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BASES OF SUBALGEBRAS OF $\mathbb{K}[[x]]$ AND $\mathbb{K}[x]$

A. ASSI, P. A. GARCÍA-SÁNCHEZ, AND V. MICALE

ABSTRACT. Let f_1, \dots, f_s be formal power series (respectively polynomials) in the variable x . We study the semigroup of orders of the formal series in the algebra $K[[f_1, \dots, f_s]] \subseteq K[[x]]$ (respectively the semigroup of degrees of polynomials in $K[f_1, \dots, f_s] \subseteq K[x]$). We give procedures to compute these semigroups and several applications. We prove in particular that the space curve parametrized by f_1, \dots, f_s has a flat deformation into a monomial curve.

1. INTRODUCTION

Let \mathbb{K} be a field and let $\mathbb{K}[[x]]$ be the ring of formal power series over \mathbb{K} . Let $f_1(x), \dots, f_s(x)$ be s elements of $\mathbb{K}[[x]]$ and let $R = \mathbb{K}[[f_1, \dots, f_s]]$ be the subalgebra of $\mathbb{K}[[x]]$ generated by f_1, \dots, f_s . Given $f \in R$, let $\text{o}(f)$ the order of f . The set $\text{o}(R) = \{\text{o}(f) \mid f \in R\}$ is a submonoid of \mathbb{N} , and the knowledge of a system of generators of this monoid is important for the understanding of the subalgebra R . When furthermore $\mathbb{K}[[x]]$ is an R -module of finite length, then $\text{o}(R)$ is a numerical semigroup.

A similar construction can be made in the ring of polynomials $\mathbb{K}[x]$. More precisely let $f_1(x), \dots, f_s(x)$ be s elements of $\mathbb{K}[x]$ and let $A = \mathbb{K}[f_1, \dots, f_s]$ be the subalgebra of $\mathbb{K}[x]$ generated by f_1, \dots, f_s . Given $f \in A$, let $\text{d}(f)$ the degree of f . The set $\text{d}(A) = \{\text{d}(f) \mid f \in A\}$ is a submonoid $\text{d}(A)$ of \mathbb{N} , and the knowledge of a system of generators of this monoid is important for the understanding of the subalgebra A . When furthermore $\mathbb{K}[x]$ is an A -module of finite length, then $\text{d}(A)$ is a numerical semigroup.

A numerical semigroup S is a submonoid of the set of nonnegative integers under addition such that the $\mathbb{N} \setminus S$ is finite, or equivalently, $\text{gcd}(S) = 1$ (the greatest common divisor of the elements of S), see for instance [16]. In this case, there exists a minimum $c \in S$ such that $c + \mathbb{N} \subseteq S$. We call this element the *conductor* of S , and denote it by $c(S)$ (the motivation of this name and others coming from Algebraic Geometry is explained in [3, 8]).

Assume that f_i is a monomial x^{a_i} for every $i \in \{1, \dots, s\}$. Then $\text{o}(R)$ (respectively $\text{d}(A)$) is generated by a_1, \dots, a_s . In this case, $R \simeq \mathbb{K}[[X_1, \dots, X_s]]/T$ (respectively $A \simeq \mathbb{K}[X_1, \dots, X_s]/T$), where T is a prime binomial ideal and, thus, $V(T)$ is a toric variety.

Given a subalgebra $R = \mathbb{K}[[f_1, \dots, f_s]]$ (respectively $A = \mathbb{K}[f_1, \dots, f_s]$), the main objective of this paper is to describe an algorithm that calculates a generating system of $\text{o}(R)$ (respectively $\text{d}(A)$). The algorithm we present here allows us, by using the technique of homogenization, to construct a flat $\mathbb{K}[[u]]$ -module (respectively $\mathbb{K}[u]$ -module) which is a deformation of R (respectively A) to a binomial ideal. This technique is well known when $R = \mathbb{K}[[f_1, f_2]]$ and \mathbb{K} is algebraically closed field of characteristic zero (see [11] and [19]). It turns out that the same holds wherever we can associate a semigroup to the local subalgebra, and also that the same technique can be adapted to the global setting. As a particular case we prove that a plane polynomial curve has a deformation into a complete intersection monomial space curve.

The paper is organized as follows. In Section 2 we focus on the local case, namely the case of a subalgebra R of $\mathbb{K}[[x]]$. We introduce the notion of basis of R and we show how to construct such a basis. We also show that if $\text{o}(R)$ is a numerical semigroup, then every element of a reduced basis is a polynomial. In Section 3 we show how to construct a deformation from R to a toric ideal (or a formal toric variety) by using the technique of homogenization. In Section 4 we focus on the case when $R = \mathbb{K}[[f(x), g(x)]]$ and \mathbb{K} is an algebraically closed field of characteristic zero. The existence in this case of the theory of Newton-Puiseux allows us to precise the results of Sections 2 and 3. The difference with the procedure

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presented in Section 2 is that it does not rely in the computation of successive kernels. Then in Sections 5 and 6 we adapt the local results to the case of a subalgebra A of $\mathbb{K}[x]$. When $A = \mathbb{K}[f(x), g(x)]$ and \mathbb{K} is algebraically closed field of characteristic zero, a basis of A can be obtained by using the theory of approximate roots of the resultant of $X - f(x), Y - g(x)$, which is a polynomial with one place at infinity.

The procedures presented here have been implemented in **GAP** ([10]) and will be part of the forthcoming stable release of the package **numericalsgps** ([7]).

1.1. Some notation. We denote by $\langle A \rangle$ the monoid generated by A , $A \subseteq \mathbb{N}$, that is, the set $\{n_1x_1 + \dots + n_mx_m \mid m \in \mathbb{N}, n_i \in \mathbb{N}, x_i \in A \text{ for all } i \in \{1, \dots, m\}\}$.

Associated to each numerical semigroup S we can define a natural partial ordering \leq_S , where for two elements s and r in S we have $s \leq_S r$ if there exists $u \in S$ such that $r = s + u$. The set g_i of minimal elements in $S \setminus \{0\}$ with this ordering is called a *minimal set of generators* for S . The set of minimal generators is finite since for any $s \in S \setminus \{0\}$, we have $x \not\equiv y \pmod{s}$ if $x \neq y$ are minimal elements with respect to \leq_S . The cardinality of the minimal generating system is known as the *embedding dimension* of S .

2. SEMIGROUP OF A FORMAL SPACE CURVE

Let \mathbb{K} be a field. In this section we will consider rings R that are subalgebras of $\mathbb{K}[[x]]$ and such that, if we denote the integral closure of R in its quotient field by \bar{R} , then $\bar{R} = \mathbb{K}[[x]]$ and $\lambda_R(\bar{R}/R) < \infty$ where $\lambda_R(\cdot)$ is the length as R -module. Part of the results of this section are inspired in [15] and in [14]. An alternative procedure (implemented in Maple) is provided in [5]. The main difference with our approach is that we do not rely on the multiplicity sequence, and thus we do not need to perform blow-ups. Also we take intrinsic advantage in our implementation of the **GAP** package **numericalsgps** ([10, 7]).

Let $f = \sum_i c_i x^i \in \bar{R}^* = \bar{R} \setminus \{0\}$. Define $\text{supp}(f) = \{i, c_i \neq 0\}$. We call $\min \text{supp}(f)$ the *order* of f and we denote it by $\text{o}(f)$. We also set $M_o(f) = c_{\text{o}(f)} x^{\text{o}(f)}$. If $M_o(f) = x^{\text{o}(f)}$, then we shall say, by abuse of notation, that f is *monic*. We set $\text{o}(0) = +\infty$.

We denote by $\text{o}(R)$ the set of orders of elements in $R^* = R \setminus \{0\}$, that is, $\text{o}(R) = \{\text{o}(f) \mid f \in R^*\}$. We finally set $M_o(R) = \mathbb{K}[M_o(f) \mid f \in R^*]$.

Proposition 2.1. *Let f_1, f_2 be elements of \bar{R}^* and let $a = \min\{\text{o}(f_1), \text{o}(f_2)\}$.*

- (i) $a \leq \text{o}(f_1 + f_2)$.
- (ii) If $\text{o}(f_1) \neq \text{o}(f_2)$ then $a = \text{o}(f_1 + f_2)$.
- (iii) $\text{o}(f_1 f_2) = \text{o}(f_1) + \text{o}(f_2)$.

Proof. This follows easily from the definition of order. □

Proposition 2.2. [13, Lemma 3, p.486] *Let R_1 and R_2 be rings of our type such that $R_1 \subseteq R_2$ and $\text{o}(R_1) = \text{o}(R_2)$. Then $R_1 = R_2$.*

Proposition 2.3. [13, Proposition 1, p.488] *Let R be a ring of our type. Then $\lambda_R(\bar{R}/R) = |\mathbb{N} \setminus \text{o}(R)|$.*

The following two results appear in [15], and since this paper is very hard to find, we include the proofs for sake of completeness.

Proposition 2.4. [15] *Let R be a ring of our type. Then $\text{o}(R)$ is a numerical semigroup.*

Proof. Since R is a ring and by (iii) of Proposition 2.1, we have that $\text{o}(R)$ is a subsemigroup of \mathbb{N} . By $\lambda_R(\bar{R}/R) < \infty$ and Proposition 2.3, we have the proof. □

Proposition 2.5. [15] *Let R be a ring of our type. Then R contains every element $f \in \mathbb{K}[[x]]$ of order $\text{o}(f) \geq c(\text{o}(R))$.*

Proof. Use Proposition 2.2 with $R_1 = R$ and $R_2 = R + x^{c(\text{o}(R))} \mathbb{K}[[x]]$. □

This later result allows to work with polynomials instead of series.

Let f_1, \dots, f_s be in \bar{R}^* . Let $R = \mathbb{K}[[f_1, \dots, f_s]]$ be a subalgebra of $\mathbb{K}[[x]]$ as above, that is, the integral closure of R in its quotient field is $\bar{R} = \mathbb{K}[[x]]$ and $\lambda_R(\bar{R}/R) < \infty$.

Under these hypotheses on R , we have that $\mathfrak{o}(R)$ is a numerical semigroup (Proposition 2.4).

We say that the set $\{f_1, \dots, f_s\} \subset R^*$ is a *basis* of R if $\mathfrak{o}(R) = \langle \mathfrak{o}(f_1), \dots, \mathfrak{o}(f_s) \rangle$. The set $\{f_1, \dots, f_s\}$ is a basis of R if and only if $M_{\mathfrak{o}}(R) = \mathbb{K}[M_{\mathfrak{o}}(f_1), \dots, M_{\mathfrak{o}}(f_s)]$.

Proposition 2.6. *Let the notations be as above. Given $f(x) \in \mathbb{K}[[x]]$, there exist $g(x) \in R$ and $r(x) \in \mathbb{K}[[x]]$ such that the following conditions hold.*

- (i) $f(x) = g(x) + r(x) = \sum_{\alpha} c_{\alpha} f_1^{\alpha_1} \dots f_s^{\alpha_s} + r(x)$.
- (ii) If $g(x) \neq 0$ (respectively $r(x) \neq 0$), then $\mathfrak{o}(g) \geq \mathfrak{o}(f)$ (respectively $\mathfrak{o}(r) \geq \mathfrak{o}(f)$).
- (iii) Either $r(x) = 0$ or $\text{supp}(r(x)) \subseteq \mathbb{N} \setminus \langle \mathfrak{o}(f_1), \dots, \mathfrak{o}(f_s) \rangle$.

Proof. The assertion is clear if $f \in \mathbb{K}$. Suppose that $f \notin \mathbb{K}$ and let $f(x) = \sum_{i \geq p} c_i x^i$ with $p = \mathfrak{o}(f) \geq 0$.

- (1) If $p \notin \langle \mathfrak{o}(f_1), \dots, \mathfrak{o}(f_s) \rangle$, then we set $g^1 = 0$, $r^1 = c_p x^p$ and $f^1 = f - c_p x^p$.
- (2) If $p \in \langle \mathfrak{o}(f_1), \dots, \mathfrak{o}(f_s) \rangle$, then $c_p x^p = c_{\theta} M_0(f_1)^{\theta_1} \dots M_0(f_s)^{\theta_s}$. We set $g^1 = c_{\theta} f_1^{\theta_1} \dots f_s^{\theta_s}$, $r^1 = 0$ and $f^1 = f - g^1$.

In such a way that $f = f^1 + g^1 + r^1$, $g^1 \in R$, either $r^1 = 0$ or $\text{supp}(r^1) \subseteq \mathbb{N} \setminus \langle \mathfrak{o}(f_1), \dots, \mathfrak{o}(f_s) \rangle$, and if $f^1 \neq 0$, then $\mathfrak{o}(f^1) > \mathfrak{o}(f) = p$. Then we restart with f^1 . We construct in this way sequences $(f^k)_{k \geq 1}, (g^k)_{k \geq 1}, (r^k)_{k \geq 1}$ such that for all $k \geq 1$, $f = f^k + \sum_{i=1}^k g^i + \sum_{i=1}^k r^i$, and $\mathfrak{o}(f) < \mathfrak{o}(f^1) < \dots < \mathfrak{o}(f^k)$, $\sum_{i=1}^k g^i \in R$, $\text{supp}(\sum_{i=1}^k r^i) \subseteq \mathbb{N} \setminus \langle \mathfrak{o}(f_1), \dots, \mathfrak{o}(f_s) \rangle$ and for all $i < j \leq k$, if $g^i \neq 0 \neq g^j$ (respectively $r^i \neq 0 \neq r^j$), then $\mathfrak{o}(f) \leq \mathfrak{o}(g^i) < \mathfrak{o}(g^j)$ (respectively $\mathfrak{o}(f) \leq \mathfrak{o}(r^i) < \mathfrak{o}(r^j)$). Clearly $\lim_{k \rightarrow +\infty} f^k = 0$. Hence, if $g = \lim_{k \rightarrow +\infty} \sum_{i=1}^k g^i$ and $r = \lim_{k \rightarrow +\infty} \sum_{i=1}^k r^i$, then $f = g + r$ and g, r satisfy the conditions above. \square

We denote the series $r(x)$ of the proposition above by $R_{\mathfrak{o}}(f, \{f_1, \dots, f_s\})$. This series depend strongly on step (2) of the proof of Proposition 2.6. Let for example $f_1 = x^6, f_2 = x^4 + x^5, f_3 = x^2 + x^5$, and let $f = x^4$. We have $f = f_2 - x^5 = f_3^2 - f_1 f_2 - 2x^7 + x^{11}$. We shall see that $r(x)$ becomes unique if f_1, \dots, f_s is a basis of R (see Proposition 2.8).

Proposition 2.7. *The set $\{f_1, \dots, f_s\}$ is a basis of R if and only if $R_{\mathfrak{o}}(f, \{f_1, \dots, f_s\}) = 0$ for all $f \in R$.*

Proof. Suppose that $\{f_1, \dots, f_s\}$ is a basis of R and let $f \in R$. Let $r(x) = R_{\mathfrak{o}}(f, \{f_1, \dots, f_s\})$. Then $r(x) \in R$. If $r \neq 0$, then $\mathfrak{o}(r) \in \langle \mathfrak{o}(f_1), \dots, \mathfrak{o}(f_s) \rangle$, which is a contradiction.

Conversely, suppose that $\{f_1, \dots, f_s\}$ is not a basis of R and let $0 \neq f \in R$ such that $\mathfrak{o}(f) \notin \langle \mathfrak{o}(f_1), \dots, \mathfrak{o}(f_s) \rangle$. We have $R_{\mathfrak{o}}(f, \{f_1, \dots, f_s\}) \neq 0$, which contradicts the hypothesis. \square

Proposition 2.8. *If $\{f_1, \dots, f_s\}$ is a basis of R then for all $f \in \mathbb{K}[[x]]$, $R_{\mathfrak{o}}(f, \{f_1, \dots, f_s\})$ is unique.*

Proof. Suppose that $f = g_1(x) + r_1(x) = g_2(x) + r_2(x)$ where $g_1(x), g_2(x), r_1(x), r_2(x)$ satisfy conditions (i), (ii), (iii) of Proposition 2.6. We have $r_1(x) - r_2(x) = g_2(x) - g_1(x) \in R$. If $r_1(x) - r_2(x) \neq 0$ then $\mathfrak{o}(r_1(x) - r_2(x)) \notin \langle \mathfrak{o}(f_1), \dots, \mathfrak{o}(f_s) \rangle$. This contradicts the hypothesis. \square

Let, as above, $R = \mathbb{K}[[f_1, \dots, f_s]]$. We shall suppose that f_i is monic for all $1 \leq i \leq s$. Define

$$\phi : \mathbb{K}[X_1, \dots, X_s] \longrightarrow \mathbb{K}[x], \quad \phi(X_i) = M_{\mathfrak{o}}(f_i) \text{ for all } i \in \{1, \dots, s\}.$$

Let $\{F_1, \dots, F_r\}$ be a generating system of the kernel of ϕ . We can choose all of them to be binomials. If $F_i = X_1^{\alpha_1^i} \dots X_s^{\alpha_s^i} - X_1^{\beta_1^i} \dots X_s^{\beta_s^i}$, we set $S_i = f_1^{\alpha_1^i} \dots f_s^{\alpha_s^i} - f_1^{\beta_1^i} \dots f_s^{\beta_s^i}$. Note that if $p = \sum_{k=1}^s \alpha_k^i \mathfrak{o}(f_k) = \sum_{k=1}^s \beta_k^i \mathfrak{o}(f_k)$, then $\mathfrak{o}(S_i) > p$.

Theorem 2.9. *The system $\{f_1, \dots, f_s\}$ is a basis of R if and only if $R_{\mathfrak{o}}(S_i, \{f_1, \dots, f_s\}) = 0$ for all $i \in \{1, \dots, r\}$.*

Proof. Suppose that $\{f_1, \dots, f_s\}$ is a basis of R . Since $S_i \in R$ for all $i \in \{1, \dots, r\}$, then, by Proposition 2.7, $R_{\mathfrak{o}}(S_i, \{f_1, \dots, f_s\}) = 0$.

For the sufficiency assume to the contrary that $\{f_1, \dots, f_s\}$ is not a basis of R . Then there exists $f \in R$ such that $\mathfrak{o}(f) \notin \langle \mathfrak{o}(f_1), \dots, \mathfrak{o}(f_s) \rangle$. Write

$$f = \sum_{\theta} c_{\theta} f_1^{\theta_1} \dots f_s^{\theta_s}.$$

For all $\underline{\theta}$, if $c_{\underline{\theta}} \neq 0$, we set $p_{\underline{\theta}} = \sum_{i=1}^s \theta_i \circ(f_i) = \circ(f_1^{\theta_1} \cdots f_s^{\theta_s})$. Let $p = \min\{p_{\underline{\theta}} \mid c_{\underline{\theta}} \neq 0\}$ and let $\{\underline{\theta}^1, \dots, \underline{\theta}^l\}$ be such that $p = \circ(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i})$ for all $i \in \{1, \dots, l\}$ (such a set is clearly finite). Also $p \leq \circ(f) < \infty$.

If $\sum_{i=1}^l c_{\underline{\theta}^i} M_{\circ}(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}) \neq 0$, then $p = \circ(f) \in \langle \circ(f_1), \dots, \circ(f_s) \rangle$. But this is impossible. Hence, $\sum_{i=1}^l c_{\underline{\theta}^i} M_{\circ}(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}) = 0$, and then $\sum_{i=1}^l c_{\underline{\theta}^i} X_1^{\theta_1^i} \cdots X_s^{\theta_s^i} \in \ker(\phi)$. Hence

$$\sum_{i=1}^l c_{\underline{\theta}^i} X_1^{\theta_1^i} \cdots X_s^{\theta_s^i} = \sum_{k=1}^r \lambda_k F_k$$

with $\lambda_k \in \mathbb{K}[X_1, \dots, X_s]$ for all $k \in \{1, \dots, r\}$ (recall that F_1, \dots, F_r are binomials generating $\ker(\phi)$). This implies that

$$\sum_{i=1}^l c_{\underline{\theta}^i} f_1^{\theta_1^i} \cdots f_s^{\theta_s^i} = \sum_{k=1}^r \lambda_k(f_1, \dots, f_s) S_k.$$

From the hypothesis $R_{\circ}(S_k, \{f_1, \dots, f_s\}) = 0$. Hence there is an expression of S_k of the form $S_k = \sum_{\underline{\beta}^k} c_{\underline{\beta}^k} f_1^{\beta_1^k} \cdots f_s^{\beta_s^k}$ with $\circ(f_1^{\beta_1^k} \cdots f_s^{\beta_s^k}) \geq \circ(S_k)$.

So by replacing $\sum_{i=1}^l c_{\underline{\theta}^i} f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}$ with $\sum_{k=1}^r \lambda_k(f_1, \dots, f_s) \sum_{\underline{\beta}^k} c_{\underline{\beta}^k} f_1^{\beta_1^k} \cdots f_s^{\beta_s^k}$ in the expression of f , we can rewrite f as $f = \sum_{\underline{\theta}'} c_{\underline{\theta}'} f_1^{\theta_1'} \cdots f_s^{\theta_s'}$ with $\min\{\circ(f_1^{\theta_1'} \cdots f_s^{\theta_s'}) \mid c_{\underline{\theta}'} \neq 0\} > p$.

Since $\circ(f) < +\infty$, this process will stop, yielding a contradiction. \square

Algorithm 2.10. Let the notations be as above.

1. If $R_{\circ}(S_k, \{f_1, \dots, f_s\}) = 0$ for all $k \in \{1, \dots, r\}$, then $\{f_1, \dots, f_s\}$ is a basis of R .
2. If $r(x) = R_{\circ}(S_k, \{f_1, \dots, f_s\}) \neq 0$ for some $k \in \{1, \dots, r\}$, and if $M_{\circ}(r(x)) = ax^q$, then we set $f_{s+1} = \frac{1}{a}r(x)$, and we restart with $\{f_1, \dots, f_{s+1}\}$. Note that in this case,

$$\langle \circ(f_1), \dots, \circ(f_s) \rangle \subsetneq \langle \circ(f_1), \dots, \circ(f_s), \circ(f_{s+1}) \rangle \subseteq \circ(R).$$

This process will stop, giving a basis of R , because the complement of $\circ(R)$ in \mathbb{N} is finite.

Observe that $r(x)$ is not in general a polynomial. So we must use a trick to compute it, or at least the relevant part of it. This is accomplished by using Proposition 2.5. If in the current step of the algorithm $\langle \circ(f_1), \dots, \circ(f_s) \rangle$ is a numerical semigroup, then we compute its conductor, say c . Then $c \geq c(\circ(R))$. To compute $R_{\circ}(f, \{f_1, \dots, f_s\})$ we do the following. Let $p = \circ(f)$.

1. If $p \geq c$, then return 0. We implicitly assume that x^a is in our generating set for $a \in c + \mathbb{N}$ (though we do not store them).
2. If $p \in \langle \circ(f_1), \dots, \circ(f_s) \rangle$, then $M_{\circ}(f) = \sum_{\underline{\theta}} c_{\underline{\theta}} M_{\circ}(f_1)^{\theta_1} \cdots M_{\circ}(f_s)^{\theta_s}$. Set $f = f - \sum_{\underline{\theta}} c_{\underline{\theta}} f_1^{\theta_1} \cdots f_s^{\theta_s}$, and call recursively $R_{\circ}(f, \{f_1, \dots, f_s\})$ (the process will stop because the order of the new f is larger, and eventually will become bigger than c after a finite number of steps).
3. If $p \notin \langle \circ(f_1), \dots, \circ(f_s) \rangle$, then return f .

If $\langle \circ(f_1), \dots, \circ(f_s) \rangle$ is not a numerical semigroup, let d be its greatest common divisor. Set $c = dc(\langle \circ(f_1), \dots, \circ(f_s) \rangle / d)$. In this case we proceed as follows.

1. If $p \geq c$, then return f . We cannot ensure here that f will be reduced to zero, so we add it just in case.
2. If $p \in \langle \circ(f_1), \dots, \circ(f_s) \rangle$, then $M_{\circ}(f) = \sum_{\underline{\theta}} c_{\underline{\theta}} M_{\circ}(f_1)^{\theta_1} \cdots M_{\circ}(f_s)^{\theta_s}$. Set $f = f - \sum_{\underline{\theta}} c_{\underline{\theta}} f_1^{\theta_1} \cdots f_s^{\theta_s}$, and call recursively $R_{\circ}(f, \{f_1, \dots, f_s\})$.
3. If $p \notin \langle \circ(f_1), \dots, \circ(f_s) \rangle$, then return f . One might check first if d does not divide p , because in this case for sure $p \notin \langle \circ(f_1), \dots, \circ(f_s) \rangle$.

Observe that by adding the conditions $p \geq c$, we are avoiding entering in an eventual infinite loop.

Suppose that $\{f_1, \dots, f_s\}$ is a basis of R . Also suppose that for all $i \in \{1, \dots, s\}$, f_i is monic. We say that $\{f_1, \dots, f_s\}$ is a *minimal basis* of R if $\circ(f_1), \dots, \circ(f_s)$ generate minimally the semigroup $\circ(R)$. We say that $\{f_1, \dots, f_s\}$ is a *reduced basis* of R if $\text{supp}(f_i(x) - M_0(f_i)) \subseteq \mathbb{N} \setminus \circ(R)$. Let $i \in \{1, \dots, s\}$. If $\circ(f_i) \in \langle \circ(f_1), \dots, \circ(f_{i-1}), \circ(f_{i+1}), \dots, \circ(f_s) \rangle$, then $\{f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_s\}$ is also a basis of R . Furthermore,

by applying the division process of Proposition 2.6 to $f_i - M_o(f_i)$, we can always construct a reduced basis of R .

Corollary 2.11. *The algebra R has a unique minimal reduced basis.*

Proof. Let $\{f_1, \dots, f_s\}$ and $\{g_1, \dots, g_{s'}\}$ be two minimal reduced bases of R . Hence s is the embedding dimension of $o(R)$, and the same holds for s' ; whence they are equal. Let $i = 1$. There exists j_1 such that $o(f_1) = o(g_{j_1})$, because minimal generating systems of numerical semigroups are unique. If $f_1 - g_{j_1} \neq 0$, then $o(f_1 - g_{j_1}) \notin o(R)$ (the basis is reduced), which is a contradiction because $f_1 - g_{j_1} \in R$. The same argument shows that $\{f_1, \dots, f_s\} = \{g_1, \dots, g_s\}$ \square

Remark 2.12. Let $R = \mathbb{K}[[f_1, \dots, f_s]]$ and assume that f_i is monic for all $1 \leq i \leq s$. Also assume that $o(f_1) \leq o(f_2) \leq \dots \leq o(f_s)$. Set $n = o(f_1)$ and let $f_1 = x^n + \sum_{i>n} c_i^1 x^i$. By an analytic change of variables, we may assume that $f_1 = x^n$, hence, up to an analytic isomorphism, we may assume that $R = \mathbb{K}[[x^n, f_2, \dots, f_s]]$. In particular, we may assume that R has a minimal reduced basis of the form $x^n, g_2(x), \dots, g_{s'}(x)$.

Example 2.13. Let $R = \mathbb{K}[[x^4 + x^5, x^6, x^{15} + x^{16} + \sum_{n \geq 20} x^n]]$, with \mathbb{K} a field of characteristic zero. Then R is a one-dimensional ring. Since the conductor of $\langle 4, 6, 15 \rangle$ is 18, we have that $\lambda_R(\mathbb{K}[[X]]/R) < \infty$ and we know by Proposition 2.5 that $R = \mathbb{K}[[x^4 + x^5, x^6, x^{15} + x^{16}]]$. Let us denote $x^4 + x^5$ by f_1 , x^6 by f_2 and $x^{15} + x^{16}$ by f_3 . The kernel of $\phi : \mathbb{K}[X_1, X_2, X_3] \rightarrow \mathbb{K}[x]$, $\phi(X_1) = x^4$, $\phi(X_2) = x^6$ and $\phi(X_3) = x^{15}$ is generated by $X_1^3 - X_2^2, X_3^2 - X_2^5$, hence we get $S_1 = x^{13} + x^{14} + \frac{1}{3}x^{15}$ and $S_2 = x^{31} + \frac{1}{2}x^{32}$. As $13 \notin \langle 4, 6, 15 \rangle$, we add it as $f_4 = S_1$. We do not care about S_2 , because the conductor of $\langle 4, 6, 15 \rangle$ is 18.

Now, the conductor of $\langle 4, 6, 13, 15 \rangle$ is 12. If we compute a system of generators of the kernel of $\varphi : \mathbb{K}[X_1, X_2, X_3, X_4] \rightarrow \mathbb{K}[x]$, $\varphi(X_1) = x^4$, $\varphi(X_2) = x^6$, $\varphi(X_3) = x^{15}$ and $\varphi(X_4) = x^{13}$, then all the elements S_i have orders greater than 12, and so the algorithm ends. We conclude that $o(R) = \langle 4, 6, 13, 15 \rangle$.

We have implemented this algorithm in the `numericalsgps` ([7]) `GAP` ([10]) package. Next we illustrate how to compute this semigroup with the functions we have implemented (that will be available in the next release of the package).

```
gap> x:=X(Rationals,"x");;
gap> l:=[x^4+x^5,x^6,x^15+x^16];;
gap> s:=SemigroupOfValuesOfCurve_Local(1);;
gap> MinimalGeneratingSystem(s);
[ 4, 6, 13, 15 ]
gap> SemigroupOfValuesOfCurve_Local(1,13);
x^13
```

Remark 2.14. It is known (cf. [3, Section II.1]) that there exist relations between algebraic characteristics and invariants of the semigroup $o(R)$ and the ring R . Hence, in the Example 2.13, from $o(R) = \langle 4, 6, 13, 15 \rangle = \{0, 4, 6, 8, 10, 12, \dots\}$, we deduce that the conductor of $o(R)$ is 12. The conductor of R in \bar{R} is precisely $(x^{c(o(R))})$ ([3]). We have that $\lambda_R(\bar{R}/R) = |[0, c(o(R)) - 1] \cap (\mathbb{N} \setminus o(R))| = 7$ counts the number of gaps of $o(R)$ ([13, Proposition 1]). The integer $\lambda_R(\bar{R}/R)$ is the degree of singularity of R , and this is why the genus of $o(R)$ is called by some authors the degree of singularity of the semigroup (see [3, 13]). In the setting of Weierstrass numerical semigroups the genus coincides with the geometrical genus of the curve used to define the semigroup ([6]).

The number of sporadic elements of $o(R)$ is $\lambda_R(R/(R : \bar{R})) = |[0, c(o(R)) - 1] \cap o(R)| = 5$.

The ring R is Cohen-Macaulay and its type is less than or equal to $\#T(o(R)) = 3$, where for a numerical semigroup Γ , $T(\Gamma) = \{x \in \mathbb{Z} \setminus \Gamma \mid x + \Gamma^* \subseteq \Gamma\}$ ([3, Proposition II.1.16]). According also to [3, Proposition II.1.16], equality holds if and only if $o(R : \mathfrak{m}) = T(o(R))$. However $T(o(R)) = \{2, 9, 11\}$ and $2 \notin o(R : \mathfrak{m})$. So in our example we get an strict inequality.

Example 2.15. Let $R = \mathbb{K}[[x^4, x^6 + x^7, x^{13} + a_{14}x^{14} + a_{15}x^{15} + \dots]]$ with \mathbb{K} a field. Using the same argument as in the Example 2.13, we find that if $\text{char } \mathbb{K} \neq 2$, we have that if $a_{15} - a_{14} + 1/2 = 0$, then $\{x^4, x^6 + x^7, x^{13}\}$ is the reduced basis of R . Furthermore, since $\langle 4, 6, 13 \rangle$ is a symmetric numerical semigroup (the number of nonnegative integers not in the semigroup equals the conductor divided by

two), then, by [12], R is Gorenstein. Finally $\lambda_R(\bar{R}/R) = 8$. Otherwise if $a_{15} - a_{14} + 1/2 \neq 0$, then $\{x^4, x^6 + x^7, x^{13}, x^{15}\}$ is the reduced basis of R with R a non Gorenstein ring. Furthermore $\lambda_R(\bar{R}/R) = 7$.

Otherwise, if $\text{char } \mathbb{K} = 2$, then the reduced basis of R is $\{x^4, x^6 + x^7, x^{13}, x^{15}\}$ and R is not a Gorenstein ring. Here, $\lambda_R(\bar{R}/R) = 7$.

Example 2.16. Let $R = \mathbb{K}\langle\langle x^8, x^{12} + x^{14} + x^{15} \rangle\rangle$, with \mathbb{K} a field of characteristic zero. Using the same argument as in the Example 2.13, we have that

$$\left\{ x^8, x^{12} + x^{14} + x^{15}, x^{26} + x^{27} + x^{29} - \frac{1}{2}x^{31}, \right. \\ \left. x^{53} + \frac{1}{2}x^{55} - \frac{1}{2}x^{57} - \frac{1}{8}x^{63} + \frac{25}{8}x^{67} - \frac{95}{32}x^{71} - \frac{15}{16}x^{75} - \frac{135}{32}x^{83} \right\}$$

is the reduced basis of the Gorenstein ring R . Furthermore, we have $\lambda_R(\bar{R}/R) = 42$.

```
gap> l:= [x^8, x^12+x^14+x^15];;
gap> SemigroupOfValuesOfCurve_Local(1, "basis");
[ x^8, x^15+x^14+x^12, -1/2*x^31+x^29+x^27+x^26,
-135/32*x^83-15/16*x^75-95/32*x^71+25/8*x^67-1/8*x^63-1/2*x^57+1/2*x^55+x^53 ]
```

Example 2.17. The following battery of examples was provided by Lance Bryant as a test for our algorithm.

```
gap> l:= [ [ x^6, x^8+x^9, x^19 ], [ x^7, x^9+x^10, x^19, x^31 ], [ x^7, x^21+x^28+x^33 ],
[ x^4, x^6+x^7, x^13 ], [ x^6, x^8+x^11, x^10+2*x^13, x^21 ], [ x^5, -x^18-x^21, -x^23, -x^26 ],
[ x^5, -x^18-x^21, -x^26 ], [ x^5, -x^18-x^21, x^23-x^26 ], [ x^6, x^9+x^10, x^19 ],
[ x^7, x^9+x^10, x^19 ], [ x^8, x^9+x^10, x^19 ], [ x^7, x^9+x^10, x^17, x^19 ] ];;
gap> List(1, i->MinimalGeneratingSystem(SemigroupOfValuesOfCurve_Local(i)));
[ [ 6, 8, 19, 29 ], [ 7, 9, 19, 29, 31 ], [ 7, 33 ], [ 4, 6, 13, 15 ],
[ 6, 8, 10, 21, 23, 25 ], [ 5, 18, 26, 39, 47 ], [ 5, 18, 26, 39, 47 ],
[ 5, 18, 26, 39, 47 ], [ 6, 9, 19, 20 ], [ 7, 9, 19, 29 ], [ 8, 9, 19, 30 ],
[ 7, 9, 17, 19, 29 ] ]
```

Remark 2.18. We do not know a priori if $\bar{R} \neq \mathbb{K}\langle\langle x \rangle\rangle$ and we do not have a general procedure to check it. If the algorithm is called with such an R , it will eventually not stop.

3. DEFORMATION TO A TORIC IDEAL

Let the notations be as in Section 2. Given $f(x) = \sum_{i \geq p} c_i x^i \in \mathbb{K}\langle\langle x \rangle\rangle$, we set $H_f(u, x) = \sum_{i \geq p} c_i u^{i-p} x^i$. In particular, if we consider the linear form $L : \mathbb{N}^2 \rightarrow \mathbb{N}, L(a, b) = b - a$, then H_f is L -homogeneous of order p , that is, $L(i - p, i) = p$ for all $i \in \text{supp}(f)$. We set $H_R = \mathbb{K}\langle\langle u, H_f \mid f \in R \rangle\rangle$. With these notations we have the following.

Proposition 3.1. *The set $\{f_1, \dots, f_s\}$ is a basis of R if and only if $H_R = \mathbb{K}\langle\langle u, H_{f_1}, \dots, H_{f_s} \rangle\rangle$.*

Proof. Note first that if $H_g \in H_R$ for some $g \in \mathbb{K}\langle\langle x \rangle\rangle$, then $g \in R$. Suppose that $\{f_1, \dots, f_s\}$ is a basis of R and let $f(x) \in R$. Write $H_f(u, x) = \sum_{i \geq p} c_i u^{i-p} x^i$. We have $M_o(f) = c_p x^p = c_p \prod_{i=1}^s M_o(f_i)^{p_i^k}$, hence

$$H_f - c_p \prod_{i=1}^s H_{f_i}^{p_i^k} = u^q H_{f^1}$$

with $f^1 \in R$ and either $f^1 = 0$, or $\text{o}(f^1) > p$. In the second case we restart with f^1 . A similar argument as in Proposition 2.6 proves our assertion.

Conversely, suppose that $H_R = \mathbb{K}\langle\langle u, H_{f_1}, \dots, H_{f_s} \rangle\rangle$ and let $f \in R$. Let $P(X_0, X_1, \dots, X_s) \in \mathbb{K}\langle\langle X_0, X_1, \dots, X_s \rangle\rangle$ such that $H_f = P(u, H_{f_1}, \dots, H_{f_s})$. Setting $u = 0$, we get that $M_o(f) = P(M_o(f_1), \dots, M_o(f_s)) \in \mathbb{K}\langle\langle M_o(f_1), \dots, M_o(f_s) \rangle\rangle$. Hence $M_o(f) \in \mathbb{K}\langle\langle M_o(f_1), \dots, M_o(f_s) \rangle\rangle$. \square

Suppose that $\{f_1, \dots, f_s\}$ is a basis of R . Then $T = \mathbb{K}[[u]][[H_{f_1}, \dots, H_{f_s}]]$ is a $\mathbb{K}[[u]]$ -module. When $u = 1$ (respectively $u = 0$), we get $T|_{u=1} = R$ (respectively $T|_{u=0} = \mathbb{K}[[M(f_1), \dots, M(f_s)]]$). Hence we get a deformation from R to $\mathbb{K}[[M_o(f_1), \dots, M_o(f_s)]]$. More precisely let

$$\psi : \mathbb{K}[[X_1, \dots, X_s]] \longrightarrow R = \mathbb{K}[[f_1, \dots, f_s]]$$

and

$$H_\psi : \mathbb{K}[[u]][[X_1, \dots, X_s]] \longrightarrow T = \mathbb{K}[[u]][[H_{f_1}, \dots, H_{f_s}]]$$

be the morphisms of rings such that $H_\psi(u) = u$, $\psi(X_i) = f_i$ and $H_\psi(X_i) = H_{f_i}$ for all $i \in \{1, \dots, s\}$. Let, as in Section 2,

$$S_i = f_1^{\alpha_i^1} \dots f_s^{\alpha_i^s} - f_1^{\beta_i^1} \dots f_s^{\beta_i^s} = \sum_{\theta^i} c_{\theta^i}^i f_1^{\theta_1^i} \dots f_s^{\theta_s^i}, 1 \leq i \leq r$$

with $o(f_1^{\theta_1^i} \dots f_s^{\theta_s^i}) = D_{\theta^i}^i > \sum_{k=1}^s \alpha_k^i o(f_k) = \sum_{k=1}^s \beta_k^i o(f_k) = p_i$. Let I (respectively J) be the ideal generated by $(G_i = X_1^{\alpha_i^1} \dots X_s^{\alpha_i^s} - X_1^{\beta_i^1} \dots X_s^{\beta_i^s} - \sum_{\theta^i} c_{\theta^i}^i X_1^{\theta_1^i} \dots X_s^{\theta_s^i})_{1 \leq i \leq r}$ (respectively $(H_i = X_1^{\alpha_i^1} \dots X_s^{\alpha_i^s} - X_1^{\beta_i^1} \dots X_s^{\beta_i^s} - \sum_{\theta^i} u^{D_{\theta^i}^i - p_i} c_{\theta^i}^i X_1^{\theta_1^i} \dots X_s^{\theta_s^i})_{1 \leq i \leq r}$) in $\mathbb{K}[[X_1, \dots, X_s]]$ (respectively $\mathbb{K}[[u]][[X_1, \dots, X_s]]$).

We shall consider on \mathbb{N}^s (respectively, \mathbb{N}^{s+1}) the linear form

$$O(\theta_1, \dots, \theta_s) = \sum_{i=1}^s \theta_i o(f_i)$$

(respectively $O_h(\theta_0, \theta_1, \dots, \theta_s) = -\theta_0 + \sum_{i=1}^s \theta_i o(f_i)$).

Given a monomial $X_1^{\theta_1} \dots X_s^{\theta_s}$ (respectively $u^{\theta_0} X_1^{\theta_1} \dots X_s^{\theta_s}$), we set $O(X_1^{\theta_1} \dots X_s^{\theta_s}) = O(\theta_1, \dots, \theta_s)$ (respectively $O_h(u^{\theta_0} X_1^{\theta_1} \dots X_s^{\theta_s}) = O_h(\theta_0, \theta_1, \dots, \theta_s)$).

For $G = \sum_{\theta} c_{\theta} X_1^{\theta_1} \dots X_s^{\theta_s}$ (respectively $H = \sum_{\theta} c_{\theta} u^{\theta_0} X_1^{\theta_1} \dots X_s^{\theta_s}$), we say that G (respectively H) is O -homogeneous of order a (respectively O_h -homogeneous of order b) if $O(\theta_1, \dots, \theta_s) = a$ (respectively $O_h(\theta_0, \theta_1, \dots, \theta_s) = b$) for all $(\theta_1, \dots, \theta_s)$ (respectively $(\theta_0, \theta_1, \dots, \theta_s)$) such that $c_{\theta} \neq 0$. More generally let $G = \sum_{k \geq 0} G_{p_k}$ where $p_0 < p_1 < \dots$ and G_{p_k} is O -homogeneous of order p_k . We set $O(G) = p_0$. We also set $\text{in}(G) = G_{p_0}$ and we call it the *initial form* of G .

We finally set $O(0) = +\infty$, and we recall that $O(G) = +\infty$ if and only if $G = 0$.

Lemma 3.2. *With the standing notations and hypothesis, the kernel of ψ is generated by I .*

Proof. Let, as in Section 2, F_1, \dots, F_r be a generating system of the kernel of the morphism

$$\phi : \mathbb{K}[[X_1, \dots, X_s]] \longrightarrow \mathbb{K}[[x]], \quad \phi(X_i) = M_o(f_i)$$

for all $i \in \{1, \dots, s\}$. In particular F_i is O -homogeneous of order $o(f_i)$ for all $i \in \{1, \dots, s\}$, and $\mathbb{K}[[X_1, \dots, X_s]]/(F_1, \dots, F_r) \simeq \mathbb{K}[[M_o(f_1), \dots, M_o(f_s)]]$.

For all $i \in \{1, \dots, r\}$, $\psi(G_i) = 0$. Hence $I \subseteq \ker(\psi)$.

For the other inclusion, let $G = \sum_{\theta} c_{\theta} X_1^{\theta_1} \dots X_s^{\theta_s} \in \ker(\psi)$. Write $G = \sum_{k \geq 0} c_{\theta^k} X_1^{\theta_1^k} \dots X_s^{\theta_s^k}$ where $O(\theta_1^0, \dots, \theta_s^0) \leq O(\theta_1^1, \dots, \theta_s^1) \leq \dots$. Since $\psi(G) = 0$, we have that

$$\sum_{k \geq 0} c_{\theta^k} f_1^{\theta_1^k} \dots f_s^{\theta_s^k} = 0.$$

In particular $\sum_{k, O(\theta^k) = O(\theta^0)} c_{\theta^k} M_o(f_1)^{\theta_1^k} \dots M_o(f_s)^{\theta_s^k} = 0$, and consequently $\sum_{k, O(\theta^k) = O(\theta^0)} c_{\theta^k} X_1^{\theta_1^k} \dots X_s^{\theta_s^k} \in \ker(\phi)$. This implies that

$$\sum_{k, O(\theta^k) = O(\theta^0)} c_{\theta^k} X_1^{\theta_1^k} \dots X_s^{\theta_s^k} = \sum_{i=1}^r \lambda_i^0 F_i$$

for some $\lambda_i^0 \in \mathbb{K}[[X_1, \dots, X_s]]$, $i \in \{1, \dots, r\}$, with λ_i^0 is O -homogeneous of order $O(G) - O(F_i)$. Hence

$$\sum_{k, O(\theta^k) = O(\theta^0)} c_{\theta^k} f_1^{\theta_1^k} \dots f_s^{\theta_s^k} = \sum_{i=1}^r \lambda_i^0(f_1, \dots, f_s) S_i.$$

Let $G^1 = G - \sum_{i=1}^r \lambda_i^0 G_i$. We have $G^1 \in \ker(\psi)$. If $G^1 \neq 0$, then $O(G) < O(G^1)$. Then we restart with G^1 . We construct in the same way G^2 , and $\lambda_1^1, \dots, \lambda_r^1$ such that $G^1 = G^2 + \sum_{j=1}^r \lambda_j^1 G_j$ with $O(G) < O(G^1) < O(G^2)$, λ_i^j O -homogeneous and $O(\lambda_i^0) < O(\lambda_i^1)$ for all $i \in \{1, \dots, r\}$. If we continue in this way, we get that for all $k \geq 0$,

$$G = G^{k+1} + \sum_{i=1}^r (\lambda_i^0 + \lambda_i^1 + \dots + \lambda_i^k) G_i,$$

with $O(G) < O(G^1) < \dots < O(G^{k+1})$, λ_i^j O -homogeneous, and $O(\lambda_i^0) < O(\lambda_i^1) < \dots < O(\lambda_i^k)$ for all $i \in \{1, \dots, r\}$ and for all $j \in \{1, \dots, k\}$. If $G^k = 0$ for some k , then we are done. Otherwise, let $\lambda_i = \sum_{k=0}^{\infty} \lambda_i^k$, and let $\tilde{G} = \lim_{k \rightarrow +\infty} G^k$. We have $\tilde{G} = 0$ and $G = \sum_{i=1}^r \lambda_i G_i$. This proves our assertion. \square

Let the notations be as above. Let $G = \sum_{\theta} c_{\theta} X_1^{\theta_1} \dots X_s^{\theta_s} \in \mathbb{K}[[X_1, \dots, X_s]]$ and write $G = \sum_{i \geq 0} G_{p_i}$ with $p_0 < p_1 < \dots$ and G_{p_i} O -homogeneous. We set $H_G = \sum_{i \geq 0} u^{p_i - p_0} G_{p_i}$, in such a way that H_G is O_h -homogeneous of order p_0 . Given an ideal S of $\mathbb{K}[[X_1, \dots, X_s]]$, we set $\text{in}(S) = (\text{in}(G) \mid G \in S \setminus \{0\})$. We also denote by $H_S = (H_G \mid G \in S \setminus \{0\}) \mathbb{K}[[u, X_1, \dots, X_s]]$. With these notations we have $\text{in}(S_i) = F_i$ and $H_{G_i} = H_i$ for all $i \in \{1, \dots, r\}$.

Lemma 3.3. *Let the notations be as above. We have $\text{in}(I) = (F_1, \dots, F_r)$, and $H_I = (H_1, \dots, H_r) = J$.*

Proof. The first assertion follows from the proof of Lemma 3.2. To prove the second assertion, let $H \in H_I$ and assume that H is O_h -homogeneous. We have $G = H(1, X_1, \dots, X_s) \in I$. Furthermore, $H = u^e H_G$ for some $e \geq 0$. Write $H_G = \text{in}(G) + H^1$ where $H^1(0, X_1, \dots, X_s) = 0$. We have $\text{in}(G) = \sum_{i=1}^r \lambda_i F_i$ where λ_i is O -homogeneous of order $p_0 - O(F_i)$. Let $H_{G^1} = H_G - \sum_{i=1}^r \lambda_i H_{G_i} = H_G - \sum_{i=1}^r \lambda_i H_i$. Then $H_{G^1} \in H_I$ is O_h -homogeneous and $O_h(H_G) < O_h(H_{G^1})$. Now we restart with H_{G^1} . We prove in this way that $H \in (H_1, \dots, H_r)$. \square

Let $H = \sum_{\theta} c_{\theta} u^{\theta} X_1^{\theta_1} \dots X_s^{\theta_s} \in \ker(H_{\psi})$. Write $H = \sum_k H^k$, where H^k is O_h -homogeneous. For all k , we have $H_{\psi}(H^k) = 0$. Setting $G_k = H^k(1, X_1, \dots, X_s)$, we have $\psi(G_k) = 0$. This implies that $G_k \in I$. Hence $H_{G_k} \in (H_1, \dots, H_r)$ by Lemma 3.3. But $H^k = u^{e_k} H_{G_k}$ for some $e_k \in \mathbb{N}$. Consequently $H^k \in (H_1, \dots, H_r)$. Finally $H \in (H_1, \dots, H_r)$, which proves that $\ker(H_{\psi}) \subseteq J$. As the inclusion $J \subseteq \ker(H_{\psi})$ is obvious, we conclude that $J = \ker(H_{\psi})$.

Now the morphism

$$\mathbb{K}[[u]] \longrightarrow \mathbb{K}[[u]][[X_1, \dots, X_s]]/J$$

is flat because u is not a zero divisor. Hence we get a family of formal space curves parametrized by u that gives us a deformation from $\mathbb{K}[[X_1, \dots, X_r]]/I$ to $\mathbb{K}[[X_1, \dots, X_r]]/(F_1, \dots, F_r)$.

In particular we get the following.

Theorem 3.4. *Every formal space curve of \mathbb{K}^l , parametrized by $Y_1 = g_1(x), \dots, Y_l = g_l(x)$ has a deformation into a formal monomial curve of \mathbb{K}^r for some positive integer r .*

4. BASIS OF $\mathbb{K}[[f(x), g(x)]]$

In this section we study the particular case of a subalgebra R of $\mathbb{K}[[x]]$ generated by two elements, and see that a different approach can be considered to study $\mathfrak{o}(R)$, with some interesting applications.

Let $f(x) = \sum_{i \geq n} a_i x^i$ and $g(x) = \sum_{j \geq m} b_j x^j$ be two elements of $\mathbb{K}[[x]]$ and suppose, without loss of generality, that the following conditions hold:

- (1) $a_n = b_m = 1$,
- (2) $n \leq m$,
- (3) the greatest common divisor of $\text{supp}(f(x)) \cup \text{supp}(g(x))$ is equal to 1 (in particular for all $d > 1$, $f(x), g(x) \notin \mathbb{K}[[x^d]]$).

Let the notations be as in Section 2, in particular $R = \mathbb{K}[[f, g]]$. By the analytic change of variables $f(x) = \tilde{x}^n$, we may assume that $R = \mathbb{K}[[x^n, g(x)]]$. Let $F(X, Y)$ be the x -resultant of $X - x^n, Y - g(x)$, that is, $F(X, Y)$ is the generator of the kernel of the map $\rho : \mathbb{K}[[X, Y]] \longrightarrow \mathbb{K}[[x]]$, $\rho(X) = x^n$ and $\rho(Y) = g(x)$.

Since $\mathbb{K}[[f, g]] = \mathbb{K}[[f, g - f^k]]$ for all $k \geq 1$, then we shall assume that $n < m$ and also that n does not divide m . Given a nonzero element $G(X, Y) \notin (F(X, Y))\mathbb{K}[[X, Y]]$, we set $\text{int}(F, G) = \text{o}(G(f(x), g(x)))$. Condition (3) implies that the set of $\text{int}(F, G), G(X, Y) \notin (F(X, Y))\mathbb{K}[[X, Y]]$, is a numerical semigroup. We denote it by $\Gamma(F)$. We have the following.

Proposition 4.1. $\text{o}(R) = \Gamma(F)$.

Proof. We have $a \in \Gamma(F)$ if and only if $a = \text{o}(G(f(x), g(x)))$ for some $G(X, Y) \in \mathbb{K}[[X, Y]]$ if and only if $a \in \text{o}(R)$. \square

Suppose that \mathbb{K} is algebraically closed with characteristic zero, and let $d_1 = n$, $m_1 = \inf\{i \in \text{supp}(g) \mid d_1 \nmid i\}$, that is, $m_1 = m$, and $d_2 = \gcd(n, m_1)$. For all $k \geq 2$ we set $m_k = \inf\{i \in \text{supp}(g) \mid d_k \nmid i\}$ and $d_{k+1} = \gcd(d_k, m_k)$. It follows that there exists $h \geq 1$ such that $d_{h+1} = 1$. The set $\{m_1, \dots, m_h\}$ is called the *set of Newton-Puiseux exponents* of $F(X, Y)$. Let $e_k = \frac{d_k}{d_{k+1}}$ for all $1 \leq k \leq h$ and define the sequence $(r_k)_{0 \leq k \leq h}$ as follows: $r_0 = n, r_1 = m$, and for all $2 \leq k \leq h, r_k = r_{k-1}e_{k-1} + m_k - m_{k-1}$. With these notations we have the following (see [1]).

- (1) $\Gamma(F) = \text{o}(R)$ is generated by $\{r_0, r_1, \dots, r_h\}$.
- (2) $r_k d_k < r_{k+1} d_{k+1}$ for all $k \in \{1, \dots, h-1\}$.
- (3) $\Gamma(F) = \text{o}(R)$ is free with respect to the arrangement (r_0, \dots, r_h) . More precisely, let $e_k = \frac{d_k}{d_{k+1}}$ for all $k \in \{1, \dots, h\}$. Then $e_k r_k \in \langle r_0, \dots, r_{k-1} \rangle$.
- (4) $C = \sum_{k=1}^h (e_k - 1)r_k - n + 1$ is the conductor of $\Gamma(F) = \text{o}(R)$.

Example 4.2. Let $f = x^7$ and $g = x^4 + x^2$. The above resultant is then $F = y^7 - 7x^2y^3 - x^4 - 14x^2y^2 - 7x^2y - x^2$. Then $\Gamma(F) = \text{o}(R) = \langle 2, 7 \rangle$.

```
gap> Resultant(x-t^7, y-t^4-t^2, t);
y^7-7*x^2*y^3-x^4-14*x^2*y^2-7*x^2*y-x^2
gap> s:=SemigroupOfValuesOfCurve_Local([t^7, t^4+t^2]);
<Modular numerical semigroup satisfying 7x mod 14 <= x >
gap> MinimalGeneratingSystem(last);
[ 2, 7 ]
gap> IsFreeNumericalSemigroup(s);
true
```

Let the notations be as above. For all $k \geq 2$, let $G_k(X, Y) \in \mathbb{K}[[X, Y]]$ such that $\text{o}(G_k(x^n, g(x))) = r_k$. It follows from [1] that $\deg_Y G_k = \frac{n}{d_k}$. If $g_k(x) = G_k(x^n, g(x))$, then we have the following.

Proposition 4.3. *The set $\{x^n, g, g_2, \dots, g_h\}$ is a basis of R , that is, $R = \mathbb{K}[[x^n, g, g_2, \dots, g_h]]$ and $M_o(R) = \mathbb{K}[[x^n, x^m, x^{r_2}, \dots, x^{r_h}]]$.*

Note that, by a similar argument as in Section 2, we may assume that $f = x^n, g = x^m + \sum_{i \in G(\Gamma(F))} c_i^1 x^i$, and for all $k \geq 2, g_k = x^{r_k} + \sum_{i \in G(\Gamma(F))} c_i^k x^i$, where $G(\Gamma(F)) = \{j \in \mathbb{N} \mid j \notin \Gamma(F)\}$ is the set of gaps of $\Gamma(F)$.

Let the notations be as in Section 3. The morphism

$$D : \mathbb{K}[[u]] \longrightarrow T = \mathbb{K}[[u]][[H_f, H_g, H_{g_2}, \dots, H_{g_h}]]$$

gives us a deformation of $T|_{u=1} = R = \mathbb{K}[[f(x), g(x), g_2(x), \dots, g_h(x)]]$ to $T|_{u=0} = \mathbb{K}[[x^n, x^m, x^{r_2}, \dots, x^{r_h}]]$. Note that, since $\langle n, m, r_2, \dots, r_h \rangle$ is free with respect to the given arrangement, then it is a complete intersection (see for instance [16]). For all $k \in \{1, \dots, h\}$, write $e_k r_k = \sum_{i=0}^{k-1} \theta_i^k r_i$ with $0 \leq \theta_i^k < e_i$ for all $i \in \{1, \dots, k-1\}$. If B is the ideal of $\mathbb{K}[[X_0, X_1, \dots, X_h]]$ generated by

$$\{X_1^{e_2} - X_0^{\frac{m}{d_2}}, X_2^{e_2} - X_0^{\theta_0^2} X_1^{\theta_1^2}, \dots, X_h^{e_h} - X_0^{\theta_0^h} X_1^{\theta_1^h} \dots X_{h-1}^{\theta_{h-1}^h}\}$$

then

$$\mathbb{K}[[x^n, x^m, x^{r_2}, \dots, x^{r_h}]] \simeq \mathbb{K}[[X_0, X_1, \dots, X_h]]/B.$$

Let $\bar{F}(X, Y)$ be the x -resultant of $X - x^n, y - g(x)$. By hypothesis, $\bar{F}(X, Y)$ is a polynomial. Furthermore, $\bar{F}(X, Y) = Y^n + a_1(X)Y^{n-1} + \dots + a_n(X)$ with $\text{o}(a_i(X)) > i$ for all $2 \leq i \leq n$. Set $G_{h+1} = \bar{F}$ and for all $k \geq 1$, let

$$G_{k+1} = G_k^{e_k} - X^{\theta_0^k} \prod_{i=1}^{k-1} G_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k X^{\alpha_0^k} G_1^{\alpha_1^k} \dots G_k^{\alpha_k^k},$$

where the following conditions hold:

- (1) for all $i \in \{1, \dots, k-1\}$, $0 \leq \theta_i^k < e_i$;
- (2) for all $\underline{\alpha}^k$, if $c_{\underline{\alpha}^k}^k \neq 0$, then for all $i \in \{1, \dots, k\}$, $0 \leq \alpha_i^k < e_i$;
- (3) for all $\underline{\alpha}^k$, if $c_{\underline{\alpha}^k}^k \neq 0$, then $\alpha_0^k n + \sum_{i=1}^k \alpha_i^k r_i = D_i^k > e_k r_k = \theta_0^k r_0 + \sum_{i=1}^{k-1} \theta_i^k r_i$.

It follows from Section 3. that if I (respectively J) is the ideal generated by

$$(X_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} X_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k X_0^{\alpha_0^k} X_1^{\alpha_1^k} \dots X_k^{\alpha_k^k})_{1 \leq k \leq h}$$

(respectively $(X_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} X_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k u^{D_i^k - e_k r_k} X_0^{\alpha_0^k} X_1^{\alpha_1^k} \dots X_k^{\alpha_k^k})_{1 \leq k \leq h}$) in $\mathbb{K}[X_0, \dots, X_h]$ (respectively $\mathbb{K}[u][X_0, \dots, X_h]$), then

$$R = \mathbb{K}[x^n, g(x), g_2(x), \dots, g_h(x)] \simeq \mathbb{K}[X_0, X_1, \dots, X_h]/I$$

and

$$\mathbb{K}[u][x^n, H_g, H_{g_2}, \dots, H_{g_h}] \simeq \mathbb{K}[u][X_0, X_1, \dots, X_h]/J.$$

Furthermore, $\mathbb{K}[u][X_0, X_1, \dots, X_h]/J$ is a flat $\mathbb{K}[u]$ -module. This gives us a family of formal space curves parametrized by u which is a deformation from $\mathbb{K}[X_0, X_1, \dots, X_h]/I$ to the formal toric variety $\mathbb{K}[X_0, X_1, \dots, X_h]/B$. The later being a complete intersection, we get the following.

Theorem 4.4. *Every irreducible singularity of a plane curve $X = f(x), Y = g(x)$ of \mathbb{K}^2 has a deformation into a formal monomial complete intersection curve of \mathbb{K}^{h+1} for some $h \geq 1$.*

Example 4.5. Let $f(x) = x^4, g(x) = x^6 + x^7$. The minimal polynomial of $(f(x), g(x))$ is given by:

$$F(X, Y) = Y^4 - 2X^3Y^2 + X^6 - 4X^5Y - X^7 = (Y^2 - X^3)^2 - 4X^5Y - X^7$$

Let $r_0 = 4 = d_1, r_1 = 6 = m_1$ and $G_1 = Y$. We have $d_2 = \text{gcd}(6, 4) = 2$, hence $m_2 = 7$. It follows that $r_2 = 13$. Note that if $G_2 = Y^2 - X^3$, then $g_2(x) = G_2(f(x), g(x)) = 2x^{13} + x^{14}$. Hence $\Gamma(F) = \text{o}(R) = \langle 4, 6, 13 \rangle$ and $\{f(x), g(x), g_2(x)\}$ is a basis of R . Let us double check it.

```
gap> SemigroupOfValuesOfCurve_Local([x^4, x^6+x^7], "basis");
[ x^4, x^7+x^6, -1/2*x^15+x^13 ]
```

(Observe that the output is different, since this is a reduced basis: we change $2x^{13} + x^{14}$ with $x^{13} + \frac{1}{2}x^{14}$, and then using that $14 = 2 \times 4 + 6$, we replace this last polynomial with $x^{13} - \frac{1}{2}x^{15}$.)

Consequently, $H_R = \mathbb{K}[u, x^4, x^6 + ux^7, 2x^{13} + ux^{14}]$. With the notations above, $e_1 = 3, e_2 = 2$, hence $\mathbb{K}[x^4, x^6, x^{13}] \simeq T = \mathbb{K}[X_0, X_1, X_2]/(X_1^2 - X_0^3, X_2^2 - X_0^5 X_1)$, and

$$\mathbb{K}[u] \longrightarrow \mathbb{K}[u][X_0, X_1, X_2]/(X_1^2 - X_0^3, X_2^2 - 4X_0^5 X_1 - u^2 X_0^7)$$

gives us a deformation from R to T (we can also change X_2 with $\frac{1}{2}X_2$, and then $B = (X_1^2 - X_0^3, X_2^2 - 4X_0^4 X_1)$).

5. SEMIGROUP OF A POLYNOMIAL CURVE

Let \mathbb{K} be a field and let $f_1(x), \dots, f_s(x)$ be s polynomials of $\mathbb{K}[x]$. Let $A = \mathbb{K}[f_1, \dots, f_s]$ be a subalgebra of $\mathbb{K}[x]$, and assume, without loss of generality, that f_i is monic for all $i \in \{1, \dots, s\}$. Given $f(x) = \sum_{i=0}^p c_i x^i \in A$, with $c_p \neq 0$, we set $d(f) = p$ and $M(f) = c_p x^p$, the *degree* and *leading monomial*, respectively. We also define $\text{supp}(f) = \{i \mid c_i \neq 0\}$. The set $d(A) = \{d(f) \mid f \in A\}$ is a submonoid of \mathbb{N} . We shall assume that $\lambda_A(\mathbb{K}[x]/A) < \infty$. In particular $d(A)$ is a numerical semigroup. We say that $\{f_1, \dots, f_s\}$ is a *basis* of A if $\{d(f_1), \dots, d(f_s)\}$ generates $d(A)$. Clearly, $\{f_1, \dots, f_s\}$ is a basis of A if

and only if $\mathbb{K}[M(f), f \in A] = \mathbb{K}[M(f_1), \dots, M(f_s)]$. For several variables, these basis are known in the literature as SAGBI basis ([17, 4]). Since there are already algorithms in the literature to calculate a basis of A , we will not include the procedure here.

We would like just mention that if we follow a similar argument to the one used in Section 2, the sequences of degrees decrease, and thus the finiteness conditions are easier to deduce. In this setting a basis for A is unique up to constants.

6. DEFORMATION TO A TORIC IDEAL

Let the notations be as in Section 5. Given $f(x) = \sum_{i=0}^p c_i x^i \in \mathbb{K}[x]$, we set $h_f(u, x) = \sum_{i=0}^p c_i u^{p-i} x^i$, in particular, if we consider the linear form $L_h : \mathbb{N}^2 \rightarrow \mathbb{N}, L(a, b) = a + b$, then h_f is L_h -homogeneous of degree p , that is, $L_h(i, p-i) = p$ for all $i \in \text{Supp}(f)$. We set $h_A = \mathbb{K}[u, h_f \mid f \in A]$. With these notations we have the following result, and its proof is similar to that of Proposition 3.1.

Proposition 6.1. *The set $\{f_1, \dots, f_r\}$ is a basis of A if and only if $h_A = \mathbb{K}[u, h_{f_1}, \dots, h_{f_s}]$.*

Suppose that $\{f_1, \dots, f_s\}$ is a basis of A . By the inclusion morphism of rings $D : \mathbb{K}[u] \rightarrow B = \mathbb{K}[u, h_{f_1}, \dots, h_{f_s}]$, B is a $\mathbb{K}[u]$ -module. When $u = 1$ (respectively $u = 0$), we get $B|_{u=1} = A$ (respectively $B|_{u=0} = \mathbb{K}[M(f_1), \dots, M(f_s)]$). Hence we get a deformation from A to $\mathbb{K}[M(f_1), \dots, M(f_s)]$. More precisely let

$$\psi : \mathbb{K}[X_1, \dots, X_s] \rightarrow \mathbb{K}[f_1, \dots, f_s]$$

and

$$h_\psi : \mathbb{K}[u][X_1, \dots, X_s] \rightarrow \mathbb{K}[u][h_{f_1}, \dots, h_{f_s}]$$

be the morphisms of rings such that $h_\psi(u) = u$, $\psi(X_i) = f_i$ and $h_\psi(X_i) = h_{f_i}$ for all $i = 1, \dots, s$. For all $i = 1, \dots, r$, let

$$S_i = f_1^{\alpha_1^i} \dots f_s^{\alpha_s^i} - f_1^{\beta_1^i} \dots f_s^{\beta_s^i} = \sum_{\theta^i} c_{\theta^i}^i f_1^{\theta_1^i} \dots f_s^{\theta_s^i}$$

with $d(f_1^{\theta_1^i} \dots f_s^{\theta_s^i}) = D_{\theta^i}^i > \sum_{k=1}^s \alpha_k^i d(f_k) = \sum_{k=1}^s \beta_k^i d(f_k) = p_i$. Let I (respectively J) be the ideal generated by $(G_i = X_1^{\alpha_1^i} \dots X_s^{\alpha_s^i} - X_1^{\beta_1^i} \dots X_s^{\beta_s^i} - \sum_{\theta^i} c_{\theta^i}^i X_1^{\theta_1^i} \dots X_s^{\theta_s^i})_{1 \leq i \leq r}$ (respectively $(H_i = X_1^{\alpha_1^i} \dots X_s^{\alpha_s^i} - X_1^{\beta_1^i} \dots X_s^{\beta_s^i} - \sum_{\theta^i} u^{p_i - D_{\theta^i}^i} c_{\theta^i}^i X_1^{\theta_1^i} \dots X_s^{\theta_s^i})_{1 \leq i \leq r}$) in $\mathbb{K}[X_1, \dots, X_s]$ (respectively $\mathbb{K}[u][X_1, \dots, X_s]$).

We shall consider on \mathbb{N}^s (respectively, \mathbb{N}^{s+1}) the linear form

$$D(\theta_1, \dots, \theta_s) = \sum_{i=1}^s \theta_i d(f_i)$$

(respectively $D_h(\theta_0, \theta_1, \dots, \theta_s) = \theta_0 + \sum_{i=1}^s \theta_i d(f_i)$).

Given a monomial $X_1^{\theta_1} \dots X_s^{\theta_s}$ (respectively $u^{\theta_0} X_1^{\theta_1} \dots X_s^{\theta_s}$), we set $D(X_1^{\theta_1} \dots X_s^{\theta_s}) = D(\theta_1, \dots, \theta_s)$ (respectively $D_h(u^{\theta_0} X_1^{\theta_1} \dots X_s^{\theta_s}) = D_h(\theta_0, \theta_1, \dots, \theta_s)$).

For $G = \sum_{\theta} c_{\theta} X_1^{\theta_1} \dots X_s^{\theta_s}$ (respectively $H = \sum_{\theta} c_{\theta} u^{\theta_0} X_1^{\theta_1} \dots X_s^{\theta_s}$), we say that G (respectively H) is D -homogeneous of degree a (respectively D_h -homogeneous of degree b) if $D(\theta_1, \dots, \theta_s) = a$ (respectively $D_h(\theta_0, \theta_1, \dots, \theta_s) = b$) for all $(\theta_1, \dots, \theta_s)$ (respectively $(\theta_0, \theta_1, \dots, \theta_s)$) such that $c_{\theta} \neq 0$. More generally let $G = \sum_{k=0}^m G_{p_k}$ where $p_0 > p_1 > \dots > p_m$ and G_{p_k} is D -homogeneous of degree p_k . We set $D(G) = p_0$. We also set $\text{In}(G) = G_{p_0}$ and we call it the *initial form* of G .

Lemma 6.2. *With the standing notations and hypothesis, the kernel of ψ is generated by I .*

Proof. Let F_1, \dots, F_r be a generating system of the kernel of the morphism

$$\phi : \mathbb{K}[X_1, \dots, X_s] \rightarrow \mathbb{K}[x], \phi(X_i) = M(f_i)$$

for all $i \in \{1, \dots, s\}$. In particular F_i is D -homogeneous of degree $d(f_i)$ for all $i \in \{1, \dots, s\}$, and $\mathbb{K}[X_1, \dots, X_s]/(F_1, \dots, F_r) \simeq \mathbb{K}[M(f_1), \dots, M(f_s)]$.

For all $i \in \{1, \dots, r\}$, $\psi(G_i) = 0$. Hence $I \subseteq \ker(\psi)$. For the other inclusion, let $G = \sum_{\theta} c_{\theta} X_1^{\theta_1} \cdots X_s^{\theta_s} \in \ker(\psi)$. Write $G = \sum_{k=0}^m c_{\theta^k} X_1^{\theta_1^k} \cdots X_s^{\theta_s^k}$ where $D(\theta^0) \geq D(\theta^1) \geq \dots > D(\theta^m)$. Since $\psi(G) = 0$, we have that

$$\sum_{k=0}^m c_{\theta^k} f_1^{\theta_1^k} \cdots f_s^{\theta_s^k} = 0.$$

In particular, $\sum_{k, D(\theta^k)=D(\theta^0)} c_{\theta^k} M(f_1)^{\theta_1^k} \cdots M(f_s)^{\theta_s^k} = 0$, and consequently $\sum_{k, D(\theta^k)=D(\theta^0)} c_{\theta^k} X_1^{\theta_1^k} \cdots X_s^{\theta_s^k} \in \ker(\phi)$. This implies that

$$\sum_{k, D(\theta^k)=D(\theta^0)} c_{\theta^k} X_1^{\theta_1^k} \cdots X_s^{\theta_s^k} = \sum_{i=1}^r \lambda_i^0 F_i$$

for some $\lambda_i^0 \in \mathbb{K}[X_1, \dots, X_s]$, $i \in \{1, \dots, r\}$, with λ_i^0 is D -homogeneous of degree $D(G) - D(F_i)$. Hence

$$\sum_{k, D(\theta^k)=D(\theta^0)} c_{\theta^k} f_1^{\theta_1^k} \cdots f_s^{\theta_s^k} = \sum_{i=1}^r \lambda_i^0(f_1, \dots, f_s) S_i.$$

Let $G^1 = G - \sum_{i=1}^r \lambda_i^0 G_i$. It follows that $G^1 \in \ker(\psi)$. If $G^1 \neq 0$, then $D(G) > D(G^1)$. Then we restart with G^1 . We construct in the same way G^2 , and $\lambda_1^1, \dots, \lambda_r^1$ such that $G^1 = G^2 + \sum_{j=1}^r \lambda_j^1 G_j$ with $D(G) > D(G^1) > D(G^2)$, λ_i^1 D -homogeneous and $D(\lambda_i^0) > D(\lambda_i^1)$ for all $i \in \{1, \dots, r\}$. If we continue in this way, we get that for all $k \geq 0$,

$$G = G^{k+1} + \sum_{i=1}^r (\lambda_i^0 + \lambda_i^1 + \dots + \lambda_i^k) G_i,$$

with $D(G) > D(G^1) > \dots > D(G^{k+1})$, λ_i^j D -homogeneous, and $D(\lambda_i^0) > D(\lambda_i^1) > \dots > D(\lambda_i^k)$ for all $i \in \{1, \dots, r\}$ and for all $j \in \{1, \dots, k\}$. Thus, there exists l such that $G^{l+1} = 0$. Hence $G = \sum_{i=1}^r (\lambda_i^0 + \lambda_i^1 + \dots + \lambda_i^l) G_i$. This proves our assertion. \square

Let the notations be as above. Let $G = \sum_{\theta} c_{\theta} X_1^{\theta_1} \cdots X_s^{\theta_s} \in \mathbb{K}[X_1, \dots, X_s]$ and write $G = \sum_{i=0}^m G_{p_i}$ with $p_0 > p_1 > \dots > p_m$ and G_{p_i} D -homogeneous. We set $H_G = \sum_{i=0}^m u^{p_0-p_i} G_{p_i}$, in such a way that H_G is D_h -homogeneous of degree p_0 . Given an ideal S of $\mathbb{K}[X_1, \dots, X_s]$, we set $\text{In}(S) = (\text{In}(G) \mid G \in S \setminus \{0\})$. We also denote by $H_S = (H_G \mid G \in S \setminus \{0\}) \mathbb{K}[u, X_1, \dots, X_s]$. With these notations we have $\text{In}(S_i) = F_i$ and $H_{G_i} = H_i$ for all $i \in \{1, \dots, r\}$.

Lemma 6.3. *Let the notations be as above. We have $\text{In}(I) = (F_1, \dots, F_r)$, and $H_I = (H_1, \dots, H_r) = J$.*

Proof. The first assertion follows from the proof of Lemma 6.2.

To prove the second assertion, let $H \in H_I$ and assume that H is D_h -homogeneous. We have $G = H(1, X_1, \dots, X_s) \in I$. Furthermore, $H = u^e H_G$ for some $e \geq 0$. Write $H_G = \text{In}(G) + H^1$ where $H^1(0, X_1, \dots, X_s) = 0$. We have $\text{In}(G) = \sum_{i=1}^r \lambda_i F_i$ where λ_i is D -homogeneous of degree $D(G) - D(F_i)$. Let $H_{G^1} = H_G - \sum_{i=1}^r \lambda_i H_{G_i} = H_G - \sum_{i=1}^r \lambda_i H_i$. Then $H_{G^1} \in H_I$ is D_h -homogeneous and $D_h(H_G) > D_h(H_{G^1})$. Now we restart with H_{G^1} . In this way we show that $H \in (H_1, \dots, H_r)$. \square

Let $H = \sum_{\theta} c_{\theta} u^{\theta} X_1^{\theta_1} \cdots X_s^{\theta_s} \in \ker(h_{\psi})$. Write $H = \sum_{k=0}^n H^k$ where H^k is D_h -homogeneous. For all k , we have $h_{\psi}(H^k) = 0$. Setting $G_k = H^k(1, X_1, \dots, X_s)$, we have $\psi(G_k) = 0$. This implies that $G_k \in I$, and thus $H_{G_k} \in (H_1, \dots, H_r)$ by Proposition 6.3. But $H^k = u^{e_k} H_{G_k}$ for some $e_k \in \mathbb{N}$, whence $H^k \in (H_1, \dots, H_r)$. Finally $H \in (H_1, \dots, H_r)$, which proves that $\ker(h_{\psi}) \subseteq J$. The inclusion $J \subseteq \ker(h_{\psi})$ is obvious, and we can conclude that $J = \ker(h_{\psi})$.

Now the morphism

$$\mathbb{K}[u] \longrightarrow \mathbb{K}[u][X_1, \dots, X_r]/J$$

is flat (because $p(u)$ is not a zero divisor for all $p(u) \in \mathbb{K}[u]$). Hence we get a family of polynomial space curves parametrized by u which gives us a deformation from $\mathbb{K}[X_1, \dots, X_r]/I$ to $\mathbb{K}[X_1, \dots, X_r]/(F_1, \dots, F_r)$.

In particular, we get the following analogue to Theorem 3.4, which can be seen as a geometric reinterpretation of [18, Corollary 11.6] (also [4, Corollary 6.1]).

Theorem 6.4. *Every polynomial space curve of \mathbb{K}^l , parametrized by $Y_1 = g_1(x), \dots, Y_l = g_l(x)$ has a deformation into a monomial curve of \mathbb{K}^r for some positive integer r .*

7. BASIS OF $K[f(x), g(x)]$

In [20], the case when a subalgebra A of $\mathbb{K}[x]$ has a (SAGBI) basis with two elements is treated. Here we study subalgebras generated by two elements of $\mathbb{K}[x]$, and see how a basis can be obtained by using a different approach to that of the general case, as we already did for $\mathbb{K}[x]$ in Section 4.

Let $f(x) = \sum_{i=1}^n a_i x^i$ and $g(x) = \sum_{j=1}^m b_j x^j$ be two polynomials of $\mathbb{K}[x]$ and suppose, without loss of generality, that the following conditions hold:

- (1) $a_n = b_m = 1$,
- (2) $n \geq m$,
- (3) the greatest common divisor of $\text{supp}(f(x)) \cup \text{supp}(g(x))$ is equal to 1 (in particular for all $d > 1$, $f(x), g(x) \notin \mathbb{K}[x^d]$).

Let the notations be as in Section 5, in particular $A = \mathbb{K}[f, g]$. Let also $F(X, Y)$ be the x -resultant of $X - f(x), Y - g(x)$, that is, $F(X, Y)$ is the generator of the kernel of the map $\psi : \mathbb{K}[X, Y] \rightarrow \mathbb{K}[x], \psi(X) = f(x)$ and $\psi(Y) = g(x)$. Since $\mathbb{K}[f, g] = \mathbb{K}[f, g - f]$, then we shall assume that $n > m$. Write $F(X, Y) = Y^n + c_1(X)Y^{n-1} + \dots + c_n(X)$. Given a polynomial $G(X, Y) \notin (F(X, Y))\mathbb{K}[X, Y]$, we set $\text{int}(F, G) = \deg_x G(f(x), g(x))$. Assume that \mathbb{K} is algebraically closed with characteristic zero. Let d be a divisor of n , and let G be a monic polynomial in $\mathbb{K}[X][Y]$ of degree $\frac{n}{d}$ in Y . Write $F = G^d + \alpha_1(X, Y)G^{d-1} + \dots + \alpha_d(X, Y)$ where for all $k \in \{1, \dots, d\}$, if $\alpha_k \neq 0$, then $\deg_Y \alpha_k < \frac{n}{d}$. We say that G is a d th approximate root of F if $\alpha_1 = 0$. There is a unique d th approximate root of F . We denote it by $\text{App}(F, d)$. The following results can be found in [1].

Theorem 7.1. *Under the standing hypothesis.*

- (1) $F(X, Y)$ has one place at infinity, that is the affine curve $F(X, Y) = 0$ has one point at infinity, and the projective closure of this curve in $\mathbb{P}_{\mathbb{K}}^2$ is analytically irreducible at this point.
- (2) $\{\text{int}(F, G) \mid G \in \mathbb{K}[X, Y] \setminus (F)\}$ is a numerical semigroup.
- (3) Let $D(n)$ be the set of divisors of n . The set $\{\text{int}(F, \text{App}(F, d)) \mid d \in D(n)\}$ generates $\Gamma(F)$.

We call $\{\text{int}(F, G) \mid G \in \mathbb{K}[X, Y] \setminus (F)\}$ the semigroup of F , and we denote it by $\Gamma(F)$.

Corollary 7.2. *Let the notations be as above. We have $d(A) = \Gamma(F)$.*

Proof. In fact, $h(x) \in A$ if and only if $h(x) = P(f(x), g(x))$ for some $P(X, Y) \in \mathbb{K}[X, Y]$. Hence $a \in d(A)$ if and only if $a = \text{int}(F, P), P \in \mathbb{K}[X, Y]$ which means that $a \in \Gamma(F)$. \square

Let $F(X, Y) = Y^n + c_1(X)Y^{n-1} + \dots + c_n(X)$ be as above, and assume, after a possible change of variables $X' = X, Y' = Y + \frac{c_1}{n}$, that $c_1(X, Y) = 0$ (note that this does not change A). In particular $\text{App}(F, n) = Y$. A system of generators of $\Gamma(F)$ can be found algorithmically in the following way.

Let $r_0 = d_1 = n = \text{int}(F, X), r_1 = \deg_X a_n(X) = \text{int}(F, \text{App}(F, n))$, and $d_2 = \gcd(r_0, r_1)$. We set $G_2 = \text{App}(F, d_2), r_2 = \text{int}(F, G_2) = \deg_x G_2(f(x), g(x))$, and $d_3 = \gcd(r_2, d_2)$, and so on... With these notations we have the following:

- (1) $d_1 > d_2 > \dots$ and there exists $h \geq 1$ such that $d_{h+1} = 1$;
- (2) $\Gamma(F) = d(A)$ is generated by $\{r_0, r_1, \dots, r_h\}$;
- (3) $r_k d_k > r_{k-1} d_{k-1}$ for all $k \in \{1, \dots, h\}$;
- (4) $\Gamma(F) = d(A)$ is free with respect to the arrangement (r_0, \dots, r_h) . More precisely, let $e_k = \frac{d_k}{d_{k+1}}$ for all $k \in \{1, \dots, h\}$. Then $e_k r_k \in \langle r_0, \dots, r_{k-1} \rangle$;
- (5) $C = \sum_{k=1}^h (e_k - 1)r_k - n + 1$ is the conductor of $\Gamma(F) = d(A)$.

Lemma 7.3. *If $A = \mathbb{K}[x]$, then $r_k = d_{k+1}$ for all $k = 1, \dots, h$. In particular $\deg_x G_h(f(x), g(x)) = 1$ and m divides n .*

Proof. If $A = \mathbb{K}[x]$ then $C = 0$, hence $\sum_{k=1}^h (e_k - 1)r_k = n - 1$. Since $r_k \geq d_{k+1}$, then $\sum_{k=1}^h (e_k - 1)r_k \geq n - 1$ with equality if and only if $r_k = d_{k+1}$ for all $k = 1, \dots, h$. Since $m = r_1 = d_2 = \gcd(n, m)$, then m divides n . \square

Lemma 7.4. [20, Theorem 2] *If $\gcd(n, m) = 1$, then $\{f(x), g(x)\}$ is a basis of A .*

Proof. If $\gcd(n, m) = 1$, then $\Gamma(F) = d(A) = \langle n, m \rangle$. Hence $\{f(x), g(x)\}$ is a basis of A . \square

Lemma 7.5. *Suppose that $\gcd(n, m) = p_1 \cdots p_l$ where p_i is a positive prime number for all $i \in \{1, \dots, l\}$ (and the p_i 's are not necessarily distinct). The set $\{f(x), g(x)\}$ is not a basis of A . Furthermore, if c is the cardinality of a basis of A , then $2 \leq c \leq l + 2$. In particular, if $\gcd(n, m)$ is a prime number $p > 1$, then a basis of A has either two or three elements.*

Proof. Since $\gcd(n, m) > 1$, then the first assertion is clear. On the other hand, since $d_2 = \gcd(n, m) = p_1 \cdots p_r$, we have $A \neq \mathbb{K}[x]$, and $h \leq l + 1$. Hence $\Gamma(F) = d(A)$ has at most $l + 2$ generators. The result now follows. \square

Remark 7.6. Let $\underline{r} = (r_0 = n, r_1 = m, r_2, \dots, r_h)$ be a sequence of integers and for all $k \geq 1$, let $d_k = \gcd(r_0, \dots, r_{k-1})$ and $e_k = \frac{d_k}{d_{k+1}}$. Assume that the following conditions hold:

- (1) $d_1 > d_2 > \dots > d_{h+1} = 1$;
- (2) $r_k d_k > r_{k-1} d_{k-1}$ for all $k \in \{1, \dots, h\}$;
- (3) $e_k r_k \in \langle r_0, \dots, r_{k-1} \rangle$ for all $k = 1, \dots, h$.

Such a sequence is called a δ -sequence and it is well known (see [1]) that there exists a polynomial $\tilde{F}(X, Y)$ with one place at infinity such that the semigroup $\{\text{rank}_{\mathbb{K}} \mathbb{K}[X, Y]/(\tilde{F}, G), G \notin (F)\}$ is generated by \underline{r} .

It follows from Theorem 7.1. that a polynomial curve has one place at infinity. The converse is not true in general. Abhyankar asked whether every semigroup generated by a δ -sequence (hence the semigroup of a curve with one place at infinity) is the semigroup of a polynomial curve (for example, the δ -sequence $(10, 4, 5)$ generates the semigroup $\langle 4, 5 \rangle$ which is the semigroup of the polynomial curve $A = \mathbb{K}[x^4, x^5]$). It has been proved recently that the answer is no ([9]). It would be nice to see which supplementary conditions a δ -sequence should satisfy in order to generate the semigroup of a polynomial curve.

Remark 7.7. Let $f(x)$ and $g(x)$ be as above, and let $A = \mathbb{K}[f(x), g(x)]$. Let also $F(X, Y)$ be the x -resultant of $X - f(x)$ and $Y - g(x)$. Let $r_0 = n, r_1 = m, r_2, \dots, r_h$ be the generators of $\Gamma(F)$ calculated as above. Let $1 \leq k \leq h$ and let $G_k(X, Y) = \text{App}(F, d_k)$. We have $d(G_k(f(x), g(x))) = r_k$, but G_k is not the unique polynomial with this condition (for example, $d((G_k + \lambda)(f(x), g(x))) = r_k$ for all $\lambda \neq 0$). Hence it is natural to ask the following: is there a polynomial $G(X, Y)$ (of degree $< n$ in Y) such that G is parametrized by polynomials in x such that $d(G(f(x), g(x))) = r_k$? Such a polynomial, if it exists, should be of degree $\frac{n}{d_k}$ and should have the contact with F at a characteristic exponent of F (see [1] for the definition of the characteristic exponents of a curve with one place at infinity and the notion of contact). Hence the existence of such a polynomial implies that a polynomial curve can be approximated by polynomial curves.

Let the notations be as above, in particular $F(X, Y) = Y^n + c_1(X)Y^{n-1} + \dots + c_n(X)$ is the x -resultant of $(X - f(x), Y - g(x))$. Let $G_1 = Y, G_2, \dots, G_h$ be the set of approximate roots of $F(X, Y)$ constructed algorithmically as above. In particular $r_0 = n, r_1 = m, r_2 = \text{int}(F, G_2), \dots, r_h = \text{int}(F, G_h)$ generate $d(A)$. For all $k = 2, \dots, h$, let $g_k(x) = G_k(f(x), g(x))$ and let $M(g_k) = b_{r_k} x^{r_k}$. We have $A = \mathbb{K}[f(x), g(x), g_2(x), \dots, g_h(x)]$. Furthermore, the map

$$D : \mathbb{K}[u] \longrightarrow B = \mathbb{K}[u][h_f, h_g, h_{g_2}, \dots, h_{g_h}]$$

introduced in Section 6. gives us a deformation of the polynomial curve $B|_{u=1} = A$ into $B|_{u=0} = \mathbb{K}[t^n, t^m, t^{r_2}, \dots, t^{r_h}]$. Note that, since $\langle n, m, r_2, \dots, r_h \rangle$ is free with respect to the given arrangement, then it is a complete intersection. For all $k \in \{1, \dots, h\}$, write $e_k r_k = \sum_{i=0}^{k-1} \theta_i^k r_i$ with $0 \leq \theta_i^k < e_i$ for every $i \in \{1, \dots, k-1\}$. With the notations above, if T is the ideal of $\mathbb{K}[X_0, X_1, \dots, X_h]$ generated by

$$\{X_1^{e_2} - X_0^{\frac{m}{d_2}}, X_2^{e_2} - X_0^{\theta_0^2} X_1^{\theta_1^2}, \dots, X_h^{e_h} - X_0^{\theta_0^h} X_1^{\theta_1^h} \dots X_{h-1}^{\theta_{h-1}^h}\},$$

then

$$\mathbb{K}[x^n, x^m, x^{r_2}, \dots, x^{r_h}] \simeq \mathbb{K}[X_0, X_1, \dots, X_h]/T.$$

Set $G_{h+1} = F$ and for all $k \geq 1$, let

$$G_{k+1} = G_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} G_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k X_0^{\alpha_0^k} \cdot G_1^{\alpha_1^k} \cdots G_k^{\alpha_k^k},$$

where the following conditions hold:

- (1) for all $i \in \{1, \dots, k-1\}$, $0 \leq \theta_i^k < e_i$;
- (2) for all $\underline{\alpha}^k$, if $c_{\underline{\alpha}^k}^k \neq 0$, then for all $i \in \{1, \dots, k\}$, $0 \leq \alpha_i^k < e_i$,
- (3) for all $\underline{\alpha}^k$, if $c_{\underline{\alpha}^k}^k \neq 0$, then $\alpha_0^k n + \sum_{i=1}^k \alpha_i^k r_i = D_i^k < e_k r_k = \theta_0^k r_0 + \sum_{i=1}^{k-1} \theta_i^k r_i$.

It follows from Section 6 that if I (respectively J) is the ideal generated by $(X_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} X_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k X_0^{\alpha_0^k} \cdot X_1^{\alpha_1^k} \cdots X_k^{\alpha_k^k})_{1 \leq k \leq h}$ (respectively $(X_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} X_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k u^{e_k r_k - D_i^k} X_0^{\alpha_0^k} \cdot X_1^{\alpha_1^k} \cdots X_k^{\alpha_k^k})_{1 \leq k \leq h}$) in $\mathbb{K}[X_0, \dots, X_h]$ (respectively $\mathbb{K}[u][X_0, \dots, X_h]$), then

$$A = \mathbb{K}[x^n, g(x), g_2(x), \dots, g_h(x)] \simeq \mathbb{K}[X_0, X_1, \dots, X_h]/I$$

and

$$\mathbb{K}[u][x^n, h_{g(x)}, h_{g_2(x)}, \dots, h_{g_h(x)}] \simeq \mathbb{K}[u][X_0, X_1, \dots, X_h]/J.$$

Furthermore, $\mathbb{K}[u][X_0, X_1, \dots, X_h]/J$ is a flat $\mathbb{K}[u]$ -module. This gives us a family of space curves parametrized by u which is a deformation from $\mathbb{K}[X_0, X_1, \dots, X_h]/I$ to the toric variety $\mathbb{K}[X_0, X_1, \dots, X_h]/T$. The later being a complete intersection, we get the following result.

Theorem 7.8. *Every polynomial curve $X = f(x), Y = g(x)$ of \mathbb{K}^2 has a deformation into a monomial complete intersection curve of \mathbb{K}^{h+1} for some positive integer h .*

Example 7.9. Let $f(x) = x^6 + x^3, g(x) = x^4$. The minimal polynomial of $(f(x), g(x))$ is given by:

$$F(X, Y) = Y^6 - 2X^2Y^3 - 4XY^3 - Y^3 + X^4.$$

Let $r_0 = 6 = d_1, r_1 = 4$ and $G_1 = Y$. We have $d_2 = \gcd(6, 4) = 2$, and $G_2 = \text{App}(F, 2) = Y^3 - X^2 - 2X - \frac{1}{2}$. Since $g_2(x) = G_2(f(x), g(x)) = -2x^9 - 3x^6 - 2x^3 - \frac{1}{2}$, then $r_2 = 9$ and $d_3 = 1$, hence $\Gamma(F) = \text{d}(A) = \langle 6, 4, 9 \rangle$ and $\{f(x), g(x), -g_2(x)\}$ is a basis of A . Consequently, $h_A = \mathbb{K}[u, x^6 + u^3x^3, x^4, 2x^9 + 3u^3x^6 + 2u^6x^3 + \frac{1}{2}u^9]$. Note that, with the notations above, $e_1 = 3, e_2 = 2$, hence $\mathbb{K}[x^6, x^4, 2x^9] \simeq \mathbb{K}[X_0, X_1, X_2]/(X_1^3 - X_0^2, X_2^2 - 4X_0^3) = \mathbb{K}[X_0, X_1, X_2]/T$, $\mathbb{K}[x^6 + x^3, x^4, 2x^9 - 3x^6 - 2x^3 - \frac{1}{2}] \simeq \mathbb{K}[X_0, X_1, X_2]/(X_1^3 - X_0^2 - 2X_0 - \frac{1}{2}, X_2^2 - 4X_0^3 - 5X_0^2 - 2X_0 - \frac{1}{4})$, and

$$\mathbb{K}[u] \longrightarrow \mathbb{K}[u][X_0, X_1, X_2]/(X_1^3 - X_0^2 - 2u^6X_0 - \frac{1}{2}u^9, X_2^2 - 4X_0^3 - 5u^6X_0^2 - 2u^{12}X_0 - \frac{1}{4}u^{18})$$

gives us a deformation from A to $\mathbb{K}[X_0, X_1, X_2]/T$.

The computation of the approximate roots and of $\Gamma(F)$ can be performed with the algorithm presented in [2].

```
gap> f:=y^6-2*x^2*y^3-4*x*y^3-y^3+x^4;;
gap> SemigroupOfValuesOfPlaneCurveWithSinglePlaceAtInfinity(f);
<Numerical semigroup with 3 generators>
gap> MinimalGeneratingSystem(last);
[ 4, 6, 9 ]
gap> SemigroupOfValuesOfPlaneCurveWithSinglePlaceAtInfinity(f);
[ [ 6, 4, 9 ], [ y, y^3-x^2-2*x-1/2 ] ]
```

Example 7.10. Let $f(x) = x^6 + x, g(x) = x^4$. The minimal polynomial of $(f(x), g(x))$ is given by:

$$F(X, Y) = Y^6 - 2X^2Y^3 - 4XY^2 - Y + X^4.$$

Let $r_0 = 6 = d_1, r_1 = 4$ and $G_1 = Y$. We have $d_2 = \gcd(6, 4) = 2$, and $G_2 = \text{App}(F, 2) = Y^3 - X^2$. Since $g_2(x) = G_2(f(x), g(x)) = -2x^7 - x^2$, then $r_2 = 7$ and $d_3 = 1$, hence $\Gamma(F) = \text{d}(A) = \langle 6, 4, 7 \rangle$ and $\{f(x), g(x), -g_2(x)\}$ is a basis of A . Consequently, $h_A = \mathbb{K}[u, x^6 + u^5x, x^4, 2x^7 + u^5x^2]$. Note that,

with the notations above, $e_1 = 3, e_2 = 2$, hence $\mathbb{K}[x^6, x^4, x^7] \simeq \mathbb{K}[X_0, X_1, X_2]/(X_1^3 - X_0^2, X_2^2 - X_0X_1^2) = \mathbb{K}[X_0, X_1, X_2]/T$, and

$$\mathbb{K}[u] \longrightarrow \mathbb{K}[u][X_0, X_1, X_2]/(X_1^3 - X_0^2, X_2^2 - 4X_0X_1^2 - u^{10}X_1)$$

gives us a deformation from A to $\mathbb{K}[X_0, X_1, X_2]/T$ (we can also change X_2 with $\frac{1}{2}X_2$, and then we get $(X_1^3 - X_0^2, X_2^2 - X_0X_1^2 - \frac{1}{4}u^{10}X_1)$ instead).

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