[ (p-2) \frac{\beta_*(\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} - g'(\psi_*) \right] \nabla' \psi_* \right) \\
& - \Lambda_{\beta_*} \left( (\gamma - \beta_*) \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} g(\psi_*) + \beta_* (p-2) \frac{\beta_*(\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{(\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2)^{\frac{4-p}{2}}} \psi_* \right) \\
& + e^{(\nu-\gamma)t} \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{q}{2}} \left[ 1 + qa \frac{\beta_*(\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] + O(a \|g(\psi_*)\|_{C^2}).
\end{aligned} \tag{3.30}$$

In this expression we have in particular

$$\begin{aligned}
& -\operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \left[ (p-2) \frac{\beta_* (\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} - g'(\psi_*) \right] \nabla' \psi_* \right) \\
&= (p-1) \operatorname{div}' \left[ g'(\psi_*) \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \nabla \psi_* \right] \\
&- \beta_* \operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-4}{2}} [(p-2) \beta_* \psi_* g'(\psi_*) + (p-2)(\gamma - \beta_*) g(\psi_*)] \psi_* \right) \\
&= (p-1) g''(\psi_*) \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} |\nabla \psi_*|^2 \\
&+ (p-1) g'(\psi_*) \operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \nabla \psi_* \right) \\
&- (p-2) \beta_* \operatorname{div}' \left[ \frac{((\gamma - \beta_*) g(\psi_*) \psi_* + \beta_* g'(\psi_*) \psi_*^2)}{\left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{4-p}{2}}} \nabla' \psi_* \right].
\end{aligned} \tag{3.31}$$

Using equation (3.26) satisfied by  $\psi_*$ , it follows

$$\begin{aligned}
& -\operatorname{div}' \left( \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \left[ (p-2) \frac{\beta_* (\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} - g'(\psi_*) \right] \nabla' \psi_* \right) \\
&= (p-1) \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \left( g''(\psi_*) |\nabla' \psi_*|^2 - \beta_* \Lambda_{\beta_*} g'(\psi_*) \psi_* \right) \\
&- (p-2) \beta_* \operatorname{div}' \left[ \frac{((\gamma - \beta_*) g(\psi_*) \psi_* + \beta_* g'(\psi_*) \psi_*^2)}{\left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{4-p}{2}}} \nabla' \psi_* \right].
\end{aligned} \tag{3.32}$$

Plugging this identity into the expression (3.30), we obtain after some simplifications

$$e^{-\gamma t} \mathcal{Q}[V] = \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} g(\psi_*) \mathcal{Q}_1[V] + e^{(\nu-\gamma)t} R[V] + O(a \|g(\psi_*)\|_{C^2}), \tag{3.33}$$

where

$$R[V] = e^{\nu t} \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{q}{2}} \left[ 1 + q \frac{\beta_* (a' - \beta_* a) \psi_* g(\psi_*) - a g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right], \tag{3.34}$$

and

$$\begin{aligned}
\mathcal{Q}_1[V] &= (\gamma - \Lambda_{\beta_*})(\gamma - \beta_*) \left[ 1 + (p-2) \frac{\beta_*^2 \psi_*^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] - (p-1) \beta_* \Lambda_{\beta_*} \frac{\psi_* g'(\psi_*)}{g(\psi_*)} \\
&+ [(p-4) \beta_* \Lambda_{\beta_*} \psi_* - 2 \Delta' \psi_*] \left( \gamma - \beta_* \left( 1 - \frac{\psi_* g'(\psi_*)}{g(\psi_*)} \right) \right) \frac{\beta_* \psi_*}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} + (p-1) \frac{g''(\psi_*)}{g(\psi_*)} |\nabla' \psi_*|^2 \\
&- (p-2) \left[ \frac{\psi_* g'(\psi_*)}{g(\psi_*)} ((\beta_* + 1) \gamma - \beta_* \Lambda_{\beta_*} + \beta_*) + \gamma - \beta_* + \beta_* \frac{\psi_*^2 g''(\psi_*)}{g(\psi_*)} \right] \frac{|\nabla' \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2}.
\end{aligned} \tag{3.35}$$

In this expression the difficult term to treat is  $[(p-4) \beta_* \Lambda_{\beta_*} \psi_* - 2 \Delta' \psi_*]$  because of its lack of sign. However  $\Delta' \psi_* = O(\psi_*)$  by (6.20).

*Step 2: The perturbation method and computation with  $g(\psi_*) = \psi_*$ .* With such a choice of function  $g$

$$\begin{aligned}
\mathcal{Q}_1[V] &= (\gamma - \Lambda_{\beta_*})(\gamma - \beta_*) \left[ 1 + (p-2) \frac{\beta_*^2 \psi_*^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] \\
&- (p-2) [(\gamma - \Lambda_{\beta_*}) \beta_* + 2\gamma] \frac{|\nabla' \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2}.
\end{aligned} \tag{3.36}$$

Equivalently

$$\begin{aligned}
\mathcal{Q}_1[V] &= \left[ 1 + (p-2) \frac{\beta_*^2 \psi_*^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] (\gamma^2 - (\Lambda_{\beta_*} + \beta_*) \gamma) \\
&- \left[ (p-2)(\beta_* + 2) \frac{|\nabla' \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} + 0(\psi_*^2) \right]
\end{aligned}$$

and finally

$$\mathcal{Q}_1[V] = \left[ 1 + (p-2) \frac{\beta_*^2 \psi_*^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] \gamma [\gamma - (\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2)) + 0(\psi_*^2)]. \tag{3.37}$$

Using the fact that  $\beta > \frac{N-1}{p-1}$  if  $1 < p < 2$  and  $1 < \beta < \frac{N-1}{p-1}$  if  $2 < p < N$  (see Appendix II), we have

$$\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2) \geq \begin{cases} \Lambda_{\beta_*} + \beta_*(p-1) & \text{if } p \geq 2 \\ N + 3(p-2) > N-3 & \text{if } 1 < p < 2. \end{cases} \tag{3.38}$$

When  $N = 2$ , we have explicitly  $\beta_* = \frac{1+2\sqrt{p^2-3p+3}}{3(p-1)}$  (see [19, Th 3.3]). Therefore for all  $N \geq 2$  and  $p > 1$ , there holds

$$\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2) > 0. \tag{3.39}$$

We fix  $\epsilon_0 > 0$  such that, whenever  $\psi_* \leq \epsilon_0$ , there holds

$$\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2) + 0(\psi_*^2) > \frac{1}{2} (\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2)). \tag{3.40}$$

If we fix  $\gamma_0 > 0$  such that

$$\gamma_0 < \min \left\{ \frac{1}{2} (\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2)), \nu, \beta_* \right\}, \quad (3.41)$$

we obtain

$$\mathcal{Q}_1[V] \leq -\min\{1, p-1\}\gamma B^2 \quad \forall 0 < \gamma \leq \gamma_0, \quad (3.42)$$

whenever  $\psi_* \leq \epsilon_0$ , for some  $B$  depending only on  $p, q$  and  $N$  (through  $\psi_*$  and  $\nu$ ), which yields, in the same range of value of  $\psi_*$ ,

$$\left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} g(\psi_*) \mathcal{Q}_1[V] \leq -c_0 \psi_* \quad \forall 0 < \gamma \leq \gamma_0, \quad (3.43)$$

for some  $c_0 > 0$  depending on  $N, p, q$ . This estimate is valid whatever is  $p$ , but only in a neighborhood of  $\psi_* = 0$ . If we replace  $g(\psi_*) = \psi_*$  by  $g_k(\psi_*) = \psi_* e^{-k\psi_*}$  for  $0 < k < 1$ , and denote by  $\mathcal{Q}_{1,k}[V]$  the corresponding expression of  $\mathcal{Q}_1[V]$  which becomes now  $\mathcal{Q}_{1,0}[V]$ . We define similarly  $\mathcal{Q}_k[V]$ , and  $\mathcal{Q}[V]$  becomes  $\mathcal{Q}_0[V]$ . Since  $g'_k(\psi_*) = e^{-k\psi_*} - k g_k(\psi_*)$  and  $g''_k = -2k e^{-k\psi_*} + k^2 g_k(\psi_*)$ , we obtain

$$\begin{aligned} \mathcal{Q}_{1,k}[V] &= \mathcal{Q}_{1,0}[V] + k(p-1)\beta_* \Lambda_{\beta_*} \psi_* + (p-1) \left( -\frac{2k}{\psi_*} + k^2 \right) |\nabla' \psi_*|^2 \\ &\quad + (2-p)\beta_* (-2k + k^2) \psi_* + O(\psi_*^2) \end{aligned} \quad (3.44)$$

Because  $\nabla' \psi_*$  vanishes only at the North pole  $\mathbf{e}_N$ , there exists  $k_0 \in (0, 1]$  such that

$$k(1-p)\beta_* \Lambda_{\beta_*} \psi_* + (p-1) \left( \frac{2k}{\psi_*} - k^2 \right) |\nabla' \psi_*|^2 \geq \frac{1}{2} (2-p)_+ \beta_* (-2k + k^2) \psi_* \quad \forall k \leq k_0$$

whenever  $\psi_* \leq \epsilon_0$  which yields

$$\left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} g_k(\psi_*) \mathcal{Q}_{1,k}[V] \leq -c_1 k \quad \forall k \leq k_0 \quad (3.45)$$

for some  $c_1 = c_1(N, p, q, \epsilon_0)$ . There exists  $c_2 = c_2(N, p, q) > 0$  such that

$$\left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{q}{2}} \left[ 1 + q e^{\gamma t} \frac{\beta_* (\gamma - \beta_*) \psi_* g_k(\psi_*) - g'_k(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] \leq c_2 \quad (3.46)$$

in  $S_+^{N-1} \times (-\infty, \ln \delta]$ . Moreover

$$O(a \|g(\psi_*)\|_{C^2}) \leq e^{\gamma t} c_k \quad (3.47)$$

for some  $c_k = c_k(N, p, q) > 0$ . We derive from (3.45)-(3.48)

$$e^{-\gamma t} \mathcal{Q}_k[V] \leq -c_1 k + c_2 e^{(\nu-\gamma)t} + e^{\gamma t} c_k \quad \forall k \leq k_0 \quad (3.48)$$

Thus there exists  $T_k \leq \ln \delta$  such that  $\mathcal{Q}_k[V] \leq 0$ , for all  $t \leq T_k$  and provided  $\psi_* \leq \epsilon_0$ . This local estimate will be used in the construction of the sub-solution when  $p \geq 2$ .

Step 3: The case  $1 < p < 2$ . Since the function  $\psi^*$  depends only on the azimuthal angle  $\theta \in (0; \frac{\pi}{2}]$  we will write  $\psi_*(\theta) = \psi_*(\theta)$  and  $\nabla' \psi_*(\sigma) = \psi_{*\theta}(\theta) \mathbf{n}$  where  $\mathbf{n}$  is the downward unit vector tangent to  $S^{N-1}$  in the hyperplane going through  $\sigma$  and the poles. From (6.8),

$$(p-4)\beta_* \Lambda_{\beta_*} \psi - 2\Delta' \psi_* = (p-2) \left( \beta_* \Lambda_{\beta_*} \psi_* + 2 \frac{\beta_*^2 \psi_* + \psi_{*\theta\theta}}{\beta_*^2 \psi_*^2 + \psi_{*\theta}^2} \right), \quad (3.49)$$

since  $\psi_{*\theta}^2 = |\nabla' \psi_*|^2$  and thus

$$\begin{aligned} & ((p-4)\beta_* \Lambda_{\beta_*} \psi_* - 2\Delta' \psi_*) \frac{\beta_* \gamma \psi_*}{\beta_*^2 \psi_*^2 + \psi_{*\theta}^2} \\ &= (p-2) \gamma \left( \Lambda_{\beta_*} \frac{\beta_*^2 \psi_*^2}{\beta_*^2 \psi_*^2 + \psi_{*\theta}^2} + 2\beta_* \frac{\beta_*^2 \psi_* + \psi_{*\theta\theta} \psi_*}{(\beta_*^2 \psi_*^2 + \psi_{*\theta}^2)^2} \psi_{*\theta}^2 \right). \end{aligned} \quad (3.50)$$

From Theorem 6.1-Step 4 we know that  $\beta_*^2 \psi_* + \psi_{*\theta\theta} \geq 0$ , thus the contribution of this term to  $\mathcal{Q}_1[V]$  is nonpositive. We replace this expression in  $\mathcal{Q}_1[V]$  with  $g(\psi_*) = \psi_*$ , then

$$\begin{aligned} \mathcal{Q}_1[V] &= (\gamma - \Lambda_{\beta_*})(\gamma - \beta_*) \left( 1 + (p-2) \frac{\beta_*^2 \psi_*^2}{\beta_*^2 \psi_*^2 + \psi_{*\theta}^2} \right) - \Lambda_{\beta_*} \beta_* (p-1) \\ &+ (p-2) \gamma \Lambda_{\beta_*} \frac{\beta_*^2 \psi_*^2}{\beta_*^2 \psi_*^2 + \psi_{*\theta}^2} - (p-2) ((\beta_* + 2)\gamma - \Lambda_{\beta_*} \beta_*) \frac{\psi_{*\theta}^2}{\beta_*^2 \psi_*^2 + \psi_{*\theta}^2} \\ &+ 2\beta_* (p-2) \frac{\beta_*^2 \psi_*^2 + \psi_{*\theta\theta} \psi_*}{(\beta_*^2 \psi_*^2 + \psi_{*\theta}^2)^2} \psi_{*\theta}^2 \\ &\leq \gamma \left( 1 + (p-2) \frac{\beta_*^2 \psi_*^2}{\beta_*^2 \psi_*^2 + \psi_{*\theta}^2} \right) (\gamma - \Lambda_{\beta_*} - \beta_*) - (p-2) \gamma \frac{(\beta_* + 2) \psi_{*\theta}^2 - \Lambda_{\beta_*} \beta_*^2 \psi_*^2}{\beta_*^2 \psi_*^2 + \psi_{*\theta}^2} \\ &\leq \gamma \left( 1 + (p-2) \frac{\beta_*^2 \psi_*^2}{\beta_*^2 \psi_*^2 + \psi_{*\theta}^2} \right) \left( \gamma - \left( \Lambda_{\beta_*} + \beta_* + (p-2) \frac{(\beta_* + 2) \psi_{*\theta}^2 - \Lambda_{\beta_*} \beta_*^2 \psi_*^2}{(p-1) \beta_*^2 \psi_*^2 + \psi_{*\theta}^2} \right) \right). \end{aligned} \quad (3.51)$$

We can write

$$\begin{aligned} & \Lambda_{\beta_*} + \beta_* + (p-2) \frac{(\beta_* + 2) \psi_{*\theta}^2 - \Lambda_{\beta_*} \beta_*^2 \psi_*^2}{(p-1) \beta_*^2 \psi_*^2 + \psi_{*\theta}^2} \\ &= \frac{(\Lambda_{\beta_*} + (p-1) \beta_*) \beta_*^2 \psi_*^2 + (\Lambda_{\beta_*} + \beta_* (p-1) + 2(p-2)) \psi_{*\theta}^2}{(p-1) \beta_*^2 \psi_*^2 + \psi_{*\theta}^2} \quad (3.52) \\ &\geq c_0 (\Lambda_{\beta_*} + \beta_* (p-1) + 2(p-2)). \end{aligned}$$

This expression  $\Lambda_{\beta_*} + \beta_* (p-1) + 2(p-2)$  is always positive: obviously if  $N \geq 3$  and by using the explicit expression of  $\beta_*$  if  $N = 2$ . Thus there exists  $\gamma_0$  and  $c_1 > 0$  such that  $\mathcal{Q}_1[V] < -c_1$  for  $0 < \gamma \leq \gamma_0$ . The perturbation method of Step 2, is valid in the whole range of values of  $\psi_*$  and we derive from (3.42)-(3.43) that (3.48) holds for all  $k \leq k_0$  and  $t \leq T_k$ . Therefore  $\mathcal{Q}_k[V] \leq 0$ .

*Step 4: The case  $p \geq 2$ .* For  $c > 0$  to be fixed and  $\psi_* \geq \epsilon_0$ ,  $\gamma \in (0, \gamma_0]$ , we take  $g(\psi_*) = c\psi_*^{1-\frac{\gamma}{\beta_*}}$ . Then we derive from (3.35):

$$\begin{aligned} \mathcal{Q}_1[V] &= (\gamma - \Lambda_{\beta_*})(\gamma - \beta_*) \frac{(p-1)\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2} - (p-1)\beta_*\Lambda_{\beta_*} \left(1 - \frac{\gamma}{\beta_*}\right) \\ &\quad - (p-1) \frac{\gamma(\beta_* - \gamma)}{\beta_*^2} \psi_*^{-1-\frac{\gamma}{\beta_*}} |\nabla'\psi_*|^2 - (p-2)(\beta_* - \gamma)(\gamma - \Lambda_{\beta_*}) \frac{|\nabla'\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2} \\ &= (1-p) \left[ \gamma(\beta_* - \gamma) + \frac{\gamma(\beta_* - \gamma)}{\beta_*^2} \psi_*^{-1-\frac{\gamma}{\beta_*}} |\nabla'\psi_*|^2 \right]. \end{aligned} \quad (3.53)$$

For  $k \leq k_0$  we fix  $c$  such that  $c\epsilon_0^{1-\frac{\gamma}{\beta_*}} = \epsilon_0 e^{-k\epsilon_0} \iff c = \epsilon_0^{\frac{\gamma}{\beta_*}} e^{-k\epsilon_0}$  and we define  $g$  by

$$g(\psi_*) = \min \left\{ \psi_* e^{-k\psi_*}, \epsilon_0^{\frac{\gamma}{\beta_*}} e^{-k\epsilon_0} \psi_*^{1-\frac{\gamma}{\beta_*}} \right\} = \begin{cases} \psi_* e^{-k\psi_*} & \text{if } 0 \leq \psi_* \leq \epsilon_0 \\ \epsilon_0^{\frac{\gamma}{\beta_*}} e^{-k\epsilon_0} \psi_*^{1-\frac{\gamma}{\beta_*}} & \text{if } \epsilon_0 \leq \psi_* \leq 1, \end{cases} \quad (3.54)$$

and we set  $V(t, \sigma) = \psi^*(\sigma) - a(t)g(\psi_*(\sigma))$  with  $(t, \sigma) \in (-\infty, T_k] \times S_+^{N-1}$  and define  $\tilde{u}(r, \sigma) = r^{-\beta_*}(\psi^*(\sigma) - a(e^t)g(\psi_*(\sigma)))$  accordingly for  $(r, \sigma) \in (-\infty, e^{T_k}] \times S_+^{N-1}$ . Since  $\psi_*$  is a decreasing function the coincidence set  $\{\sigma \in S_+^{N-1} : \psi_*(\sigma) = \epsilon_0\}$  is a circular cone  $\Sigma_{\theta_0}$  with vertex 0, axis  $\mathbf{e}_N$  and angle  $\theta_0$ . We set  $R_0 = e^{T_k}$

$$\begin{aligned} \Gamma_1 &= \left\{ x = (r, \theta) \in B_{R_0}^+ : \theta_0 < \theta < \frac{\pi}{2} \right\} = \left\{ (r, \sigma) \in [0, R_0] \times S_+^{N-1} : 0 < \psi_*(\sigma) < \epsilon_0 \right\}, \\ \Gamma_2 &= \left\{ x = (r, \theta) \in B_{R_0}^+ : 0 < \theta < \theta_0 \right\} = \left\{ (r, \sigma) \in [0, R_0] \times S_+^{N-1} : \epsilon_0 < \psi_*(\sigma) < 1 \right\}, \end{aligned}$$

and define

$$\begin{aligned} \tilde{u}(r, \sigma) &= r^{-\beta_*} (\psi_*(\sigma) - r^\gamma g(\psi_*(\sigma))) \\ &= \begin{cases} u_1(r, \sigma) = r^{-\beta_*} (1 - r^\gamma e^{-k\psi_*(\sigma)}) \psi_*(\sigma) & \text{if } (r, \theta) \in \Gamma_1 \\ u_2(r, \sigma) = r^{-\beta_*} \left( 1 - r^\gamma \epsilon_0^{\frac{\gamma}{\beta_*}} e^{-k\epsilon_0} (\psi_*(\sigma))^{1-\frac{\gamma}{\beta_*}} \right) \psi_*(\sigma) & \text{if } (r, \theta) \in \Gamma_2. \end{cases} \end{aligned}$$

The function  $\tilde{u}$  is a subsolution separately on  $\Gamma_1$  and  $\Gamma_2$  and is Lipschitz continuous in  $\overline{\Omega} \setminus \{0\}$ . If we denote by  $g_1$  and  $g_2$  the restriction of  $g$  to  $\Gamma_1$  and  $\Gamma_2$  respectively, that is to  $\Omega_1$  and  $\Omega_2$ , then  $g_1'(\epsilon_0) > g_2'(\epsilon_0) > 0$ . Let  $\zeta \in C_c^1(B_{R_0}^+)$  which vanishes in neighborhoods of 0 and  $\partial B_{R_0}^+$ ,  $\zeta \geq 0$ , then

$$\int_{\Gamma_i} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla \zeta dx + \int_{\Omega_i} |\nabla \tilde{u}|^q \zeta dx \leq \int_{\Sigma_{\theta_0}} |\nabla u_i|^{p-2} \partial_{\mathbf{n}_i} u_i \zeta dS, \quad (3.55)$$

where  $\mathbf{n}_i$  is the normal unit vector on  $\Sigma_{\theta_0}$  outward from  $\Gamma_i$ . Actually,  $\mathbf{n}_2 = -\mathbf{n}_1 = \mathbf{n}$  thus

$$\nabla \tilde{u} = \tilde{u}_r \mathbf{e} + r^{-\beta_*-1} (1 - r^\gamma g'(\psi_*)) \nabla' \psi_* = \tilde{u}_r \mathbf{e} + r^{-\beta_*-1} (1 - r^\gamma g'(\psi_*)) \psi_{*\theta} \mathbf{n}.$$

and on  $\Sigma_{\theta_0}$ ,

$$\nabla \tilde{u} = \begin{cases} \tilde{u}_r \mathbf{e} - r^{-\beta_*-1}(1 - r^\gamma g'_1(\epsilon_0)) \psi_{*\theta} \mathbf{n} & \text{in } \Gamma_1 \\ \tilde{u}_r \mathbf{e} + r^{-\beta_*-1}(1 - r^\gamma g'_2(\epsilon_0)) \psi_{*\theta} \mathbf{n} & \text{in } \Gamma_2 \end{cases}$$

Therefore

$$|\nabla u_1|^{p-2} \partial_{\mathbf{n}_1} u_1 = -r^{-\beta_*-1}(1 - r^\gamma g'_1(\epsilon_0)) \left( \tilde{u}_r^2 + r^{-2\beta_*-2}(1 - r^\gamma g'_1(\epsilon_0))^2 \psi_{*\theta}^2 \right)^{\frac{p-2}{2}} \psi_{*\theta} \quad \text{in } \Gamma_1$$

and

$$|\nabla u_2|^{p-2} \partial_{\mathbf{n}_2} u_2 = r^{-\beta_*-1}(1 - r^\gamma g'_2(\epsilon_0)) \left( \tilde{u}_r^2 + r^{-2\beta_*-2}(1 - r^\gamma g'_2(\epsilon_0))^2 \psi_{*\theta}^2 \right)^{\frac{p-2}{2}} \psi_{*\theta} \quad \text{in } \Gamma_2.$$

By adding the two inequalities (3.55)

$$\int_{\Omega} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla \zeta dx + \int_{\Omega} |\nabla \tilde{u}|^q \zeta dx \leq \int_{\Sigma_{\theta_0}} \left( |\nabla u_1|^{p-2} \partial_{\mathbf{n}_1} u_1 + |\nabla u_2|^{p-2} \partial_{\mathbf{n}_2} u_2 \right) \zeta dS. \quad (3.56)$$

By monotonicity of the function  $X \mapsto (\tilde{u}_r^2 + X^2)^{\frac{p-2}{2}}$  and since

$$r^{-\beta_*-1}(1 - r^\gamma g'_2(\epsilon_0)) \geq r^{-\beta_*-1}(1 - r^\gamma g'_1(\epsilon_0)) \geq 0,$$

we derive

$$\begin{aligned} & r^{-\beta_*-1}(1 - r^\gamma g'_2(\epsilon_0)) \left( \tilde{u}_r^2 + r^{-2\beta_*-2}(1 - r^\gamma g'_2(\epsilon_0))^2 \psi_{*\theta}^2 \right)^{\frac{p-2}{2}} \\ & \geq r^{-\beta_*-1}(1 - r^\gamma g'_1(\epsilon_0)) \left( \tilde{u}_r^2 + r^{-2\beta_*-2}(1 - r^\gamma g'_1(\epsilon_0))^2 \psi_{*\theta}^2 \right)^{\frac{p-2}{2}} \end{aligned}$$

We derive that the right-hand side of (3.56) is nonpositive because  $\psi_{*\theta} \leq 0$ , and therefore  $\tilde{u}$  is a positive subsolution of (1.1) in  $B_{R_0}^+$  dominated by  $\Psi_*$  and satisfying (3.24).  $\square$

*Proof of Theorem 3.6.* Let  $M = \max\{\Psi_*(x) : x \in \partial B_{R_0}^+\}$ , then  $M = R_0^{-\beta_*}$ . The function  $u^*$  defined by

$$u^*(x) = \begin{cases} (\tilde{u}(x) - M)_+ & \text{if } x \in B_{R_0}^+ \\ 0 & \text{if } x \in \Omega \setminus B_{R_0}^+, \end{cases}$$

is indeed a subsolution of (1.1) in whole  $\Omega$  where it satisfies  $u^* \leq \Psi_*$  and it vanishes on  $\partial\Omega \setminus \{0\}$ . Since  $\Phi_*$  is a positive p-harmonic function in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies (3.21), it is supersolution of (1.1) and therefore it dominates  $u^*$ . Therefore there exists a solution  $u$  of (1.1) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies  $u^* \leq u \leq \Phi_*$ . This implies that (3.20) holds with  $k = 1$  and we conclude with Lemma 3.8.  $\square$

When  $p = N$  the statement of Theorem 3.6 holds without the assumption on  $\partial\Omega$ .

**Theorem 3.10** *Assume  $N - 1 < q < N + \frac{N}{2}$  and  $\Omega$  be a bounded  $C^2$  domain such that  $0 \in \partial\Omega$ . Then for any  $k > 0$  there exists a unique positive solution  $u := u_k$  of (3.18) in  $\Omega$ , which belongs to  $C^1(\overline{\Omega} \setminus \{0\})$ , vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies uniformly with respect to  $\sigma \in S_+^{N-1}$*

$$\lim_{\substack{x \rightarrow 0 \\ x/|x| \rightarrow \sigma}} |x| u_k(x) = \sigma. \quad (3.57)$$

Since  $p = N$ ,  $\beta_* = 1$  and  $\psi_*(\sigma) = \frac{x_N}{|x|} = \cos \theta_{N-1}$  with the identification of  $\sigma$  and  $\theta_{N-1} := \theta$ . We recall that if  $\omega \in \mathbb{R}^N$  and  $\mathcal{I}_\omega$  denotes the inversion of center  $\omega$  and power 1, i.e.  $\mathcal{I}_\omega(x) = \omega + \frac{x-\omega}{|x-\omega|^2}$ , then  $\tilde{u} = u \circ \mathcal{I}_\omega$  satisfies (3.19).

**Lemma 3.11** *Assume  $\Omega$  be a bounded  $C^2$  domain such that  $0 \in \partial\Omega$ . Then there exists a unique  $N$ -harmonic function  $\Phi_*$  in  $\Omega$ , which vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies*

$$\lim_{\substack{x \rightarrow 0 \\ x/|x| \rightarrow \sigma}} |x| \Phi_*(x) = \sigma, \quad (3.58)$$

*uniformly with respect to  $\sigma \in S_+^{N-1}$ .*

*Proof.* Uniqueness is standard. Let  $\omega = -\mathbf{e}_N \in \overline{\Omega}^c$ , with the notations of the proof of Theorem 3.5,  $\omega' = -\omega$ ,  $a = -\frac{1}{2}\mathbf{e}_N$  and  $a' = -a$ . We can assume that the balls  $B_{\frac{1}{2}}(a)$  and  $B_{\frac{1}{2}}(a')$  are tangent to  $\partial\Omega$  at 0 and respectively subset of  $\Omega^c$  and  $\Omega$ . The function  $x \mapsto \Psi(x) = -\frac{x_N}{|x|^2}$  which is  $N$ -harmonic in  $\mathbb{R}_-^N$  and vanishes on  $\partial\mathbb{R}_-^{N-1} \setminus \{0\}$  is transformed by the inversion  $\mathcal{I}_{\omega'}$  of center  $\omega'$  and power 1 into the function  $\Psi_{\omega'} = \Psi \circ \mathcal{I}_{\omega'}$  which is positive and  $N$ -harmonic in  $B_{\frac{1}{2}}(a')$  and vanishes on  $\partial B_{\frac{1}{2}}(a') \setminus \{0\}$ . The function  $\hat{\Psi} = -\Psi$  which is  $N$ -harmonic in  $\mathbb{R}_+^N$  and vanishes on  $\partial\mathbb{R}_+^{N-1} \setminus \{0\}$  is transformed by the inversion  $\mathcal{I}_\omega$  of center  $\omega$  and power 1 into the function  $\Psi_\omega = \hat{\Psi} \circ \mathcal{I}_\omega$  which is positive and  $N$ -harmonic in  $B_{\frac{1}{2}}^c(a)$  and vanishes on  $\partial B_{\frac{1}{2}}(a) \setminus \{0\}$ . For  $\epsilon > 0$  we denote by  $\Phi_\epsilon$  the solution of

$$\begin{aligned} -\Delta_N \Phi_\epsilon &= 0 && \text{in } \Omega \cap B_\epsilon^c \\ \Phi_\epsilon &= 0 && \text{in } (B_{\frac{1}{2}}^c(a') \cap \partial B_\epsilon) \cup (\partial\Omega \cap B_\epsilon^c) \\ \Phi_\epsilon &= \Psi_{\omega'} && \text{in } B_{\frac{1}{2}}(a') \cap \partial B_\epsilon. \end{aligned} \quad (3.59)$$

If  $0 < \epsilon' < \epsilon$ ,  $\Phi_{\epsilon'} \geq \Psi_{\omega'}$  in  $B_{\frac{1}{2}}(a') \cap \partial B_{\epsilon'}$ , thus  $\Phi_{\epsilon'} \geq \Phi_{\epsilon'}$  in  $\Omega \cap B_{\epsilon'}^c$ . We also denote by  $\hat{U}_\epsilon$  the solution of

$$\begin{aligned} -\Delta_N \hat{\Phi}_\epsilon &= 0 && \text{in } \Omega \cap B_\epsilon^c \\ \hat{\Phi}_\epsilon &= 0 && \text{in } \partial\Omega \cap B_\epsilon^c \\ \hat{\Phi}_\epsilon &= \Psi_\omega && \text{in } \Omega \cap \partial B_\epsilon^c. \end{aligned} \quad (3.60)$$



In the same way as above

$$0 < \epsilon' < \epsilon \implies \hat{\Phi}_{\epsilon'} \leq \hat{\Phi}_{\epsilon} \quad \text{in } \Omega \cap \partial B_{\epsilon}^c$$

Using the explicit form of  $\Psi$ ,  $\mathcal{I}_{\omega} : x \mapsto \omega + \frac{x-\omega}{|x-\omega|^2}$  and  $\mathcal{I}_{\omega'} : x \mapsto \omega' + \frac{x-\omega'}{|x-\omega'|^2}$  we see that

$$\Psi_{\omega'} \lfloor_{B_{\frac{1}{2}}(a') \cap \partial B_{\epsilon}} \leq \frac{1+\epsilon}{1-\epsilon} \Psi_{\omega} \lfloor_{B_{\frac{1}{2}}(a') \cap \partial B_{\epsilon}},$$

thus

$$\Phi_{\epsilon} \leq \frac{1+\epsilon}{1-\epsilon} \hat{\Phi}_{\epsilon} \quad \text{in } \Omega \cap B_{\epsilon}^c.$$

Letting  $\epsilon \rightarrow 0$  we conclude that  $\Phi_{\epsilon}$  converges uniformly in  $\bar{\Omega} \setminus \{0\}$  to  $\Phi_*$  which vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies (3.58).  $\square$

The proof of the next statement is similar to the one of Lemma 3.8 up to some minor modifications.

**Lemma 3.12** *Let the assumptions on  $q$  and  $\Omega$  of Theorem 3.10 be satisfied. If for some  $k > 0$  there exists a solution  $u_k$  of (3.18) in  $\Omega$ , which belongs to  $C^1(\bar{\Omega} \setminus \{0\})$ , vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies (3.57), then for any  $k > 0$  there exists such a solution.*

**Lemma 3.13** *Under the assumptions of Theorem 3.10 there exists a Lipschitz continuous nonnegative subsolution  $\tilde{u}$  of (3.18) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ , is smaller than  $\Phi_*$  and satisfies*

$$\lim_{\substack{x \rightarrow 0 \\ x/|x| \rightarrow \sigma}} |x| \tilde{u}(x) = \sigma, \quad (3.61)$$

uniformly with respect to  $\sigma \in S_+^{N-1}$ .

*Proof.* Let  $\tau > 0$  to be fixed and let  $w$  be the solution of

$$-\Delta_N w + |\nabla w|^q = 0 \quad \text{in } B_2^- \quad (3.62)$$

which vanishes on  $\partial B_2^- \setminus \{0\}$  and satisfies

$$\lim_{\substack{x \rightarrow 0 \\ x/|x| \rightarrow \sigma}} |x| w(x) = \sigma \quad (3.63)$$

in the  $C^1$ -topology of  $S_-^{N-1}$ . Its existence follows from Theorem 3.6 and this function is dominated by the N-harmonic function  $\Phi_*$  corresponding to this domain obtain in Lemma 3.11. By  $\mathcal{I}_{\omega'}$ , the half-ball  $B_2^-$  is transform into the lunule  $G = B_{\frac{1}{2}}(a') \setminus B_{\frac{2}{3}}(\frac{4}{3}\omega')$  and  $\tilde{w} = w \circ \mathcal{I}_{\omega'}$  satisfies

$$-\Delta_N \tilde{w} + |x - \omega'|^{2(q-N)} |\nabla \tilde{w}|^q = 0 \quad \text{in } G. \quad (3.64)$$

Since  $|x - \omega'| \leq 1$  in  $G$ ,  $-\Delta_N \tilde{w} + |\nabla \tilde{w}|^q \leq 0$  in  $G$ . We extend  $\tilde{w}$  by 0 in  $\Omega \setminus G$  and the resulting function  $\tilde{u}$  is a subsolution of (3.18) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ , is smaller than the N-harmonic function  $\Phi_*$  obtained in Lemma 3.11, and satisfies (3.61).  $\square$

The proof of Theorem 3.10 is similar to the one of Theorem 3.6.

## 4 Classification of boundary singularities

We assume that  $\Omega \subset \mathbb{R}^N$  is a  $C^2$  domain and  $0 \in \partial\Omega$ . Furthermore, in order to avoid extremely technical computations, we shall assume either that  $\partial\Omega$  is flat near 0 or  $p = N$ . We denote  $\mathbb{R}_+^N = \{x = (x', x_N), x_N > 0\}$  and suppose that the tangent plane to  $\partial\Omega$  at 0 is  $\partial\mathbb{R}_+^N = \{x = (x', 0)\}$  and the normal inward unit vector at 0 is  $\mathbf{e}_N$ , therefore  $\mathbf{n} = -\mathbf{e}_N$  in the sequel. We denote by  $\omega_s$  the unique positive solution of (3.1) in  $S_+^{N-1}$  and by  $U_s$  the corresponding singular solution of (3.1) in  $\mathbb{R}_+^N$  defined by

$$U_s(x) = |x|^{-\beta_q} \omega_s\left(\frac{x}{|x|}\right). \quad (4.65)$$

We denote by  $\psi_*$  a the unique positive solution of (3.2) with maximum 1 and by  $\Psi_*$  the corresponding p-harmonic function

$$\Psi_*(x) = |x|^{-\beta_*} \psi_*\left(\frac{x}{|x|}\right). \quad (4.66)$$

**Proposition 4.1** *Assume  $N - 1 < q < N - \frac{1}{2}$  and  $\Omega$  is a bounded  $C^2$  domain with  $0 \in \partial\Omega$ . Then for any  $k > 0$  there exists a unique  $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$ ,  $u > 0$  solution of (3.18) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies*

$$\lim_{\substack{x \rightarrow 0 \\ x \in \Omega}} d^2(x) \frac{u_k(x)}{x} = k. \quad (4.67)$$

Next we can study the link between weak and strong singularities.

**Proposition 4.2** *Under the assumptions of Proposition ?? there exists  $\lim_{k \rightarrow \infty} u_k = u_\infty$  which is the unique element of  $C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$  vanishes on  $\partial B_\delta^+ \setminus \{0\}$ , satisfies (1.1) in  $B_\delta^+$  and*

$$\lim_{x \rightarrow 0} \frac{u_\infty(x)}{U_s(x)} = 1. \quad (4.68)$$

*Proof.* Uniqueness follows from (4.68) and the maximum principle. For existence, since the mapping  $k \mapsto u_k$  is increasing and  $u_k \leq U_s$ , there exists  $\lim_{k \rightarrow \infty} u_k := u_\infty \leq U_s$  and  $u_\infty \in C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$ . It vanishes on  $\partial B_\delta^+ \setminus \{0\}$  and satisfies (1.1) in  $B_\delta^+$ . In order to take into account the domain  $B_\delta^+$  in the notations, we set

$u_k = u_{k,\delta}$ . Since the mapping  $\delta \mapsto u_{k,\delta}$  is also increasing and  $u_{k,\delta} \leq k\Psi_*$ , there also exists  $\lim_{\delta \rightarrow \infty} u_{k,\delta} := u_{k,\infty} \leq k\Psi_*$ . Then, for all  $\ell > 0$ ,

$$T_\ell[u_{k,\delta}](x) = \ell^{\beta_q} u_{k,\delta}(\ell x) = u_{k\ell^{\beta_q}, \ell^{-1}\delta}(x). \quad (4.69)$$

Letting  $k \rightarrow \infty$ , we obtain

$$T_\ell[u_{\infty,\delta}](x) = \ell^{\beta_q} u_{\infty,\delta}(\ell x) = u_{\infty, \ell^{-1}\delta}(x), \quad (4.70)$$

and letting  $\delta \rightarrow \infty$ , we obtain

$$T_\ell[u_{\infty,\infty}](x) = \ell^{\beta_q} u_{\infty,\infty}(\ell x) = u_{\infty,\infty}(x). \quad (4.71)$$

This implies that

$$u_{\infty,\infty}(r, \sigma) = r^{-\beta_q} \omega'_s(\sigma), \quad (4.72)$$

and  $\omega'_s$  is a positive solution of problem (3.1). Therefore  $\omega'_s = \omega_s$  by Theorem 3.2. If we let  $\ell \rightarrow 0$  in (4.69) and take  $|x| = 1$ ,  $x = \sigma$ , we derive

$$\lim_{\ell \rightarrow 0} \ell^{\beta_q} u_{\infty,\delta}(\ell, \sigma) = \lim_{\ell \rightarrow 0} u_{\infty, \ell^{-1}\delta}(1, \sigma) = u_{\infty,\infty}(1, \sigma) = \omega_s(\sigma). \quad (4.73)$$

This convergence holds in  $C^1(\overline{S_+^{N-1}})$  because of Lemma ???. This implies (4.68).  $\square$

Similarly we have

**Proposition 4.3** *Under the assumptions of Proposition 4.1 there exists  $\lim_{k \rightarrow \infty} u_k = u_\infty$  which is the unique element of  $C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$  which vanishes on  $\partial B_\delta^+ \setminus \{0\}$ , satisfies (3.18) in  $\Omega$  and*

$$\lim_{x \rightarrow 0} \frac{u_\infty(x)}{U_s(x)} = 1. \quad (4.74)$$

**Theorem 4.4** *Assume  $1 < p < N$ ,  $\beta_q > \beta_*$  and  $\partial\Omega \cap B_\delta = \{x = (x', 0) : |x'| < \delta\}$ , for some  $\delta > 0$ . If  $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$  is a positive solution of (1.1) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ , then we have the following alternative*

(i) *either there exists  $k \geq 0$  such that*

$$\lim_{x \rightarrow 0} \frac{u(x)}{\Psi_*(x)} = k, \quad (4.75)$$

(ii) *or*

$$\lim_{x \rightarrow 0} \frac{u(x)}{U_s(x)} = 1. \quad (4.76)$$

*Proof. Step 1.* Assume

$$\liminf_{x \rightarrow 0} \frac{u(x)}{\Psi_*(x)} < \infty, \quad (4.77)$$

then we claim that (??) holds. We first note that if 4.77 holds, there also holds

$$\liminf_{x \rightarrow 0} \frac{u(x)}{u_1(x)} < \infty, \quad (4.78)$$

where  $u_1$  is the solution of (1.1) obtained in Proposition ?? with  $k = 1$ . If  $\{x_n\}$  is converging to 0 and such that

$$\liminf_{x \rightarrow 0} \frac{u(x)}{u_1(x)} = k = \lim_{n \rightarrow 0} \frac{u(x_n)}{u_1(x_n)},$$

there also holds by the boundary Harnack inequality (2.34)

$$\frac{u(x_n)}{u_1(x_n)} = \frac{u(x_n)}{dx_n} \frac{d(x_n)}{u_1(x_n)} \geq c_5^{-2} \frac{u(x)}{u_1(x)} \quad \forall x \text{ s.t. } |x| = |x_n|.$$

This implies in particular

$$u(x) \leq c_5^2(k + \epsilon_n)u_1(x) \quad \forall x \text{ s.t. } |x| = |x_n|$$

where  $\{\epsilon_n\}$  is converging to  $0_+$ , and by comparison principle

$$u(x) \leq Ku_1(x) \quad \forall x \in \mathbb{R}_+^N \text{ s.t. } |x_n| \leq |x| \leq \frac{\delta}{2},$$

for some  $K > 0$  and all  $n \in \mathbb{N}_*$ . Therefore

$$\limsup_{x \rightarrow 0} \frac{u(x)}{u_1(x)} < \infty. \quad (4.79)$$

We can assume that  $k \neq 0$ , otherwise (4.75) holds with  $k = 0$  and actually  $u$  remains bounded near 0. As a consequence, there exists  $K > 0$  such that

$$u(x) \leq K\Psi_*(x) \quad \forall x \in B_{\frac{\delta}{2}}^+. \quad (4.80)$$

Let  $m = \max\{u(x) : |x| = \delta\}$ . For  $0 < \tau < \delta$  we denote by  $k_\tau$  the minimum of the  $\kappa > 0$  such that  $u(x) \leq \kappa\Psi_*(x) + m$  for  $\tau \leq |x| \leq \delta$ . Then  $u(x) \leq k_\tau\Psi_*(x) + m$ , and either the graphs of the mappings  $u(\cdot)$  and  $k_\tau\Psi_*(\cdot) + m$  are tangent at some  $x_\tau \in B_\delta^+ \setminus \overline{B}_\tau^+$ , or they are tangent on the boundary of the domain, and the only possibility is that they are tangent on  $|x| = \tau$ . Since

$$|\nabla\Psi_*(x)|^2 = |x|^{-2(\tau+1)} (\tau^2\psi^2 + |\nabla\eta|^2),$$

it never vanishes. If we set  $w = u - (k_\tau\Psi_*(x) + m)$ , then

$$-\mathcal{L}w + |\nabla u|^q = 0 \quad (4.81)$$

where the operator

$$\mathcal{L} = \sum_{i,j} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial x}{\partial x_j} \right)$$

is uniformly elliptic in a neighborhood of  $x_\tau$  (see [16, Lemma 1.3]). Furthermore  $w \leq 0$  and  $w(x_\tau) = 0$  by the strong maximum principle  $\nabla u(x_\tau)$  must vanishes, which contradicts the fact that  $\nabla u(x_\tau) = \nabla w(x_\tau)$  by the tangency condition, and  $\nabla w(x_\tau) \neq 0$ . Therefore  $|x_\tau| = \tau$  and  $x_\tau \notin \partial\mathbb{R}_+^N$ . If  $\tau' < \tau$ ,  $k_\tau \leq k_{\tau'}$ , and we set  $k = \lim_{\tau \rightarrow 0} k_\tau$ , which is finite because of (4.80). There exists  $\{\tau_n\}$  such that  $\sigma_n := \tau^{-1}x_{\tau_n} \rightarrow \sigma_0$ . Furthermore

$$r^{\beta_*}u(r, \sigma) \leq k_\tau \psi_*(\sigma) + mr^{\beta_*} \quad \text{if } \tau \leq r \leq \delta \quad \text{and} \quad \tau^{\beta_*}u(\tau, \sigma_\tau) = k_\tau \psi_*(\sigma_\tau) + mr^{\beta_*}. \quad (4.82)$$

Put

$$u_\tau(x) = \tau^{\beta_*}u(\tau x) \quad (4.83)$$

Then

$$-\Delta u_\tau + \tau^{p-q-\beta_*(q+1-q)} |\nabla u_\tau|^q = 0 \quad \text{in } B_{\frac{\delta}{\tau}}^+ \setminus \{0\}$$

and, by (4.80),

$$0 \leq u_\tau(x) \leq K|x|^{-\beta_*} \quad \text{in } B_{\frac{\delta}{2\tau}}^+ \setminus \{0\}.$$

By Lemma ??, there holds the set of functions  $\{u_\tau(\cdot)\}$  is relatively compact in the  $C_{loc}^1$  topology of  $\overline{\mathbb{R}_+^N} \setminus \{0\}$ . Therefore, there exists a sequence  $\{\tau'_n\} \subset \{\tau_n\}$  converging to 0, and a positive p-harmonic function  $v$  in  $\mathbb{R}_+^N$ , continuous in  $\overline{\mathbb{R}_+^N} \setminus \{0\}$  and vanishing on  $\partial\mathbb{R}_+^N \setminus \{0\}$ , such that  $u_{\tau'_n} \rightarrow v$ , and  $v$  satisfies (4.80) in  $\overline{\mathbb{R}_+^N} \setminus \{0\}$ . By Theorem 5.1  $v$  is separable, therefore there exists  $k^*$  such that  $v = k^* \Psi_*$ . In particular,

$$\lim_{\tau'_n \rightarrow 0} u_{\tau'_n}(1, \sigma) = k^* \psi_*(\sigma) \quad (4.84)$$

in the  $C^1(\overline{S_+^{N-1}})$  topology. Combining (4.82), (4.83) and (4.84) we conclude that  $k^* = k$  and

$$\lim_{\tau'_n \rightarrow 0} \tau_n^{\beta_*} u_{\tau'_n}(1, \sigma) = k \psi_*(\sigma) \quad (4.85)$$

Using Proposition ??, it is equivalent to

$$\lim_{\tau'_n \rightarrow 0} \frac{u(\tau'_n, \sigma)}{u_k(\tau'_n, \sigma)} = 1 \quad (4.86)$$

uniformly on  $S_+^{N-1}$ . For any  $\epsilon > 0$ , there exists  $n_\epsilon > 0$  such that  $n \geq n_\epsilon$  implies

$$u_{k-\epsilon}(\tau'_n, \sigma) \leq u(\tau'_n, \sigma) \leq u_{k+\epsilon}(\tau'_n, \sigma)$$

By comparison principle,

$$u_{k-\epsilon} \leq u \leq u_{k+\epsilon} + m \quad \text{in } B_\delta^+ \setminus B_{\tau'_n}^+, \quad (4.87)$$

and finally

$$u_{k-\epsilon} \leq u \leq u_{k+\epsilon} + m \quad \text{in } B_\delta^+, \quad (4.88)$$

Since  $\epsilon$  is arbitrary and using again Proposition ??, it implies

$$\lim_{r \rightarrow 0} \frac{u(r, \sigma)}{\Psi_*(r, \sigma)} = k, \quad (4.89)$$

locally uniformly on  $S^{N-1}$ . But since the convergence holds in  $C^1(\overline{S_+^{N-1}})$ , (4.75) follows.

*Step 2.* Assume

$$\lim_{x \rightarrow 0} \frac{u(x)}{\Psi_*(x)} = \infty. \quad (4.90)$$

For any  $0 < \epsilon < \delta$  and  $k > 0$ , there holds

$$u_k(x) \leq u(x) \leq v_\epsilon(x) \quad \text{in } B_\delta^+ \setminus B_\epsilon^+ \quad (4.91)$$

where  $v_\epsilon$  has been defined in (3.12), and letting  $\epsilon \rightarrow 0$  and  $k \rightarrow \infty$ , we derive

$$u_\infty(x) \leq u(x) \leq v_0(x) \quad \text{in } B_\delta^+ \setminus \{0\}. \quad (4.92)$$

We have seen in Theorem 3.3 that  $v_0$  is a separable solution of (1.1) in  $\mathbb{R}_+^N$  which vanishes on  $\partial\mathbb{R}_+^N \setminus \{0\}$ , therefore  $v_0(x) = U_s(x)$ . This implies

$$u_\infty(x) \leq u(x) \leq |x|^{-\beta_q} \omega_s\left(\frac{x}{|x|}\right) \quad \text{in } B_\delta^+ \setminus \{0\}. \quad (4.93)$$

We conclude using Proposition 4.2.  $\square$

When  $p = N$ , the assumption that  $\partial\Omega$  is an hyperplane near 0 can be removed and the following result holds. Its proof is similar and based upon the use of Theorem 3.10.

**Theorem 4.5** *Assume  $N - 1 < q < N - \frac{1}{2}$ . If  $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$  is a positive solution of (3.18) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ , then we have the following alternative*

- (i) either there exists  $k \geq 0$  such that (4.75) holds,
- (ii) or (4.76) holds.

## 5 Appendix I: Positive $p$ -harmonic functions in a half space

We recall that  $\mathbb{R}_+^N = \{x = (x', x_N) : x_N > 0\}$ . In this section we prove the following rigidity result.

**Theorem 5.1** *Assume  $1 < p \leq N$  and  $u \in C^1(\mathbb{R}_+^N) \cap C(\overline{\mathbb{R}_+^N} \setminus \{0\})$  is a positive  $p$ -harmonic function which vanishes on  $\partial\mathbb{R}_+^N \setminus \{0\}$  and such that  $|x|^{\beta_*} u(x)$  is bounded. Then there exists  $k \geq 0$  such that*

$$u(x) = k\Psi_*(x) \quad \forall x \in \mathbb{R}_+^N. \quad (5.1)$$

*Proof.* Since  $|x|_*^\beta u(x)$  is bounded,  $|x|^{\beta_*+1} \nabla u(x)$  is also bounded and there exists  $m > 0$  such that  $u(x) \leq m\Psi_*(x)$  in  $B_\delta^+$ . We denote by  $k$  the infimum of the  $c > 0$  such that  $u(x) \leq c\Psi_S(x)$ . Then

$$0 \leq u(x) \leq k\Psi_*(x) \quad \forall x \in \mathbb{R}_+^N \setminus \{0\} \quad (5.2)$$

and we assume that  $k > 0$  otherwise  $u = 0$ . Assume that the graphs over  $\mathbb{R}_+^N$  of the functions  $x \mapsto u(x)$  and  $x \mapsto k\Psi_S(x)$  are tangent at some point  $x_0 \in \mathbb{R}_+^N$  or  $x_0 \in \partial\mathbb{R}_+^N \setminus \{0\}$ . Since  $\nabla\Psi_S$  never vanishes in  $\overline{\mathbb{R}_+^N} \setminus \{0\}$  it follows from the strong maximum principle or Hopf boundary lemma that  $u = k\Psi_*$ . If the two graphs are not tangent in  $\overline{\mathbb{R}_+^N} \setminus \{0\}$ , either they are asymptotically tangent at 0, or at  $\infty$ .

(i) In the first case there exists two sequences  $\{k_n\}$  increasing to  $k$  and  $\{x_n\} \subset \mathbb{R}_+^N$  converging to zero such that  $\frac{u(x_n)}{\Psi_*(x_n)} = k_n$ . We set  $r_n = |x_n|$  and  $u_{r_n}(x) = r_n^{\beta_*} u(r_n x)$ . Then  $u_{r_n}$  is p-harmonic and positive and  $0 < u_{r_n}(x) \leq k|x|^{-\beta_*} \psi_*(\frac{x}{|x|})$ ; therefore

$$|\nabla u_{r_n}(x)| \leq C|x|^{-\beta_*-1} \quad \text{and} \quad |\nabla u_{r_n}(x) - \nabla u_{r_n}(x')| \leq C|x|^{-\beta_*-1-\alpha} |x - x'|^\alpha \quad (5.3)$$

for  $0 < |x| \leq |x'|$  and some constants  $C > 0$  and  $\alpha \in (0, 1)$ . Up to a subsequence, we can assume that  $u_{r_n} \rightarrow U$  in the  $C_{loc}^1$  topology of  $\overline{\mathbb{R}_+^N} \setminus \{0\}$  and  $\frac{x_n}{r_n} \rightarrow \xi \in S_+^{N-1}$ . The function  $U$  is p-harmonic and positive in  $\mathbb{R}_+^N$  and satisfies  $0 \leq U \leq k\Psi_*$  in  $\mathbb{R}_+^N$  and  $U(\xi) = k\Psi_*(\xi)$  if  $\xi \in S_+^{N-1}$  or  $U_{x_N}(\xi) = k\Psi_{Sx_N}(\xi)$  if  $\xi \in \partial S_+^{N-1}$ . It follows from the strong maximum principle or Hopf boundary lemma that  $U = k\Psi_*$ . Therefore  $u_{r_n} \rightarrow k\Psi_*$  and in particular

$$\lim_{r_n \rightarrow 0} \frac{r_n^{\beta_*} u(r_n, \sigma)}{\psi_*(\sigma)} = k \quad \text{uniformly on } S_+^{N-1} \quad (5.4)$$

For any  $\epsilon > 0$ , there exists  $n_\epsilon \in \mathbb{N}_*$  such that for  $n \geq n_\epsilon$ ,  $(k - \epsilon)\Psi_*(x) \leq u(x) \leq (k + \epsilon)\Psi_*(x)$  if  $|x| = r_n$ . Which implies  $(k - \epsilon)\Psi_*(x) \leq u(x) \leq (k + \epsilon)\Psi_*$  for  $|x| \geq r_n$  and therefore in  $\mathbb{R}^N$ . Since  $\epsilon$  is arbitrary, we deduce that  $u = k\Psi_*$ .

(ii) if the two graphs are tangent at infinity, there exist two sequences  $\{k_n\}$  increasing to  $k$  and  $\{x_n\}$  such that  $r_n = |x_n| \rightarrow \infty$  with  $u(x_n) = k_n\Psi_*(x_n)$  and

$$\lim_{r_n \rightarrow \infty} \frac{r_n^{\beta_*} u(r_n, \sigma)}{\psi_*(\sigma)} = k \quad \text{uniformly on } S_+^{N-1}. \quad (5.5)$$

Therefore we look at the supremum of the  $c > 0$  such that  $u \geq c\Psi_*$ . If the set of such  $c$  is empty, it would mean that

$$\inf_{x \in \mathbb{R}_+^N} \frac{u(x)}{\Psi_*(x)} = 0.$$

Clearly, if this infimum is achieved at some point, the strong maximum principle or Hopf boundary lemma imply  $u \equiv 0$ , contradicting (5.5), and this relation prevents

also this infimum be achieved at infinity. We are left with the case where there exists a sequence  $\{z_n\} \subset \mathbb{R}_+^N$ , converging to 0, such that

$$\lim_{n \rightarrow \infty} \frac{u(z_n)}{\Psi_*(z_n)} = 0. \quad (5.6)$$

By boundary Harnack inequality [5, th 2.11], there exists  $c > 0$  such that

$$c^{-1} \frac{u(z)}{\Psi_*(z)} \leq \frac{u(z_n)}{\Psi_*(z_n)} \leq c \frac{u(z)}{\Psi_*(z)} \quad \forall z \in \mathbb{R}_+^N \text{ s.t. } |z| = |z_n| \quad (5.7)$$

Combining (5.6) and (5.7), we derive that

$$\lim_{n \rightarrow \infty} \sup_{|z|=|z_n|} \frac{u(z)}{\Psi_*(z)} = 0, \quad (5.8)$$

Denoting by  $\epsilon_n$  the infimum in the above relation, we obtain that  $u \leq \epsilon_n \Psi_*$  in  $\mathbb{R}_+^N \setminus B_{\epsilon_n}$  and finally  $u = 0$ , contradiction. Thus we are left with the case where there exists  $k' \in (0, k]$  which is the supremum of the  $c > 0$  such that  $u \geq c \Psi_*$ . In particular  $u \geq k' \Psi_*$ . Remembering that  $u \leq k \Psi_*$  we obtain that  $k = k'$  implies  $u = k \Psi_*$ .

Next we assume that  $k' < k$ . Clearly the graphs of  $u$  and  $k' \Psi_*$  cannot be tangent in  $\overline{\mathbb{R}_+^N}$ , because of strong maximum principle or Hopf boundary lemma. They cannot be tangent at infinity because of (5.5). Therefore there exists two sequences  $\{k'_n\}$  increasing to  $k'$  and  $\{x'_n\} \subset \mathbb{R}_+^N$  converging to 0 such that  $\frac{u(x'_n)}{\Psi_*(x'_n)} = k'_n$ . As in Case (i) we obtain that

$$\lim_{r'_n \rightarrow 0} \frac{r_n'^{\beta_*} u(r'_n, \sigma)}{\psi_*(\sigma)} = k' \quad \text{uniformly on } S_+^{N-1} \quad (5.9)$$

where  $r'_n = |x'_n|$ , and finally derive that  $u = k' \Psi_*$ , a contradiction with (5.5). Therefore  $k = k'$ , which ends the proof.  $\square$

*Remark.* In the case  $p = N$  the result holds under the weaker assumption  $\lim_{|x| \rightarrow \infty} u(x) = 0$ . This is due to the fact that this condition implies by regularity

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{\omega_s(\frac{x}{|x|})} = 0$$

and therefore

$$u(x) \leq m \Psi_*(x) \quad \forall x \text{ s.t. } |x| \geq 1$$

where  $m = \max_{|x|=1} \frac{u(x)}{\omega_s(\frac{x}{|x|})}$ . Using the inversion  $x \mapsto \frac{x}{|x|^2}$ , we obtain that the estimate  $u \leq m \Psi_*$  holds  $\mathbb{R}^N$ , and we conclude by Theorem 5.1.

*Remark.* We conjecture that the rigidity result holds under the mere condition

$$\lim_{|x| \rightarrow \infty} |x|^{-\tilde{\beta}_*} u(x) = 0, \quad (5.10)$$



were  $\tilde{\beta}_*$  is the exponent (positive) corresponding to the regular spherical p-harmonic function under the form

$$\tilde{\Psi}_* = |x|^{\tilde{\beta}_*} \tilde{\psi}_*\left(\frac{x}{|x|}\right), \quad (5.11)$$

see [29], [25]. When  $p = N$ ,  $\tilde{\beta}_* = 1$ .

## 6 Appendix II: Estimates on $\beta_*$

When  $N = 2$  and  $1 < p \leq 2$ , it is proved in [19] that

$$\beta_* = \frac{3 - p + 2\sqrt{p^2 - 5p + 7}}{3(p - 1)}. \quad (6.1)$$

Up to now no estimate is known when  $N > 2$  except in the cases  $p = 2$  where  $\beta_* = N - 1$  and  $p = N$  where  $\beta_* = 1$ , besides the classical one

$$\beta_* > \frac{N - p}{p - 1}, \quad (6.2)$$

valid when  $p < N$ . In this section we prove the following result

**Theorem 6.1** *Assume  $1 < p < N$ . Then the following estimates hold*

$$1 < p < 2 \implies \beta_* > \frac{N - 1}{p - 1} \quad (6.3)$$

$$2 < p < N \implies \max\left\{1, \frac{N - p}{p - 1}\right\} < \beta_* < \frac{N - 1}{p - 1} \quad (6.4)$$

*Remark.* It is worth noticing that when  $p = 2$  or  $p = N$ , there holds  $\beta_* = \frac{N-1}{p-1}$ .

*Proof of Theorem 6.1.* We consider the following set of spherical coordinates in  $\mathbb{R}_+^N$  with  $x = (x_1, \dots, x_N)$

$$\begin{aligned} x_1 &= r \sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_2 \sin \theta_1 \\ x_2 &= r \sin \theta_{N-1} \sin \theta_{N-2} \dots \sin \theta_2 \cos \theta_1 \\ &\vdots \\ x_{N-1} &= r \sin \theta_{N-1} \cos \theta_{N-2} \\ x_N &= r \cos \theta_{N-1} \end{aligned} \quad (6.5)$$

with  $\theta_1 \in [0, 2\pi]$  and  $\theta_k \in [0, \pi]$  for  $k = 2, \dots, N - 2$  and  $\theta_{N-1} \in [0, \frac{\pi}{2}]$ . Under this representation, a solution  $\omega$  of (3.2) verifies

$$\begin{aligned} &-\frac{1}{\sin^{N-2} \theta_{N-1}} \left[ \sin^{N-2} \theta_{N-1} \left( \beta_*^2 \omega^2 + \omega_{\theta_{N-1}}^2 + \frac{1}{\sin^2 \theta_{N-1}} |\nabla_{\theta'} \omega|^2 \right)^{\frac{p-2}{2}} \omega_{\theta_{N-1}} \right]_{\theta_{N-1}} \\ &-\frac{1}{\sin^2 \theta_{N-1}} \operatorname{div}_{\theta'} \left[ \sin^{N-2} \theta_{N-1} \left( \beta_*^2 \omega^2 + \omega_{\theta_{N-1}}^2 + \frac{1}{\sin^2 \theta_{N-1}} |\nabla_{\theta'} \omega|^2 \right)^{\frac{p-2}{2}} \nabla_{\theta'} \omega \right] \\ &= \beta_* \Lambda_{\beta_*} \left[ \sin^{N-2} \theta_{N-1} \left( \beta_*^2 \omega^2 + \omega_{\theta_{N-1}}^2 + \frac{1}{\sin^2 \theta_{N-1}} |\nabla_{\theta'} \omega|^2 \right)^{\frac{p-2}{2}} \omega \right] \end{aligned} \quad (6.6)$$

where  $\nabla_{\theta'}$  and  $div_{\theta'}$  denotes respectively the spherical gradient the divergence in variables  $\theta' = (\theta_1, \dots, \theta_{N-2})$  parametrizing  $S^{N-2}$  and  $\Lambda_{\beta_*} = \beta_*(p-1) + p - N$ . If  $\omega$  is the unique positive solution of (3.2) (up to homothety), it depends only on  $\theta_{N-1}$  and is  $C^\infty$ . For simplicity we set  $\theta_{N-1} = \theta \in [0, \frac{\pi}{2}]$  and  $\omega = \omega(\theta)$  satisfies

$$-\frac{1}{\sin^{N-2}\theta} \left[ \sin^{N-2}\theta (\beta_*^2\omega^2 + \omega_\theta^2)^{\frac{p-2}{2}} \omega_\theta \right]_\theta = \beta_*\Lambda_{\beta_*} \left[ \sin^{N-2}\theta (\beta_*^2\omega^2 + \omega_\theta^2)^{\frac{p-2}{2}} \omega \right]_{\theta} \quad \text{in } (0, \frac{\pi}{2})$$

$$\omega(\frac{\pi}{2}) = 0, \quad \omega_\theta(0) = 0. \quad (6.7)$$

*Step 1: The eigenvalue identity.* Equation (6.7) can also be written under the form

$$-\omega_{\theta\theta} - (N-2)\cot\theta\omega_\theta - (p-2)\frac{\beta_*^2\omega + \omega_{\theta\theta}}{\beta_*^2\omega^2 + \omega_\theta^2}\omega_\theta^2 = \beta_*\Lambda_{\beta_*}\omega \quad (6.8)$$

Since

$$-\int_0^{\frac{\pi}{2}} (\omega_{\theta\theta} + (N-2)\cot\theta\omega_\theta) \cos\theta \sin^{N-2}\theta d\theta = (N-1) \int_0^{\frac{\pi}{2}} \omega \cos\theta \sin^{N-2}\theta d\theta,$$

we multiplying by  $\cos\theta \sin^{N-2}\theta$  and integrate on  $(0, \frac{\pi}{2})$ . Noticing that

$$\beta_*\Lambda_{\beta_*} + 1 - N = (p-1) \left( \beta_* - \frac{N-1}{p-1} \right) (\beta_* + 1)$$

we derive

$$(2-p) \int_0^{\frac{\pi}{2}} \frac{\beta_*^2\omega + \omega_{\theta\theta}}{\beta_*^2\omega^2 + \omega_\theta^2} \omega_\theta^2 \omega \cos\theta \sin^{N-2}\theta d\theta$$

$$= (p-1) \left( \beta_* - \frac{N-1}{p-1} \right) (\beta_* + 1) \int_0^{\frac{\pi}{2}} \omega \cos\theta \sin^{N-2}\theta d\theta. \quad (6.9)$$

*Step 2: Elliptic coordinates and reduction.* Writting  $\omega(\theta) = \omega(0) + a\theta^2 + o(\theta^2)$ ,  $\omega_\theta(\theta) = 2a\theta + o(\theta)$  and  $\omega_{\theta\theta}(\theta) = 2a + o(1)$ , then  $-Na = \beta_*\Lambda_{\beta_*}$ . This implies that  $\omega$  is decreasing near 0. It is immediate that it cannot have a local minimum in  $(0, \frac{\pi}{2})$ , therefore it remains decreasing in the whole interval. We parametrize the ellipse

$$E_r = \{(x, y) : x > 0, y < 0, x^2 + \beta^{-2}y^2 = r^2\}$$

and set  $\omega = r \cos\phi$  and  $-\omega_\theta = \beta r \sin\phi$ , with  $\phi = \phi(\theta)$  and  $r = r(\theta)$ ;  $r$  and  $\phi$  are  $C^2$  functions. Hence  $r_\theta \cos\phi - r \sin\phi \phi_\theta = -\beta r \sin\phi$ , then  $r_\theta \cos\phi = (\phi_\theta - \phi)r \sin\phi$  and  $r_\theta = (\phi_\theta - \beta)r \tan\phi$ . Plugging this into (6.9), we derive

$$-\left( (p-1)\frac{r_\theta}{r} + \phi_\theta \cot\phi + (N-2)\cot\theta \right) + \Lambda_{\beta_*} \cot\phi = 0, \quad (6.10)$$

and finally

$$(p-1)(\phi_\theta - \beta) \tan\phi + (\phi_\theta - \Lambda_{\beta_*}) \cot\phi = (2-N) \cot\theta. \quad (6.11)$$

*Step 3: Estimates on  $\phi_\theta$ .* We can write (6.11) under the equivalent form

$$(p-1)(\phi_\theta - \beta) \tan^2 \phi + \phi_\theta - \Lambda_{\beta_*} = (2-N) \frac{\cos \theta \sin \phi}{\cos \phi \sin \theta}. \quad (6.12)$$

Since

$$\lim_{\theta \rightarrow 0} \frac{\sin \phi}{\sin \theta} = \lim_{\theta \rightarrow 0} \frac{\cos \phi}{\cos \theta} \phi_\theta = \phi_\theta(0)$$

we derive  $\phi_\theta(0) - \Lambda_{\beta_*} = (2-N)\phi_\theta(0)$  and thus

$$\phi_\theta(0) = \frac{\Lambda_{\beta_*}}{N-1}. \quad (6.13)$$

Similarly, the expansion of  $\phi(\theta)$  near  $\theta = \frac{\pi}{2}$  yields to

$$\phi_\theta(\frac{\pi}{2}) = \beta_*. \quad (6.14)$$

Since  $p < N$ ,  $\Lambda_{\beta_*}/(N-1) < \beta_*$ . We claim now that

$$\phi_\theta(\theta) < \beta_* \quad \forall \theta \in (0, \frac{\pi}{2}). \quad (6.15)$$

If  $\Lambda_{\beta_*} \leq \beta_*$ , then

$$(2-N) \cot \theta = (p-1)(\phi_\theta - \beta) \tan \phi + (\phi_\theta - \Lambda_{\beta_*}) \cot \phi \geq ((p-1) \tan \phi + \cot \phi)(\phi_\theta - \beta)$$

thus (6.15) holds.

Next we assume  $\beta_* < \Lambda_{\beta_*}$ . It means  $0 < (p-2)\beta_* - (N-p)$  and thus  $p > 2$ . We claim that

$$\beta_* \leq \frac{N-2}{p-2}. \quad (6.16)$$

We proceed by contradiction and assume

$$\beta_* > \frac{N-2}{p-2}. \quad (6.17)$$

Then

$$(p-2) \left( \beta_*^2 - \frac{N-p}{p-2} \beta_* - \frac{N-2}{p-2} \right) = (p-2)(\beta_* + 1) \left( \beta_* - \frac{N-2}{p-2} \right) > 0.$$

Equivalently

$$\beta_*(\Lambda_{\beta_*} - \beta_*) > N-2.$$

Since

$$\lim_{\theta \rightarrow \frac{\pi}{2}} \cot \theta \tan \phi = \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\cos \theta}{\cos \phi} = \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\sin \theta}{\phi_\theta \sin \phi} = \frac{1}{\beta_*}$$

and

$$\begin{aligned} (p-1)(\phi_\theta(\theta) - \beta_*) \tan^2 \phi &= \Lambda_{\beta_*} - \phi_\theta(\theta) + (2-N) \frac{\cos \theta \sin \phi}{\cos \phi \sin \theta} \\ &= \frac{1}{\beta_*} (\beta_*(\Lambda_{\beta_*} - \beta_*) + 2 - N) + o(1) \end{aligned}$$

if follows that (6.17) is equivalent to the fact that there exists  $\epsilon > 0$  such that  $\phi_\theta(\theta) > \frac{\pi}{2}$  for any  $\theta \in [\frac{\pi}{2} - \epsilon, \frac{\pi}{2})$ . Since  $\phi_\theta(0) < \beta_*$ , there exists  $\bar{\theta} \in (0, \frac{\pi}{2})$  such that  $\phi_\theta(\bar{\theta}) = \beta$  and  $\phi_{\theta\theta}(\bar{\theta}) \geq 0$ . We compute  $\phi_{\theta\theta}$  and get

$$(p-1)\phi_\theta(\theta)(\phi_\theta(\theta) - \beta_*) \sec^2 \phi(\theta) + (p-1)\phi_{\theta\theta}(\theta) \tan \phi(\theta) + \phi_{\theta\theta}(\theta) \cot \phi(\theta) - \phi_\theta(\theta)(\phi_\theta(\theta) - \Lambda_{\beta_*}) \csc^2 \phi(\theta) = (N-2) \csc^2 \theta$$

Hence, at  $\theta = \bar{\theta}$

$$(p-1)\phi_{\theta\theta}(\bar{\theta}) ((p-1) \tan \phi(\bar{\theta}) + \cot \phi(\bar{\theta})) = \beta_*(\beta_* - \Lambda_{\beta_*}) \csc^2 \phi(\bar{\theta}) + (N-2) \csc^2 \bar{\theta}$$

From (6.11),

$$\cot \phi(\bar{\theta}) = \frac{N-2}{\Lambda_{\beta_*} - \beta_*} \cot \bar{\theta}$$

Therefore

$$\begin{aligned} & (p-1)\phi_{\theta\theta}(\bar{\theta}) ((p-1) \tan \phi(\bar{\theta}) + \cot \phi(\bar{\theta})) \\ &= \left( 1 + \left( \frac{N-2}{\Lambda_{\beta_*} - \beta_*} \right)^2 \cot^2 \bar{\theta} \right) \beta_*(\beta_* - \Lambda_{\beta_*}) + (N-2)(1 + \cot^2 \bar{\theta}) \\ &= \beta_*(\beta_* - \Lambda_{\beta_*}) + N-2 - \left( \frac{(N-2)^2}{\Lambda_{\beta_*} - \beta_*} + 2 - N \right) \cot^2 \bar{\theta} \\ &= -(p-2)(\beta_* + 1) \left( \beta_* - \frac{N-2}{p-2} \right) - \frac{N-2}{\Lambda_{\beta_*} - \beta_*} (\beta_*(N-1) - \Lambda_{\beta_*}) \cot^2 \bar{\theta} \\ &< 0, \end{aligned} \tag{6.18}$$

using (6.17) and the fact that  $N > p$ . This is a contradiction, thus (6.16) holds and there exists  $\epsilon > 0$  such that  $\phi_\theta < \beta_*$  in  $[\frac{\pi}{2} - \epsilon, \frac{\pi}{2})$ .

We claim now that  $\phi_\theta < \beta_*$  in  $[0, \frac{\pi}{2})$ . If it is not true, there exist  $\theta_1 < \theta_2 < \frac{\pi}{2}$  such that  $\phi_\theta(\theta_1) = \phi_\theta(\theta_2) = \beta_*$ ,  $\phi_{\theta\theta}(\theta_1) > 0$ ,  $\phi_{\theta\theta}(\theta_2) < 0$ , since Cauchy-Lipschitz theorem applies to equation (6.11) in  $(0, \frac{\pi}{2})$ . We put  $\phi(\theta_1) = \phi_1$  and  $\phi(\theta_2) = \phi_2$ . Using the equation satisfied by  $\phi_{\theta\theta}$ , we obtain for  $i = 1, 2$ ,

$$\begin{aligned} & ((p-1) \tan \phi_i + \cot \phi_i) \phi_{\theta\theta}(\theta_i) \\ &= (2-p)(\beta_* + 1) \left( \beta_* - \frac{N-2}{p-2} \right) - \frac{N-2}{\Lambda_{\beta_*} - \beta_*} (\beta_*(N-1) - \Lambda_{\beta_*}) \cot^2 \theta_i. \end{aligned} \tag{6.19}$$

Since  $\cot$  is decreasing in  $(0, \frac{\pi}{2})$ ,  $\cot^2 \theta_1 > \cot^2 \theta_2$ , hence

$$0 < ((p-1) \tan \phi_1 + \cot \phi_1) \phi_{\theta\theta}(\theta_1) < ((p-1) \tan \phi_2 + \cot \phi_2) \phi_{\theta\theta}(\theta_2) < 0,$$

a contradiction. Therefore  $\phi_\theta < \beta_*$  in  $(0, \frac{\pi}{2})$ .

*Step 4: End of the proof.* Since  $r^2 = \beta_*^2 \omega^2 + \omega_\theta^2$ ,  $r_\theta = r(\phi_\theta - \beta_*) \tan \phi$ , thus

$$rr_\theta = (\beta_*^2 \omega + \omega_{\theta\theta}) \omega_\theta = r(\phi_\theta - \beta_*) \tan \phi.$$

Since  $\omega_\theta < 0$  on  $(0, \frac{\pi}{2})$ , it follows from Step 3 that  $\beta_*^2\omega + \omega_{\theta\theta} > 0$  and finally

$$\int_0^{\frac{\pi}{2}} \frac{\beta_*^2\omega + \omega_{\theta\theta}}{\beta_*^2\omega^2 + \omega_\theta^2} \omega_\theta^2 \omega \cos \theta \sin^{N-2} \theta d\theta > 0.$$

The conclusion follows from (6.9).  $\square$

*Remark.* Since  $\omega_\theta(\frac{\pi}{2}) = -c^2 < 0$ , it follows  $\omega(\theta) = -\omega_\theta(\theta) \cot \theta + O(\frac{\pi}{2} - \theta)$  and from the eigenfunction equation (6.8)

$$\frac{\beta_*^2\omega + \omega_{\theta\theta}}{\beta_*^2\omega^2 + \omega_\theta^2} \omega_\theta^2 = (\beta_*^2\omega + \omega_{\theta\theta})(1 + o(1)).$$

Therefore

$$-(p-1)\omega_{\theta\theta} = (\beta_*\Lambda_{\beta_*} + (p-2)\beta_*^2 + 2 - N)\omega(1 + o(1)) \quad \text{as } \theta \rightarrow \frac{\pi}{2}$$

and since  $\Delta'\omega := \omega_{\theta\theta} + (N-2)\cot \theta \omega_\theta$

$$-\Delta'\omega = \frac{\beta_*(\beta_*(2p-3) + p - N) + (p-2)(N-2)}{p-1} \omega(1 + o(1)) \quad \text{as } \theta \rightarrow \frac{\pi}{2}.$$

Because  $\omega$  is  $C^\infty$  we obtain finally

$$|\Delta'\omega| \leq c\omega, \tag{6.20}$$

for some  $c > 0$ .

## References

- [1] Abdel Hamid H., Bidaut-Véron M.F., *On the connection between two quasilinear elliptic problems with lower terms of order 0 or 1*, Comm. Contemp. Math., to appear.
- [2] Alvino A., Ferone V., Trombetti G., *Estimates for the gradient of solutions of nonlinear elliptic equations with  $L^1$  data*, Ann. Mat. Pura Appl. 178, 129-142 (2000).
- [3] Bauman P., *Positive solutions of elliptic equations in nondivergence form and their adjoints* Ark. Mat. 22, 153-173 (1984).
- [4] Bidaut-Véron M.F., *Removable singularities and existence for a quasilinear equation*, Adv. Nonlinear Studies 3, 25-63 (2003).
- [5] Bidaut-Véron M.F., Borghol R., Véron L., *Boundary Harnack inequality and a priori estimates of singular solutions of quasilinear elliptic equations*, Calc. Var. 27, 159-177 (2006).

- [6] Bidaut-Véron M.F., Garcia Huidobro M., Véron, *Quasilinear elliptic Hamilton-Jacobi equations on complete manifolds*, C. R. Acad. Sci. Paris, Ser. I 351, 445-449 (2013).
- [7] Bidaut-Véron M.F., Garcia Huidobro M., Véron L., *Local and global behaviour of solutions of quasilinear Hamilton-Jacobi equations*, J. Funct. Anal. 267, 3294-3331 (2014).
- [8] Boccardo L., Gallouet T., *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal. 87, 149-169 (1989).
- [9] Boccardo L, Gallouet T., Orsina L., *Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data*, Ann. Inst. H. Poincaré Anal. non Lin. 13, 539-555 (1996).
- [10] Boccardo L., Murat F., Puel J., *Résultats d'existence pour certains problèmes elliptiques quasilineaires*, Ann. Scuola. Norm. Sup. Pisa 11 (2) 213-235 (1984).
- [11] Boccardo L., Segura de León S., Trombetti C., *Bounded and unbounded solutions for a class of quasi-linear elliptic problems with a quadratic gradient term*, J. Math. Pures Appl. 80 (9) (2001) 919-940.
- [12] Borghol R., Véron L., *Boundary singularities of solutions of  $N$ -harmonic equations with absorption*, J. Funct. Anal. 241, 611-637 (2006).
- [13] Borghol R., Véron L., *Boundary Singularities of  $N$ -Harmonic Functions*, Comm. in Part. Diff. Equ., 32, 1001-1015, (2007).
- [14] Dal Maso G., Murat F., Orsina L., Prignet A., *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Sup. Pisa, 28, 741-808 (1999).
- [15] Duzar F., Mingione G., *Local Lipschitz regularity for degenerate elliptic systems*, Ann. I. H. Poincaré-AN 27, 1361-1396 (2010).
- [16] Friedman A., Véron L., *Singular Solutions of Some Quasilinear Elliptic Equations*, Arch. Rat. Mec. Anal. 96, 258-287 (1986).
- [17] Gilbarg D., Trudinger N., *Elliptic Partial Differential Equations of Second Order*, Grundlehren der mathematischen Wissenschaften 224 2nd ed. Berlin, Heidelberg, New-York, Tokyo, Springer Verlag (1983).
- [18] Kroll, I. N., *The behaviour of the solutions of a certain quasilinear equation near zero cusps of the boundary*, Proc. Steklov Inst. Math. 125, 140-146 (1973).
- [19] Kichenassamy S., Véron L., *Singular solutions of the  $p$ -Laplace equation*, Math. Ann. 275, 599-615 (1986).

- [20] Lions P. L., *Quelques remarques sur les problme elliptiques quasilineaires du second ordre*, J. Anal.Math. 45, 234-254 (1985).
- [21] Murat F., Porretta A., *Stability properties, existence, and nonexistence of renormalized solutionsfor elliptic equations with measure data*, Comm. Partial Differential Equations 27, 2267-2310 (2002).
- [22] Nguyen Phuoc T., Véron L., *Boundary singularities of solutions to elliptic viscous Hamilton-Jacobi equations*, J. Funct. An. 263, 1487-1538 (2012).
- [23] Porretta A., *Nonlinear equations with natural growth terms and measure data*, E.J.D.E., Conference 9, 183-202 (2002).
- [24] Porretta A., Segura de León S., *Nonlinear elliptic equations having a gradient term with natural growth*, J. Math. Pures Appl. 85, 465-492 (2006).
- [25] Porretta A., Véron L., *Separable  $p$ -harmonic functions in a cone and related quasilinear equations on manifolds*, J. Europ. Math. Soc. 11, 1285-1305 (2009).
- [26] Porretta A., Véron L., *Separable solutions of quasilinear Lane-Emden equations*, J. Europ. Math. Soc 15, 755-774 (2013).
- [27] Pucci P., Serrin S., Zou H. *A Strong Maximum Principle and a Compact Support Principle for Singular Elliptic Inequalities*
- [28] Serrin J., *Local behaviour of solutions of quasilinear equations*, Acta Math. 111, 247-302 (1964).
- [29] Tolksdorf P., *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Part. Diff. Equ. 8, 773-817 (1983).
- [30] Tolksdorf P., *Regularity for a More General Class of Quasilinear Elliptic Equations*, J. Diff. Equ. 51, 126-150 (1984).
- [31] Trudinger N., *On Harnack type inequalities and their applications to quasilinear elliptic equations*, Comm. Pure Appl. Math. 20, 721-747 (1967).