BSDEs with monotone generator driven by Brownian and Poisson noises in a general filtration

T Kruse, A Popier

BSDEs with monotone generator driven by Brownian and Poisson noises in a general filtration

T. Kruse *, A. Popier †

June 9, 2015

Abstract

We analyze multidimensional BSDEs in a filtration that supports a Brownian motion and a Poisson random measure. Under a monotonicity assumption on the driver, the paper extends several results from the literature. We establish existence and uniqueness of solutions in L^p provided that the generator and the terminal condition satisfy appropriate integrability conditions. The analysis is first carried out under a deterministic time horizon, and then generalized to random time horizons given by a stopping time with respect to the underlying filtration. Moreover, we provide a comparison principle in dimension one.

Introduction

The notion of nonlinear backward stochastic differential equations (BSDEs for short) was introduced by Pardoux and Peng [33]. A solution of this equation, associated with a *terminal value* ξ and a *generator or driver* $f(t,\omega,y,z)$, is a couple of stochastic processes $(Y_t,Z_t)_{t\leq T}$ such that

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dW_s, \tag{1}$$

a.s. for all $t \leq T$, where W is a Brownian motion and the processes $(Y_t,Z_t)_{t\leq T}$ are adapted to the natural filtration of W.

In their seminal work [33], Pardoux and Peng proved existence and uniqueness of a solution under suitable assumptions, mainly square integrability of ξ and of the process $(f(t,\omega,0,0))_{t\leq T}$, on the one hand, and, the Lipschitz property w.r.t. (y,z) of the generator f, on the other hand. Since this first result, BSDEs have proved to be a powerful tool for formulating and solving a lot of mathematical problems arising for example in finance (see e.g. [2, 13, 36]), stochastic control and differential games (see e.g. [15, 16]), or partial differential equations (see e.g. [31, 32]).

*University of Duisburg-Essen, Thea-Leymann-Str. 9, 45127 Essen, Germany, e-mail: thomas.kruse@uni-due.de

†Université du Maine, Laboratoire Manceau de Mathématiques, Avenue Olivier Messiaen, 72085 Le Mans, Cedex 9, France, e-mail: alexandre.popier@univ-lemans.fr

We would like to thank two anonymous referees for helpful comments and suggestions. Thomas Kruse acknowledges the financial support from the French Banking Federation through the Chaire "Markets in Transition".

Main results

The aim of this paper is to establish existence and uniqueness of solutions to BSDE in a general filtration that supports a Brownian motion W and an independent Poisson random measure π. We consider the following multi-dimensional BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \psi_s)ds - \int_t^T \int_{\mathcal{U}} \psi_s(u)\widetilde{\pi}(du, ds) - \int_t^T Z_s dW_s - \int_t^T dM_s. \qquad (2)$$

The solution is given by the usual triple (Y, Z, ψ) and also an orthogonal local martingale M which can not be reconstructed by the integrals w.r.t. to the Brownian and Poisson noise. We assume that the generator f is monotonic (one-sided Lipschitz continuous) w.r.t. the y-variable and Lipschitz continuous w.r.t. to z and ψ. Under the condition that the data ξ and $f(t, 0, 0, 0)$ are in L^p, $p > 1$, we provide existence and uniqueness results in L^p spaces (the precise defintion will be given in Section 1).

Further contributions are a comparison result in dimension one and existence and uniqueness when the terminal time is a non necessarily bounded stopping time.

Related literature

There are already a lot of works which provide existence and uniqueness results under weaker assumptions than the ones of Pardoux and Peng [33] or El Karoui et al [12]. A huge part of the literature focuses on weakening the Lipschitz property of the coefficient f w.r.t. the y-variable. For example, Briand and Carmona [4] and Pardoux [31] consider the case of a monotonic generator w.r.t. y with different growth conditions. There have been relatively few papers which deal with the problem of existence and uniqueness of solutions in the case where the coefficients are not square integrable. El Karoui et al. [13] and Briand et al. [5] have proved existence and uniqueness of a solution for the standard BSDE (1) in the case where the data belong only to L^p for some $p \geq 1$.

Another strand of research in the theory of BSDEs concerns the underlying filtration. In [33] the filtration is generated by the Brownian motion W. Since the work of Tang and Li [39], a lot of papers (see e.g. [1, 3, 26, 29, 37] or the books of Situ [38] or recently of Delong [9]) treat the case where the filtration is generated by the Brownian motion W and a Poisson random measure π independent of W. In most of these papers, the generator f is supposed to be Lipschitz in y, even if the monotonic case is mentioned (see [37]) and all coefficients are square integrable. Yao [41] studies the L^p case, $p > 1$, and gives existence and uniqueness result in the case where the generator is monotone but with at most linear growth w.r.t. y. Li and Wei [24] give existence und uniqueness results for a fully coupled forward backward SDE under some monotonicity condition and L^p coefficients, $p \geq 2$. Note that this monotonicity condition involves the coefficients of the forward diffusion and is not the same as the assumption imposed on the generator in this paper. An extension to BSDEs driven by a continuous local martingale X and an integer-valued random measure π has been studied by Xia [40]. Xia supposes that the filtration satisfies the representation property with respect to X and π and that the driver is Lipschitz continuous and square integrable.

For more general filtrations, the representation property of a local martingale is no more true (see Section III.4 in [17]) and an additional (orthogonal) martingale term has to be introduced in the definition of a solution. This approach was developed in the seminal work of El Karoui and Huang [11] and by Carbone et al. [6] for càdlàg martingales. The filtration $\mathbb{F}$ is supposed to be complete, right continuous and quasi-left continuous. For a given square integrable martingale X ($\langle X\rangle$ denotes the predictable projection of the quadratic variation), the BSDE (1) becomes

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) d\langle X\rangle_s - \int_t^T Z_s dX_s - M_T + M_t. \tag{3}$$

The solution is now the triple (Y, Z, M) where M is a square integrable martingale orthogonal to X. Øksendal and Zhang [28] analyse BSDE of the form (3) where f does not depend on z, and apply to insider finance (see also Ceci et al. [7]). Liang et al. [25] also obtain results for a particular class of BSDE (3) on an arbitrary filtered probability space. In these papers, existence and uniqueness of the solution of (3) is proved for a Lipschitz continuous function f and under square integrability condition (in [28] the monotone case is treated but f does not depend on z). The Hilbertian structure of $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ is used in Cohen and Elliott [8] (see also [19]). If $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ is a separable Hilbert space, then an orthogonal basis of martingales can be introduced instead of X and there is no orthogonal additional term M in (3). Z becomes a sequence of predictable processes. The special case of a Lévy noise is treated before by Nualart and Schoutens [27]: the orthogonal basis of martingales is explicitely given by the Teugels martingales.

In very recent papers, Klimsiak has developed the results concerning BSDEs in this general framework in two directions. First for reflected BSDE ([18, 19]), and secondly for parabolic equations ([20, 21]) with measure data.

Main contributions

Let us outline the main contributions of our paper compared to the existing literature.

First of all our paper generalizes many results from the works [1, 3, 26, 29, 37, 39] dealing with a filtration generated by the Brownian motion and the Poisson random measure since we allow for a more general filtration.

Moreover we provide existence and uniqueness of solutions in L^p-spaces, $p > 1$. In the case where the generator depends on the stochastic integrand w.r.t. a Poisson random measure, the case when $p < 2$ has to be handled carefully and can not be treated as in [5]. Indeed in this case Burkholder-Davis-Gundy inequality with $p/2 < 1$ does not apply and the $L^{p/2}$-norm of the predictable projection cannot be controlled by the $L^{p/2}$-norm of the quadratic variation (see Inequality (25) and [23]). Yao [41] obtains similar results but for a generator with at most linear growth w.r.t. y (and for a filtration generated by W and π). Klimsiak [19] considers L^p solutions of BSDE, with $p \neq 2$, in a general filtration but where the driver only depends on y.

Compared to [6] or [40], our assumptions are in some sense more restrictive as we assume that the continuous part of the given martingale X of BSDE (3) is a Brownian motion W and the random measure associated to the jumps of X is a Poisson random measure π.

However we weaken the assumptions on the driver and on the terminal condition: the generator is only supposed to be monotone and the terminal condition is allowed to be only L^p-integrable. To the best of our knowledge, there is no existence and uniquenesss result for multi-dimensional BSDE with L^p coefficients in a general filtration. The generalization of our results for BSDE of the form (3) requires some sophisticated integrability conditions to take account of the predictable projection $\langle X \rangle$ of the quadratic variation of X. Therefore it is left for future research.

Moreover we provide a comparison principle and existence and uniqueness in the case with random terminal time for BSDE of type (2). The proof of the comparison principle generalizes the arguments of [35] to the situation where the filtration is not only generated by Brownian and Poisson noise.

Finally our setting is important for the control problem we study in the paper [22] (see also [14]). The control problem arises in mathematical finance and models the optimal liquidation of a financial position in an illiquid market. In [14] the authors consider the case when the filtration is generated by a Brownian motion and a independent Poisson measure. In [22] we do not impose any condition on the filtration generated by the market (except right-continuity, completeness and quasi-left continuity) and the Poisson random measure represents the limit orders of the trading strategy. The optimal strategy is determined by a solution of a BSDE of the form (2) where the generator depends on the Poisson random measure.

Decomposition of the paper

The paper is decomposed as follows. In the first section, we give the mathematical setting and the main results of this paper. In the second part, we consider square integrable coefficients and we prove existence and uniqueness of the solution. To prove it we mainly follow the scheme of [31] with suitable modifications. In the next part, we extend the result to L^p coeffcients for any $p > 1$. For $p > 2$, the existence is derived from the existence in the L^2 case with the right a priori estimate. For $1 < p < 2$, an extra computation has to be made since the function $x \mapsto |x|^p$ is not smooth in this case. We have to extend Lemma 2.2 in [5] or Proposition 2.1 in [18] to our framework. In the last two sections, we add two extensions: the comparison result in dimension one, and existence and uniqueness when the terminal time is a stopping time. Note that the comparison principle requires an extra condition when the generator depends on the jump part (see the counterexample in [1]). But instead of using Girsanov's theorem to obtain the comparison between two solutions, we generalize the argument of [35], which is less restrictive. This point will be crucial in [22].

1 Settings and main results

Let us consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})$. The filtration is assumed to be complete, right continuous and quasi-left continuous, which means that for every sequence (τ_n) of $\mathbb{F}$ stopping times such that $\tau_n \nearrow \tilde{\tau}$ for some stopping time $\tilde{\tau}$ we have $\bigvee_{n \in \mathbb{N}} \mathcal{F}_{\tau_n} = \mathcal{F}_{\tilde{\tau}}$.. Without loss of generality we suppose that all semimartingales have right

continuous paths with left limits. We assume that $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$ supports a k-dimensional Brownian motion W and a Poisson random measure π with intensity $\mu(du)dt$ on the space $\mathcal{U} \subset \mathbb{R}^m \setminus \{0\}$. The measure μ is σ-finite on $\mathcal{U}$ such that

$$\int_{\mathcal{U}} (1 \wedge |u|^2)\mu(du) < +\infty.$$

The compensated Poisson random measure $\widetilde{\pi}(du, dt) = \pi(du, dt) - \mu(du)dt$ is a martingale w.r.t. the filtration $\mathbb{F}$.

In this paper for a given $T \geq 0$, we denote:

- $\mathcal{P}$: the predictable σ-field on $\Omega \times [0, T]$ and

  $$\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathcal{U})$$

  where $\mathcal{B}(\mathcal{U})$ is the Borelian σ-field on $\mathcal{U}$.

- On $\widetilde{\Omega} = \Omega \times [0, T] \times \mathcal{U}$, a function that is $\widetilde{\mathcal{P}}$-measurable, is called predictable. $G_{loc}(\pi)$ is the set of $\widetilde{\mathcal{P}}$-mesurables functions ψ on $\widetilde{\Omega}$ such that for any $t \geq 0$ a.s.

  $$\int_0^t \int_{\mathcal{U}} (|\psi_s(u)|^2 \wedge |\psi_s(u)|)\mu(du) < +\infty.$$

- $\mathcal{D}$ (resp. $\mathcal{D}(0, T)$): the set of all predictable processes on $\mathbb{R}_+$ (resp. on $[0, T]$). $L^2_{loc}(W)$ is the subspace of $\mathcal{D}$ such that for any $t \geq 0$ a.s.

  $$\int_0^t |Z_s|^2 ds < +\infty.$$

- $\mathcal{M}_{loc}$: the set of càdlàg local martingales orthogonal to W and $\widetilde{\pi}$. If $M \in \mathcal{M}_{loc}$ then

  $$[M, W^i]_t = 0, 1 \leq i \leq k \qquad [M, \widetilde{\pi}(A, .)]_t = 0$$

  for all $A \in \mathcal{B}(\mathcal{U})$. In other words, $\mathbb{E}(\Delta M * \pi|\widetilde{\mathcal{P}}) = 0$, where the product $*$ denotes the integral process (see II.1.5 in [17]). Roughly speaking, the jumps of M and π are independent.

- $\mathcal{M}$ is the subspace of $\mathcal{M}_{loc}$ of martingales.

We refer to [17] (see also [3]) for details on random measures and stochastic integrals. As explained above, the filtration $\mathbb{F}$ supports the Brownian motion W and the Poisson random measure π.

Lemma 1 (Lemma III.4.24 in [17]) *Every local martingale has a decomposition*

$$\int_0^{\cdot} Z_s dW_s + \int_0^{\cdot} \int_{\mathcal{U}} \psi_s(u)\widetilde{\pi}(du, ds) + M$$

where $M \in \mathcal{M}_{loc}$, $Z \in L^2_{loc}(W)$, $\psi \in G_{loc}(\mu)$.

Now to define the solution of our BSDE, let us introduce the following spaces for $p \geq 1$.

- $\mathbb{D}^p(0,T)$ is the space of all adapted càdlàg processes X such that

$$\mathbb{E}\left(\sup_{t\in[0,T]}|X_t|^p\right)<+\infty.$$

  For simplicity, $X_*=\sup_{t\in[0,T]}|X_t|$.

- $\mathbb{H}^p(0,T)$ is the subspace of all processes $X\in\mathcal{D}(0,T)$ such that

$$\mathbb{E}\left[\left(\int_0^T|X_t|^2dt\right)^{p/2}\right]<+\infty.$$

- $\mathbb{M}^p(0,T)$ is the subspace of $\mathcal{M}$ of all martingales such that

$$\mathbb{E}\left[([M]_T)^{p/2}\right]<+\infty.$$

- $\mathbb{L}^p_\pi(0,T)=\mathbb{L}^p_\pi(\Omega\times(0,T)\times\mathcal{U})$: the set of processes $\psi\in G_{loc}(\mu)$ such that

$$\mathbb{E}\left[\left(\int_0^T\int_{\mathcal{U}}|\psi_s(u)|^2\mu(du)ds\right)^{p/2}\right]<+\infty.$$

- $\mathbb{L}^p_\mu=\mathbb{L}^p(\mathcal{U},\mu;\mathbb{R}^d)$: the set of measurable functions $\psi:\mathcal{U}\to\mathbb{R}^d$ such that

$$\|\psi\|^p_{\mathbb{L}^p_\mu}=\int_{\mathcal{U}}|\psi(u)|^p\mu(du)<+\infty.$$

- $\mathcal{T}$: the set of all finite stopping times and $\mathcal{T}_T$ the set of all stopping times with values in $[0,T]$.

- $\mathcal{E}^p(0,T)=\mathbb{D}^p(0,T)\times\mathbb{H}^p(0,T)\times\mathbb{L}^p_\pi(0,T)\times\mathbb{M}^p(0,T)$.

We consider the following BSDE (2):

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s,\psi_s)ds-\int_t^T\int_{\mathcal{U}}\psi_s(u)\widetilde{\pi}(du,ds)-\int_t^T Z_sdW_s-\int_t^T dM_s.$$

Here, the random variable ξ is $\mathcal{F}_T$-measurable with values in $\mathbb{R}^d$ ($d\geq1$) and the generator $f:\Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^{d\times k}\times\mathbb{L}^2_\mu\to\mathbb{R}^d$ is a random function, measurable with respect to $Prog\times\mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(\mathbb{R}^{d\times k})\times\mathcal{B}(\mathbb{L}^2_\mu)$ where $Prog$ denotes the sigma-field of progressive subsets of $\Omega\times[0,T]$.

The unknowns are (Y,Z,ψ,M) such that

- Y is progressively measurable and càdlàg with values in $\mathbb{R}^d$;

- $Z\in L^2_{loc}(W)$, with values in $\mathbb{R}^{d\times k}$;

- $\psi\in G_{loc}(\mu)$ with values in $\mathbb{R}^d$;

- $M\in\mathcal{M}_{loc}$ with values in $\mathbb{R}^d$.

On $\mathbb{R}^d$, $|.|$ denotes the Euclidean norm and $\mathbb{R}^{d\times k}$ is identified with the space of real matrices with d rows and k columns. If $z \in \mathbb{R}^{d\times k}$, we have $|z|^2 = \text{Trace}(zz^*)$. If M is a $\mathbb{R}^d$-valued martingale in $\mathcal{M}$, the bracket process $[M]_t$ is

$$[M]_t = \sum_{i=1}^{d} [M^i]_t,$$

where M^i is the i-th component of the vector M.

Throughout the paper, the following assumptions on the generator f are denoted by $(\mathbf{H_{ex}})$.

(H1) For every $t \in [0,T]$, $z \in \mathbb{R}^{d\times k}$ and every $\psi \in \mathbb{L}^2_\mu$ the mapping $y \in \mathbb{R}^d \mapsto f(t,y,z,\psi)$ is continuous. Moreover there exists a constant α such that

$$\langle f(t,y,z,\psi) - f(t,y',z,\psi), y-y'\rangle \leq \alpha |y-y'|^2.$$

(H2) For every $r > 0$ the mapping $(\omega,t) \mapsto \sup_{|y|\leq r} |f(t,y,0,0) - f(t,0,0,0)|$ belongs to $L^1(\Omega\times[0,T],\mathbb{P}\otimes m)$.

(H3) f is Lipschitz continuous w.r.t. z and ψ: there exists a constant K such that for any t and y, for any $z,z' \in \mathbb{R}$ and ψ,ψ' in $\mathbb{L}^2(\mu)$

$$|f(t,y,z,\psi) - f(t,y,z',\psi')| \leq K(|z-z'| + \|\psi-\psi'\|_{\mathbb{L}^2_\mu}).$$

We can suppose w.l.o.g. that $\alpha = 0$. Indeed if (Y,Z,ψ,M) is a solution of (2) then $(\bar{Y},\bar{Z},\bar{\psi},\bar{M})$ with

$$\bar{Y}_t = e^{\alpha t}Y_t, \quad \bar{Z}_t = e^{\alpha t}Z_t, \quad \bar{\psi}_t = e^{\alpha t}\psi_t, \quad d\bar{M}_t = e^{\alpha t}dM_t$$

satisfies an analogous BSDE with terminal condition $\bar{\xi} = e^{\alpha T}\xi$ and generator

$$\bar{f}(t,y,z,\psi) = e^{\alpha t}f(t,e^{-\alpha t}y,e^{-\alpha t}z,e^{-\alpha t}\psi) - \alpha y.$$

$\bar{f}$ satisfies assumptions $(\mathbf{H_{ex}})$ with $\alpha = 0$. Hence in the rest of this paper, we will suppose that $\alpha = 0$.

Our main results can be summarized as follows. Under Assumptions $(\mathbf{H_{ex}})$ and if for some $p > 1$

$$\mathbb{E}\left(|\xi|^p + \int_0^T |f(t,0,0,0)|^p dt\right) < +\infty,$$

there exists a unique solution (Y,Z,ψ,M) in $\mathcal{E}^p(0,T)$ to the BSDE (2). The comparison principle holds for this BSDE. Moreover with a suitable conditions (see (H5') and (H6)) the terminal time T can be replaced by a stopping time τ.

2 L^2 solutions

Let us begin with the following definition:

Definition 1 (L^2-solution) *We say that (Y, Z, ψ, M) is a L^2-solution of the BSDE (ξ, f) on $[0, T]$ if*

- $(Y, Z, \psi, M) \in \mathcal{E}^2(0, T)$;
- *and Equation (2) is satisfied $\mathbb{P} \otimes dt$-a.s.*

In the rest of this section, a solution (Y, Z, ψ, M) will be supposed to be in $\mathcal{E}^2(0, T)$. Now we want to prove existence of the solution of the BSDE with data (ξ, f). For this purpose, we will add the integrability conditions:

(H4) $\mathbb{E}(|\xi|^2) + \mathbb{E}\int_0^T |f(t, 0, 0, 0)|^2 dt < +\infty$.

Some a priori estimates are needed. Note that the following results are modifications of the results obtained in [29].

Lemma 2 *Let $(Y, Z, \psi, M) \in \mathcal{E}^2(0, T)$ be a solution of BSDE (2). Then*

$$\mathbb{E}\left(\int_0^T |Z_s|^2 ds + \int_0^T \int_{\mathcal{U}} |\psi_s(u)|^2 \mu(du) ds + [M]_T\right) \leq C\mathbb{E}\left(\sup_{t\in[0,T]} |Y_t|^2 + \int_0^T |f(t, 0, 0, 0)|^2 dt\right)$$

for some constant C depending only on K^2 and T.

Proof. Let $\tau \in \mathcal{T}_T$ and by Itô's formula on $|Y_t|^2$:

$$\begin{aligned}
&|Y_{\tau\wedge t}|^2 + \int_{\tau\wedge t}^{\tau} |Z_s|^2 ds + \int_{\tau\wedge t}^{\tau} \int_{\mathcal{U}} |\psi_s(u)|^2 \mu(du) ds + [M]_\tau - [M]_{\tau\wedge t} \\
&\quad = |Y_\tau|^2 + 2\int_{\tau\wedge t}^{\tau} Y_s f(s, Y_s, Z_s, \psi_s) ds - 2\int_{\tau\wedge t}^{\tau} Y_{s^-} Z_s dW_s \\
&\qquad -2\int_{\tau\wedge t}^{\tau} Y_{s^-} dM_s - \int_{\tau\wedge t}^{\tau} \int_{\mathcal{U}} \left(|Y_{s^-} + \psi_s(u)|^2 - |Y_{s^-}|^2\right) \widetilde{\pi}(du, ds). \qquad (4)
\end{aligned}$$

But from (H1) and (H3):

$$yf(t, y, z, \psi) \leq |y| \left(K|z| + K\|\psi\|_{\mathbb{L}^2_\mu} + |f(t, 0, 0, 0)|\right).$$

Hence with $t = 0$ and Young's inequality:

$$\begin{aligned}
&\frac{1}{2}\int_0^\tau |Z_s|^2 ds + \frac{1}{2}\int_0^\tau \int_{\mathcal{U}} |\psi_s(u)|^2 \mu(du) ds + [M]_\tau \\
&\quad \leq ((4K^2 + 1)T + 1) \sup_{t\in[0,T]} |Y_t|^2 + \int_0^T |f(s, 0, 0, 0)|^2 ds - 2\int_{\tau\wedge t}^{\tau} Y_{s^-} Z_s dW_s \\
&\qquad -2\int_0^\tau Y_{s^-} dM_s - \int_0^\tau \int_{\mathcal{U}} \left(|Y_{s^-} + \psi_s(u)|^2 - |Y_{s^-}|^2\right) \widetilde{\pi}(du, ds).
\end{aligned}$$

Moreover from the assumptions on Y, Z, ψ and M, the stochastic integral terms w.r.t. W, M and $\widetilde{\pi}$ are martingales. Now take $\tau = T$ and we can take the expectation on both parts:

$$\mathbb{E}\left(\frac{1}{2}\int_0^T |Z_s|^2 ds + \frac{1}{2}\int_0^T \int_{\mathcal{U}} |\psi_s(u)|^2 \mu(du)ds + [M]_T\right)$$
$$\leq (4K^2+2)T\mathbb{E}\left(\sup_{t\in[0,T]} |Y_t|^2\right) + \mathbb{E}\int_0^T |f(s,0,0,0)|^2 ds$$

which achieves the proof. □

Lemma 3 *Let (Y,Z,ψ,M) be a solution of BSDE (ξ,f) with the same conditions as in Lemma 2. Then*

$$\mathbb{E}\left(\sup_{t\in[0,T]} |Y_t|^2\right) \leq C\mathbb{E}\left(|\xi|^2 + \int_0^T |f(t,0,0,0)|^2 dt\right)$$

for some constant C depending only on K and T.

Proof. For $0 \leq t \leq T$, let us apply Itô formula (4) to $e^{\beta s}|Y_s|^2$ where β will be chosen later. We have:

$$\begin{aligned}
e^{\beta t}|Y_t|^2 &= e^{\beta T}|Y_T|^2 + 2\int_t^T e^{\beta s} Y_s f(s,Y_s,Z_s,\psi_s)ds \\
&- \int_t^T \beta e^{\beta s}|Y_s|^2 ds - \int_t^T \int_{\mathcal{U}} e^{\beta s}|\psi_s(u)|^2 \mu(du)ds \\
&- \int_t^T e^{\beta s} d[M]_s - \int_t^T e^{\beta s}|Z_s|^2 ds \\
&- 2\int_t^T e^{\beta s} Y_{s^-} dM_s - 2\int_t^T e^{\beta s} Y_{s^-} Z_s dW_s \\
&- \int_t^T e^{\beta s}\int_{\mathcal{U}} \left(|Y_{s^-} + \psi_s(u)|^2 - |Y_{s^-}|^2\right)\widetilde{\pi}(du,ds).
\end{aligned}$$

From the assumptions on f, we have for any $\varepsilon > 0$

$$\begin{aligned}
Y_s f(s,Y_s,Z_s,\psi_s) &= Y_s(f(s,Y_s,Z_s,\psi_s) - f(s,0,Z_s,\psi_s)) + Y_s(f(s,0,Z_s,\psi_s) - f(s,0,0,\psi_s)) \\
&+ Y_s(f(s,0,0,\psi_s) - f(s,0,0,0)) + Y_s f(s,0,0,0) \\
&\leq |Y_s||f(s,0,0,0)| + K|Y_s|\|\psi_s\|_{\mathbb{L}^2_\mu} + K|Y_s||Z_s| \\
&\leq \frac{(1+2K^2)}{2\varepsilon}|Y_s|^2 + \frac{\varepsilon}{2}(|f(s,0,0,0)|^2 + \|\psi_s\|^2_{\mathbb{L}^2_\mu} + |Z_s|^2). \qquad (5)
\end{aligned}$$

We take $\varepsilon = 1/2$ and we obtain:

$$\begin{aligned}
&e^{\beta t}|Y_t|^2 + \frac{1}{2}\int_t^T e^{\beta s}|Z_s|^2 ds + \frac{1}{2}\int_t^T e^{\beta s}\|\psi_s\|^2_{\mathbb{L}^2_\mu} ds + \frac{1}{2}\int_t^T e^{\beta s} d[M]_s \\
&\leq e^{\beta T}|Y_T|^2 + \frac{1}{2}\int_t^T e^{\beta s}|f(s,0,0,0)|^2 ds \\
&\quad - \int_t^T \left(\beta - 2(1+2K^2)\right) e^{\beta s}|Y_s|^2 ds + \Gamma_{t,T}
\end{aligned}$$

where $\Gamma_{t,\cdot}$ is a local martingale starting at zero at time t. Fix $\beta = 2(1+2K^2)$ and we have:

$$e^{\beta t}|Y_t|^2 + \frac{1}{2}\int_t^T e^{\beta s}\|\psi_s\|^2_{\mathbb{L}^2_\mu} ds + \frac{1}{2}\int_t^T e^{\beta s} d[M]_s$$
$$\leq e^{\beta T}|Y_T|^2 + \frac{1}{2}\int_t^T e^{\beta s}|f(s,0,0,0)|^2 ds + \Gamma_{t,T}. \tag{6}$$

Since all local martingales are true martingales, we deduce that

$$\sup_{t\in[0,T]} \mathbb{E}|Y_t|^2 + \mathbb{E}\int_0^T |Z_s|^2 ds + \mathbb{E}\int_0^T \|\psi_s\|^2_{\mathbb{L}^2_\mu} ds + \mathbb{E}[M]_T$$
$$\leq C\mathbb{E}\left(|\xi|^2 + \int_0^T |f(t,0,0,0)|^2 dt\right). \tag{7}$$

Now with $\beta = 2(1+2K^2)$ we write the Itô formula in a different way:

$$\begin{aligned} e^{\beta t}|Y_t|^2 &= e^{\beta T}|Y_T|^2 + 2\int_t^T e^{\beta s} Y_s f(s, Y_s, Z_s, \psi_s) ds \\ &- \int_t^T \beta e^{\beta s}|Y_s|^2 ds - \int_t^T \int_{\mathcal{U}} e^{\beta s}|\psi_s(u)|^2 \pi(du, ds) \\ &- \int_t^T e^{\beta s} d[M]_s - \int_t^T e^{\beta s}|Z_s|^2 ds \\ &- 2\int_t^T e^{\beta s} Y_{s^-} dM_s - 2\int_t^T e^{\beta s} Y_{s^-} Z_s dW_s \\ &- 2\int_t^T e^{\beta s}\int_{\mathcal{U}} Y_{s^-}\psi_s(u)\widetilde{\pi}(du, ds) \end{aligned}$$

which gives with (5):

$$e^{\beta t}|Y_t|^2 \leq e^{\beta T}|Y_T|^2 + \frac{1}{2}\int_t^T e^{\beta s}|f(s,0,0,0)|^2 ds + \frac{1}{4}\int_t^T e^{\beta s}\|\psi_s\|^2_{\mathbb{L}^2_\mu} ds$$
$$-2\int_t^T e^{\beta s} Y_{s^-} dM_s - 2\int_t^T e^{\beta s} Y_{s^-} Z_s dW_s - 2\int_t^T e^{\beta s}\int_{\mathcal{U}} Y_{s^-}\psi_s(u)\widetilde{\pi}(du, ds) \tag{8}$$

Next we apply the Burkholder-Davis-Gundy inequality to obtain

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t e^{\beta s} Y_{s^-} dM_s\right| \leq c\mathbb{E}\left(\int_0^T e^{2\beta s}|Y_{s^-}|^2 d[M]_s\right)^{1/2}$$
$$\leq \frac{1}{8}\mathbb{E}\left(\sup_{t\in[0,T]} e^{\beta t}|Y_t|^2\right) + 2c^2\mathbb{E}\int_0^T e^{\beta s} d[M]_s. \tag{9}$$

By the same arguments we have

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t e^{\beta s} Y_{s^-} Z_s dW_s\right| \leq c\mathbb{E}\left(\int_0^T e^{2\beta s}|Y_{s^-}|^2|Z_s|^2 ds\right)^{1/2}$$
$$\leq \frac{1}{8}\mathbb{E}\left(\sup_{t\in[0,T]} e^{\beta t}|Y_t|^2\right) + 2c^2\mathbb{E}\int_0^T e^{\beta s}|Z_s|^2 ds. \tag{10}$$

Finally the same result holds for the martingale

$$\int_0^{\cdot}\int_{\mathcal{U}} (Y_{s^-}\psi_s(u))\,\widetilde{\pi}(du,ds),$$

with

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t e^{\beta s}\int_{\mathcal{U}}(Y_{s^-}\psi_s(u))\,\widetilde{\pi}(du,ds)\right| \leq c\mathbb{E}\left(\int_0^T e^{2\beta s}|Y_{s^-}|^2\int_{\mathcal{U}}|\psi_s(u)|^2\pi(du,ds)\right)^{1/2}$$
$$\leq \frac{1}{8}\mathbb{E}\left(\sup_{t\in[0,T]} e^{\beta t}|Y_t|^2\right) + 2c^2\mathbb{E}\int_0^T e^{\beta s}\|\psi_s\|^2_{\mathbb{L}^2_{\mu}}ds. \tag{11}$$

Now coming back to (8), and using estimates (9), (10), (11) and (7), we deduce that there exists C depending on K and T such that

$$\mathbb{E}\left(\sup_{t\in[0,T]}|Y_t|^2\right) \quad\leq\quad C\mathbb{E}\left(|\xi|^2+\int_0^T|f(t,0,0,0)|^2dt\right).$$

This achieves the proof. □

The next result is an extension of the Proposition 2.1 in [4]. For convenience let us give the result and the proof.

Lemma 4 *Let $(Y,Z,\psi,M)\in\mathcal{E}^2(0,T)$ be a solution of BSDE (2) with bounded terminal condition ξ and generator $|f(t,0,0,0)|$: there exists a constant κ such that a.s.*

$$\sup_{t\in[0,T]}|f(t,0,0,0)|+|\xi|\leq\kappa. \tag{12}$$

Then Y is also almost surely bounded: there exists a constant $\beta=2(1+2K^2)$ such that almost surely and for any $t\in[0,T]$

$$|Y_t|^2\leq\kappa^2e^{\beta(T-t)}\left(1+\frac{1}{2\beta}\right).$$

Proof. We use Inequality (6) and since the involved local martingale Γ is a martingale, taking the conditional expectation w.r.t. $\mathcal{F}_t$ leads to: a.s. for every $t\in[0,T]$

$$|Y_t|^2\leq\mathbb{E}\left[e^{\beta(T-t)}|\xi|^2+\frac{1}{2}\int_t^T e^{\beta(s-t)}|f(s,0,0,0)|^2ds\middle|\mathcal{F}_t\right].$$

Hence $Y\in\mathcal{D}^{\infty}(0,T)$. □

Now we prove a stability result.

Lemma 5 *Let now (ξ,f) and (ξ',f') be two sets of data each satisfying the above assumptions $(\mathbf{H_{ex}})$ and (H4). Let (Y,Z,ψ,M) (resp. (Y',Z',ψ',M')) denote a L^2-solution of the BSDE (2) with data (ξ,f) (resp. (ξ',f')). Define*

$$(\widehat{Y},\widehat{Z},\widehat{\psi},\widehat{M},\widehat{\xi},\widehat{f})=(Y-Y',Z-Z',\psi-\psi',M-M',\xi-\xi',f-f').$$

Then there exists a constant C depending on K^2 and T, such that

$$\mathbb{E}\left(\sup_{t\in[0,T]}|\widehat{Y}_t|^2+\int_0^T|\widehat{Z}_s|^2ds+\int_0^T\int_{\mathcal{U}}|\widehat{\psi}_s(u)|^2\mu(du)ds+[\widehat{M}]_T\right)$$
$$\leq C\mathbb{E}\left(|\widehat{\xi}|^2+\int_0^T|\widehat{f}(t,Y'_t,Z'_t,\psi'_t)|^2dt\right).$$

As a consequence of this lemma, we obtain uniqueness of the solution (Y,Z,ψ,M) for the BSDE (2) in the set $\mathcal{E}^2(0,T)$ (see also Corollary 2 in dimension $d=1$).

Proof. Let $\tau\in\mathcal{T}_T$ and by Itô's formula on $|\widehat{Y}_t|^2$:

$$\begin{aligned}&|\widehat{Y}_{\tau\wedge t}|^2+\int_{\tau\wedge t}^{\tau}|\widehat{Z}_s|^2ds+\int_{\tau\wedge t}^{\tau}\int_{\mathcal{U}}|\widehat{\psi}_s(u)|^2\mu(du)ds+[\widehat{M}]_\tau-[\widehat{M}]_{\tau\wedge t}\\&=|\widehat{Y}_\tau|^2+2\int_{\tau\wedge t}^{\tau}\widehat{Y}_s(f(s,Y_s,Z_s,\psi_s)-f'(s,Y'_s,Z'_s,\psi'_s))ds-2\int_{\tau\wedge t}^{\tau}\widehat{Y}_{s^-}\widehat{Z}_sdW_s\\&\quad-2\int_{\tau\wedge t}^{\tau}\widehat{Y}_{s^-}d\widehat{M}_s-\int_{\tau\wedge t}^{\tau}\int_{\mathcal{U}}\left(|\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|^2-|\widehat{Y}_{s^-}|^2\right)\widetilde{\pi}(du,ds).\end{aligned}$$

From the monotonicity assumption on the generator and Young's inequality, we have:

$$\begin{aligned}&|\widehat{Y}_{\tau\wedge t}|^2+\frac{1}{2}\int_{\tau\wedge t}^{\tau}|\widehat{Z}_s|^2ds+\frac{1}{2}\int_{\tau\wedge t}^{\tau}\int_{\mathcal{U}}|\widehat{\psi}_s(u)|^2\mu(du)ds+[\widehat{M}]_\tau-[\widehat{M}]_{\tau\wedge t}\\&\leq|\widehat{Y}_\tau|^2+(4K^2+1)\int_{\tau\wedge t}^{\tau}|\widehat{Y}_s|^2ds+\int_{\tau\wedge t}^{\tau}|\widehat{f}(s,Y'_s,Z'_s,\psi'_s)|^2ds-2\int_{\tau\wedge t}^{\tau}\widehat{Y}_{s^-}\widehat{Z}_sdW_s\\&\quad-2\int_{\tau\wedge t}^{\tau}\widehat{Y}_{s^-}d\widehat{M}_s-\int_{\tau\wedge t}^{\tau}\int_{\mathcal{U}}\left(|\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|^2-|\widehat{Y}_{s^-}|^2\right)\widetilde{\pi}(du,ds).\end{aligned}\tag{13}$$

With $\tau=T$ and Gronwall's lemma, we have for any $t\in[0,T]$

$$\mathbb{E}|\widehat{Y}_t|^2\leq C\mathbb{E}\left(|\widehat{\xi}|^2+\int_0^T|\widehat{f}(s,Y'_s,Z'_s,\psi'_s)|^2ds\right).$$

Then using (13) with $t=0$ and $\tau=T$ and the previous inequality we obtain

$$\mathbb{E}\left(\int_0^T|\widehat{Z}_s|^2ds+\int_0^T\int_{\mathcal{U}}|\widehat{\psi}_s(u)|^2\mu(du)ds+[\widehat{M}]_T\right)\leq C\mathbb{E}\left(|\widehat{\xi}|^2+\int_0^T|\widehat{f}(s,Y'_s,Z'_s,\psi'_s)|^2ds\right).$$

Finally take the conditional expectation w.r.t. $\mathcal{F}_t$ in (13), the supremum over $t\in[0,T]$ on both sides and applying Doob's inequality to the supremum of the $(\mathcal{F}_{\tau\wedge t},\ t\in[0,T])$ martingale on the right-hand side, we have:

$$\mathbb{E}\left(\sup_{t\in[0,T]}|\widehat{Y}_t|^2\right)\leq C\mathbb{E}\left(|\widehat{\xi}|^2+\int_0^T|\widehat{f}(s,Y'_s,Z'_s,\psi'_s)|^2ds\right).$$

This completes the proof. □

Let us modify a little the growth assumption (H2):

(H2') For every $(t,y)\in[0,T]\times\mathbb{R}^d$, $|f(t,y,0,0)|\leq|f(t,0,0,0)|+\vartheta(|y|)$ where $\vartheta:\mathbb{R}_+\to\mathbb{R}_+$ is a deterministic continuous increasing function.

Now we can prove the following result.

Proposition 1 *Under assumptions (H1)-(H2')-(H3) and (H4), there exists a unique L^2-solution (Y, Z, ψ, M) for the BSDE (2).*

Proof. The proof follows closely the arguments in [20] and [29] (see also [6] or [40] for the Lipschitz case). Therefore we only sketch it.

- **Step 1:** we assume that f is Lipschitz with w.r.t. y: there exists a constant K' such that for all (t, y, y', ψ)

$$|f(t, y, z, \psi) - f(t, y', z, \psi)| \leq K'|y - y'|. \tag{14}$$

Moreover ξ and $f(t, 0, 0, 0)$ satisfy the condition (12).

Under these assumptions, for (U, V, ϕ, N) in $\mathcal{E}^2(0, T)$, we define the following processes (Y, Z, ψ, M) as follows:

$$Y_t = \mathbb{E}\left[\xi + \int_0^T f(s, U_s, V_s, \phi_s)ds \middle| \mathcal{F}_t\right] - \int_0^t f(s, U_s, V_s, \phi_s)ds,$$

and the local martingale

$$\mathbb{E}\left[\xi + \int_0^T f(s, U_s, V_s, \phi_s)ds \middle| \mathcal{F}_t\right] - Y_0$$

can be decomposed in three parts (see Lemma 1):

$$\mathbb{E}\left[\xi + \int_0^T f(s, U_s, V_s, \phi_s)ds \middle| \mathcal{F}_t\right] - Y_0 = \int_0^t Z_s dW_s + \int_0^t \psi_s(u)\widetilde{\pi}(du, ds) + M_t$$

where $Z \in L^2_{loc}(W)$ a.s., $\psi \in G_{loc}(\pi)$ and $M \in \mathcal{M}_{loc}$. From the conditions imposed on f and ξ, it is straightforward to prove that $(Y, Z, \psi, M) \in \mathcal{E}^2(0, T)$. Moreover (Y, Z, ψ, M) is the unique solution of the BSDE

$$Y_t = \xi + \int_t^T f(s, U_s, V_s, \phi_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathcal{U}} \psi_s(u)\widetilde{\pi}(du, ds) - \int_t^T dM_s.$$

Therefore we may define the mapping $\Xi : \mathcal{E}^2(0, T) \to \mathcal{E}^2(0, T)$ by putting

$$\Xi((U, V, \phi, N)) = (Y, Z, \psi, M).$$

By standard arguments (see e.g. the proof of Theorem 55.1 in [29]) we can prove that Ξ is contractive on the Banach space $(\mathcal{E}^2(0, T), \|.\|_\beta)$ where

$$\begin{aligned} \|(Y, Z, \psi, M)\|_\beta &= \mathbb{E}\left\{ \sup_{0 \leq t \leq T} e^{\beta t}|Y_t|^2 + \int_0^T e^{\beta t}|Z_s|^2 ds \right. \\ &\qquad \left. + \int_0^T e^{\beta t} \int_{\mathcal{U}} |\psi_t(u)|^2 \mu(du)dt + \left[\int_0^\cdot e^{\beta t} dM_t\right]_T \right\}, \end{aligned}$$

with suitable constant $\beta > 0$. Consequently, Ξ has a fixed point $(Y, Z, \psi, M) \in \mathcal{E}^2(0, T)$. Therefore, (Y, Z, ψ, M) is the unique solution of the BSDE (2).

- **Step 2:** We now show how to dispense with the assumptions (14) and (12). The main result is the following.

  Lemma 6 *Under assumptions* $(\mathbf{H_{ex}})$ *and (H4), given* $(V,\phi) \in \mathbb{H}^2(0,T) \times L^2_\pi(0,T)$ *there exists a unique process* (Y,Z,ψ,M) *in* $\mathcal{E}^2(0,T)$ *such that*

  $$Y_t = \xi + \int_t^T f(s,Y_s,V_s,\phi_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathcal{U}} \psi_s(u)\widetilde{\pi}(du,ds) - \int_t^T dM_s. \quad (15)$$

  The process $f(s,y,V_s,\phi_s)$ will be denoted by $f(s,y)$.

  First we keep the boundness condition (12) and we construct of smooth approximations $(f_n,\ n \in \mathbb{N})$ of f (see proof of Proposition 2.4 in [30]). For any n, f_n is smooth and monotone in y, and thus locally Lipschitz in y. We cannot directly apply the Step 1 since f_n is not necessarily globally Lipschitz. But we just add a truncation function q_p in f_n:

  $$f_{n,p}(t,y) = f_n(t,q_p(y)), \qquad q_p(y) = py/(|y| \vee p).$$

  From the first step there exists a solution $(Y^{n,p},Z^{n,p},\psi^{n,p},M^{n,p})$ to BSDE (15) with generator $f_{n,p}$. Moreover from Lemma 4, the sequence $Y^{n,p}$ is bounded since assumption (12) holds and the upper bound on $Y^{n,p}$ does not depend on p. Thus for p large enough, $Y^{n,p}$ does not depend on p, and is denoted Y^n with the same on $(Z^{n,p},\psi^{n,p},M^{n,p}) = (Z^n,\psi^n,M^n)$. Now the sequence f_n satisfies the assumptions of the lemmas 2 and 3 with constant independent of n. Thus the sequence (Y^n,U^n,Z^n,ψ^n,M^n) is bounded:

  $$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\int_0^T \left(|Y_s^n|^2 + |U_s^n|^2 + |Z_s^n|^2 + \int_{\mathcal{U}} |\psi_s^n(u)|^2 \mu(du)\right) ds + [M^n]_T\right] \leq C$$

  where $U_t^n = f_n(t,Y_t^n)$. Therefore there exists a subsequence which converges weakly to (Y,U,Z,ψ,M). We still denote by (Y^n,U^n,Z^n,ψ^n,M^n) this subsequence. The Brownian martingale $\int_\cdot^T Z_s^n dW_s$ converges weakly in $L^2(\Omega\times[0,T])$ to $\int_\cdot^T Z_s dW_s$ (see [30]). The same trick can be applied on the Poisson martingale $\int_\cdot^T \int_{\mathcal{U}} \psi_s^n(u)\widetilde{\pi}(du,ds)$ and the orthogonal martingale M^n. Finally we identify U_t and $f(t,Y_t)$ in the same way as in [30].

  Finally we remove the condition (12) by a truncation procedure. Once again we obtain a sequence (Y^n,Z^n,ψ^n,M^n) which converges in $\mathcal{E}^2(0,T)$ to the solution (Y,Z,ψ,M) using Lemma 5 (see also proof of Proposition 2).

- **Step 3:** Using the previous lemma, then we have a mapping $\Xi : \mathcal{E}^2(0,T) \to \mathcal{E}^2(0,T)$ which to $(U,V,\phi,N) \in \mathcal{E}^2(0,T)$ associates the solution $(Y,Z,\psi,M) \in \mathcal{E}^2(0,T)$ of BSDE (15), and once again it is a contractive mapping with the norm $\|.\|_\beta$ with suitable β (same computations as in the proof of Theorem 2.2 in [31]). Hence it has a fixed point (Y,Z,ψ,M), solution of the BSDE (2).

□

Now we are able to give the main result of this part.

Theorem 1 *Under assumptions* $(\mathbf{H_{ex}})$ *and (H4), there exists a unique* L^2*-solution* (Y, Z, ψ, M) *for the BSDE* (2).

Proof. In Proposition 1 the condition (H2) was replaced by (H2'). To obtain the above result we follow the arguments of the proof of Theorem 4.2 in [5] where f is approximated by a sequence of functions f_n satisfying (H2') (and the other conditions). Indeed we first assume that ξ and $f(t,0,0,0)$ are bounded. We can construct two sequences f_n and h_n satisfying $(\mathbf{H_{ex}})$ as in [5]. To be more precise, let

- θ_r be a smooth function such that $0 \leq \theta_r \leq 1$, $\theta_r(y) = 1$ if $|y| \leq r$ and $\theta_r(y) = 0$ as soon as $|y| \geq r+1$;
- $\rho_r(t) = \sup_{|y|\leq r} |f(t,y,0,0) - f(t,0,0,0)|$;
- for each $n \in \mathbb{N}^*$, $q_n(z) = zn/(|z| \vee n)$.

Then

$$\begin{aligned} f_n(t,y,z,\psi) &= [f(t,y,q_n(z),q_n(\psi)) - f(t,0,0,0)]\,\frac{n}{\rho_{r+1}(t)\vee n} + f(t,0,0,0), \\ h_n(t,y,z,\psi) &= \theta_r(y)\,[f(t,y,q_n(z),q_n(\psi)) - f(t,0,0,0)]\,\frac{n}{\rho_{r+1}(t)\vee n} + f(t,0,0,0). \end{aligned}$$

Note that we also truncate the part on ψ in f_n and h_n truncates f_n for $|y| \geq r+1$.

For fixed r and n, h_n satisfies the conditions of Proposition 1. Hence there exists a unique solution (Y^n, Z^n, ψ^n, M^n) in $\mathcal{E}^2(0,T)$ with generator h_n and from Lemma 4, Y^n satisfies the inequality $\|Y^n\|_\infty \leq r$. Lemma 2 shows that

$$\mathbb{E}\left(\int_0^T |Z_s^n|^2 ds + \int_0^T \int_{\mathcal{U}} |\psi_s^n(u)|^2 \mu(du)ds + [M^n]_T\right) \leq r'. \tag{16}$$

Therefore if we have chosen r large enough, (Y^n, Z^n, ψ^n, M^m) is solution of the BSDE (2) with generator f_n satisfying $(\mathbf{H_{ex}})$. By Itô's formula on $U = Y^{n+i} - Y^n$, $V = Z^{n+i} - Z^n$, $\phi = \psi^{n+i} - \psi^n$, $N = M^{n+i} - M^n$:

$$\begin{aligned} &e^{4K^2t}|U_t|^2 + \frac{1}{2}\int_t^T e^{4K^2s}|V_s|^2 ds + \frac{1}{2}\int_t^T \int_{\mathcal{U}} e^{4K^2s}|\phi_s(u)|^2\mu(du)ds + \int_t^T e^{4K^2s} d[N]_s \\ &\leq 2\int_t^T e^{4K^2s} U_s(f_{n+i}(s,Y_s^n,Z_s^n,\psi_s^n) - f_n(s,Y_s^n,Z_s^n,\psi_s^n))ds - 2\int_t^T e^{4K^2s}U_sV_sdW_s \\ &\quad -2\int_t^T e^{4K^2s}U_{s^-}dN_s - \int_t^T e^{4K^2s}\int_{\mathcal{U}} \left(|U_{s^-}+\phi_s(u)|^2 - |U_{s^-}|^2\right)\widetilde{\pi}(du,ds) \\ &\leq 4r\int_t^T e^{4K^2s}|f_{n+i}(s,Y_s^n,Z_s^n,\psi_s^n) - f_n(s,Y_s^n,Z_s^n,\psi_s^n)|ds - 2\int_t^T e^{4K^2s}U_sV_sdW_s \\ &\quad -2\int_t^T e^{4K^2s}U_{s^-}dN_s - \int_t^T e^{4K^2s}\int_{\mathcal{U}} \left(|U_{s^-}+\phi_s(u)|^2 - |U_{s^-}|^2\right)\widetilde{\pi}(du,ds) \end{aligned}$$

since $\|U\|_\infty \leq 2r$. Using the Burkholder-Davis-Gundy (BDG) inequality we get for some constant depending on K and T:

$$\mathbb{E}\left(\sup_{t\in[0,T]}|U_t|^2 + \int_0^T |V_s|^2 ds + \int_0^T\int_{\mathcal{U}} |\phi_s(u)|^2\mu(du)ds + [N]_T\right)$$
$$\leq Cr\mathbb{E}\int_0^T |f_{n+i}(s,Y_s^n,Z_s^n,\psi_s^n) - f_n(s,Y_s^n,Z_s^n,\psi_s^n)|ds.$$

Since $\|Y^n\|_\infty \leq r$, from the definition of f_n we have

$$\begin{aligned}
&|f_{n+i}(s,Y_s^n,Z_s^n,\psi_s^n) - f_n(s,Y_s^n,Z_s^n,\psi_s^n)| \\
&\quad\leq 2K|Z_s^n|\mathbf{1}_{|Z_s^n|>n} + 2K\|\psi_s^n\|_{L^2}\mathbf{1}_{\|\psi_s^n\|_{L^2}>n} + 2K|Z_s^n|\mathbf{1}_{\rho_{r+1}(s)>n} \\
&\qquad +2K\|\psi_s^n\|_{L^2}\mathbf{1}_{\rho_{r+1}(s)>n} + 2\rho_{r+1}(s)\mathbf{1}_{\rho_{r+1}(s)>n}.
\end{aligned}$$

Since $\rho_{r+1} \in L^1(\Omega\times[0,T],\mathbb{P}\otimes m)$ (Assumption (H2)), and $(Z^n,\psi^n) \in \mathbb{H}^2(0,T)\times L^2_\mu$ uniformly w.r.t. n (Inequality (16)), this implies that (Y^n,Z^n,ψ^n,M^m) is a Cauchy sequence in $\mathcal{E}^2(0,T)$.

The general case will be obtained by a truncation procedure on ξ and $f(t,0,0,0)$ and the inequality of Lemma 5. □

3 Existence in L^p

The following proposition was proved in the Lipschitz case without jumps in [13], Section 5, or in [5] for the Brownian filtration, for any $p > 1$.

Proposition 2 (L^p-estimates, $p \geq 2$) *We assume that f satisfies ($\mathbf{H_{ex}}$). For $p \geq 2$, if we have*

$$\mathbb{E}\left(|\xi|^p + \int_0^T |f(t,0,0,0)|^p dt\right) < +\infty, \tag{H5}$$

then the solution (Y,Z,ψ,M) belongs to $\mathcal{E}^p(0,T)$. Moreover there exists a constant C depending only on K^2, p and T such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t|^p + \left(\int_0^T |Z_s|^2 ds\right)^{p/2} + \left(\int_0^T\int_{\mathcal{U}} |\psi_s(u)|^2\mu(du)ds\right)^{p/2} + [M]_T^{p/2}\right]$$
$$\leq C\mathbb{E}\left(|\xi|^p + \int_0^T |f(s,0,0,0)|^p ds\right). \tag{17}$$

Proof. Under this condition (H5) on ξ and $f(t,0,0,0)$, we know that there exists a unique solution (Y,Z,ψ,M) which belongs to $\mathcal{E}^2(0,T)$. We want to show that (Y,Z,ψ,M) in fact belongs to $\mathcal{E}^p(0,T)$.

From the proof of Theorem 1 (or Proposition 1), the solution (Y,Z,ψ,M) is obtained as the limit of a sequence (Y^n,Z^n,ψ^n,M^n), solution of BSDE (2) but with bounded coefficients ξ^n and $f^n(t,0,0,0)$. We prove that convergence also holds in $\mathcal{E}^p(0,T)$ by proving the counterpart of Lemma 5 in $\mathcal{E}^p$. For any $(m,n) \in \mathbb{N}^2$ we denote

$$(\widehat{Y},\widehat{Z},\widehat{\psi},\widehat{M},\widehat{\xi},\widehat{f}) = (Y^m - Y^n, Z^m - Z^n, \psi^m - \psi^n, M^m - M^n, \xi^m - \xi^n, f^m - f^n).$$

Step 1: we prove that the sequence Y^n converges in $\mathbb{D}^p(0,T)$ to Y. Since $p \geq 2$ we can apply Itô formula with the C^2-function $\theta(y) = |y|^p$ to the process $\widehat{Y}$. Note that

$$\frac{\partial \theta}{\partial y_i}(y) = p y_i |y|^{p-2}, \frac{\partial^2 \theta}{\partial y_i \partial y_j}(y) = p|y|^{p-2}\delta_{i,j} + p(p-2) y_i y_j |y|^{p-4}$$

where $\delta_{i,j}$ is the Kronecker delta. Therefore for every $0 \leq t \leq T$ we have:

$$\begin{aligned}
|\widehat{Y}_t|^p = |\widehat{\xi}|^p &+ \int_t^T p\widehat{Y}_s |\widehat{Y}_s|^{p-2}(f^m(s, Y_s^m, Z_s^m, \psi_s^m) - f^n(s, Y_s^n, Z_s^n, \psi_s^n))ds \\
&- p\int_t^T \widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2} d\widehat{M}_s - p\int_t^T \widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}\widehat{Z}_s dW_s \\
&- p\int_t^T \int_{\mathcal{U}} \left(\widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}\widehat{\psi}_s(u)\right)\widetilde{\pi}(du, ds) - \frac{1}{2}\int_t^T \text{Trace}\left(D^2\theta(\widehat{Y}_s)\widehat{Z}_s\widehat{Z}_s^*\right) ds \\
&- \int_t^T \int_{\mathcal{U}} \left(|\widehat{Y}_{s^-} + \widehat{\psi}_s(u)|^p - |\widehat{Y}_{s^-}|^p - p\widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}\widehat{\psi}_s(u)\right)\pi(du, ds) \\
&- \frac{1}{2}\int_t^T \sum_{1\leq i,j\leq d} \frac{\partial^2\theta}{\partial y_i \partial y_j}(\widehat{Y}_s) d[\widehat{M}^i, \widehat{M}^j]_t^c \\
&- \sum_{t<s\leq T} \left(|\widehat{Y}_{s^-} + \Delta\widehat{M}_s|^p - |\widehat{Y}_{s^-}|^p - p\widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}\Delta\widehat{M}_s\right). \qquad (18)
\end{aligned}$$

The notation $[M]^c$ denotes the continuous part of the bracket process $[M]$. First remark that for a non negative symmetric matrix $\Gamma \in \mathbb{R}^{d\times d}$

$$\sum_{1\leq i,j\leq d} D^2\theta(y)_{i,j}\Gamma_{i,j} = p|y|^{p-2}\text{Trace}(\Gamma) + p(p-2)|y|^{p-4}(y^*)\Gamma y \geq p|y|^{p-2}\text{Trace}(\Gamma),$$

and thus

$$\text{Trace}(D^2\theta(y)zz^*) \geq p|y|^{p-2}|z|^2.$$

Moreover using Taylor formula (and Lemma A.4 in [41] for the last inequality) we have

$$\begin{aligned}
\theta(x+y) - \theta(x) - \nabla\theta(x)y &= \int_0^1 yD^2\theta(x+ry)y(1-r)dr \\
&= p|y|^2\int_0^1 (1-r)|x+ry|^{p-2}dr + p(p-2)\int_0^1 (y(x+ry))^2|x+ry|^{p-4}(1-r)dr \\
&\geq p|y|^2\int_0^1 (1-r)|x+ry|^{p-2}dr \geq p(p-1)3^{1-p}|y|^2|x|^{p-2}.
\end{aligned}$$

Therefore we deduce that

$$\begin{aligned}
&\frac{1}{2}\int_t^T \sum_{1\leq i,j\leq d} \frac{\partial^2\theta}{\partial y_i\partial y_j}(\widehat{Y}_s) d[\widehat{M}^i, \widehat{M}^j]_t^c + \sum_{t<s\leq T}\left(|\widehat{Y}_{s^-} + \Delta\widehat{M}_s|^p - |\widehat{Y}_{s^-}|^p - p\widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}\Delta\widehat{M}_s(z)\right) \\
&\quad \geq \frac{p}{2}\int_t^T |\widehat{Y}_s|^{p-2} d[\widehat{M}]_s^c + p(p-1)3^{1-p}\sum_{t<s\leq T}|\widehat{Y}_s|^{p-2}|\Delta\widehat{M}_s|^2 \geq \kappa_p\int_t^T |\widehat{Y}_s|^{p-2} d[\widehat{M}]_s
\end{aligned}$$

where $\kappa_p = \min(p/2, p(p-1)3^{1-p}) > 0$. Now the Poisson part in (18) can be written as follows:

$$
\begin{aligned}
&-p\int_t^T\int_{\mathcal{U}}\left(\widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}\widehat{\psi}_s(u)\right)\widetilde{\pi}(du,ds)\\
&\quad-\int_t^T\int_{\mathcal{U}}\left(|\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|^p-|\widehat{Y}_{s^-}|^p-p\widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}\widehat{\psi}_s(u)\right)\pi(du,ds)\\
&=-\int_t^T\int_{\mathcal{U}}\left(|\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|^p-|\widehat{Y}_{s^-}|^p-p\widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}\widehat{\psi}_s(u)\right)\mu(du)ds\\
&\quad-\int_t^T\int_{\mathcal{U}}\left(|\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|^p-|\widehat{Y}_{s^-}|^p\right)\widetilde{\pi}(du,ds)\\
&\leq -p(p-1)3^{1-p}\int_t^T|\widehat{Y}_{s^-}|^{p-2}\|\widehat{\psi}_s\|^2_{L^2_\mu}ds-\int_t^T\int_{\mathcal{U}}\left(|\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|^p-|\widehat{Y}_{s^-}|^p\right)\widetilde{\pi}(du,ds).
\end{aligned}
$$

Then (18) becomes

$$
\begin{aligned}
&|\widehat{Y}_t|^p+\kappa_p\int_t^T|\widehat{Y}_s|^{p-2}|\widehat{Z}_s|^2ds+\kappa_p\int_t^T|\widehat{Y}_{s^-}|^{p-2}d[\widehat{M}]_s+\kappa_p\int_t^T|\widehat{Y}_{s^-}|^{p-2}\|\widehat{\psi}_s\|^2_{L^2_\mu}ds\\
&\quad\leq|\widehat{\xi}|^p+\int_t^Tp\widehat{Y}_s|\widehat{Y}_s|^{p-2}(f^m(s,Y^m_s,Z^m_s,\psi^m_s)-f^n(s,Y^n_s,Z^n_s,\psi^n_s))ds\\
&\quad-p\int_t^T\widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}d\widehat{M}_s-p\int_t^T\widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}\widehat{Z}_sdW_s\\
&\quad-\int_t^T\int_{\mathcal{U}}\left(|\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|^p-|\widehat{Y}_{s^-}|^p\right)\widetilde{\pi}(du,ds).
\end{aligned}
$$

From the assumptions on f^m, we still have (5) and we choose $\varepsilon=\frac{\kappa_p}{p}$. We obtain

$$
\begin{aligned}
&|\widehat{Y}_t|^p+\frac{\kappa_p}{2}\int_t^T|\widehat{Y}_s|^{p-2}|\widehat{Z}_s|^2ds+\frac{\kappa_p}{2}\int_t^T|\widehat{Y}_{s^-}|^{p-2}d[\widehat{M}]_s+\frac{\kappa_p}{2}\int_t^T|\widehat{Y}_{s^-}|^{p-2}\|\widehat{\psi}_s\|^2_{L^2_\mu}ds\\
&\quad\leq|\widehat{\xi}|^p+\frac{p^2}{2\kappa_p}(2K^2+1)\int_t^T|\widehat{Y}_s|^pds+\frac{\kappa_p}{2}\int_t^T|\widehat{Y}_s|^{p-2}|\widehat{f}(s,Y^n_s,Z^n_s,\psi^n_s)|^2ds\\
&\quad-p\int_t^T\widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}d\widehat{M}_s-p\int_t^T\widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}\widehat{Z}_sdW_s\\
&\quad-\int_t^T\int_{\mathcal{U}}\left(|\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|^p-|\widehat{Y}_{s^-}|^p\right)\widetilde{\pi}(du,ds).
\end{aligned}
$$

Using Young's inequality, we finally have

$$
\begin{aligned}
&|\widehat{Y}_t|^p+\frac{\kappa_p}{2}\int_t^T|\widehat{Y}_s|^{p-2}|\widehat{Z}_s|^2ds+\frac{\kappa_p}{2}\int_t^T|\widehat{Y}_{s^-}|^{p-2}d[\widehat{M}]_s+\frac{\kappa_p}{2}\int_t^T|\widehat{Y}_{s^-}|^{p-2}\|\widehat{\psi}_s\|^2_{L^2_\mu}ds\\
&\quad\leq|\widehat{\xi}|^p+\left[\frac{p^2}{2\kappa_p}(2K^2+1)+\frac{\kappa_p(p-2)}{2p}\right]\int_t^T|\widehat{Y}_s|^pds+\frac{\kappa_p}{p}\int_t^T|\widehat{f}(s,Y^n_s,Z^n_s,\psi^n_s)|^pds\\
&\quad-p\int_t^T\widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}d\widehat{M}_s-p\int_t^T\widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}\widehat{Z}_sdW_s\\
&\quad-\int_t^T\int_{\mathcal{U}}\left(|\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|^p-|\widehat{Y}_{s^-}|^p\right)\widetilde{\pi}(du,ds).
\end{aligned}
\tag{19}
$$

Note that the three local martingales in the previous inequality are true martingales. Indeed since Y^m and Y^n are in $\mathbb{D}^\infty(0,T)$ and M^m and M^n are in $\mathbb{M}^2(0,T)$, the local martingale

$$\int_0^\cdot \widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}d\widehat{M}_s$$

is a true martingale and we can apply the Burkholder-Davis-Gundy inequality to obtain

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t \widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}d\widehat{M}_s\right| &\leq c_p\mathbb{E}\left(\int_0^T |\widehat{Y}_{s^-}|^{2p-2}d[\widehat{M}]_s\right)^{1/2}\\ &\leq \frac{1}{4p}\mathbb{E}\left(\sup_{t\in[0,T]}|\widehat{Y}_t|^p\right)+pc_p^2\mathbb{E}\int_0^T |\widehat{Y}_{s^-}|^{p-2}d[\widehat{M}]_s. \end{aligned} \tag{20}$$

By the same arguments we have

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t \widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}\widehat{Z}_s dW_s\right| &\leq c_p\mathbb{E}\left(\int_0^T |\widehat{Y}_{s^-}|^{2p-2}|\widehat{Z}_s|^2 ds\right)^{1/2}\\ &\leq \frac{1}{4p}\mathbb{E}\left(\sup_{t\in[0,T]}|\widehat{Y}_t|^p\right)+pc_p^2\mathbb{E}\int_0^T |\widehat{Y}_s|^{p-2}|\widehat{Z}_s|^2 ds. \end{aligned} \tag{21}$$

Finally the same result holds for the martingale

$$\int_0^\cdot\int_{\mathcal{U}}\left(\widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}\widehat{\psi}_s(u)\right)\widetilde{\pi}(du,ds),$$

with

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t\int_{\mathcal{U}}\left(\widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}\widehat{\psi}_s(u)\right)\widetilde{\pi}(du,ds)\right| &\leq c_p\mathbb{E}\left(\int_0^T |\widehat{Y}_{s^-}|^{2p-2}\int_{\mathcal{U}}|\widehat{\psi}_s(u)|^2\pi(du,ds)\right)^{1/2}\\ &\leq \frac{1}{4p}\mathbb{E}\left(\sup_{t\in[0,T]}|\widehat{Y}_t|^p\right)+pc_p^2\mathbb{E}\int_0^T |\widehat{Y}_{s^-}|^{p-2}\|\widehat{\psi}_s\|^2_{L^2_\mu}ds. \end{aligned} \tag{22}$$

Now we come to the conclusion. Using (19) we can take expectations and obtain for every $0\leq t\leq T$:

$$\mathbb{E}|\widehat{Y}_t|^p \leq \mathbb{E}|\widehat{\xi}|^p+\left[\frac{p^2}{2\kappa_p}(2K^2+1)+\frac{\kappa_p(p-2)}{2p}\right]\mathbb{E}\int_t^T|\widehat{Y}_s|^p ds+\frac{\kappa_p}{p}\mathbb{E}\int_t^T|\widehat{f}(s,Y^n_s,Z^n_s,\psi^n_s)|^p ds,$$

hence by Gronwall's lemma

$$\mathbb{E}|\widehat{Y}_t|^p \leq C\mathbb{E}\left(|\widehat{\xi}|^p+\mathbb{E}\int_0^T|\widehat{f}(s,Y^n_s,Z^n_s,\psi^n_s)|^p ds\right)$$

for some constant C depending on K, p and T. From this and (19) again we also deduce that

$$\begin{aligned}&\mathbb{E}\int_0^T|\widehat{Y}_s|^p ds+\mathbb{E}\int_0^T|\widehat{Y}_s|^{p-2}|Z_s|^2 ds+\mathbb{E}\int_0^T|\widehat{Y}_{s^-}|^{p-2}d[\widehat{M}]_s+\mathbb{E}\int_0^T|\widehat{Y}_{s^-}|^{p-2}\|\widehat{\psi}_s\|^2_{L^2_\mu}ds\\ &\qquad\leq C\mathbb{E}\left(|\widehat{\xi}|^p+\mathbb{E}\int_0^T|\widehat{f}(s,Y^n_s,Z^n_s,\psi^n_s)|^p ds\right).\end{aligned}$$

Let us come back to (18) and use the convexity of the function θ and Estimate (5) with $\varepsilon = \kappa_p/p$, to deduce that:

$$
\begin{aligned}
|\widehat{Y}_t|^p \leq |\widehat{\xi}|^p &+ \left[\frac{p^2}{2\kappa_p}(2K^2+1) + \frac{\kappa_p(p-2)}{2p}\right]\int_t^T |\widehat{Y}_s|^p ds + \frac{\kappa_p}{p}\int_t^T |\widehat{f}(s,Y_s^n,Z_s^n,\psi_s^n)|^p ds \\
&+\frac{\kappa_p}{2}\int_t^T |\widehat{Y}_s|^{p-2}|\widehat{Z}_s|^2 ds + \frac{\kappa_p}{2}\int_t^T |\widehat{Y}_{s^-}|^{p-2}\|\widehat{\psi}_s\|^2_{L^2_\mu} ds \\
&-p\int_t^T \widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2} d\widehat{M}_s - p\int_t^T \widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}\widehat{Z}_s dW_s \\
&-p\int_t^T\int_{\mathcal{U}} \left(\widehat{Y}_{s^-}|\widehat{Y}_{s^-}|^{p-2}\widehat{\psi}_s(u)\right)\widetilde{\pi}(du,ds).
\end{aligned}
$$

Now using estimates (20), (21) and (22), we get:

$$
\mathbb{E}\left(\sup_{t\in[0,T]}|\widehat{Y}_t|^p\right) \leq C\mathbb{E}\left(|\widehat{\xi}|^p + \mathbb{E}\int_0^T |\widehat{f}(s,Y_s^n,Z_s^n,\psi_s^n)|^p ds\right).
$$

Therefore the limit process Y belongs to $\mathbb{D}^p(0,T)$.

Step 2: We adopt the arguments of the proof of Lemma 2 (see also Lemma 3.1 in [5]) to prove that:

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_0^T |\widehat{Z}_s|^2 ds\right)^{p/2} + \left(\int_0^T\int_{\mathcal{U}} |\widehat{\psi}_s(u)|^2\mu(du)ds\right)^{p/2} + [\widehat{M}]_T^{p/2}\right] \\
\leq C\mathbb{E}\left[\sup_{t\in[0,T]}|\widehat{Y}_t|^p + \int_0^T |\widehat{f}(s,Y_s^n,Z_s^n,\psi_s^n)|^p ds\right]. \qquad (23)
\end{aligned}
$$

This estimate gives the convergence of (Z^n,ψ^n,M^n) in the desired integrability space. Indeed let $\tau_k \in \mathcal{T}_T$ defined by:

$$
\tau_k = \inf\left\{t\in[0,T], \int_0^t |\widehat{Z}_r|^2 dr + \int_0^t\int_{\mathcal{U}} |\widehat{\psi}_s(u)|^2\pi(du,ds) + [\widehat{M}]_t \geq k\right\}\wedge T.
$$

By Itô's formula on $|\widehat{Y}_t|^2$:

$$
\begin{aligned}
&|\widehat{Y}_0|^2 + \int_0^{\tau_k}|\widehat{Z}_s|^2 ds + \int_0^{\tau_k}\int_{\mathcal{U}}|\widehat{\psi}_s(u)|^2\pi(du,ds) + [\widehat{M}]_{\tau_k} \\
&= |\widehat{Y}_{\tau_k}|^2 + 2\int_0^{\tau_k}\widehat{Y}_s\left(f^m(s,Y_s^m,Z_s^m,\psi_s^m) - f^n(s,Y_s^n,Z_s^n,\psi_s^n)\right)ds \\
&\quad -2\int_0^{\tau_k}\widehat{Y}_{s^-}\widehat{Z}_s dW_s - 2\int_0^{\tau_k}\widehat{Y}_{s^-}d\widehat{M}_s - 2\int_0^{\tau_k}\int_{\mathcal{U}}\widehat{Y}_{s^-}\widehat{\psi}_s(u)\widetilde{\pi}(du,ds).
\end{aligned}
$$

Once again with a straightforward modification of estimate (5):

$$
\begin{aligned}
&\frac{1}{2}\int_0^{\tau_k}|\widehat{Z}_s|^2 ds + \int_0^{\tau_k}\int_{\mathcal{U}}|\widehat{\psi}_s(u)|^2\pi(du,ds) + [\widehat{M}]_{\tau_k} \\
&\leq |\widehat{Y}_*|^2 + \frac{((1+1/\varepsilon)K^2+1)}{2}\int_0^T|\widehat{Y}_s|^2 ds + \frac{1}{2}\int_0^T|\widehat{f}(s,Y_s^n,Z_s^n,\psi_s^n)|^2 ds \\
&\quad +\frac{\varepsilon}{2}\int_0^{\tau_k}\int_{\mathcal{U}}|\widehat{\psi}_s(u)|^2\mu(du)ds \\
&\quad -2\int_0^{\tau_k}\widehat{Y}_{s^-}\widehat{Z}_s dW_s - 2\int_0^{\tau_k}\widehat{Y}_{s^-}d\widehat{M}_s - 2\int_0^{\tau_k}\int_{\mathcal{U}}\widehat{Y}_{s^-}\widehat{\psi}_s(u)\widetilde{\pi}(du,ds)
\end{aligned}
$$

where $\widehat{Y}_* = \sup_{t\in[0,T]} |\widehat{Y}_t|$. It follows that

$$\begin{aligned}
&\left(\int_0^{\tau_k} |\widehat{Z}_s|^2 ds\right)^{p/2} + \left(\int_0^{\tau_k}\int_{\mathcal{U}} |\widehat{\psi}_s(u)|^2 \pi(du,ds)\right)^{p/2} + [\widehat{M}]_{\tau_k}^{p/2} \\
&\leq C_p \left[\left(1 + \frac{((1+1/\varepsilon)K^2+1)T}{2}\right)^{p/2} |\widehat{Y}_*|^p + \left(\int_0^T |\widehat{f}(s, Y_s^n, Z_s^n, \psi_s^n)|^2 ds\right)^{p/2}\right] \\
&\quad + C_p \varepsilon^{p/2} \left(\int_0^{\tau_k}\int_{\mathcal{U}} |\widehat{\psi}_s(u)|^2 \mu(du) ds\right)^{p/2} + C_p \left|\int_0^{\tau_k}\int_{\mathcal{U}} \widehat{Y}_{s^-}\widehat{\psi}_s(u)\widetilde{\pi}(du,ds)\right|^{p/2} \\
&\quad + C_p \left[\left|\int_0^{\tau_k} \widehat{Y}_{s^-}\widehat{Z}_s dW_s\right|^{p/2} + \left|\int_0^{\tau_k} \widehat{Y}_{s^-} d\widehat{M}_s\right|^{p/2}\right]. \qquad (24)
\end{aligned}$$

Since $p/2 \geq 1$, we can apply the Burkholder-Davis-Gundy inequality to obtain

$$\begin{aligned}
C_p\mathbb{E}\left|\int_0^{\tau_k} \widehat{Y}_{s^-} d\widehat{M}_s\right| &\leq d_p\mathbb{E}\left[\left(\int_0^{\tau_k} |\widehat{Y}_{s^-}|^2 d[\widehat{M}]_s\right)^{p/4}\right] \leq \frac{d_p^2}{4}\mathbb{E}\left(|\widehat{Y}_*|^p\right) + \frac{1}{2}[\widehat{M}]_{\tau_k}^{p/2}, \\
C_p\mathbb{E}\left|\int_0^{\tau_k} \widehat{Y}_{s^-}\widehat{Z}_s dW_s\right| &\leq d_p\mathbb{E}\left[\left(\int_0^{\tau_k} |\widehat{Y}_{s^-}|^2 |\widehat{Z}_s|^2 ds\right)^{p/4}\right] \\
&\leq \frac{d_p^2}{4}\mathbb{E}\left(|\widehat{Y}_*|^p\right) + \frac{1}{2}\mathbb{E}\left[\left(\int_0^{\tau_k} |\widehat{Z}_s|^2 ds\right)^{p/2}\right], \\
C_p\mathbb{E}\left|\int_0^{\tau_k}\int_{\mathcal{U}} \widehat{Y}_{s^-}\widehat{\psi}_s(u)\widetilde{\pi}(du,ds)\right| &\leq d_p\mathbb{E}\left[\left(\int_0^{\tau_k} |\widehat{Y}_{s^-}|^2 \int_{\mathcal{U}} |\widehat{\psi}_s(u)|^2 \pi(du,ds)\right)^{p/4}\right] \\
&\leq \frac{d_p^2}{4}\mathbb{E}\left(|\widehat{Y}_*|^p\right) + \frac{1}{2}\mathbb{E}\left[\left(\int_0^{\tau_k} \|\widehat{\psi}_s\|_{L^2_\mu}^2 \pi(du,ds)\right)^{p/2}\right].
\end{aligned}$$

Hence coming back to (24) and taking the expectation

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left(\int_0^{\tau_k} |\widehat{Z}_s|^2 ds\right)^{p/2} + \frac{1}{2}\mathbb{E}\left(\int_0^{\tau_k}\int_{\mathcal{U}} |\widehat{\psi}_s(u)|^2 \pi(du,ds)\right)^{p/2} + \frac{1}{2}\mathbb{E}[\widehat{M}]_{\tau_k}^{p/2} \\
&\leq C_{p,K,T,\varepsilon}\ \mathbb{E}|\widehat{Y}_*|^p + C_p\mathbb{E}\left[\left(\int_0^T |\widehat{f}(s, Y_s^n, Z_s^n, \psi_s^n)|^2 ds\right)^{p/2}\right] \\
&\quad + C_p\varepsilon^{p/2}\mathbb{E}\left(\int_0^{\tau_k}\int_{\mathcal{U}} |\widehat{\psi}_s(u)|^2 \mu(du) ds\right)^{p/2}.
\end{aligned}$$

Finally we use that for some constant $e_p > 0$

$$\mathbb{E}\left(\int_0^{\tau_k}\int_{\mathcal{U}} |\widehat{\psi}_s(u)|^2 \mu(du) ds\right)^{p/2} \leq e_p\mathbb{E}\left(\int_0^{\tau_k}\int_{\mathcal{U}} |\widehat{\psi}_s(u)|^2 \pi(du,ds)\right)^{p/2}, \qquad (25)$$

(see [23, 10]) and thus we can choose ε sufficiently small and depending only on p such that:

$$\begin{aligned}
&\mathbb{E}\left(\int_0^{\tau_k} |\widehat{Z}_s|^2 ds\right)^{p/2} + \mathbb{E}\left(\int_0^{\tau_k}\int_{\mathcal{U}} |\widehat{\psi}_s(u)|^2 \mu(du) ds\right)^{p/2} + \mathbb{E}[\widehat{M}]_{\tau_k}^{p/2} \\
&\leq \widetilde{C}_{p,K,T,\varepsilon}\ \mathbb{E}|\widehat{Y}_*|^p + \widetilde{C}_p\mathbb{E}\left[\left(\int_0^T |\widehat{f}(s, Y_s^n, Z_s^n, \psi_s^n)|^2 ds\right)^{p/2}\right]. \qquad (26)
\end{aligned}$$

We can let k go to $+\infty$ in order to have estimate (23).

Step 3: The inequality (17) can be deduced from the previous steps: we just replace $(Y^m, Z^m, \psi^m, M^m, f^m)$ by (Y, Z, ψ, M, f) and $(Y^n, Z^n, \psi^n, M^n, f)$ by $(0, 0, 0, 0, 0)$. □

Now we consider the case where $p \in [1, 2)$. The main difference is that we cannot directly apply the Itô formula to $\theta(y) = |y|^p$. The next result is an extension of the Meyer-Itô formula and as mentioned in [5], it is likely that this result already appeared somewhere. A version of this result is given in Lemma 2.2 in [5] without jumps or in Proposition 2.1 in [18] in dimension one. We denote by $\check{x} = |x|^{-1}x\mathbf{1}_{x\neq 0}$.

Lemma 7 *We consider the $\mathbb{R}^d$-valued semimartingale $(X_t)_{t\in[0,T]}$ defined by*

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t Z_s dW_s + \int_0^t \int_{\mathcal{U}} \psi_s(u)\widetilde{\pi}(du, ds) + M_t,$$

such that $t \mapsto K_t$ belongs to $L^1_{loc}(0, +\infty)$ a.s., $Z \in L^2_{loc}(W)$, $\psi \in G_{loc}(\pi)$ and $M \in \mathcal{M}_{loc}$. Then for any $p \geq 1$, we have

$$\begin{aligned}
|X_t|^p &= |X_0|^p + \frac{1}{2}L(t)\mathbf{1}_{p=1} + p\int_0^t |X_s|^{p-1}\check{X}_s K_s ds + p\int_0^t |X_s|^{p-1}\check{X}_s Z_s dW_s \\
&+ p\int_0^t |X_{s^-}|^{p-1}\check{X}_{s^-} dM_s + p\int_0^t |X_s|^{p-1}\check{X}_s \int_{\mathcal{U}} \psi_s(u)\widetilde{\pi}(du, ds) \\
&+ \int_0^t \int_{\mathcal{U}} \left[|X_{s^-} + \psi_s(u)|^p - |X_{s^-}|^p - p|X_{s^-}|^{p-1}\check{X}_{s^-}\psi_s(u)\right] \pi(du, ds) \\
&+ \sum_{0<s\leq t} \left[|X_{s^-} + \Delta M_s|^p - |X_{s^-}|^p - p|X_{s^-}|^{p-1}\check{X}_{s^-}\Delta M_s\right] \\
&+ \frac{p}{2}\int_0^t |X_s|^{p-2}\mathbf{1}_{X_s\neq 0}\left\{(2-p)\left[|Z_s|^2 - (\check{X}_s)^* Z_s Z_s^* \check{X}_s\right] + (p-1)|Z_s|^2\right\} ds \\
&+ \frac{p}{2}\int_0^t |X_s|^{p-2}\mathbf{1}_{X_s\neq 0}\left\{(2-p)\left[d[M]^c_s - (\check{X}_s)^* d[M, M]^c_s \check{X}_s\right] + (p-1)d[M]^c_s\right\}. \quad (27)
\end{aligned}$$

The process $(L(t),\ t \in [0; T])$ is continuous, nondecreasing with $L_0 = 0$ and increases only on the boundary of the random set $\{t \in [0; T]; X_{t^-} = X_t = 0\}$.

Proof. Since in the case $p \in [1, 2)$ the function θ is not smooth enough to apply Itô's formula we use an approximation. Let $\varepsilon > 0$ and let us consider the function $u_\varepsilon(y) = (|y|^2 + \varepsilon^2)^{1/2}$. It is a smooth function and we have

$$\frac{\partial u_\varepsilon^p}{\partial y_i}(y) = p y_i u_\varepsilon(y)^{p-2}, \frac{\partial^2 u_\varepsilon^p}{\partial y_i \partial y_j}(y) = p u_\varepsilon(y)^{p-2}\delta_{i,j} + p(p-2)y_i y_j u_\varepsilon(y)^{p-4}.$$

We apply Itô's formula to X:

$$
\begin{aligned}
u_\varepsilon(X_t)^p = u_\varepsilon(X_0)^p &+ \int_0^t p u_\varepsilon(X_s)^{p-2} X_s K_s ds + p\int_0^t u_\varepsilon(X_s)^{p-2} X_s Z_s dW_s \\
&+ p\int_0^t u_\varepsilon(X_{s^-})^{p-2} X_{s^-} dM_s + p\int_0^t u_\varepsilon(X_{s^-})^{p-2} X_{s^-} \int_{\mathcal{U}} \psi_s(u)\widetilde{\pi}(du,ds) \\
&+ \frac{1}{2}\int_0^t \mathrm{Trace}\left(D^2(u_\varepsilon^p)(X_s) Z_s Z_s^*\right) ds \\
&+ \int_0^t\int_{\mathcal{U}} \left(u_\varepsilon(X_{s^-}+\psi_s(u))^p - u_\varepsilon(X_{s^-})^p - pX_{s^-}u_\varepsilon(X_{s^-})^{p-2}\psi_s(u)\right)\pi(du,ds) \\
&+ \frac{1}{2}\int_0^t \sum_{1\le i,j\le d} \frac{\partial^2 u_\varepsilon^p}{\partial y_i \partial y_j}(X_s) d[M^i,M^j]_s^c \\
&+ \sum_{0<s\le t} \left(u_\varepsilon(X_{s^-}+\Delta M_s)^p - u_\varepsilon(X_{s^-})^p - pX_{s^-}u_\varepsilon(X_{s^-})^{p-2}\Delta M_s\right). \qquad (28)
\end{aligned}
$$

Now we have to pass to the limit when ε goes to 0. As in [5] for the terms involving the first derivatives of u_ε we have

$$
\begin{aligned}
\int_0^t u_\varepsilon(X_s)^{p-2} X_s K_s ds &\longrightarrow \int_0^t |X_s|^{p-1}\check{X}_s K_s ds \\
\int_0^t u_\varepsilon(X_s)^{p-2} X_s Z_s dW_s &\longrightarrow \int_0^t |X_s|^{p-1}\check{X}_s Z_s dW_s \\
\int_0^t u_\varepsilon(X_s)^{p-2} X_s \int_{\mathcal{U}} \psi_s(u)\widetilde{\pi}(du,ds) &\longrightarrow \int_0^t |X_s|^{p-1}\check{X}_s \int_{\mathcal{U}} \psi_s(u)\widetilde{\pi}(du,ds) \\
\int_0^t u_\varepsilon(X_{s^-})^{p-2} X_{s^-} dM_s &\longrightarrow \int_0^t |X_{s^-}|^{p-1}\check{X}_{s^-} dM_s.
\end{aligned}
$$

Moreover by the same arguments (convexity of u_ε and Fatou's lemma) the two following terms

$$
\begin{aligned}
&\int_0^t\int_{\mathcal{U}} \left[u_\varepsilon(X_{s^-}+\psi_s(u))^p - u_\varepsilon(X_{s^-})^p - pX_{s^-}u_\varepsilon(X_{s^-})^{p-2}\psi_s(u)\right]\pi(du,ds) \\
&\sum_{0<s\le t} \left[u_\varepsilon(X_{s^-}+\Delta M_s)^p - u_\varepsilon(X_{s^-})^p - pX_{s^-}u_\varepsilon(X_{s^-})^{p-2}\Delta M_s\right]
\end{aligned}
$$

converge, at least in probability, to

$$
\begin{aligned}
&\int_0^t\int_{\mathcal{U}} \left[|X_{s^-}+\psi_s(u)|^p - |X_{s^-}|^p - p|X_{s^-}|^{p-1}\check{X}_{s^-}\psi_s(u)\right]\pi(du,ds) \\
&\sum_{0<s\le t} \left[|X_{s^-}+\Delta M_s|^p - |X_{s^-}|^p - p|X_{s^-}|^{p-1}\check{X}_{s^-}\Delta M_s\right].
\end{aligned}
$$

Now for a non negative symmetric matrix $\Gamma \in \mathbb{R}^{d\times d}$

$$
\begin{aligned}
&\sum_{1\le i,j\le d} D^2\theta(y)_{i,j}\Gamma_{i,j} = pu_\varepsilon(y)^{p-2}\mathrm{Trace}(\Gamma) + p(p-2)u_\varepsilon(y)^{p-4}(y^*)\Gamma y \\
&= p(2-p)\left(\frac{|y|}{u_\varepsilon(y)}\right)^{4-p}|y|^{p-2}\left[\mathrm{Trace}(\Gamma) - (\check{y})^*\Gamma\check{y}\right]\mathbf{1}_{y\neq 0} \\
&\quad + p(p-1)\left(\frac{|y|}{u_\varepsilon(y)}\right)^{4-p}|y|^{p-2}\mathrm{Trace}(\Gamma)\mathbf{1}_{y\neq 0} + p\varepsilon^2 u_\varepsilon(y)^{p-4}\mathrm{Trace}(\Gamma). \qquad (29)
\end{aligned}
$$

We have the following properties:

- $\text{Trace}(\Gamma) \geq (\check{y})^* \Gamma \check{y}$,
- $\frac{|y|}{u_\varepsilon(y)} \nearrow \mathbf{1}_{y \neq 0}$ as $\varepsilon \searrow 0$.

For $\Gamma_s = Z_s Z_s^*$, by monotone convergence we obtain that

$$\int_0^t \left(\frac{|X_s|}{u_\varepsilon(X_s)} \right)^{4-p} |X_s|^{p-2} \left\{ (2-p) \left[|Z_s|^2 - (\check{X}_s)^* Z_s Z_s^* \check{X}_s \right] + (p-1)|Z_s|^2 \right\} \mathbf{1}_{X_s \neq 0} ds$$

converges $\mathbb{P}$-a.s. for all $0 \leq t \leq T$ to

$$\int_0^t |X_s|^{p-2} \left\{ (2-p) \left[|Z_s|^2 - (\check{X}_s)^* Z_s Z_s^* \check{X}_s \right] + (p-1)|Z_s|^2 \right\} \mathbf{1}_{X_s \neq 0} ds.$$

And for the integral w.r.t. the matrix $[M, M]^c = ([M^i, M^j]^c_t,\ 1 \leq i, j \leq d)$ we have the same result and the convergence to

$$\int_0^t |X_s|^{p-2} \mathbf{1}_{X_s \neq 0} \left\{ (2-p) \left[d[M]^c_s - (\check{X}_s)^* d[M, M]^c_s \check{X}_s \right] + (p-1) d[M]^c_s \right\},$$

where $[M]^c = \sum_{i=1}^d [M^i, M^i]^c$. There is one remaining term in (28):

$$C^p_\varepsilon(t) = p\varepsilon^2 \int_0^t u_\varepsilon(X_s)^{p-4} \left[|Z_s|^2 ds + d[M]^c_s \right].$$

It follows from (28) and the considerations above that this term converges to a process $L^p(t)$. By the same arguments as in [5], we can prove that $L^p(t) = 0$ if $p > 1$. Indeed if $p \geq 4$, $u_\varepsilon(X_s)^{p-4}$ converges in $L^1(\Omega \times (0,T))$ and if $1 < p < 4$, using Hölder inequality with $\theta = (4-p)/3 \in (0,1)$:

$$C^p_\varepsilon(t) \leq p \left(\int_0^t \varepsilon^2 u_\varepsilon(X_s)^{-3} \left[|Z_s|^2 ds + d[M]^c_s \right] \right)^\theta \left(\int_0^t \varepsilon^2 \left[|Z_s|^2 ds + d[M]^c_s \right] \right)^{1-\theta}.$$

Since the first term in the right-hand side converges to $L^1(t)$, $C^p_\varepsilon(t)$ tends to zero.

Let us denote by $L(t)$ the process $L^1(t)$ and we proceed almost as in Chapter IV.7 (see Theorem 69) in [34]. By letting ε tend to zero in (28) we obtain that L satisfies (27). By identifying the jumps on both sides of the equation it follows that L is continuous. Moreover, L is non decreasing in time. Now let us set $A = \{t \in [0;T];\ X_{t^-} = X_t = 0\}$. If t is in the interior of A, then there exists $\delta > 0$ such that $X_s = 0$ whenever $|t-s| \leq \delta$ and the quadratic variation of X is constant on the interval $[t-\delta; t+\delta]$ and then $Z_s = 0$ and $[M]_s = 0$ almost everywhere on this interval. Hence L does not increase in the interior of A. Now assume that t is in the interior of the complement of A. Since L is continuous, the associated measure dL is diffusive and does not charge any countable set. In particular, as X is càdlàg , dL does not charge the points where X jumps. Hence, we can assume that $X_t = X_{t-}$. Then there exists some $\delta > 0$ such that $X_s \neq 0$ for $|t-s| < \delta$. Consequently, $L(s) = L(t)$ for $|t-s| < \delta$, which completes the proof. □

As a byproduct to the proof we obtain the following lemma.

Lemma 8 *For $p \in (1,2)$, for any $t \geq 0$*

$$\int_0^t \mathbf{1}_{X_s=0} \left[|Z_s|^2 ds + d[M]_s^c \right] = 0.$$

Proof. Indeed C_ε^p can be written as follows:

$$\begin{aligned} C_\varepsilon^p(t) &= p\varepsilon^2 \int_0^t u_\varepsilon(X_s)^{p-4} \left[|Z_s|^2 ds + d[M]_s^c \right] \\ &= p\varepsilon^2 \int_0^t \left(|X_s|^2 + \varepsilon^2 \right)^{p/2-2} \mathbf{1}_{X_s \neq 0} \left[|Z_s|^2 ds + d[M]_s^c \right] \\ &\quad + p\varepsilon^{p-2} \int_0^t \mathbf{1}_{X_s=0} \left[|Z_s|^2 ds + d[M]_s^c \right]. \end{aligned}$$

Hence C_ε^p can converge to zero if and only if the last term is zero. □

Corollary 1 *If (Y, Z, ψ, M) is a solution of BSDE (2), $p \in [1,2)$, $c(p) = \frac{p(p-1)}{2}$ and $0 \leq t \leq r \leq T$, then:*

$$\begin{aligned} |Y_t|^p \leq |Y_r|^p &+ p \int_t^r |Y_s|^{p-1} \check{Y}_s f(s, Y_s, Z_s, \psi_s) ds - p \int_t^r |Y_s|^{p-1} \check{Y}_s Z_s dW_s \\ &- p \int_t^r |Y_{s^-}|^{p-1} \check{Y}_{s^-} dM_s - p \int_t^r |Y_s|^{p-1} \check{Y}_s \int_{\mathcal{U}} \psi_s(u) \widetilde{\pi}(du, ds) \\ &- \int_t^r \int_{\mathcal{U}} \left[|Y_{s^-} + \psi_s(u)|^p - |Y_{s^-}|^p - p|Y_{s^-}|^{p-1} \check{Y}_{s^-} \psi_s(u) \right] \pi(du, ds) \\ &- \sum_{0<t\leq r} \left[|Y_{s^-} + \Delta M_s|^p - |Y_{s^-}|^p - p|Y_{s^-}|^{p-1} \check{Y}_{s^-} \Delta M_s \right] \\ &- c(p) \int_t^r |Y_s|^{p-2} |Z_s|^2 \mathbf{1}_{Y_s \neq 0} ds - c(p) \int_t^r |Y_s|^{p-2} \mathbf{1}_{Y_s \neq 0} d[M]_s^c. \end{aligned}$$

Moreover if $p \in (1,2)$, then $\int_0^t \mathbf{1}_{Y_s=0} \left[|Z_s|^2 ds + d[M]_s^c \right] = 0$.

Proof. A direct consequence of Lemmas 7 and 8. □

Lemma 9 *For $p \in [1,2)$, the non-decreasing process involving the jumps of Y controls the quadratic variation as follows:*

$$\begin{aligned} &\sum_{0<s\leq t} \left[|Y_{s^-} + \Delta M_s|^p - |Y_{s^-}|^p - p|Y_{s^-}|^{p-1} \check{Y}_{s^-} \Delta M_s \right] \\ &\quad \geq c(p) \sum_{0<s\leq t} |\Delta M_s|^2 \left(|Y_{s^-}|^2 \vee |Y_{s^-} + \Delta M_s|^2 \right)^{p/2-1} \mathbf{1}_{|Y_{s^-}| \vee |Y_{s^-} + \Delta M_s| \neq 0}. \end{aligned}$$

The same holds for the jumps due to the Poisson random measure.

Proof. We proceed as in the proof of Proposition 2 and we use the approximation of Lemma 7. Using Taylor expansion we obtain

$$\begin{aligned}
&\sum_{0<s\leq t}\left[u_\varepsilon(Y_{s^-}+\Delta M_s)^p-u_\varepsilon(Y_{s^-})^p-pY_{s^-}u_\varepsilon(Y_{s^-})^{p-2}\Delta M_s\right]\\
&=\sum_{0<s\leq t}\int_0^1(1-a)\Delta M_s D^2(u_\varepsilon(Y_{s^-}+a\Delta M_s)^p)\Delta M_s da\\
&=p\sum_{0<s\leq t}\int_0^1(1-a)|\Delta M_s|^2u_\varepsilon(Y_{s^-}+a\Delta M_s)^{p-2}da\\
&\quad+p(p-2)\sum_{0<s\leq t}\int_0^1(1-a)\langle\Delta M_s,Y_{s^-}+a\Delta M_s\rangle^2u_\varepsilon(Y_{s^-}+a\Delta M_s)^{p-4}da\\
&\geq p(p-1)\sum_{0<s\leq t}|\Delta M_s|^2\int_0^1(1-a)u_\varepsilon(Y_{s^-}+a\Delta M_s)^{p-2}da.
\end{aligned}$$

Since $|Y_{s^-}+a\Delta M_s|=|(1-a)Y_{s^-}+a(Y_{s^-}+\Delta M_s)|\leq|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|$, we obtain:

$$\begin{aligned}
&\sum_{0<s\leq t}\left[u_\varepsilon(Y_{s^-}+\Delta M_s)^p-u_\varepsilon(Y_{s^-})^p-pY_{s^-}u_\varepsilon(Y_{s^-})^{p-2}\Delta M_s\right]\\
&\quad\geq\frac{p(p-1)}{2}\sum_{0<s\leq t}|\Delta M_s|^2\left(|Y_{s^-}|^2\vee|Y_{s^-}+\Delta M_s|^2+\varepsilon^2\right)^{p/2-1}.
\end{aligned}$$

Passing to the limit as ε goes to zero, we obtain:

$$\begin{aligned}
&\sum_{0<s\leq t}\left[|Y_{s^-}+\Delta M_s|^p-|Y_{s^-}|^p-p|Y_{s^-}|^{p-1}\check{Y}_{s^-}\Delta M_s\right]\\
&\quad\geq\frac{p(p-1)}{2}\sum_{0<s\leq t}|\Delta M_s|^2\left(|Y_{s^-}|^2\vee|Y_{s^-}+\Delta M_s|^2\right)^{p/2-1}\mathbf{1}_{|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|\neq0}.
\end{aligned}$$

This achieves the proof ot the lemma. □

Remark 1 *If $p\geq2$, then the conclusions of Corollary 1 and of Lemma 9 hold with $c(p)=p/2$.*

From now on, we assume that $p\in(1,2)$. The proof of the existence of a unique solution of BSDE (2) in the space $\mathcal{E}^p(0,T)$ is based on the following technical result. This estimates are also proved in [19], Proposition 5.3, but in dimension 1. Moreover this estimate looks very similar to Inequality (17). The main difference is that for $p<2$, or $p/2<1$, the compensator of a martingale does not control the predictable projection (see [23] and the counterexample therein). We say that the condition (C) holds if $\mathbb{P}$-a.s.

$$\langle\check{y},f(t,y,z,\psi)\rangle\leq f_t+\alpha|y|+K|z|+K\|\psi\|_{L^2_\mu},$$

with $K\geq0$ and f_t is a non-negative progressively measurable process. Let us denote $F=\int_0^T f_r dr$.

Proposition 3 *Let the assumption (C) hold and let be (Y, Z, ψ, M) be a solution of BSDE (2) and assume moreover that F^p is integrable and $Y \in \mathbb{D}^p(0,T)$. Then (Z, ψ, M) belongs to $\mathbb{H}^p(0,T) \times L^p_\pi(0,T) \times \mathbb{M}^p(0,T)$ and there exists a constant C depending on p, K and T such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t|^p + \left(\int_0^T |Z_t|^2 dt\right)^{p/2} + ([M]_T)^{p/2} + \left(\int_0^T\int_{\mathcal{U}} |\psi_s(u)|^2 \pi(du,ds)\right)^{p/2} + \left(\int_0^T\int_{\mathcal{U}} |\psi_s(u)|^2 \mu(du)ds\right)^{p/2}\right] \leq C\mathbb{E}\left[|\xi|^p + \left(\int_0^T f_r dr\right)^p\right].$$

Once again let us emphasize that the dependence of f w.r.t. ψ implies that we have to control the two expectations containing the term ψ. A crucial point in the proof of Proposition 2 was Inequality (25). Now in the case $p < 2$ we can not control (see [23], Section 4) the expectation of the predictable projection:

$$\mathbb{E}\left(\int_0^T\int_{\mathcal{U}} |\psi_s(u)|^2 \pi(du,ds)\right)^{p/2}$$

with the expectation of the quadratic variation:

$$\mathbb{E}\left(\int_0^T\int_{\mathcal{U}} |\psi_s(u)|^2 \pi(du,ds)\right)^{p/2}.$$

Proof. For some $a \in \mathbb{R}$, let us define $\widetilde{Y}_t = e^{at}Y_t$, $\widetilde{Z}_t = e^{at}Z_t$, $\widetilde{\psi}_t = e^{at}\psi_t$ and $d\widetilde{M}_t = e^{at}dM_t$. $(\widetilde{Y}, \widetilde{Z}, \widetilde{\psi}, \widetilde{M})$ satisfies an analogous BSDE with terminal condition $\widetilde{\xi} = e^{aT}\xi$ and generator

$$\widetilde{f}(t,y,z,\psi) = e^{at}f(t, e^{-at}y, e^{-at}z, e^{-at}\psi) - ay.$$

$\widetilde{f}$ satisfies assumptions $(\mathbf{H_{ex}})$ and (C) with $\widetilde{K} = K$ and $\widetilde{\alpha} = \alpha - a$. We choose a large enough such that

$$\widetilde{\alpha} + 2K^2/(p-1) \leq 0.$$

Since we are working on a compact time interval, the integrability conditions are equivalent with or without the superscript $\widetilde{}$. We omit the superscript $\widetilde{}$ for notational convenience.
Step 1: We prove first that if $\alpha + 2K^2/(p-1) \leq 0$, there exists a constant κ_p such that

$$\mathbb{E}(Y_*^p) \leq \kappa_p \mathbb{E}(X),$$

where

$$Y_* = \sup_{t\in[0,T]}|Y_t|, \quad \text{and} \quad X = |\xi|^p + p\int_0^T |Y_s|^{p-1} f_s ds.$$

We apply Corollary 1 for $\tau \in \mathcal{T}_T$:

$$
\begin{aligned}
&|Y_{t\wedge\tau}|^p + c(p)\int_{t\wedge\tau}^{\tau} |Y_s|^{p-2}|Z_s|^2 \mathbf{1}_{Y_s\neq 0} ds + c(p)\int_{t\wedge\tau}^{\tau} |Y_s|^{p-2}\mathbf{1}_{Y_s\neq 0} d[M]^c_s \\
&\leq |Y_\tau|^p + p\int_{t\wedge\tau}^{\tau} |Y_s|^{p-1}\check{Y}_s f(s,Y_s,Z_s,\psi_s)ds - p\int_{t\wedge\tau}^{\tau} |Y_s|^{p-1}\check{Y}_s Z_s dW_s \\
&\quad -p\int_{t\wedge\tau}^{\tau} |Y_{s^-}|^{p-1}\check{Y}_{s^-} dM_s - p\int_{t\wedge\tau}^{\tau} |Y_s|^{p-1}\check{Y}_s \int_{\mathcal{U}} \psi_s(u)\widetilde{\pi}(du,ds) \\
&\quad -\int_{t\wedge\tau}^{\tau}\int_{\mathcal{U}} \left[|Y_{s^-}+\psi_s(u)|^p - |Y_{s^-}|^p - p|Y_{s^-}|^{p-1}\check{Y}_{s^-}\psi_s(u)\right]\pi(du,ds) \\
&\quad -\sum_{t\wedge\tau<s\leq\tau} \left[|Y_{s^-}+\Delta M_s|^p - |Y_{s^-}|^p - p|Y_{s^-}|^{p-1}\check{Y}_{s^-}\Delta M_s\right].
\end{aligned}
$$

With the assumption on f this becomes

$$
\begin{aligned}
&|Y_{t\wedge\tau}|^p + c(p)\int_{t\wedge\tau}^{\tau} |Y_s|^{p-2}|Z_s|^2 \mathbf{1}_{Y_s\neq 0} ds + c(p)\int_{t\wedge\tau}^{\tau} |Y_s|^{p-2}\mathbf{1}_{Y_s\neq 0} d[M]^c_s \\
&\leq |Y_\tau|^p + p\int_{t\wedge\tau}^{\tau} \left(|Y_s|^{p-1}f_s + \alpha|Y_s|^p\right)ds + pK\int_{t\wedge\tau}^{\tau} |Y_s|^{p-1}|Z_s|ds \\
&\quad +pK\int_{t\wedge\tau}^{\tau} |Y_s|^{p-1}\|\psi_s\|_{L^2_\mu} ds - p\int_{t\wedge\tau}^{\tau} |Y_s|^{p-1}\check{Y}_s Z_s dW_s \\
&\quad -p\int_{t\wedge\tau}^{\tau} |Y_{s^-}|^{p-1}\check{Y}_{s^-} dM_s - p\int_{t\wedge\tau}^{\tau} |Y_s|^{p-1}\check{Y}_s \int_{\mathcal{U}} \psi_s(u)\widetilde{\pi}(du,ds) \\
&\quad -\int_{t\wedge\tau}^{\tau}\int_{\mathcal{U}} \left[|Y_{s^-}+\psi_s(u)|^p - |Y_{s^-}|^p - p|Y_{s^-}|^{p-1}\check{Y}_{s^-}\psi_s(u)\right]\pi(du,ds) \\
&\quad -\sum_{t\wedge\tau<s\leq\tau} \left[|Y_{s^-}+\Delta M_s|^p - |Y_{s^-}|^p - p|Y_{s^-}|^{p-1}\check{Y}_{s^-}\Delta M_s\right]
\end{aligned}
$$

Moreover

$$
\begin{aligned}
pK|Y_s|^{p-1}|Z_s| &\leq \frac{pK^2}{p-1}|Y_s|^p + \frac{c(p)}{2}|Y_s|^{p-2}|Z_s|^2\mathbf{1}_{Y_s\neq 0} \\
pK|Y_s|^{p-1}\|\psi_s\|_{L^2_\mu} &\leq \frac{pK^2}{p-1}|Y_s|^p + \frac{c(p)}{2}|Y_s|^{p-2}\|\psi_s\|^2_{L^2_\mu}\mathbf{1}_{Y_s\neq 0}
\end{aligned}
$$

and from the previous lemma

$$
\begin{aligned}
&\int_{t\wedge\tau}^{\tau}\int_{\mathcal{U}} \left[|Y_{s^-}+\psi_s(u)|^p - |Y_{s^-}|^p - p|Y_{s^-}|^{p-1}\check{Y}_{s^-}\psi_s(u)\right]\pi(du,ds) \\
&\quad \geq c(p)\int_{t\wedge\tau}^{\tau}\int_{\mathcal{U}} |\psi_s(u)|^2 \left(|Y_{s^-}|^2 \vee |Y_{s^-}+\psi_s(u)|^2\right)^{p/2-1}\mathbf{1}_{|Y_{s^-}|\vee|Y_{s^-}+\psi_s(u)|\neq 0}\pi(du,ds).
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{t\wedge\tau<s\leq\tau} \left[|Y_{s^-}+\Delta M_s|^p - |Y_{s^-}|^p - p|Y_{s^-}|^{p-1}\check{Y}_{s^-}\Delta M_s\right] \\
&\quad \geq c(p)\sum_{t\wedge\tau<s\leq\tau} |\Delta M_s|^2 \left(|Y_{s^-}|^2 \vee |Y_{s^-}+\Delta M_s|^2\right)^{p/2-1}\mathbf{1}_{|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|\neq 0}
\end{aligned}
$$

Therefore we deduce the following inequality:

$$
\begin{aligned}
&|Y_{t\wedge\tau}|^p + \frac{c(p)}{2}\int_{t\wedge\tau}^{\tau}|Y_s|^{p-2}|Z_s|^2\mathbf{1}_{Y_s\neq 0}ds + c(p)\int_{t\wedge\tau}^{\tau}|Y_s|^{p-2}\mathbf{1}_{Y_s\neq 0}d[M]_s^c\\
&+c(p)\sum_{t\wedge\tau<s\leq\tau}\left(|Y_{s^-}|^2\vee|Y_{s^-}+\Delta M_s|^2\right)^{p/2-1}\mathbf{1}_{|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|\neq 0}|\Delta M_s|^2\\
&+c(p)\int_{t\wedge\tau}^{\tau}\int_{\mathcal{U}}|\psi_s(u)|^2\left(|Y_{s^-}|^2\vee|Y_{s^-}+\psi_s(u)|^2\right)^{p/2-1}\mathbf{1}_{|Y_{s^-}|\vee|Y_{s^-}+\psi_s(u)|\neq 0}\pi(du,ds)\\
&-\frac{c(p)}{2}\int_{t\wedge\tau}^{\tau}|Y_s|^{p-2}\|\psi_s\|^2_{L^2_\mu}\mathbf{1}_{Y_s\neq 0}ds\\
&\leq |Y_\tau|^p + p\int_{t\wedge\tau}^{\tau}\left(|Y_s|^{p-1}f_s+\alpha|Y_s|^p\right)ds + p\int_{t\wedge\tau}^{\tau}\frac{2K^2}{p-1}|Y_s|^p ds\\
&\quad -p\int_{t\wedge\tau}^{\tau}|Y_s|^{p-1}\check{Y}_s\left(Z_s dW_s + dM_s + \int_{\mathcal{U}}\psi_s(u)\widetilde{\pi}(du,ds)\right). \qquad (30)
\end{aligned}
$$

At the very beginning of this proof we suppose that $\alpha+\frac{2K^2}{p-1}\leq 0$. Thus the term $(\alpha+\frac{2K^2}{p-1})\int_{t\wedge\tau}^{\tau}|Y_s|^p ds$ disappears. Let us define τ_k as a fundamental sequence of stopping times for the local martingale

$$
\int_0^{\cdot}|Y_s|^{p-1}\check{Y}_s\left(Z_s dW_s + dM_s + \int_{\mathcal{U}}\psi_s(u)\widetilde{\pi}(du,ds)\right).
$$

Let

$$
\hat{\tau}_k = \inf\left\{t\geq 0,\quad \int_0^t\int_{\mathcal{U}}|\psi_s(u)|^2\left(|Y_{s^-}|^2\vee|Y_s|^2\right)^{p/2-1}\mathbf{1}_{|Y_{s^-}|\vee|Y_s|\neq 0}\pi(du,ds)\geq k\right\}\wedge T.
$$

We take $\tau=\tau_k\wedge\hat{\tau}_k$. Now we have:

$$
\begin{aligned}
&\mathbb{E}\int_0^{\tau}\int_{\mathcal{U}}|\psi_s(u)|^2\left(|Y_{s^-}|^2\vee|Y_{s^-}+\psi_s(u)|^2\right)^{p/2-1}\mathbf{1}_{|Y_{s^-}|\vee|Y_{s^-}+\psi_s(u)|\neq 0}\pi(du,ds)\\
&\quad=\mathbb{E}\int_0^{\tau}\int_{\mathcal{U}}|\psi_s(u)|^2\left(|Y_{s^-}|^2\vee|Y_s|^2\right)^{p/2-1}\mathbf{1}_{|Y_{s^-}|\vee|Y_s|\neq 0}\pi(du,ds)\\
&\quad=\mathbb{E}\int_0^{\tau}\int_{\mathcal{U}}|\psi_s(u)|^2|Y_s|^{p-2}\mathbf{1}_{Y_s\neq 0}\mu(du)ds \qquad (31)
\end{aligned}
$$

the last equality coming from the localization due to $\hat{\tau}_k$ and since the set $\{s\geq 0,\ Y_s\neq Y_{s^-}\}$ is countable. Taking the expectation in (30), we get

$$
\begin{aligned}
&\mathbb{E}(|Y_{t\wedge\tau}|^p) + \frac{c(p)}{2}\mathbb{E}\int_0^{\tau}|Y_s|^{p-2}|Z_s|^2\mathbf{1}_{Y_s\neq 0}ds + c(p)\mathbb{E}\int_0^{\tau}|Y_s|^{p-2}\mathbf{1}_{Y_s\neq 0}d[M]_s^c\\
&\quad+c(p)\mathbb{E}\sum_{0<s\leq\tau}\left(|Y_{s^-}|^2\vee|Y_{s^-}+\Delta M_s|^2\right)^{p/2-1}\mathbf{1}_{|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|\neq 0}|\Delta M_s|^2\\
&\quad+\frac{c(p)}{2}\mathbb{E}\int_t^{\tau}\int_{\mathcal{U}}|\psi_s(u)|^2\left(|Y_{s^-}|^2\vee|Y_{s^-}+\psi_s(u)|^2\right)^{p/2-1}\mathbf{1}_{|Y_{s^-}|\vee|Y_{s^-}+\psi_s(u)|\neq 0}\pi(du,ds)\\
&\leq\mathbb{E}\left(|Y_\tau|^p + p\int_0^{\tau}|Y_s|^{p-1}f_s ds\right). \qquad (32)
\end{aligned}
$$

We also obtain:

$$\frac{c(p)}{2}\mathbb{E}\int_0^\tau\int_{\mathcal{U}}|\psi_s(u)|^2|Y_s|^{p-2}\mathbf{1}_{Y_s\neq 0}\mu(du)ds\leq\mathbb{E}\left(|Y_\tau|^p+p\int_0^\tau|Y_s|^{p-1}f_sds\right).$$

Recall that X is the quantity

$$X=|\xi|^p+p\int_0^T|Y_s|^{p-1}f_sds.$$

Then we can pass to the limit on k in (32), and we obtain the same estimate for $\tau=T$ and $\mathbb{E}(X)$ on the right-hand side. Let us split the local martingale of (30) into three terms:

$$\begin{aligned}\Gamma_t &= \int_0^t|Y_s|^{p-1}\check{Y}_sZ_sdW_s,\\ \Theta_t &= \int_0^t|Y_s|^{p-1}\check{Y}_sdM_s,\quad \Xi_t=\int_0^t|Y_s|^{p-1}\check{Y}_s\int_{\mathcal{U}}\psi_s(u)\widetilde{\pi}(du,ds).\end{aligned}$$

Then using (30) and BDG inequality

$$\mathbb{E}(Y_*^p)\leq\mathbb{E}(X)+k_p\mathbb{E}\left([\Gamma]_T^{1/2}+[\Theta]_T^{1/2}+[\Xi]_T^{1/2}\right).$$

The bracket $[\Gamma]_T^{1/2}$ can be handled as in [5]:

$$k_p\mathbb{E}\left([\Gamma]_T^{1/2}\right)\leq\frac{1}{6}\mathbb{E}(Y_*^p)+\frac{3k_p^2}{2}\mathbb{E}\left(\int_0^T|Y_s|^{p-2}|Z_s|^2\mathbf{1}_{Y_s\neq 0}ds\right).$$

For the other terms we have

$$\begin{aligned}k_p\mathbb{E}\left([\Xi]_T^{1/2}\right) &\leq k_p\mathbb{E}\left(Y_*^{p/2}\left(\int_0^T|Y_s|^{p-2}|\psi_s|^2\mathbf{1}_{Y_s\neq 0}\pi(du,ds)\right)^{1/2}\right)\\ &\leq \frac{1}{6}\mathbb{E}(Y_*^p)+\frac{3k_p^2}{2}\mathbb{E}\left(\int_0^T|Y_s|^{p-2}\|\psi_s\|_{L^2_\mu}^2\mathbf{1}_{Y_s\neq 0}ds\right),\end{aligned}$$

and for $[\Theta]$ since $p>1$

$$\begin{aligned}k_p\mathbb{E}\left([\Theta]_T^{1/2}\right) &\leq k_p\mathbb{E}\left[\left(\int_0^T\left(|Y_{s^-}|^2\vee|Y_{s^-}+\Delta M_s|^2\right)^{p-1}\mathbf{1}_{|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|\neq 0}d[M]_s\right)^{1/2}\right]\\ &\leq k_p\mathbb{E}\left[\left(\sup_{s\in[0,T]}\left(|Y_{s^-}|^2\vee|Y_{s^-}+\Delta M_s|^2\right)^{p/2}\right)^{1/2}\right.\\ &\qquad\left.\left(\int_0^T\left(|Y_{s^-}|^2\vee|Y_{s^-}+\Delta M_s|^2\right)^{p/2-1}\mathbf{1}_{|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|\neq 0}d[M]_s\right)^{1/2}\right]\\ &\leq \frac{1}{6}\mathbb{E}(Y_*^p)+\frac{3k_p^2}{2}\mathbb{E}\left(\int_0^T|Y_{s^-}|^{p-2}\mathbf{1}_{|Y_{s^-}|\neq 0}d[M]_s^c\right.\\ &\qquad\left.+\sum_{0<s\leq T}\left(|Y_{s^-}|^2\vee|Y_{s^-}+\Delta M_s|^2\right)^{p/2-1}\mathbf{1}_{|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|\neq 0}|\Delta M_s|^2\right).\end{aligned}$$

We deduce that there exists a constant depending only on p such that

$$\mathbb{E}(Y_*^p) \quad \leq \quad \kappa_p \mathbb{E}\left(X\right).$$

Step 2: Let us derive now a priori estimates for the martingale part of the BSDE. We use Corollary 1:

$$\begin{aligned}
\mathbb{E}\left(\int_0^T |Z_s|^2 ds\right)^{p/2} &= \mathbb{E}\left(\int_0^T \mathbf{1}_{Y_s\neq 0}|Z_s|^2 ds\right)^{p/2} \\
&= \mathbb{E}\left(\int_0^T (Y_s)^{2-p}(Y_s)^{p-2}\mathbf{1}_{Y_s\neq 0}|Z_s|^2 ds\right)^{p/2} \\
&\leq \mathbb{E}\left[(Y_*)^{p(2-p)/2}\left(\int_0^T (Y_s)^{p-2}\mathbf{1}_{Y_s\neq 0}|Z_s|^2 ds\right)^{p/2}\right] \\
&\leq \left\{\mathbb{E}\left[(Y_*)^p\right]\right\}^{(2-p)/2}\left\{\mathbb{E}\int_0^T (Y_s)^{p-2}\mathbf{1}_{Y_s\neq 0}|Z_s|^2 ds\right\}^{p/2} \\
&\leq \frac{2-p}{2}\mathbb{E}\left[(Y_*)^p\right] + \frac{p}{2}\mathbb{E}\int_0^T (Y_s)^{p-2}\mathbf{1}_{Y_s\neq 0}|Z_s|^2 ds \qquad (33)
\end{aligned}$$

where we have used Hölder's and Young's inequality with $\frac{2-p}{2} + \frac{p}{2} = 1$. With Inequality (32) we deduce:

$$\mathbb{E}\left(\int_0^T |Z_s|^2 ds\right)^{p/2} \leq \frac{2-p}{2}\mathbb{E}\left(|Y_*|^p\right) + \frac{p}{2}\mathbb{E}\int_0^T |Y_s|^{p-2}\mathbf{1}_{Y_s\neq 0}|Z_s|^2 ds \leq \widetilde{\kappa}_p \mathbb{E}(X).$$

The same argument can be used to control $[M]^c$. For the pure-jump part of $[M]$ we have using the function u_ε defined in the proof of Lemma 7:

$$\begin{aligned}
&\mathbb{E}\left(\sum_{0<s\leq T}|\Delta M_s|^2\right)^{p/2} \\
&= \mathbb{E}\left(\sum_{0<s\leq T}\left(u_\varepsilon(|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|)\right)^{2-p}\left(u_\varepsilon(|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|)\right)^{p-2}|\Delta M_s|^2\right)^{p/2} \\
&\leq \mathbb{E}\left[(u_\varepsilon(Y_*))^{p(2-p)/2}\left(\sum_{0<s\leq T}\left(u_\varepsilon(|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|)\right)^{p-2}|\Delta M_s|^2\right)^{p/2}\right] \\
&\leq \left\{\mathbb{E}\left[(u_\varepsilon(Y_*))^p\right]\right\}^{(2-p)/2} \\
&\qquad \times\left\{\mathbb{E}\left(\sum_{0<s\leq T}\left(u_\varepsilon(|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|)\right)^{p-2}|\Delta M_s|^2\right)\right\}^{p/2} \\
&\leq \frac{2-p}{2}\mathbb{E}\left[(u_\varepsilon(Y_*))^p\right] + \frac{p}{2}\mathbb{E}\left(\sum_{0<s\leq T}\left(u_\varepsilon(|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|)\right)^{p-2}|\Delta M_s|^2\right)
\end{aligned}$$

Let ε go to zero. We use a convergence result, which is a direct consequence of the proof of Lemma 9:

$$\begin{aligned}
&\mathbb{E}\left(\sum_{0<s\leq T}|\Delta M_s|^2\right)^{p/2}\\
&\leq \frac{2-p}{2}\mathbb{E}\left(|Y_*|^p\right)+\frac{p}{2}\mathbb{E}\left(\sum_{0\leq s<T}\left(|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|\right)^{p-2}\mathbf{1}_{|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|\neq 0}|\Delta M_s|^2\right)\\
&\leq \widetilde{\kappa}_p\mathbb{E}(X).
\end{aligned}$$

The same argument shows that

$$\mathbb{E}\left(\int_0^T\int_{\mathcal{U}}|\psi_s(u)|^2\pi(du,ds)\right)^{p/2} \quad\leq\quad \widetilde{\kappa}_p\mathbb{E}(X).$$

For the estimate of

$$\mathbb{E}\left(\int_0^T\int_{\mathcal{U}}|\psi_s(u)|^2\mu(du)ds\right)^{p/2}$$

we follow the same scheme with a localization argument and Equality (31) to obtain that

$$\lim_{\varepsilon\to 0}\mathbb{E}\int_0^T\int_{\mathcal{U}}\left(u_\varepsilon(|Y_{s^-}|)^{p-2}|\psi_s(u)|^2\mu(du)ds=\mathbb{E}\int_0^T\int_{\mathcal{U}}|Y_{s^-}|^{p-2}\mathbf{1}_{Y_{s^-}\neq 0}|\psi_s(u)|^2\mu(du)ds.\right.$$

Step 3: Now we prove the wanted estimate. Recall that we have found a constant $\hat{\kappa}_p$ such that

$$\begin{aligned}
\mathbb{E}\Bigg[&|Y_*|^p+\left(\int_0^T Z_s^2ds\right)^{p/2}+[M]_T^{p/2}\\
&+\left(\int_0^T\int_{\mathcal{U}}|\psi_s(u)|^2\mu(du)ds\right)^{p/2}+\left(\int_0^T\int_{\mathcal{U}}|\psi_s(u)|^2\pi(du,ds)\right)^{p/2}\Bigg]\leq\hat{\kappa}_p\mathbb{E}(X)
\end{aligned}$$

where

$$X=|\xi|^p+p\int_0^T|Y_s|^{p-1}f_sds.$$

But Young's inequality leads to

$$p\hat{\kappa}_p\int_0^T|Y_s|^{p-1}f_sds\leq p\hat{\kappa}_p|Y_*|^{p-1}\int_0^T f_sds\leq\frac{1}{2}\mathbb{E}\left(|Y_*|^p\right)+d_p\left(\int_0^T f_sds\right)^p.$$

Therefore we have proved that for a such that $\alpha+\frac{3K^2}{(p-1)}\leq a$, then

$$\begin{aligned}
\mathbb{E}\Bigg[&\left(\sup_{t\in[0,T]}e^{at}|Y_t|^p\right)+\left(\int_0^T e^{2as}Z_s^2ds\right)^{p/2}+\left(\int_0^T\int_{\mathcal{U}}e^{2as}|\psi_s(u)|^2\pi(du,ds)\right)^{p/2}\\
&+\left(\int_0^T e^{2as}d[M]_s\right)^{p/2}\Bigg]\leq C\mathbb{E}\left[e^{aT}|\xi|^p+\left(\int_0^T e^{ar}f_rdr\right)^p\right]
\end{aligned}$$

where C just depends on p. This gives the desired estimate. □

Theorem 2 *Under Assumptions* $(\mathbf{H_{ex}})$ *and (H5), there exists a unique solution* (Y, Z, ψ, M) *in* $\mathcal{E}^p(0,T)$ *to the BSDE* (2)*. Moreover for some constant* $C = C_{p,K,T}$

$$\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t|^p + \left(\int_0^T |Z_t|^2 dt\right)^{p/2} + \left(\int_0^T\int_{\mathcal{U}} |\psi_s(u)|^2 \pi(du,ds)\right)^{p/2} + ([M]_T)^{p/2}\right]$$
$$\leq C\mathbb{E}\left[|\xi|^p + \left(\int_0^T |f(r,0,0,0)|dr\right)^p\right].$$

Proof. As for Theorem 1, we follow the proof of the second step of Theorem 4.2 in [5]. We truncate ξ and $f(t,0,0,0)$ to obtain ξ_n and f_n with $\|\xi_n\|_\infty \leq n$ and $|f_n(t,0,0,0)| \leq n$:

$$\xi_n = q_n(\xi), \qquad f_n(t,y,z,\psi) = f(t,y,z,\psi) - f(t,0,0,0) + q_n(f(t,0,0,0)),$$

with $q_n(x) = xn/(|x|\vee n)$. Thanks to Theorem 1, we have a unique solution (Y^n, Z^n, ψ^n, M^n) in $\mathcal{E}^2$, and thus in $\mathcal{E}^p$ for any $p > 1$. Now for any m and n:

$$\begin{aligned} f_m(t,Y_t^m,Z_t^m,\psi_t^m) - f_n(t,Y_t^n,Z_t^n,\psi_t^n) &= f(t,Y_t^m,Z_t^m,\psi_t^m) - f(t,Y_t^n,Z_t^m,\psi_t^n) \\ &+ f(t,Y_t^n,Z_t^m,\psi_t^m) - f(t,Y_t^n,Z_t^n,\psi_t^m) + f(t,Y_t^n,Z_t^n,\psi_t^m) - f(t,Y_t^n,Z_t^n,\psi_t^n) \\ &+ q_m(f(t,0,0,0)) - q_n(f(t,0,0,0)) \end{aligned}$$

Hence

$$\begin{aligned} &\langle \frac{(Y_t^m - Y_t^n)}{|Y_t^m - Y_t^n|}\mathbf{1}_{Y_t^m - Y_t^n \neq 0}, f_m(t,Y_t^m,Z_t^m,\psi_t^m) - f_n(t,Y_t^n,Z_t^n,\psi_t^n)\rangle \\ &\quad \leq |q_m(f(t,0,0,0)) - q_n(f(t,0,0,0))| + K|Z_t^m - Z_t^n| + K\|\psi_t^m - \psi_t^n\|_{L^2_\mu}. \end{aligned}$$

This inequality is Assumption (C) in Proposition 3 with $\alpha = 0$. This proposition shows that

$$\begin{aligned} \mathbb{E}\Bigg[&\sup_{t\in[0,T]}|Y_t^m - Y_t^n|^p + \left(\int_0^T |Z_s^m - Z_s^n|^2 ds\right)^{p/2} + ([M^m - M^n]_T)^{p/2} \\ &+ \left(\int_0^T\int_{\mathcal{U}} |\psi_s^m(u) - \psi_s^n(u)|^2 \pi(du,ds)\right)^{p/2} + \left(\int_0^T\int_{\mathcal{U}} |\psi_s^m(u) - \psi_s^n(u)|^2 \mu(du)ds\right)^{p/2}\Bigg] \\ &\leq C\mathbb{E}\left[|\xi_m - \xi_n|^p + \left(\int_0^T |q_m(f(r,0,0,0)) - q_n(f(r,0,0,0))|dr\right)^p\right]. \end{aligned}$$

Thus (Y^n, Z^n, ψ^n, M^n) is a Cauchy sequence in $\mathcal{E}^p$ and the conclusion follows. □

4 Comparison Principle

In this section we give some results which are derived from the previous sections. In the first part we assume that $d = 1$ and aim at comparing two solutions Y^1 and Y^2 of the BSDE (2) with coefficients (ξ^1, f^1) and (ξ^2, f^2). As in the papers of Barles et al. [1], Royer [37], Situ [38] or Quenez & Sulem [35], we have to restrict the dependence of f w.r.t. ψ. Some monotonicity w.r.t. ψ is necessary. The following set of conditions will be denoted by $(\mathbf{H_{comp}})$. The three conditions (H1) to (H3) hold but assumption (H3) is replaced by:

(H3') f is Lipschitz continuous w.r.t. z with constant K and for each $(y,z,\psi,\phi)\in\mathbb{R}\times\mathbb{R}^k\times(L^2_\mu)^2$, there exists a predictable process $\kappa=\kappa^{y,z,\psi,\phi}:\Omega\times[0,T]\times\mathcal{U}\to\mathbb{R}$ such that:

$$f(t,y,z,\psi)-f(t,y,z,\phi)\leq\int_{\mathcal{U}}(\psi(u)-\phi(u))\kappa_t^{y,z,\psi,\phi}(u)\mu(du)$$

with $\mathbb{P}\otimes m\otimes\mu$-a.e. for any (y,z,ψ,ψ'),

- $-1\leq\kappa_t^{y,z,\psi,\phi}(u)$
- $|\kappa_t^{y,z,\psi,\phi}(u)|\leq\vartheta(u)$ where $\vartheta\in L^2_\mu$.

Note that ($\mathbf{H_{comp}}$) implies ($\mathbf{H_{ex}}$). Indeed if (H3') is true we also have:

$$f(t,y,z,\psi)-f(t,y,z,\phi)\geq\int_{\mathcal{U}}(\psi(u)-\phi(u))\kappa_t^{y,z,\phi,\psi}(u)\mu(du)$$

by changing the role of ψ and ϕ in κ and thus

$$|f(t,y,z,\psi)-f(t,y,z,\phi)|\leq\|\vartheta\|_{L^2_\mu}\|\psi-\phi\|_{L^2_\mu}.$$

We follow the line of argument of [35]. In particular we consider the Doléans-Dade exponential local martingale: Let α, β be predictable processes integrable w.r.t. dt and dW_t, respectively. Let γ be a predictable process defined on $[0,T]\times\Omega\times\mathbb{R}$ integrable w.r.t. $\tilde{\pi}(du,ds)$. For any $0\leq t\leq s\leq T$ let E be the solution of

$$dE_{t,s}=E_{t,s^-}\left[\beta_s dW_s+\int_{\mathcal{U}}\gamma_s(u)\widetilde{\pi}(du,ds)\right],\qquad E_{t,t}=1,$$

and let Γ be the solution of

$$d\Gamma_{t,s}=\Gamma_{t,s^-}\left[\alpha_s ds+\beta_s dW_s+\int_{\mathcal{U}}\gamma_s(u)\widetilde{\pi}(du,ds)\right],\qquad \Gamma_{t,t}=1. \tag{34}$$

Of course $\Gamma_{t,s}=\exp\left(\int_t^s\alpha_r dr\right)E_{t,s}$ and

$$E_{t,s}=\exp\left(\int_t^s\beta_r dW_r-\frac{1}{2}\int_t^s\beta_r^2 dr\right)\prod_{t<r\leq s}(1+\gamma_r(\Delta X_r))e^{-\gamma_r(\Delta X_r)}$$

with $X_t=\int_0^t\int_{\mathcal{U}}u\pi(du,ds)$.

Lemma 10 *Assume that the processes $|\beta|$ and $\|\gamma\|_{L^2_\mu}$ are bounded and that α is bounded from above. Let (Y,Z,ψ,M) be the solution of the following linear BSDE:*

$$\begin{aligned}Y_t &= \xi+\int_t^T\left[f_s+\alpha_s Y_s+\beta_s Z_s+\int_{\mathcal{U}}\gamma_s(u)\psi_s(u)\mu(du)\right]ds\\ &- \int_t^T\int_{\mathcal{U}}\psi_s(u)\widetilde{\pi}(du,ds)-\int_t^T Z_s dW_s-\int_t^T dM_s.\end{aligned} \tag{35}$$

Then Γ is q-integrable for any $q\geq 2$, and the solution (Y,Z,ψ,M) belongs to $\mathcal{E}^p(0,T)$ if

$$\mathbb{E}\left(|\xi|^p+\int_0^T|f_s|^p ds\right)<+\infty.$$

Moreover

$$Y_t=\mathbb{E}\left[\Gamma_{t,T}\xi+\int_t^T\Gamma_{t,s}f_s ds\middle|\mathcal{F}_t\right].$$

Proof. The integrability of $\Gamma_{t,s}$ is given in [35], Proposition A.1 and by Doob's inequality:

$$\mathbb{E}\sup_{s\in[t,T]}|\Gamma_{t,s}|^q \leq C_q \sup_{s\in[t,T]}\mathbb{E}|\Gamma_{t,s}|^q \leq C_q\mathbb{E}|\Gamma_{t,T}|^q < +\infty.$$

We follow the arguments of the proof of Theorem 3.4 in [35]. Let (Y,Z,ψ,M) be a solution. For $0\leq t\leq s\leq T$ set

$$\phi_s = Y_s\Gamma_{t,s} + \int_t^s \Gamma_{t,r}f_r dr.$$

Then by integration by parts we obtain

$$\begin{aligned}
d\phi_s &= \Gamma_{t,s^-}dY_s + Y_{s^-}d\Gamma_{t,s} + d[\Gamma_{t,.},Y]_s + \Gamma_{t,s}f_s ds \\
&= \Gamma_{t,s^-}\left(-f_s - \alpha_s Y_s - \beta_s Z_s - \int_{\mathcal{U}}\gamma_s(u)\psi_s(u)\mu(du)\right)ds \\
&\quad +\Gamma_{t,s^-}\int_{\mathcal{U}}\psi_s(u)\widetilde{\pi}(du,ds) + \Gamma_{t,s^-}Z_s dW_s + \Gamma_{t,s^-}dM_s. \\
&+ Y_{s^-}\Gamma_{t,s^-}\left(\alpha_s ds + \beta_s dW_s + \int_{\mathcal{U}}\gamma_s(u)\widetilde{\pi}(du,ds)\right) \\
&+ \Gamma_{t,s^-}\beta_s Z_s ds + \Gamma_{t,s^-}\int_{\mathcal{U}}\psi_s(u)\gamma_s(u)\pi(du,ds) + \Gamma_{t,s}f_s ds \\
&= \Gamma_{t,s^-}\int_{\mathcal{U}}(\psi_s(u) + Y_{s^-}\gamma_s(u) + \psi_s(u)\gamma_s(u))\widetilde{\pi}(du,ds) \\
&+ \Gamma_{t,s^-}(Z_s + Y_s\beta_s)dW_s + \Gamma_{t,s^-}dM_s.
\end{aligned}$$

From the assumptions made on the coefficients, we obtain that ϕ is a martingale and thus

$$\phi_t = Y_t = \mathbb{E}\left[\phi_T\middle|\mathcal{F}_t\right] = \mathbb{E}\left[Y_T\Gamma_{t,T} + \int_t^T \Gamma_{t,r}f_r dr\middle|\mathcal{F}_t\right].$$

□

The next proposition is a modification of Theorem 4.2 in [35] (see also Theorem 252 in [38]).

Proposition 4 *We consider a generator f_1 satisfying $(\mathbf{H_{ex}})$ and we ask f_2 to verify $(\mathbf{H_{comp}})$. Let ξ^1 and ξ^2 be two terminal conditions for BSDEs (2) driven respectively by f_1 and f_2. Denote by (Y^1,Z^1,ψ^1,M^1) and (Y^2,Z^2,ψ^2,M^2) the respective solutions in some space $\mathcal{E}^p(0,T)$ with $p>1$. If $\xi^1\leq\xi^2$ and $f_1(t,Y_t^1,Z_t^1,\psi_t^1)\leq f_2(t,Y_t^1,Z_t^1,\psi_t^1)$, then a.s. for any $t\in[0,T]$, $Y_t^1\leq Y_t^2$.*

Proof. As usual we set

$$\widehat{Y} = Y^2 - Y^1, \quad \widehat{Z} = Z^2 - Z^1, \quad \widehat{\psi} = \psi^2 - \psi^1, \quad \widehat{M} = M^2 - M^1.$$

Then $(\widehat{Y},\widehat{Z},\widehat{\psi},\widehat{M})$ satisfies:

$$\widehat{Y}_t = \widehat{\xi} + \int_t^T h_s ds - \int_t^T\int_{\mathcal{U}}\widehat{\psi}_s(u)\widetilde{\pi}(du,ds) - \int_t^T\widehat{Z}_s dW_s - \int_t^T d\widehat{M}_s,$$

where

$$h_s = f_2(Y_s^2,Z_s^2,\psi_s^2) - f_1(Y_s^1,Z_s^1,\psi_s^1).$$

Now we define

$$\begin{aligned} f_s &= f_2(Y_s^1, Z_s^1, \psi_s^1) - f_1(Y_s^1, Z_s^1, \psi_s^1) \\ \alpha_s &= \frac{f_2(Y_s^2, Z_s^1, \psi_s^1) - f_2(Y_s^1, Z_s^1, \psi_s^1)}{\widehat{Y}_s} \mathbf{1}_{\widehat{Y}_s \neq 0} \\ \beta_s &= \frac{f_2(Y_s^2, Z_s^2, \psi_s^1) - f_2(Y_s^2, Z_s^1, \psi_s^1)}{\widehat{Z}_s} \mathbf{1}_{\widehat{Z}_s \neq 0} \end{aligned}$$

then

$$\begin{aligned} h_s &= f_s + \alpha_s \widehat{Y}_s + \beta_s \widehat{Z}_s + f_2(Y_s^2, Z_s^2, \psi_s^2) - f_2(Y_s^2, Z_s^2, \psi_s^1) \\ &\geq f_s + \alpha_s \widehat{Y}_s + \beta_s \widehat{Z}_s + \int_{\mathcal{U}} \kappa_s^{Y_s^2, Z_s^2, \psi_s^1, \psi_s^2} \widehat{\psi}_s(u) \mu(du) \end{aligned}$$

since f_2 satisfies $(\mathbf{H_{comp}})$. Moreover since f_2 is Lipschitz continuous w.r.t. z, $|\beta|$ is bounded by K, whereas from Assumption (H1), α is bounded from above. Moreover, the process $\kappa_s^{Y_s^2, Z_s^2, \psi_s^1, \psi_s^2}$ is controlled by $\vartheta \in L^2_\mu$. Therefore the process Γ defined by (34) is q-integrable for any $q \geq 2$ and

$$\widehat{Y}_t \geq \mathbb{E}\left[\Gamma_{t,T} \widehat{\xi} + \int_t^T \Gamma_{t,s} f_s ds \middle| \mathcal{F}_t \right].$$

To conclude recall that since $-1 \leq \kappa_t^{y,z,\psi,\phi}(u)$, $\Gamma_{t,s} \geq 0$ a.s. and by assumptions, $\widehat{\xi} \geq 0$ and $f_s \geq 0$. Therefore $\widehat{Y}_t \geq 0$ and the conclusion follows. □

Note that the conditions $(\mathbf{H_{ex}})$ are just imposed on f^1 to ensure existence of a solution (Y^1, Z^1, ψ^1, M^1). This proposition gives again uniqueness of the solution.

Corollary 2 *Assume* $(\mathbf{H_{comp}})$ *and (H4) (resp. (H5)). Then there exists at most one solution* (Y, Z, ψ, M) *of BSDE* (ξ, f) *in* $\mathcal{E}^2(0,T)$ *(resp.* $\mathcal{E}^p(0,T)$*).*

5 Random terminal times

We come back to the general multidimensional case but we assume that τ is a stopping time for the filtration $\mathbb{F}$, which need not be bounded. Assumptions $(\mathbf{H_{ex}})$ still hold with a monotonicity constant α and a Lipschitz constant K. (H2) is replaced by:

$$\forall r > 0,\ \forall n \in \mathbb{N}, \quad \sup_{|y| \leq r} (|f(t, y, 0, 0) - f(t, 0, 0, 0)|) \in L^1(\Omega \times (0, n)). \tag{H2"}$$

We assume that $1 < p$ and condition (H4) (or (H4')) is replaced by the following one: for some $\rho \in \mathbb{R}$ such that

$$\rho > \nu = \alpha + \frac{K^2}{(p-1) \wedge 1},$$

we have

$$\mathbb{E}\left[e^{p\rho\tau} |\xi|^p + \int_0^\tau e^{p\rho t} |f(t, 0, 0, 0)|^p dt \right] < +\infty. \tag{H5'}$$

The constant α appears in (H1) and K in (H3). We will need the following additional assumption

$$\xi \text{ is } \mathcal{F}_\tau - \text{measurable and } \mathbb{E}\left[\int_0^\tau e^{p\rho t} |f(t, \xi_t, \eta_t, \gamma_t)|^p dt \right] < +\infty, \tag{H6}$$

where $\xi_t = \mathbb{E}(\xi|\mathcal{F}_t)$ and (η, γ, N) are given by the martingale representation:

$$\xi = \mathbb{E}(\xi) + \int_0^\infty \eta_s dW_s + \int_0^\infty \int_{\mathcal{U}} \gamma_s(u)\widetilde{\pi}(du, ds) + N_\tau$$

with

$$\mathbb{E}\left[\left(\int_0^\infty |\eta_s|^2 ds + \int_0^\infty \int_{\mathcal{U}} |\gamma_s(u)|^2 \mu(du)ds + [N]_\tau\right)^{p/2}\right] < +\infty.$$

Definition 2 *A process $(Y, Z, \psi, M) = (Y_t, Z_t, \psi_t, M_t)_{t\geq 0}$, such that Y is progressively measurable and càdlàg and $(Z, \psi, M) \in \mathcal{D}(0,T) \times \mathcal{P} \times \mathcal{M}_{loc}$, with values in $\mathbb{R}^d \times \mathbb{R}^{d\times k} \times \mathbb{R}^d \times \mathbb{R}^d$ is a solution to the BSDE (2) with random terminal time τ with data $(\xi; f)$ if on the set $\{t \geq \tau\}$ $Y_t = \xi$ and $Z_t = \psi_t = M_t = 0$, $\mathbb{P}$-a.s., $t \mapsto f(t, Y_t, Z_t, \psi_t)\mathbf{1}_{t\leq T}$ belongs to $L^1_{loc}(0, \infty)$ for any $T \geq 0$, Z belongs to $L^2_{loc}(W)$, ψ belongs to $G_{loc}(\pi)$ and, $\mathbb{P}$-a.s., for all $0 \leq t \leq T$,*

$$\begin{aligned} Y_{t\wedge\tau} &= Y_{T\wedge\tau} + \int_{t\wedge\tau}^{T\wedge\tau} f(s, Y_s, Z_s, \psi_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s \\ &- \int_{t\wedge\tau}^{T\wedge\tau} \int_{\mathcal{U}} \psi_s(u)\widetilde{\pi}(du, ds) - \int_{t\wedge\tau}^{T\wedge\tau} dM_s. \end{aligned} \tag{36}$$

Proposition 5 *Under conditions (H1), (H2"), (H3), (H5') and (H6), the BSDE (36) has at most one solution satisfying*

$$\begin{aligned} &\mathbb{E}\left[e^{p\rho(t\wedge\tau)}|Y_{t\wedge\tau}|^p + \int_0^{T\wedge\tau} e^{p\rho s}|Y_s|^p ds + \int_0^{T\wedge\tau} e^{p\rho s}|Y_s|^{p-2}|Z_s|^2 \mathbf{1}_{Y_s\neq 0} ds\right] \\ &+\mathbb{E}\left[\int_0^{T\wedge\tau} e^{p\rho s}|Y_s|^{p-2}\mathbf{1}_{Y_s\neq 0}\|\psi_s\|^2_{L^2_\mu} ds + \int_0^{T\wedge\tau} e^{p\rho s}|Y_s|^{p-2}\mathbf{1}_{Y_s\neq 0} d[M]^c_s\right] \\ &+\mathbb{E}\left[\sum_{0<s\leq T\wedge\tau} e^{p\rho s}|\Delta M_s|^2 \left(|Y_{s^-}|^2 \vee |Y_{s^-} + \Delta M_s|^2\right)^{p/2-1} \mathbf{1}_{|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|\neq 0}\right] \\ &< +\infty. \end{aligned} \tag{37}$$

Proof. Assume that there exist two solutions (Y, Z, ψ, M) and (Y', Z', ψ', M') satisfying (37) and let

$$\widehat{Y}_t = Y_t - Y'_t, \quad \widehat{Z}_t = Z_t - Z'_t, \quad \widehat{\psi}_t = \psi_t - \psi'_t, \quad \widehat{M}_t = M_t - M'_t.$$

Let us denote $c(p) = \frac{p}{2}((p-1)\wedge 1)$. From Corollary 1, Lemma 9 and Remark 1 we have for $0 \le t \le T$

$$
\begin{aligned}
&e^{p\rho(t\wedge\tau)}|\widehat{Y}_{t\wedge\tau}|^p + c(p)\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|\widehat{Y}_s|^{p-2}|\widehat{Z}_s|^2\mathbf{1}_{\widehat{Y}_s\neq 0}ds + c(p)\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|\widehat{Y}_s|^{p-2}\mathbf{1}_{\widehat{Y}_s\neq 0}d[\widehat{M}]^c_s\\
&+c(p)\sum_{t\wedge\tau<s\le T\wedge\tau} e^{p\rho s}\left(|\widehat{Y}_{s^-}|^2\vee|\widehat{Y}_{s^-}+\Delta\widehat{M}_s|^2\right)^{p/2-1}\mathbf{1}_{|\widehat{Y}_{s^-}|\vee|\widehat{Y}_{s^-}+\Delta\widehat{M}_s|\neq 0}|\Delta\widehat{M}_s|^2\\
&+c(p)\int_{t\wedge\tau}^{T\wedge\tau}\int_{\mathcal{U}} e^{p\rho s}\left(|\widehat{Y}_{s^-}|^2\vee|\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|^2\right)^{p/2-1}\mathbf{1}_{|\widehat{Y}_{s^-}|\vee|\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|\neq 0}|\widehat{\psi}_s(u)|^2\pi(du,ds)\\
&\le e^{p\rho(T\wedge\tau)}|\widehat{Y}_{T\wedge\tau}|^p\\
&\quad+p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\left(|\widehat{Y}_s|^{p-1}\check{\widehat{Y}}_s(f(s,Y_s,Z_s,\psi_s)-f(s,Y'_s,Z'_s,\psi'_s))-\rho|\widehat{Y}_s|^p\right)ds\\
&\quad-p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|\widehat{Y}_s|^{p-1}\check{\widehat{Y}}_s\widehat{Z}_s dW_s - p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|\widehat{Y}_{s^-}|^{p-1}\check{\widehat{Y}}_{s^-}d\widehat{M}_s\\
&\quad-p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|\widehat{Y}_s|^{p-1}\check{\widehat{Y}}_s\int_{\mathcal{U}}\widehat{\psi}_s(u)\widetilde{\pi}(du,ds). \qquad (38)
\end{aligned}
$$

From the assumption on f and Young's inequality we deduce that

$$
\begin{aligned}
|\widehat{y}|^{p-1}\check{\widehat{y}}(f(s,y,z,\psi)-f(s,y',z',\psi')-\rho|\widehat{y}|^p &\le \left(\alpha+\frac{K^2}{(p-1)\wedge 1}-\rho\right)|\widehat{y}|^p\\
&+\frac{(p-1)\wedge 1}{2}|\widehat{y}|^{p-2}\mathbf{1}_{\widehat{y}\neq 0}|\widehat{z}|^2+\frac{(p-1)\wedge 1}{2}|\widehat{y}|^{p-2}\mathbf{1}_{\widehat{y}\neq 0}\|\widehat{\psi}\|_{L^2}\\
&\le \frac{(p-1)\wedge 1}{2}|\widehat{y}|^{p-2}\mathbf{1}_{\widehat{y}\neq 0}|\widehat{z}|^2+\frac{(p-1)\wedge 1}{2}|\widehat{y}|^{p-2}\mathbf{1}_{\widehat{y}\neq 0}\|\widehat{\psi}\|_{L^2}.
\end{aligned}
$$

Note that from the integrability conditions on the solution every local martingale involved in (38) is a uniformly integrable martingale. Moreover using a localization argument (see Equation (31)), the two terms:

$$
\int_{t\wedge\tau}^{T\wedge\tau}\int_{\mathcal{U}} e^{p\rho s}\left(|\widehat{Y}_{s^-}|^2\vee|\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|^2\right)^{p/2-1}\mathbf{1}_{|\widehat{Y}_{s^-}|\vee|\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|\neq 0}|\widehat{\psi}_s(u)|^2\pi(du,ds)
$$

and

$$
\int_{t\wedge\tau}^{T\wedge\tau}\int_{\mathcal{U}} e^{p\rho s}|\widehat{Y}_s|^{p-2}\mathbf{1}_{\widehat{Y}_s\neq 0}\|\widehat{\psi}_s(u)\|_{L^2}\mu(du)ds
$$

have the same expectation. Hence taking the expectation in (38) we obtain:

$$
\mathbb{E}e^{p\rho(t\wedge\tau)}|\widehat{Y}_{t\wedge\tau}|^p \le \mathbb{E}e^{p\rho(T\wedge\tau)}|\widehat{Y}_{T\wedge\tau}|^p.
$$

If we replace ρ by ρ' with $\alpha+\frac{K^2}{(p-1)\wedge 1} < \rho' < \rho$ we obtain the same result, and thus we get for any $0 \le t \le T$

$$
\mathbb{E}e^{p\rho'(t\wedge\tau)}|\widehat{Y}_{t\wedge\tau}|^p \le e^{p(\rho'-\rho)T}\mathbb{E}e^{p\rho(T\wedge\tau)}|\widehat{Y}_{T\wedge\tau}|^p.
$$

We let T go to infinity to obtain $\widehat{Y}_t = 0$.

Therefore (Y,Z,ψ,M) and (Y',Z',ψ',M') satisty BSDE (36) and $Y = Y'$. Thus we have the same martingale parts and by orthogonality, $\widehat{Z} = \widehat{\psi} = \widehat{M} = 0$. Uniqueness of the solution is proved. □

Proposition 6 *Under conditions (H1), (H2"), (H3), (H5') and (H6), the BSDE* (36) *has a solution satisfying*

$$\begin{aligned}
&\mathbb{E}\left[e^{p\rho(t\wedge\tau)}|Y_{t\wedge\tau}|^p+\int_0^{T\wedge\tau}e^{p\rho s}|Y_s|^p ds+\int_0^{T\wedge\tau}e^{p\rho s}|Y_s|^{p-2}|Z_s|^2\mathbf{1}_{Y_s\neq 0}ds\right]\\
&+\mathbb{E}\left[\int_0^{T\wedge\tau}e^{p\rho s}|Y_s|^{p-2}\mathbf{1}_{Y_s\neq 0}\|\psi_s\|^2_{L^2_\mu}ds+\int_0^{T\wedge\tau}e^{p\rho s}|Y_s|^{p-2}\mathbf{1}_{Y_s\neq 0}d[M]^c_s\right]\\
&+\mathbb{E}\left[\sum_{0<s\leq T\wedge\tau}e^{p\rho s}|\Delta M_s|^2\left(|Y_{s^-}|^2\vee|Y_{s^-}+\Delta M_s|^2\right)^{p/2-1}\mathbf{1}_{|Y_{s^-}|\vee|Y_{s^-}+\Delta M_s|\neq 0}\right]\\
&\leq C\mathbb{E}\left[e^{p\rho\tau}|\xi|^p+\int_0^\tau e^{p\rho s}|f(s,0,0,0)|^p ds\right]. \qquad (39)
\end{aligned}$$

Moreover

$$\begin{aligned}
&\mathbb{E}\left(\int_0^\tau e^{2\rho s}|Z_s|^2ds\right)^{p/2}+\mathbb{E}\left(\int_0^\tau e^{2\rho s}\int_{\mathcal{U}}|\psi_s(u)|^2\mu(du)ds\right)^{p/2}+\mathbb{E}\left(\int_0^\tau e^{2\rho s}d[M]_s\right)^{p/2}\\
&\leq C\mathbb{E}\left[e^{p\rho\tau}|\xi|^p+\int_0^\tau e^{p\rho s}|f(s,0,0,0)|^p ds\right]. \qquad (40)
\end{aligned}$$

The constant C depends only on p, K and α.

Proof. We follow the proof of Theorem 4.1 in [31]. For each $n\in\mathbb{N}$ we construct a solution $\{(Y^n,Z^n,\psi^n,M^n),\ t\geq 0\}$ as follows. By Theorem 2, on the interval $[0,n]$:

$$\begin{aligned}
Y^n_t &= \mathbb{E}(\xi|\mathcal{F}_n)+\int_t^n\mathbf{1}_{[0,\tau]}(s)f(s,Y^n_s,Z^n_s,\psi^n_s)ds-\int_t^n Z^n_s dW_s\\
&- \int_t^n\int_{\mathcal{U}}\psi^n_s(u)\widetilde{\pi}(du,ds)-\int_t^n dM^n_s.
\end{aligned}$$

And for $t\geq n$ (Assumption (H6)):

$$Y^n_t=\xi_t,\quad Z^n_t=\eta_t,\quad \psi^n_t(u)=\gamma_t(u),\quad M^n_t=N_t.$$

- **Step 1:** *the sequence $\{(Y^n,Z^n,\psi^n,M^n),\ t\geq 0\}$ satisfies Inequality* (39).

Using Corollary 1, Lemma 9 and Remark 1 we have for $0\leq t\leq T\leq n$

$$\begin{aligned}
&e^{p\rho(t\wedge\tau)}|Y^n_{t\wedge\tau}|^p+c(p)\int_{t\wedge\tau}^{T\wedge\tau}e^{p\rho s}|Y^n_s|^{p-2}|Z^n_s|^2\mathbf{1}_{Y^n_s\neq 0}ds+c(p)\int_{t\wedge\tau}^{T\wedge\tau}e^{p\rho s}|Y^n_s|^{p-2}\mathbf{1}_{Y^n_s\neq 0}d[M^n]^c_s\\
&+c(p)\sum_{t\wedge\tau<s\leq T\wedge\tau}e^{p\rho s}|\Delta M^n_s|^2\left(|Y^n_{s^-}|^2\vee|Y^n_{s^-}+\Delta M^n_s|^2\right)^{p/2-1}\mathbf{1}_{|Y^n_{s^-}|\vee|Y^n_{s^-}+\Delta M^n_s|\neq 0}\\
&+c(p)\int_{t\wedge\tau}^{T\wedge\tau}\int_{\mathcal{U}}e^{p\rho s}|\psi^n_s(u)|^2\left(|Y^n_{s^-}|^2\vee|Y^n_{s^-}+\psi^n_s(u)|^2\right)^{p/2-1}\mathbf{1}_{|Y^n_{s^-}|\vee|Y^n_{s^-}+\psi^n_s(u)|\neq 0}\pi(du,ds)\\
&\leq e^{p\rho(T\wedge\tau)}|Y^n_{T\wedge\tau}|^p+p\int_{t\wedge\tau}^{T\wedge\tau}e^{p\rho s}\left(|Y^n_s|^{p-1}\check{Y}^n_s f(s,Y^n_s,Z^n_s,\psi^n_s)-\rho|Y^n_s|^p\right)ds\\
&\quad -p\int_{t\wedge\tau}^{T\wedge\tau}e^{p\rho s}|Y^n_s|^{p-1}\check{Y}^n_s Z^n_s dW_s-p\int_{t\wedge\tau}^{T\wedge\tau}e^{p\rho s}|Y^n_{s^-}|^{p-1}\check{Y}^n_{s^-}dM^n_s\\
&\quad -p\int_{t\wedge\tau}^{T\wedge\tau}e^{p\rho s}|Y^n_s|^{p-1}\check{Y}^n_s\int_{\mathcal{U}}\psi^n_s(u)\widetilde{\pi}(du,ds).
\end{aligned}$$

Now with Young's inequality and for some $\delta > 0$ sufficiently small

$$
\begin{aligned}
|y|^{p-1}\check{y}f(t,y,z,\psi) \leq & \left(\alpha+\delta+\frac{K^2}{((p-1)\wedge 1-2\delta)}\right)|y|^p \\
& + \left(\frac{(p-1)\wedge 1}{2}-\delta\right)|y|^{p-2}\mathbf{1}_{y\neq 0}|z|^2 + \frac{1}{p}|f(t,0,0,0)|^p\left(\frac{p\delta}{(p-1)\wedge 1}\right)^{1-p} \\
& + \left(\frac{(p-1)\wedge 1}{2}-\delta\right)|y|^{p-2}\mathbf{1}_{y\neq 0}\|\psi\|_{L^2}. \qquad (41)
\end{aligned}
$$

We choose $\delta > 0$ such that $\alpha + 2\delta + \frac{K^2}{((p-1)\wedge 1-2\delta)} \leq \rho$ and we obtain:

$$
\begin{aligned}
& e^{p\rho(t\wedge\tau)}|Y^n_{t\wedge\tau}|^p + p\delta\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|Y^n_s|^p ds + p\delta\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|Y^n_s|^{p-2}|Z^n_s|^2\mathbf{1}_{Y^n_s\neq 0}ds \\
& + c(p)\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|Y^n_s|^{p-2}\mathbf{1}_{Y^n_s\neq 0}d[M^n]^c_s \\
& + c(p)\sum_{t\wedge\tau<s\leq T\wedge\tau} e^{p\rho s}|\Delta M^n_s|^2\left(|Y^n_{s^-}|^2\vee|Y^n_{s^-}+\Delta M^n_s|^2\right)^{p/2-1}\mathbf{1}_{|Y^n_{s^-}|\vee|Y^n_{s^-}+\Delta M^n_s|\neq 0} \\
& + c(p)\int_{t\wedge\tau}^{T\wedge\tau}\int_{\mathcal{U}} e^{p\rho s}|\psi^n_s(u)|^2\left(|Y^n_{s^-}|^2\vee|Y^n_{s^-}+\psi^n_s(u)|^2\right)^{p/2-1}\mathbf{1}_{|Y^n_{s^-}|\vee|Y^n_{s^-}+\psi^n_s(u)|\neq 0}\pi(du,ds) \\
& - p\left(\frac{(p-1)\wedge 1}{2}-\delta\right)\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|Y^n_s|^{p-2}\mathbf{1}_{Y^n_s\neq 0}\|\psi^n_s\|_{L^2}ds \\
& \leq e^{p\rho(T\wedge\tau)}|Y_{T\wedge\tau}|^p + \int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|f(s,0,0,0)|^p\left(\frac{p\delta}{(p-1)\wedge 1}\right)^{1-p}ds \\
& \quad - p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|Y^n_s|^{p-1}\check{Y}^n_s Z^n_s dW_s \\
& \quad - p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|Y^n_{s^-}|^{p-1}\check{Y}^n_{s^-}dM^n_s - p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|Y^n_s|^{p-1}\check{Y}^n_s\int_{\mathcal{U}}\psi^n_s(u)\widetilde{\pi}(du,ds).
\end{aligned}
$$

Taking the expectation we get

$$
\begin{aligned}
& \mathbb{E}\left[e^{p\rho(t\wedge\tau)}|Y^n_{t\wedge\tau}|^p + p\delta\int_0^{T\wedge\tau} e^{p\rho s}|Y^n_s|^p ds\right] \\
& + p\delta\mathbb{E}\left[\int_0^{T\wedge\tau} e^{p\rho s}|Y^n_s|^{p-2}\mathbf{1}_{Y^n_s\neq 0}\|\psi^n_s\|_{L^2}ds + \int_0^{T\wedge\tau} e^{p\rho s}|Y^n_s|^{p-2}|Z^n_s|^2\mathbf{1}_{Y^n_s\neq 0}ds\right] \\
& + c(p)\mathbb{E}\int_0^{T\wedge\tau} e^{p\rho s}|Y^n_s|^{p-2}\mathbf{1}_{Y^n_s\neq 0}d[M^n]^c_s \\
& + c(p)\mathbb{E}\left[\sum_{0<s\leq T\wedge\tau} e^{p\rho s}|\Delta M^n_s|^2\left(|Y^n_{s^-}|^2\vee|Y^n_{s^-}+\Delta M^n_s|^2\right)^{p/2-1}\mathbf{1}_{|Y^n_{s^-}|\vee|Y^n_{s^-}+\Delta M^n_s|\neq 0}\right] \\
& + c(p)\mathbb{E}\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\int_{\mathcal{U}}|\psi^n_s(u)|^2 \\
& \qquad \left[\left(|Y^n_{s^-}|^2\vee|Y^n_{s^-}+\psi^n_s(u)|^2\right)^{p/2-1}\mathbf{1}_{|Y^n_{s^-}|\vee|Y^n_{s^-}+\psi^n_s(u)|\neq 0}\pi(du,ds) - \frac{1}{2}|Y^n_s|^{p-2}\mathbf{1}_{Y^n_s\neq 0}\mu(du)ds\right] \\
& \leq \mathbb{E}\left[e^{p\rho(T\wedge\tau)}|Y^n_{T\wedge\tau}|^p + \left(\frac{p\delta}{(p-1)\wedge 1}\right)^{1-p}\int_0^{T\wedge\tau} e^{p\rho s}|f(s,0,0,0)|^p ds\right]. \qquad (42)
\end{aligned}
$$

Using an argument based on Burkholder-Davis-Gundy inequality (see Step 1 in the proof of Proposition 2 ($p \geq 2$) or Proposition 3 ($p < 2$)) we can moreover include a $\sup_{t\in[0,n]}$ inside the expectation on the left-hand side.

- **Step 2:** *the sequence* (Y^n) *converges.*

Take $m > n$ and define

$$\widehat{Y}_t = Y_t^m - Y_t^n, \quad \widehat{Z}_t = Z_t^m - Z_t^n, \quad \widehat{\psi}_t = \psi_t^m - \psi_t^n, \quad \widehat{M}_t = M_t^m - M_t^n.$$

For $n \leq t \leq m$,

$$\begin{aligned}\widehat{Y}_t &= \int_{t\wedge\tau}^{m\wedge\tau} f(s, Y_s^m, Z_s^m, \psi_s^m)ds - \int_{t\wedge\tau}^{m\wedge\tau} \widehat{Z}_s dW_s - \int_{t\wedge\tau}^{m\wedge\tau}\int_{\mathcal{U}} \widehat{\psi}_s(u)\widetilde{\pi}(du,ds) \\ &\quad -\widehat{M}_{m\wedge\tau} + \widehat{M}_{t\wedge\tau}.\end{aligned}$$

Thus for $n \leq t \leq m$,

$$\begin{aligned}
&e^{p\rho(t\wedge\tau)}|\widehat{Y}_{t\wedge\tau}|^p + c(p)\int_{t\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\widehat{Y}_s|^{p-2}|\widehat{Z}_s|^2\mathbf{1}_{\widehat{Y}_s\neq 0}ds + c(p)\int_{t\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\widehat{Y}_s|^{p-2}\mathbf{1}_{\widehat{Y}_s\neq 0}d[\widehat{M}]_s^c \\
&+c(p)\sum_{t\wedge\tau<s\leq m\wedge\tau} e^{p\rho s}|\Delta\widehat{M}_s|^2\left(|\widehat{Y}_{s^-}|^2 \vee |\widehat{Y}_{s^-}+\Delta\widehat{M}_s|^2\right)^{p/2-1}\mathbf{1}_{|\widehat{Y}_{s^-}|\vee|\widehat{Y}_{s^-}+\Delta\widehat{M}_s|\neq 0} \\
&+c(p)\int_{t\wedge\tau}^{m\wedge\tau}\int_{\mathcal{U}} e^{p\rho s}|\widehat{\psi}_s(u)|^2\left(|\widehat{Y}_{s^-}|^2 \vee |\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|^2\right)^{p/2-1}\mathbf{1}_{|\widehat{Y}_{s^-}|\vee|\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|\neq 0}\pi(du,ds) \\
&\leq p\int_{t\wedge\tau}^{m\wedge\tau} e^{p\rho s}\left(|\widehat{Y}_s|^{p-1}\check{\widehat{Y}}_s f(s,Y_s^m,Z_s^m,\psi_s^m) - \rho|\widehat{Y}_s|^p\right)ds \\
&\quad -p\int_{t\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\widehat{Y}_s|^{p-1}\check{\widehat{Y}}_s\widehat{Z}_s dW_s - p\int_{t\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\widehat{Y}_{s^-}|^{p-1}\check{\widehat{Y}}_{s^-}d\widehat{M}_s \\
&\quad -p\int_{t\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\widehat{Y}_s|^{p-1}\check{\widehat{Y}}_s\int_{\mathcal{U}}\widehat{\psi}_s(u)\widetilde{\pi}(du,ds) \\
&\leq p\int_{t\wedge\tau}^{m\wedge\tau} e^{p\rho s}\left(\alpha|\widehat{Y}_s|^p + K|\widehat{Y}_s|^{p-1}|\widehat{Z}_s| + K|\widehat{Y}_s|^{p-1}\|\widehat{\psi}_s\|_{L^2} - \rho|\widehat{Y}_s|^p\right)ds \\
&\quad +p\int_{t\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\widehat{Y}_s|^{p-1}\check{\widehat{Y}}_s f(s,\xi_s,\eta_s,\gamma_s)ds \\
&\quad -p\int_{t\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\widehat{Y}_s|^{p-1}\check{\widehat{Y}}_s\widehat{Z}_s dW_s - p\int_{t\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\widehat{Y}_{s^-}|^{p-1}\check{\widehat{Y}}_{s^-}d\widehat{M}_s \\
&\quad -p\int_{t\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\widehat{Y}_s|^{p-1}\check{\widehat{Y}}_s\int_{\mathcal{U}}\widehat{\psi}_s(u)\widetilde{\pi}(du,ds).
\end{aligned}$$

By an argument already used to control the generator (see (41)) and to obtain Inequality (42), we deduce that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[n,m]} e^{p\rho(t\wedge\tau)}|\widehat{Y}_{t\wedge\tau}|^p + \int_{n\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\widehat{Y}_s|^p ds + \int_{n\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\widehat{Y}_s|^{p-2}\mathbf{1}_{\widehat{Y}_s\neq 0} d[\widehat{M}]^c_s\right] \\
&+\mathbb{E}\left[\int_{n\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\widehat{Y}_s|^{p-2}|\widehat{Z}_s|^2\mathbf{1}_{\widehat{Y}_s\neq 0} ds + \int_{n\wedge\tau}^{m\wedge\tau}\int_{\mathcal{U}} e^{p\rho s}|\widehat{Y}_s|^{p-2}|\widehat{\psi}_s(u)|^2\mathbf{1}_{\widehat{Y}_s\neq 0}\mu(du)ds\right] \\
&+\mathbb{E}\left[\sum_{n\wedge\tau<s\leq m\wedge\tau} e^{p\rho s}|\Delta\widehat{M}_s|^2\left(|\widehat{Y}_{s^-}|^2\vee|\widehat{Y}_{s^-}+\Delta\widehat{M}_s|^2\right)^{p/2-1}\mathbf{1}_{\widehat{Y}_{s^-}\neq 0}\right] \\
&\quad\leq C\mathbb{E}\int_{n\wedge\tau}^{\tau} e^{p\rho s}|f(s,\xi_s,\eta_s,\gamma_s)|^p ds. \qquad (43)
\end{aligned}$$

By assumption the last term goes to zero as n goes to infinity. Next for $t\leq n$

$$\begin{aligned}
\widehat{Y}_t &= \widehat{Y}_n + \int_{n\wedge\tau}^{m\wedge\tau}(f(s,Y^m_s,Z^m_s,\psi^m_s)-f(s,Y^n_s,Z^n_s,\psi^n_s))ds - \int_{n\wedge\tau}^{m\wedge\tau}\widehat{Z}_s dW_s \\
&\quad - \int_{n\wedge\tau}^{m\wedge\tau}\int_{\mathcal{U}}\widehat{\psi}_s(u)\widetilde{\pi}(du,ds) - \widehat{M}_{m\wedge\tau} + \widehat{M}_{n\wedge\tau}.
\end{aligned}$$

It follows from the same argument as in the proof of Proposition 5 that

$$\begin{aligned}
\mathbb{E}e^{p\rho(t\wedge\tau)}|\widehat{Y}_{t\wedge\tau}|^p + \mathbb{E}\int_0^{\tau} e^{p\rho s}|\widehat{Y}_s|^p ds &\leq \mathbb{E}e^{p\rho(n\wedge\tau)}|\widehat{Y}_n|^p \\
&\leq C\mathbb{E}\int_{n\wedge\tau}^{\tau} e^{p\rho s}|f(s,\xi_s,\eta_s,\gamma_s)|^p ds
\end{aligned}$$

and the convergence of the sequence Y^n. Moreover from the first step the limit satisfies the a priori estimate (39).

- **Step 3:** *convergence of the martingale part* (Z^n,ψ^n,M^n).

The proof is rather different for $p\geq 2$ and $p<2$. In the first case, we follow the proof of [31], Theorem 4.1 and Proposition 2. We apply Itô's formula to $e^{2\rho s}|\widehat{Y}_s|^2$ for $n\leq t\leq m$:

$$\begin{aligned}
&\int_{t\wedge\tau}^{m\wedge\tau} e^{2\rho s}|\widehat{Z}_s|^2 ds + \int_{t\wedge\tau}^{m\wedge\tau}\int_{\mathcal{U}} e^{2\rho s}|\widehat{\psi}_s(u)|^2\pi(du,ds) + \int_{t\wedge\tau}^{m\wedge\tau} e^{2\rho s}d\widehat{M}_s \\
&= 2\int_{t\wedge\tau}^{m\wedge\tau} e^{2\rho s}\widehat{Y}_s\left(f(s,Y^m_s,Z^m_s,\psi^m_s)-f(s,\xi_s,\eta_s,\gamma_s)-\rho|\widehat{Y}_s|^2\right)ds \\
&\quad +2\int_{t\wedge\tau}^{m\wedge\tau} e^{2\rho s}\widehat{Y}_s f(s,\xi_s,\eta_s,\gamma_s)ds \\
&\quad -2\int_{t\wedge\tau}^{m\wedge\tau} e^{2\rho s}\widehat{Y}_s\widehat{Z}_s dW_s - 2\int_{t\wedge\tau}^{m\wedge\tau} e^{2\rho s}\widehat{Y}_{s^-}d\widehat{M}_s - 2\int_{t\wedge\tau}^{m\wedge\tau} e^{2\rho s}\int_{\mathcal{U}}\widehat{Y}_{s^-}\widehat{\psi}_s(u)\widetilde{\pi}(du,ds).
\end{aligned}$$

With the same arguments used to obtain (26), the assumptions on f, Burkholder-Davis-Gundy ($p/2\geq 1$) and Young's inequality lead to:

$$\begin{aligned}
&\mathbb{E}\left(\int_{n\wedge\tau}^{m\wedge\tau} e^{2\rho s}|\widehat{Z}_s|^2 ds\right)^{p/2} + \mathbb{E}\left(\int_{n\wedge\tau}^{m\wedge\tau} e^{2\rho s}\int_{\mathcal{U}}|\widehat{\psi}_s(u)|^2\mu(du)ds\right)^{p/2} + \mathbb{E}\left(\int_{n\wedge\tau}^{m\wedge\tau} e^{2\rho s}d\widehat{M}_s\right)^{p/2} \\
&\quad\leq \widetilde{C}_{p,K,T,\varepsilon}\,\mathbb{E}\left(\sup_{n\leq t} e^{p\rho s}|\widehat{Y}_s|^p\right) + \widetilde{C}_p\mathbb{E}\left[\int_{n\wedge\tau}^{\tau} e^{p\rho s}|f(s,\xi_s,\eta_s,\gamma_s)|^p ds\right].
\end{aligned}$$

Then following the same scheme but with $t \leq n$, we obtain:

$$\mathbb{E}\left(\int_0^{n\wedge\tau} e^{2\rho s}|\widehat{Z}_s|^2 ds\right)^{p/2} + \mathbb{E}\left(\int_0^{n\wedge\tau} e^{2\rho s}\int_{\mathcal{U}}|\widehat{\psi}_s(u)|^2\mu(du)ds\right)^{p/2} + \mathbb{E}\left(\int_0^{n\wedge\tau} e^{2\rho s}d\widehat{M}_s\right)^{p/2}$$
$$\leq \widetilde{C}_{p,K,T,\varepsilon}\,\mathbb{E}\left(e^{p\rho(n\wedge\tau)}|\widehat{Y}_{n\wedge\tau}|^p\right).$$

Now assume that $p \in (1,2)$. From Inequality (43) and by the proof of uniqueness we deduce that

$$\mathbb{E}\left[\int_0^{m\wedge\tau} e^{p\rho s}|\widehat{Y}_s|^{p-2}\mathbf{1}_{\widehat{Y}_s\neq 0}d[\widehat{M}]^c_s + \int_0^{m\wedge\tau} e^{p\rho s}|\widehat{Y}_s|^{p-2}|\widehat{Z}_s|^2\mathbf{1}_{\widehat{Y}_s\neq 0}ds\right]$$
$$+\mathbb{E}\left[\int_0^{m\wedge\tau}\int_{\mathcal{U}} e^{p\rho s}|\widehat{Y}_s|^{p-2}|\widehat{\psi}_s(u)|^2\mathbf{1}_{\widehat{Y}_s\neq 0}\mu(du)ds\right]$$
$$+\mathbb{E}\left[\int_0^{m\wedge\tau}\int_{\mathcal{U}} e^{p\rho s}|\widehat{\psi}_s(u)|^2\left(|\widehat{Y}_{s^-}|^2 \vee |\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|^2\right)^{p/2-1}\mathbf{1}_{|\widehat{Y}_{s^-}|\vee|\widehat{Y}_{s^-}+\widehat{\psi}_s(u)|\neq 0}\pi(du,ds)\right]$$
$$+\mathbb{E}\left[\sum_{0<s\leq m\wedge\tau} e^{p\rho s}|\Delta\widehat{M}_s|^2\left(|\widehat{Y}_{s^-}|^2 \vee |\widehat{Y}_{s^-}+\Delta\widehat{M}_s|^2\right)^{p/2-1}\mathbf{1}_{|\widehat{Y}_{s^-}|\vee|\widehat{Y}_{s^-}+\Delta\widehat{M}_s|\neq 0}\right]$$
$$\leq C\mathbb{E}\int_{n\wedge\tau}^{\tau} e^{p\rho s}|f(s,\xi_s,\eta_s,\gamma_s)|^p ds.$$

Then we can use again the argument (33) in order to have:

$$\mathbb{E}\left(\int_0^{m\wedge\tau} e^{2\rho s}|\widehat{Z}_s|^2 ds\right)^{p/2} = \mathbb{E}\left(\int_0^{m\wedge\tau} e^{2\rho s}\mathbf{1}_{\widehat{Y}_s\neq 0}|\widehat{Z}_s|^2 ds\right)^{p/2}$$
$$\leq \frac{2-p}{2}\mathbb{E}\left[\sup_{n\leq t}\left(e^{\rho p s}|\widehat{Y}_s|^p\right)\right] + \frac{p}{2}\mathbb{E}\int_0^T e^{\rho p s}|\widehat{Y}_s|^{p-2}\mathbf{1}_{\widehat{Y}_s\neq 0}|\widehat{Z}_s|^2 ds$$
$$\leq \frac{2-p}{2}\mathbb{E}\left[\sup_{n\leq t}\left(e^{\rho p s}|\widehat{Y}_s|^p\right)\right] + \frac{p}{2}C\mathbb{E}\int_{n\wedge\tau}^{\tau} e^{p\rho s}|f(s,\xi_s,\eta_s,\gamma_s)|^p ds.$$

We can repeat this for $\widehat{\psi}$ and $\widehat{M}$.

Therefore in both cases we proved that the sequence (Z^n,ψ^n,M^n) is a Cauchy sequence for the norm:

$$\mathbb{E}\left(\int_0^{\tau} e^{2\rho s}|\widehat{Z}_s|^2 ds\right)^{p/2} + \mathbb{E}\left(\int_0^{\tau} e^{2\rho s}\int_{\mathcal{U}}|\widehat{\psi}_s(u)|^2\mu(du)ds\right)^{p/2} + \mathbb{E}\left(\int_0^{\tau} e^{2\rho s}d[\widehat{M}]_s\right)^{p/2}$$

Hence it converges to (Z,ψ,M) and from the two previous steps the limit (Y,Z,ψ,M) is a solution of the BSDE (36) which satisfies (39) and (40). □

From the two previous propositions we deduce:

Theorem 3 *Under conditions (H1), (H2"), (H3), (H5') and (H6), the BSDE* (36) *has a unique solution satisfying* (39) *and* (40).

Remark 2 *As in Pardoux [31] (Exercise 4.2), one can replace the condition* $\rho > \nu = \alpha + \frac{K^2}{(p-1)}$ *by the condition* $\rho > \alpha$ *if there exists a progressively measurable process* g *such that for any* z *and* ψ

$$|f(t,0,z,\psi)| \leq g_t,$$

and

$$E\int_0^\tau e^{p\rho t}|g_t|^p dt < \infty.$$

In this case the conclusion of Theorem 3 also holds.

Indeed for $p \geq 2$ as in the proof of Proposition 2 we can obtain for every $0 \leq t \leq T$ and every $\rho > \alpha$

$$\begin{aligned}
&e^{p\rho(t\wedge\tau)}|Y_{t\wedge\tau}|^p + \kappa_p \int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|Y_s|^{p-2}|Z_s|^2 ds + \kappa_p \int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|Y_{s^-}|^{p-2} d[M]_s \\
&\quad + \kappa_p \int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|Y_{s^-}|^{p-2}\|\psi_s\|^2_{L^2_\mu} ds \\
&\leq e^{p\rho(T\wedge\tau)}|Y_{T\wedge\tau}|^p + \int_{t\wedge\tau}^{T\wedge\tau} pe^{p\rho s}\left(Y_s|Y_s|^{p-2} f(s, Y_s, Z_s, \psi_s) - \rho|Y_s|^p\right) ds \\
&- p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s} Y_{s^-}|Y_{s^-}|^{p-2} dM_s - p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s} Y_{s^-}|Y_{s^-}|^{p-2} Z_s dW_s \\
&- \int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\int_{\mathcal{U}} \left(|Y_{s^-} + \psi_s(u)|^p - |Y_{s^-}|^p\right) \widetilde{\pi}(du, ds)
\end{aligned}$$

where κ_p just depends on p. Now for any $\varepsilon > 0$

$$y|y|^{p-2} f(s, y, z, \psi) - \rho|y|^p \leq (\alpha - \rho)|y|^p + |y|^{p-1} g_s \leq (\alpha + \varepsilon - \rho)|y|^p + \frac{1}{p}\left(\frac{p\varepsilon}{p-1}\right)^{1-p} g_s^p.$$

Therefore for any $\rho > \alpha$ we choose ε such that $\rho > \alpha + \varepsilon$ and taking the expectation we have

$$\begin{aligned}
&\mathbb{E}e^{p\rho(t\wedge\tau)}|Y_{t\wedge\tau}|^p + \mathbb{E}\kappa_p \int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|Y_s|^{p-2}|Z_s|^2 ds + \kappa_p \mathbb{E}\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|Y_{s^-}|^{p-2} d[M]_s \\
&\quad + \mathbb{E}\kappa_p \int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|Y_{s^-}|^{p-2}\|\psi_s\|^2_{L^2_\mu} ds \\
&\leq \mathbb{E}e^{p\rho(T\wedge\tau)}|Y_{T\wedge\tau}|^p + \left(\frac{p\varepsilon}{p-1}\right)^{1-p} \mathbb{E}\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|g_s|^p ds.
\end{aligned}$$

The same argument can be used in the case $1 < p < 2$.

Remark 3 *In dimension one, if ξ and $f(t, 0, 0, 0)$ are non negative, the L^p-solution Y is non negative and if $f(s, 0, z, \psi) \leq 0$ for any z and ψ, the conclusion of Theorem 3 holds.*

Remark 4 *In dimension one, under the assumptions of Theorem 3 (or of the previous remarks), and with condition (H3'), then the comparison result (Proposition 4) holds.*

Indeed we can sketch the proof to obtain that for any $0 \leq t \leq T$

$$\widehat{Y}_{t\wedge\tau} \geq \mathbb{E}\left[\Gamma_{t\wedge\tau, T\wedge\tau}\widehat{Y}_{T\wedge\tau} + \int_{t\wedge\tau}^{T\wedge\tau} \Gamma_{t\wedge\tau, s} f_s ds \middle| \mathcal{F}_{t\wedge\tau}\right]$$

with suitable integrability conditions. The conclusion follows by letting T go to $+\infty$.

References

[1] G. Barles, R. Buckdahn, and É. Pardoux. Backward stochastic differential equations and integral-partial differential equations. *Stochastics Stochastics Rep.*, 60(1-2):57–83, 1997.

[2] P. Barrieu and N. El Karoui. Inf-convolution of risk measures and optimal risk transfer. *Finance Stoch.*, 9(2):269–298, 2005.

[3] D. Becherer. Bounded solutions to backward SDE's with jumps for utility optimization and indifference hedging. *Ann. Appl. Probab.*, 16(4):2027–2054, 2006.

[4] P. Briand and R. Carmona. BSDEs with polynomial growth generators. *J. Appl. Math. Stochastic Anal.*, 13(3):207–238, 2000.

[5] Ph. Briand, B. Delyon, Y. Hu, E. Pardoux, and L. Stoica. L^p solutions of backward stochastic differential equations. *Stochastic Process. Appl.*, 108(1):109–129, 2003.

[6] R. Carbone, B. Ferrario, and M. Santacroce. Backward stochastic differential equations driven by càdlàg martingales. *Teor. Veroyatn. Primen.*, 52(2):375–385, 2007.

[7] C. Ceci, A. Cretarola, and F. Russo. BSDEs under partial information and financial applications. *Stochastic Process. Appl.*, 124(8):2628–2653, 2014.

[8] S. N. Cohen and R. J. Elliott. Existence, uniqueness and comparisons for BSDEs in general spaces. *Ann. Probab.*, 40(5):2264–2297, 2012.

[9] Ł. Delong. *Backward stochastic differential equations with jumps and their actuarial and financial applications*. European Actuarial Academy (EAA) Series. Springer, London, 2013. BSDEs with jumps.

[10] K. Dzhaparidze and E. Valkeila. On the Hellinger type distances for filtered experiments. *Probab. Theory Related Fields*, 85(1):105–117, 1990.

[11] N. El Karoui and S.-J. Huang. A general result of existence and uniqueness of backward stochastic differential equations. In *Backward stochastic differential equations (Paris, 1995–1996)*, volume 364 of *Pitman Res. Notes Math. Ser.*, pages 27–36. Longman, Harlow, 1997.

[12] N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng, and M. C. Quenez. Reflected solutions of backward SDE's, and related obstacle problems for PDE's. *Ann. Probab.*, 25(2):702–737, 1997.

[13] N. El Karoui, S.G. Peng, and M.C. Quenez. Backward stochastic differential equations in finance. *Math. Finance*, 7(1):1–71, 1997.

[14] P. Graewe, U. Horst, and J. Qiu. A Non-Markovian Liquidation Problem and Backward SPDEs with Singular Terminal Conditions. *ArXiv e-prints*, 2013.

[15] S. Hamadène and J. P. Lepeltier. Backward equations, stochastic control and zero-sum stochastic differential games. *Stochastics Stochastics Rep.*, 54(3-4):221–231, 1995.

[16] S. Hamadène, J.-P. Lepeltier, and S. Peng. BSDEs with continuous coefficients and stochastic differential games. In *Backward stochastic differential equations (Paris, 1995–1996)*, volume 364 of *Pitman Res. Notes Math. Ser.*, pages 115–128. Longman, Harlow, 1997.

[17] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003.

[18] T. Klimsiak. BSDEs with monotone generator and two irregular reflecting barriers. *Bull. Sci. Math.*, 137(3):268–321, 2013.

[19] T. Klimsiak. Reflected BSDEs on Filtered Probability Spaces. *ArXiv e-prints*, August 2014.

[20] T. Klimsiak and A. Rozkosz. Dirichlet forms and semilinear elliptic equations with measure data. *J. Funct. Anal.*, 265(6):890–925, 2013.

[21] Tomasz Klimsiak. On time-dependent functionals of diffusions corresponding to divergence form operators. *J. Theoret. Probab.*, 26(2):437–473, 2013.

[22] T. Kruse and A. Popier. Minimal supersolutions for BSDEs with singular terminal condition and application to optimal position targeting. *ArXiv e-prints*, April 2015.

[23] E. Lenglart, D. Lépingle, and M. Pratelli. Présentation unifiée de certaines inégalités de la théorie des martingales. In *Seminar on Probability, XIV (Paris, 1978/1979) (French)*, volume 784 of *Lecture Notes in Math.*, pages 26–52. Springer, Berlin, 1980. With an appendix by Lenglart.

[24] J. Li and Q. Wei. L^p estimates for fully coupled FBSDEs with jumps. *Stochastic Process. Appl.*, 124(4):1582–1611, 2014.

[25] G. Liang, T. Lyons, and Z. Qian. Backward stochastic dynamics on a filtered probability space. *Ann. Probab.*, 39(4):1422–1448, 2011.

[26] M.-A. Morlais. A new existence result for quadratic BSDEs with jumps with application to the utility maximization problem. *Stochastic Process. Appl.*, 120(10):1966–1995, 2010.

[27] D. Nualart and W. Schoutens. Backward stochastic differential equations and Feynman-Kac formula for Lévy processes, with applications in finance. *Bernoulli*, 7(5):761–776, 2001.

[28] B. Øksendal and T. Zhang. Backward stochastic differential equations with respect to general filtrations and applications to insider finance. *Commun. Stoch. Anal.*, 6(4):703–722, 2012.

[29] E. Pardoux. Generalized discontinuous backward stochastic differential equations. In *Backward stochastic differential equations (Paris, 1995–1996)*, volume 364 of *Pitman Res. Notes Math. Ser.*, pages 207–219. Longman, Harlow, 1997.

[30] É. Pardoux. Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic PDEs of second order. In *Stochastic analysis and related topics, VI (Geilo, 1996)*, volume 42 of *Progr. Probab.*, pages 79–127. Birkhäuser Boston, Boston, MA, 1998.

[31] É Pardoux. BSDEs, weak convergence and homogenization of semilinear PDEs. In *Nonlinear analysis, differential equations and control (Montreal, QC, 1998)*, volume 528 of *NATO Sci. Ser. C Math. Phys. Sci.*, pages 503–549. Kluwer Acad. Publ., Dordrecht, 1999.

[32] É. Pardoux and S. Peng. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic partial differential equations and their applications (Charlotte, NC, 1991)*, volume 176 of *Lecture Notes in Control and Inform. Sci.*, pages 200–217. Springer, Berlin, 1992.

[33] É. Pardoux and S. G. Peng. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.*, 14(1):55–61, 1990.

[34] Philip E. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition, 2004. Stochastic Modelling and Applied Probability.

[35] M.-C. Quenez and A. Sulem. BSDEs with jumps, optimization and applications to dynamic risk measures. *Stochastic Process. Appl.*, 123(8):3328–3357, 2013.

[36] R. Rouge and N. El Karoui. Pricing via utility maximization and entropy. *Math. Finance*, 10(2):259–276, 2000. INFORMS Applied Probability Conference (Ulm, 1999).

[37] M. Royer. Backward stochastic differential equations with jumps and related non-linear expectations. *Stochastic Process. Appl.*, 116(10):1358–1376, 2006.

[38] Rong Situ. *Theory of stochastic differential equations with jumps and applications*. Mathematical and Analytical Techniques with Applications to Engineering. Springer, New York, 2005. Mathematical and analytical techniques with applications to engineering.

[39] S.J. Tang and X.J. Li. Necessary conditions for optimal control of stochastic systems with random jumps. *SIAM J. Control Optim.*, 32(5):1447–1475, 1994.

[40] J. Xia. Backward stochastic differential equation with random measures. *Acta Math. Appl. Sinica (English Ser.)*, 16(3):225–234, 2000.

[41] S. Yao. Lp Solutions of Backward Stochastic Differential Equations with Jumps. *ArXiv e-prints*, July 2010.