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Abdallah Assi, Pedro A. García-Sánchez, Vincenzo Micale

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BASES OF SUBALGEBRAS OF $\mathbb{K}\llbracket x \rrbracket$ AND $\mathbb{K}[x]$

A. ASSI, P. A. GARCÍA-SÁNCHEZ, AND V. MICALE

ABSTRACT. Let f_1, \dots, f_s be formal power series (respectively polynomials) in the variable x . We study the semigroup of orders of the formal series in the algebra $K\llbracket f_1, \dots, f_s \rrbracket \subseteq K\llbracket x \rrbracket$ (respectively the semigroup of degrees of polynomials in $K[f_1, \dots, f_s] \subseteq K[x]$). We give procedures to compute these semigroups and several applications.

1. INTRODUCTION

Let \mathbb{K} be a field and let $\mathbb{K}\llbracket x \rrbracket$ (respectively $\mathbb{K}[x]$) be the ring of formal power series (respectively the ring of polynomials) over \mathbb{K} . Let $f_1(x), \dots, f_s(x)$ be s elements of $\mathbb{K}\llbracket x \rrbracket$ (respectively $\mathbb{K}[x]$) and let $R = \mathbb{K}\llbracket f_1, \dots, f_s \rrbracket$ (respectively $A = \mathbb{K}[f_1, \dots, f_s]$) be the subalgebra of $\mathbb{K}\llbracket x \rrbracket$ (respectively $\mathbb{K}[x]$). Given $f \in R$ (respectively A), let $o(f)$ (respectively $d(f)$) the order (respectively the degree) of f . The set $\{o(f), f \in R\}$ (respectively $\{d(f), f \in A\}$) is a submonoid $o(R)$ (respectively $d(A)$) of \mathbb{N} , and the knowledge of a system of generators of this monoid is important for the understanding of the subalgebra R (respectively A). When furthermore $\mathbb{K}\llbracket x \rrbracket$ (respectively $\mathbb{K}[x]$) is an R -module (respectively A -module) of finite length, then $o(R)$ (respectively $d(A)$) is a numerical semigroup.

A numerical semigroup S is a submonoid of the set of nonnegative integers under addition such that the $\mathbb{N} \setminus S$ is finite, or equivalently, $\gcd(S) = 1$ (the greatest common divisor of the elements of S), see for instance [14]. In this case, there exists a minimum $c \in S$ such that $c + \mathbb{N} \subseteq S$. We call this element the *conductor* of S , and denote it by $c(S)$ (the motivation of this name and others coming from Algebraic Geometry is explained in [3, 7]).

Assume that f_i is a monomial x^{a_i} for every $i \in \{1, \dots, s\}$. Then $o(R)$ (respectively $d(A)$) is generated by a_1, \dots, a_s . In this case, $R \simeq \mathbb{K}\llbracket X_1, \dots, X_s \rrbracket / T$ (respectively $A \simeq \mathbb{K}[X_1, \dots, X_s] / T$), where T is a binomial ideal (equivalently $V(T)$ is a toric variety).

Given a subalgebra $R = \mathbb{K}\llbracket f_1, \dots, f_s \rrbracket$ (respectively $A = \mathbb{K}[f_1, \dots, f_s]$), the main objective of this paper is to describe an algorithm that calculates a generating system of $o(R)$ (respectively $d(A)$). The algorithm we present here allows us, by using the technique of homogenization, to construct a flat $\mathbb{K}\llbracket u \rrbracket$ -module (respectively $\mathbb{K}[u]$ -module) which is a deformation of R (respectively A) to a binomial ideal. This technique is well known when $R = \mathbb{K}\llbracket f_1, f_2 \rrbracket$ and \mathbb{K} is algebraically closed field of characteristic zero (see [17]). It turns out that the same holds wherever we can associate a semigroup to the local subalgebra, and also that the same technique can be adapted to the global setting. As a particular case we prove that a plane polynomial curve has a deformation into a complete intersection monomial space curve.

The paper is organized as follows. In Section 2 we focus on the local case, namely the case of a subalgebra R of $\mathbb{K}\llbracket x \rrbracket$. We introduce the notion of basis of R and we show how to construct such a basis. We also show that if $o(R)$ is a numerical semigroup, then every element of a reduced basis is a polynomial. In Section 3 we show how to construct a deformation from R to a toric ideal (or a formal toric variety) by using the technique of homogenization. In Section 4 we focus on the case when $R = \mathbb{K}\llbracket f(x), g(x) \rrbracket$ and \mathbb{K} is algebraically closed field of characteristic zero. The existence in this case of the theory of Newton-Puiseux allows us to precise the results of Sections 2 and 3. The difference with the procedure presented in Section 2 is that it does not rely in the computation of successive kernels. Then in Sections 5 and 6 we adapt the local results to the case of a subalgebra A of $\mathbb{K}[x]$. When $A = \mathbb{K}[f(x), g(x)]$ and \mathbb{K} is algebraically closed field of characteristic zero, a basis of A can be obtained by using the theory of approximate roots of the resultant of $X - f(x), Y - g(x)$, which is a polynomial with one place at infinity.

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The procedures presented here have been implemented in **GAP** ([9]) and will be part of the forthcoming stable release of the package **numericalsgps** ([6]).

1.1. Some notation. We denote by $\langle A \rangle$ the monoid generated by A , $A \subseteq \mathbb{N}$, that is, the set $\{n_1x_1 + \dots + n_mx_m \mid m \in \mathbb{N}, n_i \in \mathbb{N}, x_i \in A \text{ for all } i \in \{1, \dots, m\}\}$.

Associated to each numerical semigroup S we can define a natural partial ordering \leq_S , where for two elements s and r in S we have $s \leq_S r$ if there exists $u \in S$ such that $r = s + u$. The set g_i of minimal elements in $S \setminus \{0\}$ with this ordering is called a *minimal set of generators* for S . The set of minimal generators is finite since for any $s \in S \setminus \{0\}$, we have $x \not\equiv y \pmod{s}$ if $x \neq y$ are minimal elements with respect to \leq_S . The cardinality of the minimal generating system is known as the *embedding dimension* of S .

2. SEMIGROUP OF A FORMAL SPACE CURVE

Let \mathbb{K} be a field. In this section we will consider rings R that are subalgebras of $\mathbb{K}[[x]]$ and such that, if we denote the integral closure of R in its quotient field by \bar{R} , then $\bar{R} = \mathbb{K}[[x]]$ and $\lambda_R(\bar{R}/R) < \infty$ where $\lambda_R(\cdot)$ is the length as R -module. Part of the results of this section are inspired in [13] and in [12]. An alternative procedure (implemented in Maple) is provided in [5]. The main different with our approach is that we do not rely on the multiplicity sequence, and thus we do not need to perform blow-ups. Also we take intrinsic advantage in our implementation of the **GAP** package **numericalsgps** ([9, 6]).

Let $f = \sum_i c_i x^i \in \bar{R}^* = \bar{R} \setminus \{0\}$. Define $\text{supp}(f) = \{i, c_i \neq 0\}$. We call $\min \text{supp}(f)$ the *order* of f and we denote it by $\text{o}(f)$. We also set $M_o(f) = c_{\text{o}(f)} x^{\text{o}(f)}$. If $M_o(f) = x^{\text{o}(f)}$, then we shall say, by abuse of notation, that f is *monic*. We set $\text{o}(0) = +\infty$.

We denote by $\text{o}(R)$ the set of orders of elements in $R^* = R \setminus \{0\}$, that is, $\text{o}(R) = \{\text{o}(f) \mid f \in R^*\}$. We finally set $M_o(R) = \mathbb{K}[M_o(f), f \in R^*]$.

Proposition 2.1. *Let f_1, f_2 be elements of \bar{R}^* and let $a = \min\{\text{o}(f_1), \text{o}(f_2)\}$.*

- (i) $a \leq \text{o}(f_1 + f_2)$.
- (ii) If $\text{o}(f_1) \neq \text{o}(f_2)$ then $a = \text{o}(f_1 + f_2)$.
- (iii) $\text{o}(f_1 f_2) = \text{o}(f_1) + \text{o}(f_2)$.

Proof. This follows easily from the definition of order. □

Proposition 2.2. [11, Lemma 3, p.486] *Let R_1 and R_2 be rings of our type such that $R_1 \subseteq R_2$ and $\text{o}(R_1) = \text{o}(R_2)$. Then $R_1 = R_2$.*

Proposition 2.3. [11, Proposition 1, p.488] *Let R be a ring of our type. Then $\lambda_R(\bar{R}/R) = |\mathbb{N} \setminus \text{o}(R)|$.*

The following two results appear in [13], and since this paper is very hard to find, we include the proofs for sake of completeness.

Proposition 2.4. [13] *Let R be a ring of our type. Then $\text{o}(R)$ is a numerical semigroup.*

Proof. Since R is a ring and by (iii) of Proposition 2.1, we have that $\text{o}(R)$ is a subsemigroup of \mathbb{N} . By $\lambda_R(\bar{R}/R) < \infty$ and Proposition 2.3, we have the proof. □

Proposition 2.5. [13] *Let R be a ring of our type. Then R contains every element $f \in \mathbb{K}[[x]]$ of order $\text{o}(f) \geq c(\text{o}(R))$.*

Proof. Use Proposition 2.2 with $R_1 = R$ and $R_2 = R + x^{c(\text{o}(R))} \mathbb{K}[[x]]$. □

This later result allows to work with polynomials instead of series.

Let f_1, \dots, f_s be in \bar{R}^* . Let $R = \mathbb{K}[[f_1, \dots, f_s]]$ be a subalgebra of $\mathbb{K}[[x]]$ as above, that is, the integral closure of R in its quotient field is $\bar{R} = \mathbb{K}[[x]]$ and $\lambda_R(\bar{R}/R) < \infty$.

Under the hypotheses on R , we have that $\text{o}(R)$ is a numerical semigroup (Proposition 2.4).

We say that the set $\{f_1, \dots, f_s\} \subset R^*$ is a *basis* of R if $R = \mathbb{K}[[f_1, \dots, f_s]]$ and $\text{o}(R) = \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$. The set $\{f_1, \dots, f_s\}$ is a basis of R if and only if $M_o(R) = \mathbb{K}[M_o(f_1), \dots, M_o(f_s)]$.

Proposition 2.6. *Let the notations be as above. Given $f(x) \in \mathbb{K}[[x]]$, there exist $g(x) \in R$ and $r(x) \in \mathbb{K}[[x]]$ such that the following conditions hold.*

- (i) $f(x) = g(x) + r(x) = \sum_{\alpha} c_{\alpha} f_1^{\alpha_1} \cdots f_s^{\alpha_s} + r(x)$.
- (ii) If $g(x) \neq 0$ (respectively $r(x) \neq 0$), then $\text{o}(g) \geq \text{o}(f)$ (respectively $\text{o}(r) \geq \text{o}(f)$).
- (iii) Either $r(x) = 0$ or $\text{supp}(r(x)) \subseteq \mathbb{N} \setminus \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$.

Proof. The assertion is clear if $f \in \mathbb{K}$. Suppose that $f \notin \mathbb{K}$ and let $f(x) = \sum_{i \geq p} c_i x^i$ with $p = \text{o}(f) \geq 0$.

- (1) If $p \notin \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$, then we set $g^1 = 0$, $r^1 = c_p x^p$ and $f^1 = f - c_p x^p$.
- (2) If $p \in \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$, then $c_p x^p = c_{\underline{\theta}} M(f_1)^{\theta_1} \cdots M(f_s)^{\theta_s}$. We set $g^1 = c_{\underline{\theta}} f_1^{\theta_1} \cdots f_s^{\theta_s}$, $r^1 = 0$ and $f^1 = f - g^1$.

In such a way that $f = f^1 + g^1 + r^1$, $g^1 \in R$, either $r^1 = 0$ or $\text{supp}(r^1) \subseteq \mathbb{N} \setminus \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$, and if $f^1 \neq 0$, then $\text{o}(f^1) > \text{o}(f) = p$. Then we restart with f^1 . We construct in this way sequences $(f^k)_{k \geq 1}, (g^k)_{k \geq 1}, (r^k)_{k \geq 1}$ such that for all $k \geq 1$, $f = f^k + \sum_{i=1}^k g^i + \sum_{i=1}^k r^i$, and $\text{o}(f) < \text{o}(f^1) < \cdots < \text{o}(f^k)$, $\sum_{i=1}^k g^i \in R$, $\text{supp}(\sum_{i=1}^k r^i) \subseteq \mathbb{N} \setminus \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$ and for all $i < j \leq k$, if $g^i \neq 0 \neq g^j$ (respectively $r^i \neq 0 \neq r^j$), then $\text{o}(f) \leq \text{o}(g^i) < \text{o}(g^j)$ (respectively $\text{o}(f) \leq \text{o}(r^i) < \text{o}(r^j)$). Clearly $\lim_{k \rightarrow +\infty} f^k = 0$. Hence, if $g = \lim_{k \rightarrow +\infty} \sum_{i=1}^k g^i$ and $r = \lim_{k \rightarrow +\infty} \sum_{i=1}^k r^i$, then $f = g + r$ and g, r satisfy the conditions above. \square

We call the series $r(x)$ of the result above the *remainder* of f with respect to $\{f_1, \dots, f_s\}$ and we denote it by $R_{\text{o}}(f, \{f_1, \dots, f_s\})$.

Proposition 2.7. *The set $\{f_1, \dots, f_s\}$ is a basis of R if and only if $R_{\text{o}}(f, \{f_1, \dots, f_s\}) = 0$ for all $f \in R$.*

Proof. Suppose that $\{f_1, \dots, f_s\}$ is a basis of R and let $f \in R$. Let $r(x) = R_{\text{o}}(f, \{f_1, \dots, f_s\})$. Then $r(x) \in R$. If $r \neq 0$, then $\text{o}(r) \in \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$, which is a contradiction.

Conversely, given $0 \neq f \in R$, if $\text{o}(f) \notin \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$, then $R_{\text{o}}(f, \{f_1, \dots, f_s\}) \neq 0$, which is a contradiction. \square

Let, as above, $R = \mathbb{K}[[f_1, \dots, f_s]]$. We shall suppose that f_i is monic for all $1 \leq i \leq s$. Define

$$\phi : \mathbb{K}[X_1, \dots, X_s] \longrightarrow \mathbb{K}[x], \quad \phi(X_i) = M_{\text{o}}(f_i) \text{ for all } i \in \{1, \dots, s\}.$$

Let $\{F_1, \dots, F_r\}$ be a generating system of the kernel of ϕ . Clearly for all $i \in \{1, \dots, r\}$, F_i is a binomial. If $F_i = X_1^{\alpha_1^i} \cdots X_s^{\alpha_s^i} - X_1^{\beta_1^i} \cdots X_s^{\beta_s^i}$, we set $S_i = f_1^{\alpha_1^i} \cdots f_s^{\alpha_s^i} - f_1^{\beta_1^i} \cdots f_s^{\beta_s^i}$. Note that if $p = \sum_{k=1}^s \alpha_k^i \text{o}(f_k) = \sum_{k=1}^s \beta_k^i \text{o}(f_k)$, then $\text{o}(S_i) > p$.

Theorem 2.8. *The system $\{f_1, \dots, f_s\}$ is a basis of R if and only if $R_{\text{o}}(S_i, \{f_1, \dots, f_s\}) = 0$ for all $i \in \{1, \dots, r\}$.*

Proof. Suppose that $\{f_1, \dots, f_s\}$ is a basis of R . Since $S_i \in R$ for all $i \in \{1, \dots, r\}$, then, by Proposition 2.7, $R_{\text{o}}(S_i, \{f_1, \dots, f_s\}) = 0$.

For the sufficiency assume to the contrary that $\{f_1, \dots, f_s\}$ is not a basis of R . Then there exists $f \in R$ such that $\text{o}(f) \notin \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$. Write

$$f = \sum_{\underline{\theta}} c_{\underline{\theta}} f_1^{\theta_1} \cdots f_s^{\theta_s}.$$

For all $\underline{\theta}$, if $c_{\underline{\theta}} \neq 0$, we set $p_{\underline{\theta}} = \sum_{i=1}^s \theta_i \text{o}(f_i) = \text{o}(f_1^{\theta_1} \cdots f_s^{\theta_s})$. Let $p = \min\{p_{\underline{\theta}} \mid c_{\underline{\theta}} \neq 0\}$ and let $\{\underline{\theta}^1, \dots, \underline{\theta}^l\}$ be such that $p = \text{o}(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i})$ for all $i \in \{1, \dots, l\}$ (such a set is clearly finite). Also $p \leq \text{o}(f) < \infty$.

If $\sum_{i=1}^l c_{\underline{\theta}^i} M_{\text{o}}(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}) \neq 0$, then $p = \text{o}(f) \in \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$. But this is imposible. Hence, $\sum_{i=1}^l c_{\underline{\theta}^i} M_{\text{o}}(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}) = 0$, and then $\sum_{i=1}^l c_{\underline{\theta}^i} X_1^{\theta_1^i} \cdots X_s^{\theta_s^i} \in \ker(\phi)$. Hence

$$\sum_{i=1}^l c_{\underline{\theta}^i} X_1^{\theta_1^i} \cdots X_s^{\theta_s^i} = \sum_{k=1}^r \lambda_k F_k$$

with $\lambda_k \in \mathbb{K}[X_1, \dots, X_s]$ for all $k \in \{1, \dots, r\}$ (recall that F_1, \dots, F_r are binomials generating $\ker(\phi)$). This implies that

$$\sum_{i=1}^l c_{\underline{\theta}^i} f_1^{\theta_1^i} \cdots f_s^{\theta_s^i} = \sum_{k=1}^r \lambda_k(f_1, \dots, f_s) S_k.$$

From the hypothesis $R_o(S_k, \{f_1, \dots, f_s\}) = 0$. Hence there is an expression of S_k of the form $S_k = \sum_{\underline{\beta}^k} c_{\underline{\beta}^k} f_1^{\beta_1^k} \cdots f_s^{\beta_s^k}$ with $o(f_1^{\beta_1^k} \cdots f_s^{\beta_s^k}) \geq o(S_k)$.

So by replacing $\sum_{i=1}^l c_{\underline{\theta}^i} f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}$ with $\sum_{k=1}^r \lambda_k(f_1, \dots, f_s) \sum_{\underline{\beta}^k} c_{\underline{\beta}^k} f_1^{\beta_1^k} \cdots f_s^{\beta_s^k}$ in the expression of f , we can rewrite f as $f = \sum_{\underline{\theta}'} c_{\underline{\theta}'} f_1^{\theta_1'} \cdots f_s^{\theta_s'}$ with $\min\{o(f_1^{\theta_1'} \cdots f_s^{\theta_s'}) \mid c_{\underline{\theta}'} \neq 0\} > p$.

Since $o(f) < +\infty$, this process will stop, yielding a contradiction. \square

Algorithm 2.9. Let the notations be as above.

1. If $R_o(S_k, \{f_1, \dots, f_s\}) = 0$ for all $k \in \{1, \dots, r\}$, then $\{f_1, \dots, f_s\}$ is a basis of R .
2. If $r(x) = R_o(S_k, \{f_1, \dots, f_s\}) \neq 0$ for some $k \in \{1, \dots, r\}$, and if $M_o(r(x)) = ax^q$, then we set $f_{s+1} = \frac{1}{a}r(x)$, and we restart with $\{f_1, \dots, f_{s+1}\}$. Note that in this case,

$$\langle o(f_1), \dots, o(f_s) \rangle \subsetneq \langle o(f_1), \dots, o(f_s), o(f_{s+1}) \rangle \subseteq o(R).$$

This process will stop, giving a basis of R , because the complement of $o(R)$ in \mathbb{N} is finite.

Observe that $r(x)$ is not in general a polynomial. So we must use a trick to compute it, or at least the relevant part of it. This is accomplished by using Proposition 2.5. If in the current step of the algorithm $\langle o(f_1), \dots, o(f_s) \rangle$ is a numerical semigroup, then we compute its conductor, say c . Then $c \geq c(o(R))$. To compute $R_o(f, \{f_1, \dots, f_s\})$ we do the following. Let $p = o(f)$.

1. If $p \geq c$, then return 0. We implicitly assume that x^a is in our generating set for $a \in c + \mathbb{N}$ (though we do not store them).
2. If $p \in \langle o(f_1), \dots, o(f_s) \rangle$, then $M_o(f) = \sum_{\underline{\theta}} c_{\underline{\theta}} M_o(f_1)^{\theta_1} \cdots M_o(f_s)^{\theta_s}$. Set $f = f - \sum_{\underline{\theta}} c_{\underline{\theta}} f_1^{\theta_1} \cdots f_s^{\theta_s}$, and call recursively $R_o(f, \{f_1, \dots, f_s\})$ (the process will stop because the order of the new f is larger, and eventually will become bigger than c after a finite number of steps).
3. If $p \notin \langle o(f_1), \dots, o(f_s) \rangle$, then return f .

If $\langle o(f_1), \dots, o(f_s) \rangle$ is not a numerical semigroup, let d be its greatest common divisor. Set $c = dc(\langle o(f_1), \dots, o(f_s) \rangle / d)$. In this case we proceed as follows.

1. If $p \geq c$, then return f . We cannot ensure here that f will be reduced to zero, so we add it just in case.
2. If $p \in \langle o(f_1), \dots, o(f_s) \rangle$, then $M_o(f) = \sum_{\underline{\theta}} c_{\underline{\theta}} M_o(f_1)^{\theta_1} \cdots M_o(f_s)^{\theta_s}$. Set $f = f - \sum_{\underline{\theta}} c_{\underline{\theta}} f_1^{\theta_1} \cdots f_s^{\theta_s}$, and call recursively $R_o(f, \{f_1, \dots, f_s\})$.
3. If $p \notin \langle o(f_1), \dots, o(f_s) \rangle$, then return f . One might check first if d does not divide p , because in this case for sure $p \notin \langle o(f_1), \dots, o(f_s) \rangle$.

Observe that by adding the conditions $p \geq c$, we are avoiding entering in an eventual infinite loop.

Suppose that $\{f_1, \dots, f_s\}$ is a basis of R . Also suppose that for all $i \in \{1, \dots, s\}$, f_i is monic. We say that $\{f_1, \dots, f_s\}$ is a *minimal basis* of R if $o(f_1), \dots, o(f_s)$ generate minimally the semigroup $o(R)$. We say that $\{f_1, \dots, f_s\}$ is a *reduced basis* of R if $\text{supp}(f_i(x) - M_o(f_i)) \subseteq \mathbb{N} \setminus o(R)$. Let $i \in \{1, \dots, s\}$. If $o(f_i) \in \langle o(f_1), \dots, o(f_{i-1}), o(f_{i+1}), \dots, o(f_s) \rangle$, then $\{f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_s\}$ is also a basis of R . Furthermore, by applying the division process of Proposition 2.6 to $f_i - M_o(f_i)$, we can always construct a reduced basis of R .

Corollary 2.10. *The algebra R has a unique minimal reduced basis.*

Proof. Let $\{f_1, \dots, f_s\}$ and $\{g_1, \dots, g_{s'}\}$ be two minimal reduced bases of R . Hence s is the embedding dimension of $o(R)$, and the same holds for s' ; whence they are equal. Let $i = 1$. There exists j_1 such that $o(f_1) = o(g_{j_1})$, because minimal generating systems of numerical semigroups are unique. If $f_1 - g_{j_1} \neq 0$, then $o(f_1 - g_{j_1}) \notin o(R)$ (the basis is reduced), which is a contradiction because $f_1 - g_{j_1} \in R$. The same argument shows that $\{f_1, \dots, f_s\} = \{g_1, \dots, g_s\}$ \square

Remark 2.11. Let $R = \mathbb{K}[[f_1, \dots, f_s]]$ and assume that f_i is monic for all $1 \leq i \leq s$. Also assume that $o(f_1) \leq o(f_2) \leq \dots \leq o(f_s)$. Set $n = o(f_1)$ and let $f_1 = x^n + \sum_{i>n} c_i^1 x^i$. By an analytic change of variables, we may assume that $f_1 = x^n$, hence, up to an analytic isomorphism, we may assume that $R = \mathbb{K}[[x^n, f_2, \dots, f_s]]$. In particular, we may assume that R has a minimal reduced basis of the form $x^n, g_2(x), \dots, g_{s'}(x)$.

Example 2.12. Let $R = \mathbb{K}[[x^4 + x^5, x^6, x^{15} + x^{16}]]$, with \mathbb{K} a field of characteristic zero. Then R is a one-dimensional ring and we will show that $\lambda_R(\mathbb{K}[[X]]/R) < \infty$. Let us denote $x^4 + x^5$ by f_1 , x^6 by f_2 and $x^{15} + x^{16}$ by f_3 . After computing the kernel of $\phi : \mathbb{K}[X_1, X_2, X_3] \rightarrow \mathbb{K}[x]$, $\phi(X_1) = x^4$, $\phi(X_2) = x^6$ and $\phi(X_3) = x^{15}$, we get $S_1 = \frac{1}{3}x^{15} + x^{14} + x^{13}$ and $S_2 = \frac{1}{2}x^{32} + x^{31}$. As $13 \notin \langle 4, 6, 15 \rangle$, we add it as $f_4 = S_1$. We do not care about S_2 , because the conductor of $\langle 4, 6, 15 \rangle$ is 18.

Now, the conductor of $\langle 4, 6, 13, 15 \rangle$ is 12. If we compute the kernel of $\varphi : \mathbb{K}[X_1, X_2, X_3, X_4] \rightarrow \mathbb{K}[x]$, $\phi(X_1) = x^4$, $\phi(X_2) = x^6$, $\phi(X_3) = x^{15}$ and $\phi(X_4) = x^{13}$, then all the elements S_i have orders greater than 12, and so the algorithm ends. We conclude that $o(R) = \langle 4, 6, 13, 15 \rangle$.

We have implemented this algorithm in the `numericalsgps` ([6]) `GAP` ([9]) package. Next we illustrate how to compute this semigroup with the functions we have implemented (that will be available in the next release of the package).

```
gap> x:=X(Rationals,"x");;
gap> l:=[x^4+x^5,x^6,x^15+x^16];;
gap> s:=SemigroupOfValuesOfCurve_Local(l);;
gap> MinimalGeneratingSystem(s);
[ 4, 6, 13, 15 ]
gap> SemigroupOfValuesOfCurve_Local(l,13);
x^13
```

Remark 2.13. It is known (cf. [3, Section II.1]) that there exist relations between algebraic characters and invariants of the semigroup $o(R)$ and the ring R . Hence, in the Example 2.12, from $o(R) = \langle 4, 6, 13, 15 \rangle = \{0, 4, 6, 8, 10, 12, \dots\}$, we have that $\lambda_R(\bar{R}/R) = |[0, c(o(R)) - 1] \cap (\mathbb{N} \setminus o(R))| = 7$, $\lambda_R(R/(R : \bar{R})) = |[0, c(o(R)) - 1] \cap o(R)| = 5$, $t(R) \leq t(o(R)) = 3$, where $t(R)$ is the type of the ring R .

Example 2.14. Let $R = \mathbb{K}[[x^4, x^6 + x^7, x^{13} + a_{14}x^{14} + a_{15}x^{15} + \dots]]$ with \mathbb{K} a field. Using the same argument as in the Example 2.12, we find that if $\text{char } \mathbb{K} \neq 2$, we have that if $a_{15} - a_{14} + 1/2 = 0$, then $\{x^4, x^6 + x^7, x^{13}\}$ is the reduced basis of R . Furthermore, since $\langle 4, 6, 13 \rangle$ is a symmetric numerical semigroup (the number of nonnegative integers not in the semigroup equals the conductor divided by two), then, by [10], R is Gorenstein. Finally $\lambda_R(\bar{R}/R) = 8$. Otherwise if $a_{15} - a_{14} + 1/2 \neq 0$, then $\{x^4, x^6 + x^7, x^{13}, x^{15}\}$ is the reduced basis of R with R a non Gorenstein ring. Furthermore $\lambda_R(\bar{R}/R) = 7$.

Otherwise, if $\text{char } \mathbb{K} = 2$, then the reduced basis of R is $\{x^4, x^6 + x^7, x^{13}, x^{15}\}$ and R is not a Gorenstein ring. Here, $\lambda_R(\bar{R}/R) = 7$.

Example 2.15. Let $R = \mathbb{K}[[x^8, x^{12} + x^{14} + x^{15}]]$, with \mathbb{K} a field of characteristic zero. Using the same argument as in the Example 2.12, we have that

$$\left\{ x^8, x^{12} + x^{14} + x^{15}, x^{26} + x^{27} + x^{29} - \frac{1}{2}x^{31}, \right. \\ \left. x^{53} + \frac{1}{2}x^{55} - \frac{1}{2}x^{57} - \frac{1}{8}x^{63} + \frac{25}{8}x^{67} - \frac{95}{32}x^{71} - \frac{15}{16}x^{75} - \frac{135}{32}x^{83} \right\}$$

is the reduced basis of the Gorenstein ring R . Furthermore, we have $\lambda_R(\bar{R}/R) = 42$.

```
gap> l:=[x^8,x^12+x^14+x^15];;
gap> SemigroupOfValuesOfCurve_Local(l,"basis");
[ x^8, x^15+x^14+x^12, -1/2*x^31+x^29+x^27+x^26,
-135/32*x^83-15/16*x^75-95/32*x^71+25/8*x^67-1/8*x^63-1/2*x^57+1/2*x^55+x^53 ]
```

Example 2.16. The following battery of examples was provided by Lance Bryant as a test for our algorithm.

```

gap> l:= [ [ x^6,x^8+x^9,x^19 ], [x^7,x^9+x^10,x^19,x^31 ], [x^7,x^21+x^28+x^33 ],
[x^4,x^6+x^7,x^13 ], [x^6,x^8+x^11,x^10+2*x^13,x^21 ], [x^5,-x^18-x^21,-x^23,-x^26 ],
[x^5,-x^18-x^21,-x^26 ], [x^5,-x^18-x^21,x^23-x^26 ], [x^6,x^9+x^10,x^19 ],
[x^7,x^9+x^10,x^19 ], [x^8,x^9+x^10,x^19 ], [x^7,x^9+x^10,x^17,x^19 ] ];
gap> List(1, i->MinimalGeneratingSystem(SemigroupOfValuesOfCurve_Local(i)));
[ [ 6, 8, 19, 29 ], [ 7, 9, 19, 29, 31 ], [ 7, 33 ], [ 4, 6, 13, 15 ],
[ 6, 8, 10, 21, 23, 25 ], [ 5, 18, 26, 39, 47 ], [ 5, 18, 26, 39, 47 ],
[ 5, 18, 26, 39, 47 ], [ 6, 9, 19, 20 ], [ 7, 9, 19, 29 ], [ 8, 9, 19, 30 ],
[ 7, 9, 17, 19, 29 ] ]

```

3. DEFORMATION TO A TORIC IDEAL

Let the notations be as in Section 2. Given $f(x) = \sum_{i \geq p} c_i x^i \in \mathbb{K}[[x]]$, we set $H_f(u, x) = \sum_{i \geq p} c_i u^{i-p} x^i$. In particular, if we consider the linear form $L : \mathbb{N}^2 \rightarrow \mathbb{N}, L(a, b) = b - a$, then H_f is L -homogeneous of degree p , that is, $L(i - p, i) = p$ for all $i \in \text{supp}(f)$. We set $H_R = \mathbb{K}[[H_f, f \in R]]$. With these notations we have the following.

Proposition 3.1. *The set $\{f_1, \dots, f_s\}$ is a basis of R if and only if $H_R = \mathbb{K}[[H_{f_1}, \dots, H_{f_s}]]$.*

Proof. Suppose that $\{f_1, \dots, f_s\}$ is a basis of R and let $f(x) \in R$. Write $H_f(u, x) = \sum_{i \geq p} c_i u^{i-p} x^i$. We have $M_o(f) = c_p x^p = c_p \prod_{i=1}^s M_o(f_i)^{p_i^k}$, hence

$$H_f - c_p \prod_{i=1}^s H_{f_i}^{p_i^k} = u^q H_{f^1}$$

with $f^1 \in R$ and either $f^1 = 0$, or $\text{o}(f^1) > p$. In the second case we restart with f^1 . A similar argument as in Proposition 2.6. proves our assertion.

Conversely, suppose that $H_R = \mathbb{K}[[H_{f_1}, \dots, H_{f_s}]]$ and let $f \in R$. Let $P(X_1, \dots, X_s) \in \mathbb{K}[[X_1, \dots, X_s]]$ such that $H_f = P(H_{f_1}, \dots, H_{f_s})$. If $u = 0$, then $M_o(f) = P(M_o(f_1), \dots, M_o(f_s)) \in \mathbb{K}[[M_o(f_1), \dots, M_o(f_s)]]$, hence $M_o(f) \in \mathbb{K}[[M_o(f_1), \dots, M_o(f_s)]]$. \square

Remark 3.2. Suppose that $\{f_1, \dots, f_r\}$ is a basis of R . Then $T = \mathbb{K}[[u]][[H_{f_1}, \dots, H_{f_s}]]$ is a $\mathbb{K}[[u]]$ -module. When $u = 1$ (respectively $u = 0$), we get $T|_{u=1} = R$ (respectively $T|_{u=0} = \mathbb{K}[[M(f_1), \dots, M(f_s)]]$). Hence we get a deformation from R to $\mathbb{K}[[M_o(f_1), \dots, M_o(f_s)]]$. More precisely let

$$\psi : \mathbb{K}[[X_1, \dots, X_s]] \rightarrow R = \mathbb{K}[[f_1, \dots, f_s]]$$

and

$$H_\psi : \mathbb{K}[[u]][[X_1, \dots, X_s]] \rightarrow T = \mathbb{K}[[u]][[H_{f_1}, \dots, H_{f_s}]]$$

be the morphisms of rings such that $H_\psi(u) = u$, $\psi(X_i) = f_i$ and $H_\psi(X_i) = H_{f_i}$ for all $i \in \{1, \dots, s\}$. For all $i \in \{1, \dots, r\}$, let

$$S_i = f_1^{\alpha_1^i} \dots f_s^{\alpha_s^i} - f_1^{\beta_1^i} \dots f_s^{\beta_s^i} = \sum_{\underline{\theta}^i} c_{\underline{\theta}^i}^i f_1^{\theta_1^i} \dots f_s^{\theta_s^i}$$

with $\text{o}(f_1^{\alpha_1^i} \dots f_s^{\alpha_s^i}) = D_{\underline{\theta}^i}^i > \sum_{k=1}^s \alpha_k^i \text{o}(f_k) = \sum_{k=1}^s \beta_k^i \text{o}(f_k) = p_i$. Let I (respectively J) be the ideal generated by $(X_1^{\alpha_1^i} \dots X_s^{\alpha_s^i} - X_1^{\beta_1^i} \dots X_s^{\beta_s^i} - \sum_{\underline{\theta}^i} c_{\underline{\theta}^i}^i X_1^{\theta_1^i} \dots X_s^{\theta_s^i})_{1 \leq i \leq r}$ (respectively $(X_1^{\alpha_1^i} \dots X_s^{\alpha_s^i} - X_1^{\beta_1^i} \dots X_s^{\beta_s^i} - \sum_{\underline{\theta}^i} u^{\underline{D}_{\underline{\theta}^i}^i - p_i} c_{\underline{\theta}^i}^i X_1^{\theta_1^i} \dots X_s^{\theta_s^i})_{1 \leq i \leq r}$) in $\mathbb{K}[[X_1, \dots, X_s]]$ (respectively $\mathbb{K}[[u]][[X_1, \dots, X_s]]$).

Since $\{f_1, \dots, f_s\}$ is a basis of R , the kernel of ψ (respectively $H(\psi)$) is generated by I (respectively J). Now the morphism

$$\mathbb{K}[[u]] \rightarrow \mathbb{K}[[u]][[X_1, \dots, X_s]]/J$$

is flat because u is not a zero divisor. Hence we get a family of formal space curves parametrized by u which gives us a deformation from $\mathbb{K}[[X_1, \dots, X_r]]/I$ to $\mathbb{K}[[X_1, \dots, X_r]]/(F_1, \dots, F_r)$.

In particular we get the following.

Theorem 3.3. *Every formal space curve of \mathbb{K}^l , parametrized by $Y_1 = g_1(x), \dots, Y_l = g_l(x)$ has a deformation into a formal monomial curve of \mathbb{K}^r for some positive integer r .*

4. BASIS OF $\mathbb{K}[[f(x), g(x)]]$

In [18] the case of finding an algebra basis for $\mathbb{K}[f(x), g(x)]$ with f and g polynomials is treated. Here we will do the analogue for $\mathbb{K}[[f(x), g(x)]]$.

Let $f(x) = \sum_{i \geq n} a_i x^i$ and $g(x) = \sum_{j \geq m} b_j x^j$ be two elements of $\mathbb{K}[[x]]$ and suppose, without loss of generality, that the following conditions hold:

- (1) $a_n = b_m = 1$.
- (2) $n \leq m$.
- (3) For all $d > 1$, $f(x), g(x) \notin \mathbb{K}[[x^d]]$ (i.e. the gcd of $\text{supp}(f(x)) \cup \text{supp}(g(x)) = 1$).

Let the notations be as in Section 2, in particular $R = \mathbb{K}[[f, g]]$. By the analytic change of variables $f(x) = \tilde{x}^n$, we may assume that $R = \mathbb{K}[[x^n, g(x)]]$. Let $F(X, Y)$ be the x -resultant of $X - x^n, Y - g(x)$, that is, $F(X, Y)$ is the generator of the kernel of the map $\rho : \mathbb{K}[[X, Y]] \rightarrow \mathbb{K}[[x]]$, $\rho(X) = x^n$ and $\rho(Y) = g(x)$. Since $\mathbb{K}[[f, g]] = \mathbb{K}[[f, g - f^k]]$ for all $k \geq 1$, then we shall assume that $n < m$ and also that n does not divide m . Given a nonzero element $G(X, Y) \notin (F(X, Y))\mathbb{K}[[X, Y]]$, we set $\text{int}(F, G) = \text{o}(G(f(x), g(x)))$. The set of $\text{int}(F, G)$, $G(X, Y) \notin (F(X, Y))\mathbb{K}[[X, Y]]$, is a numerical semigroup. We denote it by $\Gamma(F)$. We have the following.

Proposition 4.1. $\text{o}(R) = \Gamma(F)$.

Proof. We have $a \in \Gamma(F)$ if and only if $a = \text{o}(G(f(x), g(x)))$ for some $G(X, Y) \in \mathbb{K}[[X, Y]]$ if and only if $a \in \text{o}(R)$. \square

Suppose that \mathbb{K} is algebraically closed with characteristic zero, and let $d_1 = n$, $m_1 = \inf\{i \in \text{supp}(g) \mid d_1 \nmid i\}$, that is, $m_1 = m$, and $d_2 = \text{gcd}(n, m_1)$. For all $k \geq 2$ we set $m_k = \inf\{i \in \text{supp}(g) \mid d_k \nmid i\}$ and $d_{k+1} = \text{gcd}(d_k, m_k)$. It follows that there exists $h \geq 1$ such that $d_{h+1} = 1$. The set $\{m_1, \dots, m_h\}$ is called the *set of Newton-Puiseux exponents* of $F(X, Y)$. Let $e_k = \frac{d_k}{d_{k+1}}$ for all $1 \leq k \leq h$ and define the sequence $(r_k)_{0 \leq k \leq h}$ as follows: $r_0 = n, r_1 = m$, and for all $2 \leq k \leq h, r_k = r_{k-1}e_{k-1} + m_k - m_{k-1}$. With these notations we have the following:

- (1) $\Gamma(F) = \text{o}(R)$ is generated by $\{r_0, r_1, \dots, r_h\}$.
- (2) $r_k d_k < r_{k+1} d_{k+1}$ for all $k \in \{1, \dots, h-1\}$.
- (3) $\Gamma(F) = \text{o}(R)$ is free with respect to the arrangement (r_0, \dots, r_h) . More precisely, let $e_k = \frac{d_k}{d_{k+1}}$ for all $k \in \{1, \dots, h\}$. Then $e_k r_k \in \langle r_0, \dots, r_{k-1} \rangle$.
- (4) $C = \sum_{k=1}^h (e_k - 1)r_k - n + 1$ is the conductor of $\Gamma(F) = \text{o}(R)$.

Example 4.2. Let $f = x^7$ and $g = x^4 + x^2$. The above resultant is then $F = y^7 - 7x^2y^3 - x^4 - 14x^2y^2 - 7x^2y - x^2$. Then $\Gamma(F) = \text{o}(R) = \langle 2, 7 \rangle$.

```
gap> Resultant(x-t^7, y-t^4-t^2,t);
y^7-7*x^2*y^3-x^4-14*x^2*y^2-7*x^2*y-x^2
gap> s:=SemigroupOfValuesOfCurve_Local([t^7,t^4+t^2]);
<Modular numerical semigroup satisfying 7x mod 14 <= x >
gap> MinimalGeneratingSystem(last);
[ 2, 7 ]
gap> IsFreeNumericalSemigroup(s);
true
```

Let the notations be as above. For all $k \geq 2$, let $G_k(X, Y) \in \mathbb{K}[[X, Y]]$ such that $\text{o}(G_k(x^n, g(x))) = r_k$. It follows from [1] that $\text{deg}_Y G_k = \frac{n}{d_k}$. If $g_k(x) = G_k(x^n, g(x))$, then we have the following.

Proposition 4.3. *The set $\{x^n, g, g_2, \dots, g_h\}$ is a basis of R , that is, $R = \mathbb{K}[[x^n, g, g_2, \dots, g_h]]$ and $M_o(R) = \mathbb{K}[[x^n, x^m, x^{r_2}, \dots, x^{r_h}]]$.*

Note that, by a similar argument as in Section 2, we may assume that $f = x^n, g = x^m + \sum_{i \in G(\Gamma(F))} c_i^1 x^i$, and for all $k \geq 2, g_k = x^{r_k} + \sum_{i \in G(\Gamma(F))} c_i^k x^i$, where $G(\Gamma(F)) = \{j \in \mathbb{N} \mid j \notin \Gamma(F)\}$ is the set of gaps of $\Gamma(F)$.

Let the notations be as in Section 3. The morphism

$$D : \mathbb{K}[[u]] \longrightarrow T = \mathbb{K}[[u]][[H_f, H_g, H_{g_2}, \dots, H_{g_h}]]$$

gives us a deformation of $T|_{u=1} = R = \mathbb{K}[[f(x), g(x), g_2(x), \dots, g_h(x)]]$ to $T|_{u=0} = \mathbb{K}[[x^n, x^m, x^{r_2}, \dots, x^{r_h}]]$. Note that, since $\langle n, m, r_2, \dots, r_h \rangle$ is free with respect to the given arrangement, then it is a complete intersection (see for instance [14]). For all $k \in \{1, \dots, h\}$, write $e_k r_k = \sum_{i=0}^{k-1} \theta_i^k r_i$ with $0 \leq \theta_i^k < e_i$ for all $i \in \{1, \dots, k-1\}$. If B is the ideal of $\mathbb{K}[X_0, X_1, \dots, X_h]$ generated by

$$\{X_1^{e_2} - X_0^{\frac{m}{d_2}}, X_2^{e_2} - X_0^{\theta_0^2} X_1^{\theta_1^2}, \dots, X_h^{e_h} - X_0^{\theta_0^h} X_1^{\theta_1^h} \dots X_{h-1}^{\theta_{h-1}^h}\}$$

then

$$\mathbb{K}[[x^n, x^m, x^{r_2}, \dots, x^{r_h}]] \simeq \mathbb{K}[[X_0, X_1, \dots, X_h]]/B.$$

Let $\bar{F}(X, Y)$ be the x -resultant of $X - x^n, y - g(x)$. By hypothesis, $\bar{F}(X, Y)$ is a polynomial. Furthermore, $\bar{F}(X, Y) = Y^n + a_1(X)Y^{n-1} + \dots + a_n(X)$ with $\text{o}(a_i(X)) > i$ for all $2 \leq i \leq n$. Set $G_{h+1} = \bar{F}$ and for all $k \geq 1$, let

$$G_{k+1} = G_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} G_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k X_0^{\alpha_0^k} G_1^{\alpha_1^k} \dots G_k^{\alpha_k^k},$$

where the following conditions hold:

- (1) for all $i \in \{0, \dots, k-1\}$, $0 \leq \theta_i^k < e_i$;
- (2) for all $\underline{\alpha}^k$, if $c_{\underline{\alpha}^k}^k \neq 0$, then for all $i \in \{1, \dots, k\}$, $0 \leq \alpha_i^k < e_i$;
- (3) for all $\underline{\alpha}^k$, if $c_{\underline{\alpha}^k}^k \neq 0$, then $\alpha_0^k n + \sum_{i=1}^k \alpha_i^k r_i = D_i^k > e_k r_k = \theta_0^k r_0 + \sum_{i=1}^{k-1} \theta_i^k r_i$.

It follows from Section 3. that if I (respectively J) is the ideal generated by

$$(X_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} X_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k X_0^{\alpha_0^k} X_1^{\alpha_1^k} \dots X_k^{\alpha_k^k})_{1 \leq k \leq h}$$

(respectively $(X_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} X_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k u^{D_i^k - e_k r_k} X_0^{\alpha_0^k} X_1^{\alpha_1^k} \dots X_k^{\alpha_k^k})_{1 \leq k \leq h}$) in $\mathbb{K}[[X_0, \dots, X_h]]$ (respectively $\mathbb{K}[[u]][[X_0, \dots, X_h]]$), then

$$R = \mathbb{K}[[x^n, g(x), g_2(x), \dots, g_h(x)]] \simeq \mathbb{K}[[X_0, X_1, \dots, X_h]]/I$$

and

$$\mathbb{K}[[u]][[x^n, H_g, H_{g_2}, \dots, H_{g_h}]] \simeq \mathbb{K}[[u]][[X_0, X_1, \dots, X_h]]/J.$$

Furthermore, $\mathbb{K}[[u]][[X_0, X_1, \dots, X_h]]/J$ is a flat $\mathbb{K}[[u]]$ -module. This gives us a family of formal space curves parametrized by u which is a deformation from $\mathbb{K}[[X_0, X_1, \dots, X_h]]/I$ to the formal toric variety $\mathbb{K}[[X_0, X_1, \dots, X_h]]/B$. The later being a complete intersection, we get the following.

Theorem 4.4. *Every irreducible singularity of a plane curve $X = f(x), Y = g(x)$ of \mathbb{K}^2 has a deformation into a formal monomial complete intersection curve of \mathbb{K}^{h+1} for some $h \geq 1$.*

Example 4.5. Let $f(x) = x^4, g(x) = x^6 + x^7$. The minimal polynomial of $(f(x), g(x))$ is given by:

$$F(X, Y) = Y^4 - 2X^3Y^2 + X^6 - 4X^5Y - X^7 = (Y^2 - X^3)^2 - 4X^5Y - X^7$$

Let $r_0 = 4 = d_1, r_1 = 6 = m_1$ and $G_1 = Y$. We have $d_2 = \text{gcd}(6, 4) = 2$, hence $m_2 = 7$. It follows that $r_2 = 13$. Note that if $G_2 = Y^2 - X^3$, then $g_2(x) = G_2(f(x), g(x)) = 2x^{13} + x^{14}$. Hence $\Gamma(F) = \text{o}(R) = \langle 4, 6, 13 \rangle$ and $\{f(x), g(x), g_2(x)\}$ is a basis of R . Let us double check it.

```
gap> SemigroupOfValuesOfCurve_Local([x^4, x^6+x^7], "basis");
[ x^4, x^7+x^6, -1/2*x^15+x^13 ]
```

(Observe that the output is different, since this is a reduced basis: we change $2x^{13} + x^4$ with $x^{13} + \frac{1}{2}x^{14}$, and then using that $14 = 2 \times 4 + 6$, we replace this last polynomial with $x^{13} - \frac{1}{2}x^{15}$.)

Consequently, $H_R = \mathbb{K}[x^4, x^6 + ux^7, 2x^{13} + ux^{14}, u^4, u^6, u^{13}]$. With the notations above, $e_1 = 3, e_2 = 2$, hence $\mathbb{K}[x^4, x^6, x^{13}] \simeq T = \mathbb{K}[X_0, X_1, X_2]/(X_1^2 - X_0^3, X_2^2 - X_0^5 X_1)$, and

$$\mathbb{K}[u] \longrightarrow \mathbb{K}[u][X_0, X_1, X_2]/(X_1^2 - X_0^3, X_2^2 - 4X_0^5 X_1 - u^2 X_0^7)$$

gives us a deformation from R to T (we can also change X_2 with $\frac{1}{2}X_2$, and then $B = (X_1^2 - X_0^3, X_2^2 - 4X_0^4 X_1)$).

5. SEMIGROUP OF A POLYNOMIAL CURVE

Let \mathbb{K} be a field and let $f_1(x), \dots, f_s(x)$ be s polynomials of $\mathbb{K}[x]$. Let $A = \mathbb{K}[f_1, \dots, f_s]$ be a subalgebra of $\mathbb{K}[x]$, and assume, without loss of generality, that f_i is monic for all $i \in \{1, \dots, s\}$. Given $f(x) = \sum_{i=0}^p c_i x^i \in A$, with $c_p \neq 0$, we set $d(f) = p$ and $M(f) = c_p x^p$, the *degree* and *leading monomial*, respectively. We also define $\text{supp}(f) = \{i \mid c_i \neq 0\}$. The set $d(A) = \{d(f) \mid f \in A\}$ is a submonoid of \mathbb{N} . We shall assume that $\lambda_A(\mathbb{K}[x]/A) < \infty$. In particular $d(A)$ is a numerical semigroup. We say that $\{f_1, \dots, f_s\}$ is a *basis* of A if $\{d(f_1), \dots, d(f_s)\}$ generates $d(A)$. Clearly, $\{f_1, \dots, f_s\}$ is a basis of A if and only if $\mathbb{K}[M(f), f \in A] = \mathbb{K}[M(f_1), \dots, M(f_s)]$. For several variables, these basis are known in the literature as SAGBI basis ([15, 4]).

The following result is the analogue to Proposition 2.6. Actually, in this section we follow a similar argument to the one used in Section 2, with the advantage that sequences of degrees decrease, and thus the finiteness conditions are easier to deduce.

Proposition 5.1. *Given $f(x) \in \mathbb{K}[x]$, there exist $g(x) \in A$ and $r(x) \in \mathbb{K}[x]$ such that the following conditions hold:*

- (1) $f(x) = g(x) + r(x) = \sum_{\alpha} c_{\alpha} f_1^{\alpha_1} \cdots f_s^{\alpha_s} + r(x)$,
- (2) if $g(x) \neq 0$ (respectively $r(x) \neq 0$), then $d(g) \leq d(f)$ (respectively $d(r) \leq d(f)$),
- (3) $\text{supp}(r(x)) \subseteq \mathbb{N} \setminus \langle d(f_1), \dots, d(f_s) \rangle$.

Proof. The assertion is clear if $f \in \mathbb{K}$. Suppose that $f \notin \mathbb{K}$ and let $f(x) = \sum_{i=0}^p c_i x^i$ with $p = d(f) > 0$.

- (1) If $p \notin \langle d(f_1), \dots, d(f_s) \rangle$, then we set $g^1 = 0, r^1 = c_p x^p$ and $f^1 = f - c_p x^p$.
- (2) If $p \in \langle d(f_1), \dots, d(f_s) \rangle$, then $c_p x^p = c_{\underline{\theta}} M(f_1)^{\theta_1} \cdots M(f_s)^{\theta_s}$. We set $g^1 = c_{\underline{\theta}} f_1^{\theta_1} \cdots f_s^{\theta_s}, r^1 = 0$ and $f^1 = f - g^1$.

In this way, $f = f^1 + g^1 + r^1, g^1 \in \mathbf{A}, \text{supp}(r^1) \subseteq \mathbb{N} \setminus \langle d(f_1), \dots, d(f_s) \rangle$, and if $f^1 \notin \mathbb{K}$, then $d(f^1) < d(f) = p$. Then we restart with f^1 . Clearly there is $k \geq 1$ such that $f^k \in \mathbb{K}$. We set $g = g^1 + \dots + g^k + f^k$ and $r = r^1 + \dots + r^k$. \square

We call the polynomial $r(x)$ of the theorem above the *remainder* of f with respect to $\{f_1, \dots, f_s\}$ and we denote it by $R(f, \{f_1, \dots, f_s\})$. This is an adaptation of the procedure called subduction in [15], with the difference that we are adding f^k to g , and thus whenever the subduction process produces a constant, the remainder is zero.

The following characterization of basis is analogous to that given in Proposition 2.7 for formal space curves; we omit the proof since it is roughly the same.

Proposition 5.2. *The set $\{f_1, \dots, f_s\}$ is a basis of A if and only if $R(f, \{f_1, \dots, f_s\}) = 0$ for all $f \in A$.*

Let the notations be as above and let

$$\phi : \mathbb{K}[X_1, \dots, X_s] \longrightarrow \mathbb{K}[x], \phi(X_i) = M(f_i), \text{ for all } i \in \{1, \dots, s\}.$$

Let $\{F_1, \dots, F_r\}$ be a generating system of the kernel of ϕ . Clearly for all $i = 1, \dots, r, F_i$ is a binomial. If $F_i = X_1^{\alpha_i} \cdots X_s^{\alpha_i} - X_1^{\beta_i} \cdots X_s^{\beta_i}$, we set $S_i = f_1^{\alpha_i} \cdots f_s^{\alpha_i} - f_1^{\beta_i} \cdots f_s^{\beta_i}$. Observe that if $d = \sum_{k=1}^s \alpha_k^i d(f_k) = \sum_{k=1}^s \beta_k^i d(f_k)$, then $d(S_i) < d$. With this we obtain the following particular case of [16, Corollary 11.5] (taking into account that the subduction process is not exactly our remainder).

Theorem 5.3. *The set $\{f_1, \dots, f_s\}$ is a basis of A if and only if $R(S_i, \{f_1, \dots, f_s\}) = 0$ for all $i \in \{1, \dots, r\}$.*

Proof. The proof goes as in Theorem 2.8, but now the inequalities are reversed and p is defined as a maximum instead of a minimum. \square

Algorithm 5.4. Let the notations be as above.

- (1) If $R(S_k(f_1, \dots, f_s), \{f_1, \dots, f_s\}) = 0$ for all $k = 1, \dots, r$, then $\{f_1, \dots, f_s\}$ is a basis of A .
- (2) If $r(x) = R(S_k(f_1, \dots, f_s), \{f_1, \dots, f_s\}) \neq 0$ for some $1 \leq k \leq r$, and if $M(r) = ax^q$, then we set $f_{s+1} = \frac{1}{a}r(x)$, and we restart with $\{f_1, \dots, f_{s+1}\}$. Note that in this case, $\langle d(f_1), \dots, d(f_s) \rangle \subsetneq \langle d(f_1), \dots, d(f_s), d(f_{s+1}) \rangle$.

This process will stop, giving a basis of A .

Suppose that $\{f_1, \dots, f_s\}$ is a basis of A . Also suppose that f_i is monic for all $i \in \{1, \dots, s\}$. We say that $\{f_1, \dots, f_r\}$ is a *minimal basis* of A if $\{d(f_1), \dots, d(f_s)\}$ minimally generates the semigroup $d(A)$. We say that $\{f_1, \dots, f_r\}$ is a *reduced basis* of A if $\text{supp}(f_i(x) - M(f_i)) \in \mathbb{N} \setminus d(A)$.

Let $i \in \{1, \dots, s\}$. If $d(f_i) \in \langle d(f_1), \dots, d(f_{i-1}), d(f_{i+1}), \dots, d(f_s) \rangle$, then $\{f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_s\}$ is also a basis of A . Furthermore, by applying the division process of Proposition 5.1 to $f_i - M(f_i)$, we can always construct a reduced basis of A . Minimal reduced basis are not unique as in the formal space curve case, but “almost” unique.

Corollary 5.5. *Up to constants, the algebra A has a unique minimal reduced basis.*

Proof. Let $\{f_1, \dots, f_s\}$ and $\{g_1, \dots, g_{s'}\}$ be two minimal reduced bases of A . Clearly $s = s'$. Let $i = 1$. There exists j_1 such that $d(f_1) = d(g_{j_1})$. If $f_1 - g_{j_1} \notin \mathbb{K}$, then $d(f_1 - g_{j_1}) \notin \Gamma(\mathbf{A})$, which is a contradiction because $f_1 - g_{j_1} \in A$. The same argument shows that for all $i \geq 2$, there exists j_i such that $f_i - g_{j_i} \in \mathbb{K}$. \square

Corollary 5.6. *Let $\{f_1, \dots, f_s\}$ be a reduced basis of A . For all $i \in \{1, \dots, s\}$, $\text{supp}(f_i - M(f_i)) \subseteq G(\Gamma A)$.*

Example 5.7. We compute $d(R)$ for $R = \mathbb{K}[x^6 + x^3, x^4 + x^2]$; $f_1 = x^6 + x^3$ and $f_2 = x^4 + x^2$. We start by computing the kernel of $\phi : \mathbb{K}[X_1, X_2] \rightarrow \mathbb{K}[x]$, with $\phi(X_1) = x^6$ and $\phi(X_2) = x^4$. This kernel is generated by $F_1 = X_1^2 - X_2^3$. Hence $S_1 = -3x^{10} + 2x^9 - 3x^8$. Since $10 \in \langle 4, 6 \rangle$, $R(S_1, \{f_1, f_2\}) = \frac{-2}{3}x^9 - x^7 - x^5$. We add $f_3 = x^9 + \frac{3}{2}x^7 + \frac{3}{2}x^5$ to our generating set.

In the next step $\phi : \mathbb{K}[X_1, X_2, X_3] \rightarrow \mathbb{K}[x]$, with $\phi(X_1) = x^6$, $\phi(X_2) = x^4$ and $\phi(X_3) = x^9$; $\ker \phi = (X_1^2 - X_2^3, X_1^3 - X_3^2)$, whence $S_1 = x^{16} - x^{15} + \frac{7}{4}x^{14} + \frac{1}{2}x^{12} + \frac{3}{4}x^{10} - \frac{1}{3}x^9$ and $S_2 = x^{10} - \frac{2}{3}x^9 + x^8$. It turns out that $R(S_1, \{f_1, f_2, f_3\}) = 0 = R(S_2, \{f_1, f_2, f_3\})$, and consequently $\{f_1, f_2, f_3\}$ is a (reduced minimal) basis of R . Also $d(R) = \langle 4, 6, 9 \rangle$.

Example 5.8. Let us compute $K[x^7, x^9 + x^{10}, x^{17}, x^{19}]$ with our implementation.

```
gap> MinimalGeneratingSystem(SemigroupOfValuesOfCurve_Global(
[x^7, x^9+x^10, x^17, x^19]);
[ 7, 10, 16, 19, 25 ]
```

6. DEFORMATION TO A TORIC IDEAL

Let the notations be as in Section 5. Given $f(x) = \sum_{i=0}^p c_i x^i \in \mathbb{K}[x]$, we set $h_f(u, x) = \sum_{i=0}^p c_i u^{p-i} x^i$. We set $h_A = \mathbb{K}[h_f \mid f \in A]$. With these notations we have the following result, and its proof is similar to that of Proposition 3.1.

Proposition 6.1. *The set $\{f_1, \dots, f_r\}$ is a basis of A if and only if $h_A = \mathbb{K}[h_{f_1}, \dots, h_{f_s}, u^{d(f_1)}, \dots, x^{d(f_s)}]$.*

Remark 6.2. Suppose that $\{f_1, \dots, f_s\}$ is a basis of A . By the inclusion morphism of rings $D : \mathbb{K}[u] \rightarrow B = \mathbb{K}[u, h_{f_1}, \dots, h_{f_s}]$, B is a $\mathbb{K}[u]$ -module. When $u = 1$ (respectively $u = 0$), we get $B|_{u=1} = A$ (respectively $B|_{u=0} = \mathbb{K}[M(f_1), \dots, M(f_s)]$). Hence we get a deformation from A to $\mathbb{K}[M(f_1), \dots, M(f_s)]$. More precisely let

$$\psi : \mathbb{K}[X_1, \dots, X_s] \rightarrow \mathbb{K}[f_1, \dots, f_s]$$

and

$$h_\psi : \mathbb{K}[u][X_1, \dots, X_s] \rightarrow \mathbb{K}[u][h_{f_1}, \dots, h_{f_s}]$$

be the morphisms of rings such that $h_\psi(u) = u$, $\psi(X_i) = f_i$ and $h_\psi(X_i) = h_{f_i}$ for all $i = 1, \dots, s$. For all $i = 1, \dots, r$, let

$$S_i = f_1^{\alpha_i} \cdots f_s^{\alpha_i} - f_1^{\beta_i} \cdots f_s^{\beta_i} = \sum_{\underline{\theta}^i} c_{\underline{\theta}^i}^i f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}$$

with $d(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}) = D_{\underline{\theta}^i}^i < \sum_{k=1}^s \alpha_k^i d(f_k) = \sum_{k=1}^s \beta_k^i d(f_k) = p_i$. Let I (respectively J) be the ideal generated by $(X_1^{\alpha_1^i} \cdots X_s^{\alpha_s^i} - X_1^{\beta_1^i} \cdots X_s^{\beta_s^i} - \sum_{\underline{\theta}^i} c_{\underline{\theta}^i}^i X_1^{\theta_1^i} \cdots X_s^{\theta_s^i})_{1 \leq i \leq r}$ (respectively $(X_1^{\alpha_1^i} \cdots X_s^{\alpha_s^i} - X_1^{\beta_1^i} \cdots X_s^{\beta_s^i} - \sum_{\underline{\theta}^i} u^{p_i - D_{\underline{\theta}^i}^i} c_{\underline{\theta}^i}^i X_1^{\theta_1^i} \cdots X_s^{\theta_s^i})_{1 \leq i \leq r}$) in $\mathbb{K}[X_1, \dots, X_s]$ (respectively $\mathbb{K}[u][X_1, \dots, X_s]$).

Since $\{f_1, \dots, f_s\}$ is a basis of A , then the kernel of ψ (respectively h_ψ) is generated by I (respectively J). Now the morphism

$$\mathbb{K}[u] \longrightarrow \mathbb{K}[u][X_1, \dots, X_r]/J$$

is flat (because $p(u)$ is not a zero divisor for all $p(u) \in \mathbb{K}[u]$). Hence we get a family of polynomial space curves parametrized by u which gives us a deformation from $\mathbb{K}[X_1, \dots, X_r]/I$ to $\mathbb{K}[X_1, \dots, X_r]/(F_1, \dots, F_r)$.

In particular we get the following analogue to Theorem 3.3.

Theorem 6.3. *Every polynomial space curve of \mathbb{K}^l , parametrized by $Y_1 = g_1(x), \dots, Y_l = g_l(x)$ has a deformation into a monomial curve of \mathbb{K}^r for some positive integer r .*

7. BASIS OF $K[f(x), g(x)]$

Let $f(x) = \sum_{i=1}^n a_i x^i$ and $g(x) = \sum_{j=1}^m b_j x^j$ be two polynomials of $\mathbb{K}[x]$ and suppose, without loss of generality, that the following conditions hold:

- (1) $a_n = b_m = 1$.
- (2) $n \geq m$,
- (3) for all $d > 1$, $f(x), g(x) \notin \mathbb{K}[x^d]$ (that is, the gcd of $\text{supp}(f(x)) \cup \text{supp}(g(x)) = 1$).

Let the notations be as in Section 5, in particular $A = \mathbb{K}[f, g]$. Let also $F(X, Y)$ be the x -resultant of $X - f(x), Y - g(x)$, that is, $F(X, Y)$ is the generator of the kernel of the map $\psi : \mathbb{K}[X, Y] \longrightarrow \mathbb{K}[x]$, $\psi(X) = f(x)$ and $\psi(Y) = g(x)$. Since $\mathbb{K}[f, g] = \mathbb{K}[f, g - f]$, then we shall assume that $n > m$. Write $F(X, Y) = Y^n + c_1(X)Y^{n-1} + \cdots + c_n(X)$. Given a polynomial $G(X, Y) \notin (F(X, Y))\mathbb{K}[X, Y]$, we set $\text{int}(F, G) = \deg_x G(f(x), g(x))$. Assume that \mathbb{K} is algebraically closed with characteristic zero. Let d be a divisor of n , and let G be a monic polynomial in $\mathbb{K}[X][Y]$ of degree $\frac{n}{d}$ in Y . Write $F = G^d + \alpha_1(X, Y)G^{d-1} + \cdots + \alpha_d(X, Y)$ where for all $k \in \{1, \dots, d\}$, if $\alpha_k \neq 0$, then $\deg_Y \alpha_k < \frac{n}{d}$. We say that G is a d th approximate root of F if $\alpha_1 = 0$. There is a unique d th approximate root of F . We denote it by $\text{App}(F, d)$. The following results can be found in [1].

Theorem 7.1. *Under the standing hypothesis.*

- (1) $F(X, Y)$ has one place at infinity, that is, for all $k \in \{1, \dots, n\}$, if $c_k(X) \neq 0$, then $\deg_X c_k(X) < k$.
- (2) $\{\text{int}(F, G) \mid G \in \mathbb{K}[X, Y] \setminus (F)\}$ is a numerical semigroup.
- (3) Let $D(n)$ be the set of divisors of n . The set $\{\text{int}(F, \text{App}(F, d)) \mid d \in D(n)\}$ generates $\Gamma(F)$.

We call $\{\text{int}(F, G) \mid G \in \mathbb{K}[X, Y] \setminus (F)\}$ the semigroup of F , and we denote it by $\Gamma(F)$.

Corollary 7.2. *Let the notations be as above. We have $d(A) = \Gamma(F)$.*

Proof. In fact, $h(x) \in A$ if and only if $h(x) = P(f(x), g(x))$ for some $P(X, Y) \in \mathbb{K}[X, Y]$. Hence $a \in d(A)$ if and only if $a = \text{int}(F, P)$, $P \in \mathbb{K}[X, Y]$ which means that $a \in \Gamma(F)$. \square

Let $F(X, Y) = Y^n + c_1(X)Y^{n-1} + \cdots + c_n(X)$ be as above, and assume, after a possible change of variables $X' = X, Y' = Y + \frac{c_1}{n}$, that $c_1(X, Y) = 0$ (note that this does not change A). In particular $\text{App}(F, n) = Y$. A system of generators of $\Gamma(F)$ can be found algorithmically in the following way.

Let $r_0 = d_1 = n = \text{int}(F, X)$, $r_1 = \deg_X a_n(X) = \text{int}(F, \text{App}(F, n))$, and $d_2 = \text{gcd}(r_0, r_1)$. We set $G_2 = \text{App}(F, d_2)$, $r_2 = \text{int}(F, G_2) = \deg_x G_2(f(x), g(x))$, and $d_3 = \text{gcd}(r_3, d_2)$, and so on... With these notations we have the following:

- (1) $d_1 > d_2 > \dots$ and there exists $h \geq 1$ such that $d_{h+1} = 1$;
- (2) $\Gamma(F) = \mathfrak{d}(A)$ is generated by $\{r_0, r_1, \dots, r_h\}$;
- (3) $r_k d_k > r_{k-1} d_{k-1}$ for all $k \in \{1, \dots, h\}$;
- (4) $\Gamma(F) = \mathfrak{d}(A)$ is free with respect to the arrangement (r_0, \dots, r_h) . More precisely, let $e_k = \frac{d_k}{d_{k+1}}$ for all $k \in \{1, \dots, h\}$. Then $e_k r_k \in \langle r_0, \dots, r_{k-1} \rangle$;
- (5) $C = \sum_{k=1}^h (e_k - 1)r_k - n + 1$ is the conductor of $\Gamma(F) = \mathfrak{d}(A)$.

Lemma 7.3. *If $A = \mathbb{K}[x]$, then $r_k = d_{k+1}$ for all $k = 1, \dots, h$. In particular $\deg_x G_h(f(x), g(x)) = 1$ and m divides n .*

Proof. If $A = \mathbb{K}[x]$ then $C = 0$, hence $\sum_{k=1}^h (e_k - 1)r_k = n - 1$. Since $r_k \geq d_{k+1}$, then $\sum_{k=1}^h (e_k - 1)r_k \geq n - 1$ with equality if and only if $r_k = d_{k+1}$ for all $k = 1, \dots, h$. Since $m = r_1 = d_2 = \gcd(n, m)$, then m divides n . \square

Lemma 7.4. [18, Theorem 2] *If $\gcd(n, m) = 1$, then $\{f(x), g(x)\}$ is a basis of A .*

Proof. If $\gcd(n, m) = 1$, then $\Gamma(F) = \mathfrak{d}(A) = \langle n, m \rangle$. Hence $\{f(x), g(x)\}$ is a basis of A . \square

Lemma 7.5. *Suppose that $\gcd(n, m) = p_1 \cdots p_l$ where p_i is a positive prime number for all $i \in \{1, \dots, l\}$ (and the p_i 's are not necessarily distinct). The set $\{f(x), g(x)\}$ is not a basis of A . Furthermore, if c is the cardinality of a basis of A , then $2 \leq c \leq l + 2$. In particular, if $\gcd(n, m)$ is a prime number $p > 1$, then a basis of A has either two or three elements.*

Proof. Since $\gcd(n, m) > 1$, then the first assertion is clear. On the other hand, since $d_2 = \gcd(n, m) = p_1 \cdots p_r$, we have $A \neq \mathbb{K}[x]$, and $h \leq l + 1$. Hence $\Gamma(F) = \mathfrak{d}(A)$ has at most $l + 2$ generators. The result now follows. \square

Remark 7.6. Let $\underline{r} = (r_0 = n, r_1 = m, r_2, \dots, r_h)$ be a sequence of integers and for all $k \geq 1$, let $d_k = \gcd(r_0, \dots, r_{k-1})$ and $e_k = \frac{d_k}{d_{k+1}}$. Assume that the following conditions hold:

- (1) $d_1 > d_2 > \dots > d_{h+1} = 1$;
- (2) $r_k d_k > r_{k-1} d_{k-1}$ for all $k \in \{1, \dots, h\}$;
- (3) $e_k r_k \in \langle r_0, \dots, r_{k-1} \rangle$ for all $k = 1, \dots, h$.

Such a sequence is called a δ -sequence and it is well known (see [1]) that there exists a polynomial $\tilde{F}(X, Y)$ with one place at infinity such that the semigroup $\{\text{rank}_{\mathbb{K}} \mathbb{K}[X, Y]/(\tilde{F}, G), G \notin (F)\}$ is generated by \underline{r} .

Abhyankar asked whether every semigroup generated by a δ -sequence is the semigroup of a polynomial curve (for example, the δ -sequence $(10, 4, 5)$ generates the semigroup $\langle 4, 5 \rangle$ which is the semigroup of the polynomial curve $A = \mathbb{K}[x^4, x^5]$). It has been proved recently that the answer is no ([8]). It would be nice to see which supplementary conditions a δ -sequence should satisfy in order to generate the semigroup of a polynomial curve.

Remark 7.7. Let $f(x)$ and $g(x)$ be as above, and let $A = \mathbb{K}[f(x), g(x)]$. Let also $F(X, Y)$ be the x -resultant of $X - f(x)$ and $Y - g(x)$. Let $r_0 = n, r_1 = m, r_2, \dots, r_h$ be the generators of $\Gamma(F)$ calculated as above. Let $1 \leq k \leq h$ and let $G_k(X, Y) = \text{App}(F, d_k)$. We have $\mathfrak{d}(G_k(f(x), g(x))) = r_k$, but G_k is not the unique polynomial with this condition (for example, $\mathfrak{d}((G_k + \lambda)(f(x), g(x))) = r_k$ for all $\lambda \neq 0$). Hence it is natural to ask the following: is there a polynomial $G(X, Y)$ (of degree $< n$ in Y) such that G is parametrized by polynomials in x ? Such a polynomial, if it exists, should be of degree $\frac{n}{d_k}$ and should have the contact with F at a characteristic exponent of F (see [1] for the definition of the characteristic exponents of a curve with one place at infinity). Hence the existence of such a polynomial implies that a polynomial curve can be approximated by polynomial curves.

Let the notations be as above, in particular $F(X, Y) = Y^n + c_1(X)Y^{n-1} + \dots + c_n(X)$ is the x -resultant of $(X - f(x), Y - g(x))$. Let $G_1 = Y, G_2, \dots, G_h$ be the set of approximate roots of $F(X, Y)$ constructed algorithmically as above. In particular $r_0 = n, r_1 = m, r_2 = \text{int}(F, G_2), \dots, r_h = \text{int}(F, G_h)$ generate $\mathfrak{d}(A)$. For all $k = 2, \dots, h$, let $g_k(x) = G_k(f(x), g(x))$ and let $M(g_k) = b_{r_k} x^{r_k}$. We have $A = \mathbb{K}[f(x), g(x), g_2(x), \dots, g_h(x)]$. Furthermore, the map

$$D : \mathbb{K}[u] \longrightarrow B = \mathbb{K}[u][h_f, h_g, h_{g_2}, \dots, h_{g_h}]$$

introduced in Remark 6.2 gives us a deformation of the polynomial curve $B|_{u=1} = A$ into $B|_{u=0} = \mathbb{K}[t^n, t^m, t^{r_2}, \dots, t^{r_h}]$. Note that, since $\langle n, m, r_2, \dots, r_h \rangle$ is free with respect to the given arrangement, then it is a complete intersection. For all $k \in \{1, \dots, h\}$, write $e_k r_k = \sum_{i=0}^{k-1} \theta_i^k r_i$ with $0 \leq \theta_i^k < e_i$ for every $i \in \{1, \dots, k-1\}$. With the notations above, if T is the ideal of $\mathbb{K}[X_0, X_1, \dots, X_h]$ generated by

$$\{X_1^{e_2} - X_0^{\frac{m}{d_2}}, X_2^{e_2} - X_0^{\theta_0^2} X_1^{\theta_1^2}, \dots, X_h^{e_h} - X_0^{\theta_0^h} X_1^{\theta_1^h} \dots X_{h-1}^{\theta_{h-1}^h}\},$$

then

$$\mathbb{K}[x^n, x^m, x^{r_2}, \dots, x^{r_h}] \simeq \mathbb{K}[X_0, X_1, \dots, X_h]/T.$$

Set $G_{h+1} = F$ and for all $k \geq 1$, let

$$G_{k+1} = G_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} G_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k X_0^{\alpha_0^k} \cdot G_1^{\alpha_1^k} \dots G_k^{\alpha_k^k},$$

where the following conditions hold:

- (1) for all $i \in \{0, \dots, k-1\}$, $0 \leq \theta_i^k < e_i$;
- (2) for all $\underline{\alpha}^k$, if $c_{\underline{\alpha}^k}^k \neq 0$, then for all $i \in \{1, \dots, k\}$, $0 \leq \alpha_i^k < e_i$,
- (3) for all $\underline{\alpha}^k$, if $c_{\underline{\alpha}^k}^k \neq 0$, then $\alpha_0^k n + \sum_{i=1}^k \alpha_i^k r_i = D_i^k < e_k r_k = \theta_0^k r_0 + \sum_{i=1}^k \theta_i^k r_i$.

It follows from Section 6 that if I (respectively J) is the ideal generated by $(X_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} X_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k X_0^{\alpha_0^k} \cdot X_1^{\alpha_1^k} \dots X_k^{\alpha_k^k})_{1 \leq k \leq h}$ (respectively $(X_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} X_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k u^{e_k r_k - D_i^k} X_0^{\alpha_0^k} \cdot X_1^{\alpha_1^k} \dots X_k^{\alpha_k^k})_{1 \leq k \leq h}$) in $\mathbb{K}[\bar{X}_0, \dots, \bar{X}_h]$ (respectively $\mathbb{K}[u][X_0, \dots, X_h]$), then

$$A = \mathbb{K}[x^n, g(x), g_2(x), \dots, g_h(x)] \simeq \mathbb{K}[X_0, X_1, \dots, X_h]/I$$

and

$$\mathbb{K}[u][x^n, h_{g(x)}, h_{g_2(x)}, \dots, h_{g_h(x)}] \simeq \mathbb{K}[u][X_0, X_1, \dots, X_h]/J.$$

Furthermore, $\mathbb{K}[u][X_0, X_1, \dots, X_h]/J$ is a flat $\mathbb{K}[u]$ -module. This gives us a family of space curve parametrized by u which is a deformation from $\mathbb{K}[X_0, X_1, \dots, X_h]/I$ to the toric variety $\mathbb{K}[X_0, X_1, \dots, X_h]/T$. The later being a complete intersection, we get the following result.

Theorem 7.8. *Every polynomial curve $X = f(x), Y = g(x)$ of \mathbb{K}^2 has a deformation into a monomial complete intersection curve of \mathbb{K}^{h+1} for some positive integer h .*

Example 7.9. Let $f(x) = x^6 + x^3, g(x) = x^4$. The minimal polynomial of $(f(x), g(x))$ is given by:

$$F(X, Y) = Y^6 - 2X^2Y^3 - 4XY^3 - Y^3 + X^4.$$

Let $r_0 = 6 = d_1, r_1 = 4$ and $G_1 = Y$. We have $d_2 = \gcd(6, 4) = 2$, and $G_2 = \text{App}(F, 2) = Y^3 - X^2 - 2X - \frac{1}{2}$. Since $g_2(x) = G_2(f(x), g(x)) = -2x^9 - 3x^6 - 2x^3 - \frac{1}{2}$, then $r_2 = 9$ and $d_3 = 1$, hence $\Gamma(F) = d(A) = \langle 6, 4, 9 \rangle$ and $\{f(x), g(x), g_2(x)\}$ is a basis of A . Consequently, $h_A = \mathbb{K}[x^6 + u^3x^3, x^4, 2x^9 + 3u^3x^6 + 2u^6x^3 + \frac{1}{2}u^9, u^6, u^4, u^9]$. Note that, with the notations above, $e_1 = 3, e_2 = 2$, hence $\mathbb{K}[x^6, x^4, 2x^9] \simeq \mathbb{K}[X_0, X_1, X_2]/(X_1^3 - X_0^2, X_2^2 - 4X_0^3) = \mathbb{K}[X_0, X_1, X_2]/T$, $\mathbb{K}[x^6 + x^3, x^4, 2x^9 - 3x^6 - 2x^3 - \frac{1}{2}] \simeq \mathbb{K}[X_0, X_1, X_2]/(X_1^3 - X_0^2 - 2X_0 - \frac{1}{2}, X_2^2 - 4X_0^3 - 5X_0^2 - 2X_0 - \frac{1}{4})$, and

$$\mathbb{K}[u] \longrightarrow \mathbb{K}[u][X_0, X_1, X_2]/(X_1^3 - X_0^2 - 2u^6X_0 - \frac{1}{2}u^9, X_2^2 - 4X_0^3 - 5u^6X_0^2 - 2u^{12}X_0 - \frac{1}{4}u^{18})$$

gives us a deformation from A to $\mathbb{K}[X_0, X_1, X_2]/T$.

The computation of the approximate roots and of $\Gamma(F)$ can be performed with the algorithm presented in [2].

```
gap> f:=y^6-2*x^2*y^3-4*x*y^3-y^3+x^4;;
gap> SemigroupOfValuesOfPlaneCurveWithSinglePlaceAtInfinity(f);
<Numerical semigroup with 3 generators>
gap> MinimalGeneratingSystem(last);
[ 4, 6, 9 ]
gap> SemigroupOfValuesOfPlaneCurveWithSinglePlaceAtInfinity(f);
```

[[6, 4, 9], [y, y^3-x^2-2*x-1/2]]

Example 7.10. Let $f(x) = x^6 + x, g(x) = x^4$. The minimal polynomial of $(f(x), g(x))$ is given by:

$$F(X, Y) = Y^6 - 2X^2Y^3 - 4XY^2 - Y + X^4.$$

Let $r_0 = 6 = d_1, r_1 = 4$ and $G_1 = Y$. We have $d_2 = \gcd(6, 4) = 2$, and $G_2 = \text{App}(F, 2) = Y^3 - X^2$. Since $g_2(x) = G_2(f(x), g(x)) = -2x^7 - x^2$, then $r_2 = 7$ and $d_3 = 1$, hence $\Gamma(F) = \text{d}(A) = \langle 6, 4, 7 \rangle$ and $\{f(x), g(x), g_2(x)\}$ is a basis of A . Consequently, $h_A = \mathbb{K}[x^6 + u^5x, x^4, 2x^7 + u^5x^2, u^6, u^4, u^7]$. Note that, with the notations above, $e_1 = 3, e_2 = 2$, hence $\mathbb{K}[x^6, x^4, x^7] \simeq \mathbb{K}[X_0, X_1, X_2]/(X_1^3 - X_0^2, X_2^2 - X_0X_1^2) = \mathbb{K}[X_0, X_1, X_2]/T$, and

$$\mathbb{K}[u] \longrightarrow \mathbb{K}[u][X_0, X_1, X_2]/(X_1^3 - X_0^2, X_2^2 - 4X_0X_1^2 - u^{10}X_1)$$

gives us a deformation from A to $\mathbb{K}[X_0, X_1, X_2]/T$ (we can also change X_2 with $\frac{1}{2}X_2$, and then $T = (X_1^3 - X_0^2, X_2^2 - 4X_0X_1^2)$).

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UNIVERSITÉ D’ANGERS, MATHÉMATIQUES, 49045 ANGERS CEDED 01, FRANCE

E-mail address: assi@univ-angers.fr

DEPARTAMENTO DE ÁLGEBRA AND CITIC-UGR, UNIVERSIDAD DE GRANADA, E-18071 GRANADA, ESPAÑA

E-mail address: pedro@ugr.es

DIPARTIMENTO DI MATEMATICA E INFORMATICA, VIALE A. DORIA, 6 - I 95125 CATANIA, ITALIA

E-mail address: vmicale@dmf.unict.it