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# BASES OF SUBALGEBRAS OF $\mathbb{K}\llbracket x \rrbracket$ AND $\mathbb{K}[x]$

A. ASSI, P. A. GARCÍA-SÁNCHEZ, AND V. MICALE

ABSTRACT. Let  $f_1, \dots, f_s$  be formal power series (respectively polynomials) in the variable  $x$ . We study the semigroup of orders of the formal series in the algebra  $K\llbracket f_1, \dots, f_s \rrbracket \subseteq K\llbracket x \rrbracket$  (respectively the semigroup of degrees of polynomials in  $K[f_1, \dots, f_s] \subseteq K[x]$ ). We give procedures to compute these semigroups and several applications.

## 1. INTRODUCTION

Let  $\mathbb{K}$  be a field and let  $\mathbb{K}\llbracket x \rrbracket$  (respectively  $\mathbb{K}[x]$ ) be the ring of formal power series (respectively the ring of polynomials) over  $\mathbb{K}$ . Let  $f_1(x), \dots, f_s(x)$  be  $s$  elements of  $\mathbb{K}\llbracket x \rrbracket$  (respectively  $\mathbb{K}[x]$ ) and let  $R = \mathbb{K}\llbracket f_1, \dots, f_s \rrbracket$  (respectively  $A = \mathbb{K}[f_1, \dots, f_s]$ ) be the subalgebra of  $\mathbb{K}\llbracket x \rrbracket$  (respectively  $\mathbb{K}[x]$ ). Given  $f \in R$  (respectively  $A$ ), let  $\mathfrak{o}(f)$  (respectively  $\mathfrak{d}(f)$ ) the order (respectively the degree) of  $f$ . The set  $\{\mathfrak{o}(f), f \in R\}$  (respectively  $\{\mathfrak{d}(f), f \in A\}$ ) is a submonoid  $\mathfrak{o}(R)$  (respectively  $\mathfrak{d}(A)$ ) of  $\mathbb{N}$ , and the knowledge of a system of generators of this monoid is important for the understanding of the subalgebra  $R$  (respectively  $A$ ). When furthermore  $\mathbb{K}\llbracket x \rrbracket$  (respectively  $\mathbb{K}[x]$ ) is an  $R$ -module (respectively  $A$ -module) of finite length, then  $\mathfrak{o}(R)$  (respectively  $\mathfrak{d}(A)$ ) is a numerical semigroup.

A numerical semigroup  $S$  is a submonoid of the set of nonnegative integers under addition such that the  $\mathbb{N} \setminus S$  is finite, or equivalently,  $\gcd(S) = 1$  (the greatest common divisor of the elements of  $S$ ), see for instance [14]. In this case, there exists a minimum  $c \in S$  such that  $c + \mathbb{N} \subseteq S$ . We call this element the *conductor* of  $S$ , and denote it by  $c(S)$  (the motivation of this name and others coming from Algebraic Geometry is explained in [3, 7]).

Assume that  $f_i$  is a monomial  $x^{a_i}$  for every  $i \in \{1, \dots, s\}$ . Then  $\mathfrak{o}(R)$  (respectively  $\mathfrak{d}(A)$ ) is generated by  $a_1, \dots, a_s$ . In this case,  $R \simeq \mathbb{K}\llbracket X_1, \dots, X_s \rrbracket / T$  (respectively  $A \simeq \mathbb{K}[X_1, \dots, X_s] / T$ ), where  $T$  is a binomial ideal (equivalently  $V(T)$  is a toric variety).

Given a subalgebra  $R = \mathbb{K}\llbracket f_1, \dots, f_s \rrbracket$  (respectively  $A = \mathbb{K}[f_1, \dots, f_s]$ ), the main objective of this paper is to describe an algorithm that calculates a generating system of  $\mathfrak{o}(R)$  (respectively  $\mathfrak{d}(A)$ ). The algorithm we present here allows us, by using the technique of homogenization, to construct a flat  $\mathbb{K}\llbracket u \rrbracket$ -module (respectively  $\mathbb{K}[u]$ -module) which is a deformation of  $R$  (respectively  $A$ ) to a binomial ideal. This technique is well known when  $R = \mathbb{K}\llbracket f_1, f_2 \rrbracket$  and  $\mathbb{K}$  is algebraically closed field of characteristic zero (see [17]). It turns out that the same holds wherever we can associate a semigroup to the local subalgebra, and also that the same technique can be adapted to the global setting. As a particular case we prove that a plane polynomial curve has a deformation into a complete intersection monomial space curve.

The paper is organized as follows. In Section 2 we focus on the local case, namely the case of a subalgebra  $R$  of  $\mathbb{K}\llbracket x \rrbracket$ . We introduce the notion of basis of  $R$  and we show how to construct such a basis. We also show that if  $\mathfrak{o}(R)$  is a numerical semigroup, then every element of a reduced basis is a polynomial. In Section 3 we show how to construct a deformation from  $R$  to a toric ideal (or a formal toric variety) by using the technique of homogenization. In Section 4 we focus on the case when  $R = \mathbb{K}\llbracket f(x), g(x) \rrbracket$  and  $\mathbb{K}$  is algebraically closed field of characteristic zero. The existence in this case of the theory of Newton-Puiseux allows us to precise the results of Sections 2 and 3. The difference with the procedure presented in Section 2 is that it does not rely in the computation of successive kernels. Then in Sections 5 and 6 we adapt the local results to the case of a subalgebra  $A$  of  $\mathbb{K}[x]$ . When  $A = \mathbb{K}[f(x), g(x)]$  and  $\mathbb{K}$  is algebraically closed field of characteristic zero, a basis of  $A$  can be obtained by using the theory of approximate roots of the resultant of  $X - f(x), Y - g(x)$ , which is a polynomial with one place at infinity.

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The procedures presented here have been implemented in **GAP** ([9]) and will be part of the forthcoming stable release of the package **numericalsgps** ([6]).

**1.1. Some notation.** We denote by  $\langle A \rangle$  the monoid generated by  $A$ ,  $A \subseteq \mathbb{N}$ , that is, the set  $\{n_1x_1 + \dots + n_mx_m \mid m \in \mathbb{N}, n_i \in \mathbb{N}, x_i \in A \text{ for all } i \in \{1, \dots, m\}\}$ .

Associated to each numerical semigroup  $S$  we can define a natural partial ordering  $\leq_S$ , where for two elements  $s$  and  $r$  in  $S$  we have  $s \leq_S r$  if there exists  $u \in S$  such that  $r = s + u$ . The set  $g_i$  of minimal elements in  $S \setminus \{0\}$  with this ordering is called a *minimal set of generators* for  $S$ . The set of minimal generators is finite since for any  $s \in S \setminus \{0\}$ , we have  $x \not\equiv y \pmod{s}$  if  $x \neq y$  are minimal elements with respect to  $\leq_S$ . The cardinality of the minimal generating system is known as the *embedding dimension* of  $S$ .

## 2. SEMIGROUP OF A FORMAL SPACE CURVE

Let  $\mathbb{K}$  be a field. In this section we will consider rings  $R$  that are subalgebras of  $\mathbb{K}[[x]]$  and such that, if we denote the integral closure of  $R$  in its quotient field by  $\bar{R}$ , then  $\bar{R} = \mathbb{K}[[x]]$  and  $\lambda_R(\bar{R}/R) < \infty$  where  $\lambda_R(\cdot)$  is the length as  $R$ -module. Part of the results of this section are inspired in [13] and in [12]. An alternative procedure (implemented in Maple) is provided in [5]. The main different with our approach is that we do not rely on the multiplicity sequence, and thus we do not need to perform blow-ups. Also we take intrinsic advantage in our implementation of the **GAP** package **numericalsgps** ([9, 6]).

Let  $f = \sum_i c_i x^i \in \bar{R}^* = \bar{R} \setminus \{0\}$ . Define  $\text{supp}(f) = \{i, c_i \neq 0\}$ . We call  $\min \text{supp}(f)$  the *order* of  $f$  and we denote it by  $\text{o}(f)$ . We also set  $M_o(f) = c_{\text{o}(f)} x^{\text{o}(f)}$ . If  $M_o(f) = x^{\text{o}(f)}$ , then we shall say, by abuse of notation, that  $f$  is *monic*. We set  $\text{o}(0) = +\infty$ .

We denote by  $\text{o}(R)$  the set of orders of elements in  $R^* = R \setminus \{0\}$ , that is,  $\text{o}(R) = \{\text{o}(f) \mid f \in R^*\}$ . We finally set  $M_o(R) = \mathbb{K}[M_o(f), f \in R^*]$ .

**Proposition 2.1.** *Let  $f_1, f_2$  be elements of  $\bar{R}^*$  and let  $a = \min\{\text{o}(f_1), \text{o}(f_2)\}$ .*

- (i)  $a \leq \text{o}(f_1 + f_2)$ .
- (ii) If  $\text{o}(f_1) \neq \text{o}(f_2)$  then  $a = \text{o}(f_1 + f_2)$ .
- (iii)  $\text{o}(f_1 f_2) = \text{o}(f_1) + \text{o}(f_2)$ .

*Proof.* This follows easily from the definition of order. □

**Proposition 2.2.** [11, Lemma 3, p.486] *Let  $R_1$  and  $R_2$  be rings of our type such that  $R_1 \subseteq R_2$  and  $\text{o}(R_1) = \text{o}(R_2)$ . Then  $R_1 = R_2$ .*

**Proposition 2.3.** [11, Proposition 1, p.488] *Let  $R$  be a ring of our type. Then  $\lambda_R(\bar{R}/R) = |\mathbb{N} \setminus \text{o}(R)|$ .*

The following two results appear in [13], and since this paper is very hard to find, we include the proofs for sake of completeness.

**Proposition 2.4.** [13] *Let  $R$  be a ring of our type. Then  $\text{o}(R)$  is a numerical semigroup.*

*Proof.* Since  $R$  is a ring and by (iii) of Proposition 2.1, we have that  $\text{o}(R)$  is a subsemigroup of  $\mathbb{N}$ . By  $\lambda_R(\bar{R}/R) < \infty$  and Proposition 2.3, we have the proof. □

**Proposition 2.5.** [13] *Let  $R$  be a ring of our type. Then  $R$  contains every element  $f \in \mathbb{K}[[x]]$  of order  $\text{o}(f) \geq c(\text{o}(R))$ .*

*Proof.* Use Proposition 2.2 with  $R_1 = R$  and  $R_2 = R + x^{c(\text{o}(R))} \mathbb{K}[[x]]$ . □

This later result allows to work with polynomials instead of series.

Let  $f_1, \dots, f_s$  be in  $\bar{R}^*$ . Let  $R = \mathbb{K}[[f_1, \dots, f_s]]$  be a subalgebra of  $\mathbb{K}[[x]]$  as above, that is, the integral closure of  $R$  in its quotient field is  $\bar{R} = \mathbb{K}[[x]]$  and  $\lambda_R(\bar{R}/R) < \infty$ .

Under the hypotheses on  $R$ , we have that  $\text{o}(R)$  is a numerical semigroup (Proposition 2.4).

We say that the set  $\{f_1, \dots, f_s\} \subset R^*$  is a *basis* of  $R$  if  $R = \mathbb{K}[[f_1, \dots, f_s]]$  and  $\text{o}(R) = \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$ . The set  $\{f_1, \dots, f_s\}$  is a basis of  $R$  if and only if  $M_o(R) = \mathbb{K}[M_o(f_1), \dots, M_o(f_s)]$ .

**Proposition 2.6.** *Let the notations be as above. Given  $f(x) \in \mathbb{K}[[x]]$ , there exist  $g(x) \in R$  and  $r(x) \in \mathbb{K}[[x]]$  such that the following conditions hold.*

- (i)  $f(x) = g(x) + r(x) = \sum_{\alpha} c_{\alpha} f_1^{\alpha_1} \cdots f_s^{\alpha_s} + r(x)$ .
- (ii) If  $g(x) \neq 0$  (respectively  $r(x) \neq 0$ ), then  $\text{o}(g) \geq \text{o}(f)$  (respectively  $\text{o}(r) \geq \text{o}(f)$ ).
- (iii) Either  $r(x) = 0$  or  $\text{supp}(r(x)) \subseteq \mathbb{N} \setminus \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$ .

*Proof.* The assertion is clear if  $f \in \mathbb{K}$ . Suppose that  $f \notin \mathbb{K}$  and let  $f(x) = \sum_{i \geq p} c_i x^i$  with  $p = \text{o}(f) \geq 0$ .

- (1) If  $p \notin \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$ , then we set  $g^1 = 0$ ,  $r^1 = c_p x^p$  and  $f^1 = f - c_p x^p$ .
- (2) If  $p \in \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$ , then  $c_p x^p = c_{\underline{\theta}} M(f_1)^{\theta_1} \cdots M(f_s)^{\theta_s}$ . We set  $g^1 = c_{\underline{\theta}} f_1^{\theta_1} \cdots f_s^{\theta_s}$ ,  $r^1 = 0$  and  $f^1 = f - g^1$ .

In such a way that  $f = f^1 + g^1 + r^1$ ,  $g^1 \in R$ , either  $r^1 = 0$  or  $\text{supp}(r^1) \subseteq \mathbb{N} \setminus \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$ , and if  $f^1 \neq 0$ , then  $\text{o}(f^1) > \text{o}(f) = p$ . Then we restart with  $f^1$ . We construct in this way sequences  $(f^k)_{k \geq 1}, (g^k)_{k \geq 1}, (r^k)_{k \geq 1}$  such that for all  $k \geq 1$ ,  $f = f^k + \sum_{i=1}^k g^i + \sum_{i=1}^k r^i$ , and  $\text{o}(f) < \text{o}(f^1) < \cdots < \text{o}(f^k)$ ,  $\sum_{i=1}^k g^i \in R$ ,  $\text{supp}(\sum_{i=1}^k r^i) \subseteq \mathbb{N} \setminus \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$  and for all  $i < j \leq k$ , if  $g^i \neq 0 \neq g^j$  (respectively  $r^i \neq 0 \neq r^j$ ), then  $\text{o}(f) \leq \text{o}(g^i) < \text{o}(g^j)$  (respectively  $\text{o}(f) \leq \text{o}(r^i) < \text{o}(r^j)$ ). Clearly  $\lim_{k \rightarrow +\infty} f^k = 0$ . Hence, if  $g = \lim_{k \rightarrow +\infty} \sum_{i=1}^k g^i$  and  $r = \lim_{k \rightarrow +\infty} \sum_{i=1}^k r^i$ , then  $f = g + r$  and  $g, r$  satisfy the conditions above.  $\square$

We call the series  $r(x)$  of the result above the *remainder* of  $f$  with respect to  $\{f_1, \dots, f_s\}$  and we denote it by  $R_{\text{o}}(f, \{f_1, \dots, f_s\})$ .

**Proposition 2.7.** *The set  $\{f_1, \dots, f_s\}$  is a basis of  $R$  if and only if  $R_{\text{o}}(f, \{f_1, \dots, f_s\}) = 0$  for all  $f \in R$ .*

*Proof.* Suppose that  $\{f_1, \dots, f_s\}$  is a basis of  $R$  and let  $f \in R$ . Let  $r(x) = R_{\text{o}}(f, \{f_1, \dots, f_s\})$ . Then  $r(x) \in R$ . If  $r \neq 0$ , then  $\text{o}(r) \in \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$ , which is a contradiction.

Conversely, given  $0 \neq f \in R$ , if  $\text{o}(f) \notin \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$ , then  $R_{\text{o}}(f, \{f_1, \dots, f_s\}) \neq 0$ , which is a contradiction.  $\square$

Let, as above,  $R = \mathbb{K}[[f_1, \dots, f_s]]$ . We shall suppose that  $f_i$  is monic for all  $1 \leq i \leq s$ . Define

$$\phi : \mathbb{K}[X_1, \dots, X_s] \longrightarrow \mathbb{K}[x], \quad \phi(X_i) = M_{\text{o}}(f_i) \text{ for all } i \in \{1, \dots, s\}.$$

Let  $\{F_1, \dots, F_r\}$  be a generating system of the kernel of  $\phi$ . Clearly for all  $i \in \{1, \dots, r\}$ ,  $F_i$  is a binomial. If  $F_i = X_1^{\alpha_1^i} \cdots X_s^{\alpha_s^i} - X_1^{\beta_1^i} \cdots X_s^{\beta_s^i}$ , we set  $S_i = f_1^{\alpha_1^i} \cdots f_s^{\alpha_s^i} - f_1^{\beta_1^i} \cdots f_s^{\beta_s^i}$ . Note that if  $p = \sum_{k=1}^s \alpha_k^i \text{o}(f_k) = \sum_{k=1}^s \beta_k^i \text{o}(f_k)$ , then  $\text{o}(S_i) > p$ .

**Theorem 2.8.** *The system  $\{f_1, \dots, f_s\}$  is a basis of  $R$  if and only if  $R_{\text{o}}(S_i, \{f_1, \dots, f_s\}) = 0$  for all  $i \in \{1, \dots, r\}$ .*

*Proof.* Suppose that  $\{f_1, \dots, f_s\}$  is a basis of  $R$ . Since  $S_i \in R$  for all  $i \in \{1, \dots, r\}$ , then, by Proposition 2.7,  $R_{\text{o}}(S_i, \{f_1, \dots, f_s\}) = 0$ .

For the sufficiency assume to the contrary that  $\{f_1, \dots, f_s\}$  is not a basis of  $R$ . Then there exists  $f \in R$  such that  $\text{o}(f) \notin \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$ . Write

$$f = \sum_{\underline{\theta}} c_{\underline{\theta}} f_1^{\theta_1} \cdots f_s^{\theta_s}.$$

For all  $\underline{\theta}$ , if  $c_{\underline{\theta}} \neq 0$ , we set  $p_{\underline{\theta}} = \sum_{i=1}^s \theta_i \text{o}(f_i) = \text{o}(f_1^{\theta_1} \cdots f_s^{\theta_s})$ . Let  $p = \min\{p_{\underline{\theta}} \mid c_{\underline{\theta}} \neq 0\}$  and let  $\{\underline{\theta}^1, \dots, \underline{\theta}^l\}$  be such that  $p = \text{o}(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i})$  for all  $i \in \{1, \dots, l\}$  (such a set is clearly finite). Also  $p \leq \text{o}(f) < \infty$ .

If  $\sum_{i=1}^l c_{\underline{\theta}^i} M_{\text{o}}(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}) \neq 0$ , then  $p = \text{o}(f) \in \langle \text{o}(f_1), \dots, \text{o}(f_s) \rangle$ . But this is impossible. Hence,  $\sum_{i=1}^l c_{\underline{\theta}^i} M_{\text{o}}(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}) = 0$ , and then  $\sum_{i=1}^l c_{\underline{\theta}^i} X_1^{\theta_1^i} \cdots X_s^{\theta_s^i} \in \ker(\phi)$ . Hence

$$\sum_{i=1}^l c_{\underline{\theta}^i} X_1^{\theta_1^i} \cdots X_s^{\theta_s^i} = \sum_{k=1}^r \lambda_k F_k$$

with  $\lambda_k \in \mathbb{K}[X_1, \dots, X_s]$  for all  $k \in \{1, \dots, r\}$  (recall that  $F_1, \dots, F_r$  are binomials generating  $\ker(\phi)$ ). This implies that

$$\sum_{i=1}^l c_{\underline{\theta}^i} f_1^{\theta_1^i} \cdots f_s^{\theta_s^i} = \sum_{k=1}^r \lambda_k(f_1, \dots, f_s) S_k.$$

From the hypothesis  $R_o(S_k, \{f_1, \dots, f_s\}) = 0$ . Hence there is an expression of  $S_k$  of the form  $S_k = \sum_{\underline{\beta}^k} c_{\underline{\beta}^k} f_1^{\beta_1^k} \cdots f_s^{\beta_s^k}$  with  $o(f_1^{\beta_1^k} \cdots f_s^{\beta_s^k}) \geq o(S_k)$ .

So by replacing  $\sum_{i=1}^l c_{\underline{\theta}^i} f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}$  with  $\sum_{k=1}^r \lambda_k(f_1, \dots, f_s) \sum_{\underline{\beta}^k} c_{\underline{\beta}^k} f_1^{\beta_1^k} \cdots f_s^{\beta_s^k}$  in the expression of  $f$ , we can rewrite  $f$  as  $f = \sum_{\underline{\theta}'^i} c_{\underline{\theta}'^i} f_1^{\theta_1'^i} \cdots f_s^{\theta_s'^i}$  with  $\min\{o(f_1^{\theta_1'^i} \cdots f_s^{\theta_s'^i}) \mid c_{\underline{\theta}'^i} \neq 0\} > p$ .

Since  $o(f) < +\infty$ , this process will stop, yielding a contradiction.  $\square$

**Algorithm 2.9.** Let the notations be as above.

1. If  $R_o(S_k, \{f_1, \dots, f_s\}) = 0$  for all  $k \in \{1, \dots, r\}$ , then  $\{f_1, \dots, f_s\}$  is a basis of  $R$ .
2. If  $r(x) = R_o(S_k, \{f_1, \dots, f_s\}) \neq 0$  for some  $k \in \{1, \dots, r\}$ , and if  $M_o(r(x)) = ax^q$ , then we set  $f_{s+1} = \frac{1}{a}r(x)$ , and we restart with  $\{f_1, \dots, f_{s+1}\}$ . Note that in this case,

$$\langle o(f_1), \dots, o(f_s) \rangle \subsetneq \langle o(f_1), \dots, o(f_s), o(f_{s+1}) \rangle \subseteq o(R).$$

This process will stop, giving a basis of  $R$ , because the complement of  $o(R)$  in  $\mathbb{N}$  is finite.

Observe that  $r(x)$  is not in general a polynomial. So we must use a trick to compute it, or at least the relevant part of it. This is accomplished by using Proposition 2.5. If in the current step of the algorithm  $\langle o(f_1), \dots, o(f_s) \rangle$  is a numerical semigroup, then we compute its conductor, say  $c$ . Then  $c \geq c(o(R))$ . To compute  $R_o(f, \{f_1, \dots, f_s\})$  we do the following. Let  $p = o(f)$ .

1. If  $p \geq c$ , then return 0. We implicitly assume that  $x^a$  is in our generating set for  $a \in c + \mathbb{N}$  (though we do not store them).
2. If  $p \in \langle o(f_1), \dots, o(f_s) \rangle$ , then  $M_o(f) = \sum_{\underline{\theta}} c_{\underline{\theta}} M_o(f_1)^{\theta_1} \cdots M_o(f_s)^{\theta_s}$ . Set  $f = f - \sum_{\underline{\theta}} c_{\underline{\theta}} f_1^{\theta_1} \cdots f_s^{\theta_s}$ , and call recursively  $R_o(f, \{f_1, \dots, f_s\})$  (the process will stop because the order of the new  $f$  is larger, and eventually will become bigger than  $c$  after a finite number of steps).
3. If  $p \notin \langle o(f_1), \dots, o(f_s) \rangle$ , then return  $f$ .

If  $\langle o(f_1), \dots, o(f_s) \rangle$  is not a numerical semigroup, let  $d$  be its greatest common divisor. Set  $c = dc(\langle o(f_1), \dots, o(f_s) \rangle / d)$ . In this case we proceed as follows.

1. If  $p \geq c$ , then return  $f$ . We cannot ensure here that  $f$  will be reduced to zero, so we add it just in case.
2. If  $p \in \langle o(f_1), \dots, o(f_s) \rangle$ , then  $M_o(f) = \sum_{\underline{\theta}} c_{\underline{\theta}} M_o(f_1)^{\theta_1} \cdots M_o(f_s)^{\theta_s}$ . Set  $f = f - \sum_{\underline{\theta}} c_{\underline{\theta}} f_1^{\theta_1} \cdots f_s^{\theta_s}$ , and call recursively  $R_o(f, \{f_1, \dots, f_s\})$ .
3. If  $p \notin \langle o(f_1), \dots, o(f_s) \rangle$ , then return  $f$ . One might check first if  $d$  does not divide  $p$ , because in this case for sure  $p \notin \langle o(f_1), \dots, o(f_s) \rangle$ .

Observe that by adding the conditions  $p \geq c$ , we are avoiding entering in an eventual infinite loop.

Suppose that  $\{f_1, \dots, f_s\}$  is a basis of  $R$ . Also suppose that for all  $i \in \{1, \dots, s\}$ ,  $f_i$  is monic. We say that  $\{f_1, \dots, f_s\}$  is a *minimal basis* of  $R$  if  $o(f_1), \dots, o(f_s)$  generate minimally the semigroup  $o(R)$ . We say that  $\{f_1, \dots, f_s\}$  is a *reduced basis* of  $R$  if  $\text{supp}(f_i(x) - M_o(f_i)) \subseteq \mathbb{N} \setminus o(R)$ . Let  $i \in \{1, \dots, s\}$ . If  $o(f_i) \in \langle o(f_1), \dots, o(f_{i-1}), o(f_{i+1}), \dots, o(f_s) \rangle$ , then  $\{f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_s\}$  is also a basis of  $R$ . Furthermore, by applying the division process of Proposition 2.6 to  $f_i - M_o(f_i)$ , we can always construct a reduced basis of  $R$ .

**Corollary 2.10.** *The algebra  $R$  has a unique minimal reduced basis.*

*Proof.* Let  $\{f_1, \dots, f_s\}$  and  $\{g_1, \dots, g_{s'}\}$  be two minimal reduced bases of  $R$ . Hence  $s$  is the embedding dimension of  $o(R)$ , and the same holds for  $s'$ ; whence they are equal. Let  $i = 1$ . There exists  $j_1$  such that  $o(f_1) = o(g_{j_1})$ , because minimal generating systems of numerical semigroups are unique. If  $f_1 - g_{j_1} \neq 0$ , then  $o(f_1 - g_{j_1}) \notin o(R)$  (the basis is reduced), which is a contradiction because  $f_1 - g_{j_1} \in R$ . The same argument shows that  $\{f_1, \dots, f_s\} = \{g_1, \dots, g_s\}$   $\square$

**Remark 2.11.** Let  $R = \mathbb{K}[[f_1, \dots, f_s]]$  and assume that  $f_i$  is monic for all  $1 \leq i \leq s$ . Also assume that  $o(f_1) \leq o(f_2) \leq \dots \leq o(f_s)$ . Set  $n = o(f_1)$  and let  $f_1 = x^n + \sum_{i>n} c_i^1 x^i$ . By an analytic change of variables, we may assume that  $f_1 = x^n$ , hence, up to an analytic isomorphism, we may assume that  $R = \mathbb{K}[[x^n, f_2, \dots, f_s]]$ . In particular, we may assume that  $R$  has a minimal reduced basis of the form  $x^n, g_2(x), \dots, g_{s'}(x)$ .

**Example 2.12.** Let  $R = \mathbb{K}[[x^4 + x^5, x^6, x^{15} + x^{16}]]$ , with  $\mathbb{K}$  a field of characteristic zero. Then  $R$  is a one-dimensional ring and we will show that  $\lambda_R(\mathbb{K}[[X]]/R) < \infty$ . Let us denote  $x^4 + x^5$  by  $f_1$ ,  $x^6$  by  $f_2$  and  $x^{15} + x^{16}$  by  $f_3$ . After computing the kernel of  $\phi : \mathbb{K}[X_1, X_2, X_3] \rightarrow \mathbb{K}[x]$ ,  $\phi(X_1) = x^4$ ,  $\phi(X_2) = x^6$  and  $\phi(X_3) = x^{15}$ , we get  $S_1 = \frac{1}{3}x^{15} + x^{14} + x^{13}$  and  $S_2 = \frac{1}{2}x^{32} + x^{31}$ . As  $13 \notin \langle 4, 6, 15 \rangle$ , we add it as  $f_4 = S_1$ . We do not care about  $S_2$ , because the conductor of  $\langle 4, 6, 15 \rangle$  is 18.

Now, the conductor of  $\langle 4, 6, 13, 15 \rangle$  is 12. If we compute the kernel of  $\varphi : \mathbb{K}[X_1, X_2, X_3, X_4] \rightarrow \mathbb{K}[x]$ ,  $\phi(X_1) = x^4$ ,  $\phi(X_2) = x^6$ ,  $\phi(X_3) = x^{15}$  and  $\phi(X_4) = x^{13}$ , then all the elements  $S_i$  have orders greater than 12, and so the algorithm ends. We conclude that  $o(R) = \langle 4, 6, 13, 15 \rangle$ .

We have implemented this algorithm in the `numericalsgps` ([6]) `GAP` ([9]) package. Next we illustrate how to compute this semigroup with the functions we have implemented (that will be available in the next release of the package).

```
gap> x:=X(Rationals,"x");;
gap> l:=[x^4+x^5,x^6,x^15+x^16];;
gap> s:=SemigroupOfValuesOfCurve_Local(l);;
gap> MinimalGeneratingSystem(s);
[ 4, 6, 13, 15 ]
gap> SemigroupOfValuesOfCurve_Local(l,13);
x^13
```

**Remark 2.13.** It is known (cf. [3, Section II.1]) that there exist relations between algebraic characters and invariants of the semigroup  $o(R)$  and the ring  $R$ . Hence, in the Example 2.12, from  $o(R) = \langle 4, 6, 13, 15 \rangle = \{0, 4, 6, 8, 10, 12, \dots\}$ , we have that  $\lambda_R(\bar{R}/R) = |[0, c(o(R)) - 1] \cap (\mathbb{N} \setminus o(R))| = 7$ ,  $\lambda_R(R/(R : \bar{R})) = |[0, c(o(R)) - 1] \cap o(R)| = 5$ ,  $t(R) \leq t(o(R)) = 3$ , where  $t(R)$  is the type of the ring  $R$ .

**Example 2.14.** Let  $R = \mathbb{K}[[x^4, x^6 + x^7, x^{13} + a_{14}x^{14} + a_{15}x^{15} + \dots]]$  with  $\mathbb{K}$  a field. Using the same argument as in the Example 2.12, we find that if  $\text{char } \mathbb{K} \neq 2$ , we have that if  $a_{15} - a_{14} + 1/2 = 0$ , then  $\{x^4, x^6 + x^7, x^{13}\}$  is the reduced basis of  $R$ . Furthermore, since  $\langle 4, 6, 13 \rangle$  is a symmetric numerical semigroup (the number of nonnegative integers not in the semigroup equals the conductor divided by two), then, by [10],  $R$  is Gorenstein. Finally  $\lambda_R(\bar{R}/R) = 8$ . Otherwise if  $a_{15} - a_{14} + 1/2 \neq 0$ , then  $\{x^4, x^6 + x^7, x^{13}, x^{15}\}$  is the reduced basis of  $R$  with  $R$  a non Gorenstein ring. Furthermore  $\lambda_R(\bar{R}/R) = 7$ .

Otherwise, if  $\text{char } \mathbb{K} = 2$ , then the reduced basis of  $R$  is  $\{x^4, x^6 + x^7, x^{13}, x^{15}\}$  and  $R$  is not a Gorenstein ring. Here,  $\lambda_R(\bar{R}/R) = 7$ .

**Example 2.15.** Let  $R = \mathbb{K}[[x^8, x^{12} + x^{14} + x^{15}]]$ , with  $\mathbb{K}$  a field of characteristic zero. Using the same argument as in the Example 2.12, we have that

$$\left\{ x^8, x^{12} + x^{14} + x^{15}, x^{26} + x^{27} + x^{29} - \frac{1}{2}x^{31}, \right. \\ \left. x^{53} + \frac{1}{2}x^{55} - \frac{1}{2}x^{57} - \frac{1}{8}x^{63} + \frac{25}{8}x^{67} - \frac{95}{32}x^{71} - \frac{15}{16}x^{75} - \frac{135}{32}x^{83} \right\}$$

is the reduced basis of the Gorenstein ring  $R$ . Furthermore, we have  $\lambda_R(\bar{R}/R) = 42$ .

```
gap> l:=[x^8,x^12+x^14+x^15];;
gap> SemigroupOfValuesOfCurve_Local(l,"basis");
[ x^8, x^15+x^14+x^12, -1/2*x^31+x^29+x^27+x^26,
-135/32*x^83-15/16*x^75-95/32*x^71+25/8*x^67-1/8*x^63-1/2*x^57+1/2*x^55+x^53 ]
```

**Example 2.16.** The following battery of examples was provided by Lance Bryant as a test for our algorithm.

```

gap> l:= [ [ x^6,x^8+x^9,x^19 ], [x^7,x^9+x^10,x^19,x^31 ], [x^7,x^21+x^28+x^33 ],
[x^4,x^6+x^7,x^13 ], [x^6,x^8+x^11,x^10+2*x^13,x^21 ], [x^5,-x^18-x^21,-x^23,-x^26 ],
[x^5,-x^18-x^21,-x^26 ], [x^5,-x^18-x^21,x^23-x^26 ], [x^6,x^9+x^10,x^19 ],
[x^7,x^9+x^10,x^19 ], [x^8,x^9+x^10,x^19 ], [x^7,x^9+x^10,x^17,x^19 ] ];
gap> List(1, i->MinimalGeneratingSystem(SemigroupOfValuesOfCurve_Local(i)));
[ [ 6, 8, 19, 29 ], [ 7, 9, 19, 29, 31 ], [ 7, 33 ], [ 4, 6, 13, 15 ],
[ 6, 8, 10, 21, 23, 25 ], [ 5, 18, 26, 39, 47 ], [ 5, 18, 26, 39, 47 ],
[ 5, 18, 26, 39, 47 ], [ 6, 9, 19, 20 ], [ 7, 9, 19, 29 ], [ 8, 9, 19, 30 ],
[ 7, 9, 17, 19, 29 ] ]

```

### 3. DEFORMATION TO A TORIC IDEAL

Let the notations be as in Section 2. Given  $f(x) = \sum_{i \geq p} c_i x^i \in \mathbb{K}[[x]]$ , we set  $H_f(u, x) = \sum_{i \geq p} c_i u^{i-p} x^i$ . In particular, if we consider the linear form  $L : \mathbb{N}^2 \rightarrow \mathbb{N}, L(a, b) = b - a$ , then  $H_f$  is  $L$ -homogeneous of degree  $p$ , that is,  $L(i - p, i) = p$  for all  $i \in \text{supp}(f)$ . We set  $H_R = \mathbb{K}[[H_f, f \in R]]$ . With these notations we have the following.

**Proposition 3.1.** *The set  $\{f_1, \dots, f_s\}$  is a basis of  $R$  if and only if  $H_R = \mathbb{K}[[H_{f_1}, \dots, H_{f_s}]]$ .*

*Proof.* Suppose that  $\{f_1, \dots, f_s\}$  is a basis of  $R$  and let  $f(x) \in R$ . Write  $H_f(u, x) = \sum_{i \geq p} c_i u^{i-p} x^i$ . We have  $M_o(f) = c_p x^p = c_p \prod_{i=1}^s M_o(f_i)^{p_i^k}$ , hence

$$H_f - c_p \prod_{i=1}^s H_{f_i}^{p_i^k} = u^q H_{f^1}$$

with  $f^1 \in R$  and either  $f^1 = 0$ , or  $\text{o}(f^1) > p$ . In the second case we restart with  $f^1$ . A similar argument as in Proposition 2.6. proves our assertion.

Conversely, suppose that  $H_R = \mathbb{K}[[H_{f_1}, \dots, H_{f_s}]]$  and let  $f \in R$ . Let  $P(X_1, \dots, X_s) \in \mathbb{K}[[X_1, \dots, X_s]]$  such that  $H_f = P(H_{f_1}, \dots, H_{f_s})$ . If  $u = 0$ , then  $M_o(f) = P(M_o(f_1), \dots, M_o(f_s)) \in \mathbb{K}[[M_o(f_1), \dots, M_o(f_s)]]$ , hence  $M_o(f) \in \mathbb{K}[[M_o(f_1), \dots, M_o(f_s)]]$ .  $\square$

**Remark 3.2.** Suppose that  $\{f_1, \dots, f_r\}$  is a basis of  $R$ . Then  $T = \mathbb{K}[[u]][[H_{f_1}, \dots, H_{f_s}]]$  is a  $\mathbb{K}[[u]]$ -module. When  $u = 1$  (respectively  $u = 0$ ), we get  $T|_{u=1} = R$  (respectively  $T|_{u=0} = \mathbb{K}[[M(f_1), \dots, M(f_s)]]$ ). Hence we get a deformation from  $R$  to  $\mathbb{K}[[M_o(f_1), \dots, M_o(f_s)]]$ . More precisely let

$$\psi : \mathbb{K}[[X_1, \dots, X_s]] \rightarrow R = \mathbb{K}[[f_1, \dots, f_s]]$$

and

$$H_\psi : \mathbb{K}[[u]][[X_1, \dots, X_s]] \rightarrow T = \mathbb{K}[[u]][[H_{f_1}, \dots, H_{f_s}]]$$

be the morphisms of rings such that  $H_\psi(u) = u$ ,  $\psi(X_i) = f_i$  and  $H_\psi(X_i) = H_{f_i}$  for all  $i \in \{1, \dots, s\}$ . For all  $i \in \{1, \dots, r\}$ , let

$$S_i = f_1^{\alpha_1^i} \dots f_s^{\alpha_s^i} - f_1^{\beta_1^i} \dots f_s^{\beta_s^i} = \sum_{\underline{\theta}^i} c_{\underline{\theta}^i}^i f_1^{\theta_1^i} \dots f_s^{\theta_s^i}$$

with  $\text{o}(f_1^{\alpha_1^i} \dots f_s^{\alpha_s^i}) = D_{\underline{\theta}^i}^i > \sum_{k=1}^s \alpha_k^i \text{o}(f_k) = \sum_{k=1}^s \beta_k^i \text{o}(f_k) = p_i$ . Let  $I$  (respectively  $J$ ) be the ideal generated by  $(X_1^{\alpha_1^i} \dots X_s^{\alpha_s^i} - X_1^{\beta_1^i} \dots X_s^{\beta_s^i} - \sum_{\underline{\theta}^i} c_{\underline{\theta}^i}^i X_1^{\theta_1^i} \dots X_s^{\theta_s^i})_{1 \leq i \leq r}$  (respectively  $(X_1^{\alpha_1^i} \dots X_s^{\alpha_s^i} - X_1^{\beta_1^i} \dots X_s^{\beta_s^i} - \sum_{\underline{\theta}^i} u^{\underline{D}_{\underline{\theta}^i}^i - p_i} c_{\underline{\theta}^i}^i X_1^{\theta_1^i} \dots X_s^{\theta_s^i})_{1 \leq i \leq r}$ ) in  $\mathbb{K}[[X_1, \dots, X_s]]$  (respectively  $\mathbb{K}[[u]][[X_1, \dots, X_s]]$ ).

Since  $\{f_1, \dots, f_s\}$  is a basis of  $R$ , the kernel of  $\psi$  (respectively  $H(\psi)$ ) is generated by  $I$  (respectively  $J$ ). Now the morphism

$$\mathbb{K}[[u]] \rightarrow \mathbb{K}[[u]][[X_1, \dots, X_s]]/J$$

is flat because  $u$  is not a zero divisor. Hence we get a family of formal space curves parametrized by  $u$  which gives us a deformation from  $\mathbb{K}[[X_1, \dots, X_r]]/I$  to  $\mathbb{K}[[X_1, \dots, X_r]]/(F_1, \dots, F_r)$ .

In particular we get the following.

**Theorem 3.3.** *Every formal space curve of  $\mathbb{K}^l$ , parametrized by  $Y_1 = g_1(x), \dots, Y_l = g_l(x)$  has a deformation into a formal monomial curve of  $\mathbb{K}^r$  for some positive integer  $r$ .*

#### 4. BASIS OF $\mathbb{K}[[f(x), g(x)]]$

In [18] the case of finding an algebra basis for  $\mathbb{K}[f(x), g(x)]$  with  $f$  and  $g$  polynomials is treated. Here we will do the analogue for  $\mathbb{K}[[f(x), g(x)]]$ .

Let  $f(x) = \sum_{i \geq n} a_i x^i$  and  $g(x) = \sum_{j \geq m} b_j x^j$  be two elements of  $\mathbb{K}[[x]]$  and suppose, without loss of generality, that the following conditions hold:

- (1)  $a_n = b_m = 1$ .
- (2)  $n \leq m$ .
- (3) For all  $d > 1$ ,  $f(x), g(x) \notin \mathbb{K}[[x^d]]$  (i.e. the gcd of  $\text{supp}(f(x)) \cup \text{supp}(g(x)) = 1$ ).

Let the notations be as in Section 2, in particular  $R = \mathbb{K}[[f, g]]$ . By the analytic change of variables  $f(x) = \tilde{x}^n$ , we may assume that  $R = \mathbb{K}[[x^n, g(x)]]$ . Let  $F(X, Y)$  be the  $x$ -resultant of  $X - x^n, Y - g(x)$ , that is,  $F(X, Y)$  is the generator of the kernel of the map  $\rho : \mathbb{K}[[X, Y]] \rightarrow \mathbb{K}[[x]]$ ,  $\rho(X) = x^n$  and  $\rho(Y) = g(x)$ . Since  $\mathbb{K}[[f, g]] = \mathbb{K}[[f, g - f^k]]$  for all  $k \geq 1$ , then we shall assume that  $n < m$  and also that  $n$  does not divide  $m$ . Given a nonzero element  $G(X, Y) \notin (F(X, Y))\mathbb{K}[[X, Y]]$ , we set  $\text{int}(F, G) = \text{o}(G(f(x), g(x)))$ . The set of  $\text{int}(F, G)$ ,  $G(X, Y) \notin (F(X, Y))\mathbb{K}[[X, Y]]$ , is a numerical semigroup. We denote it by  $\Gamma(F)$ . We have the following.

**Proposition 4.1.**  $\text{o}(R) = \Gamma(F)$ .

*Proof.* We have  $a \in \Gamma(F)$  if and only if  $a = \text{o}(G(f(x), g(x)))$  for some  $G(X, Y) \in \mathbb{K}[[X, Y]]$  if and only if  $a \in \text{o}(R)$ .  $\square$

Suppose that  $\mathbb{K}$  is algebraically closed with characteristic zero, and let  $d_1 = n$ ,  $m_1 = \inf\{i \in \text{supp}(g) \mid d_1 \nmid i\}$ , that is,  $m_1 = m$ , and  $d_2 = \text{gcd}(n, m_1)$ . For all  $k \geq 2$  we set  $m_k = \inf\{i \in \text{supp}(g) \mid d_k \nmid i\}$  and  $d_{k+1} = \text{gcd}(d_k, m_k)$ . It follows that there exists  $h \geq 1$  such that  $d_{h+1} = 1$ . The set  $\{m_1, \dots, m_h\}$  is called the *set of Newton-Puiseux exponents* of  $F(X, Y)$ . Let  $e_k = \frac{d_k}{d_{k+1}}$  for all  $1 \leq k \leq h$  and define the sequence  $(r_k)_{0 \leq k \leq h}$  as follows:  $r_0 = n, r_1 = m$ , and for all  $2 \leq k \leq h, r_k = r_{k-1}e_{k-1} + m_k - m_{k-1}$ . With these notations we have the following:

- (1)  $\Gamma(F) = \text{o}(R)$  is generated by  $\{r_0, r_1, \dots, r_h\}$ .
- (2)  $r_k d_k < r_{k+1} d_{k+1}$  for all  $k \in \{1, \dots, h-1\}$ .
- (3)  $\Gamma(F) = \text{o}(R)$  is free with respect to the arrangement  $(r_0, \dots, r_h)$ . More precisely, let  $e_k = \frac{d_k}{d_{k+1}}$  for all  $k \in \{1, \dots, h\}$ . Then  $e_k r_k \in \langle r_0, \dots, r_{k-1} \rangle$ .
- (4)  $C = \sum_{k=1}^h (e_k - 1)r_k - n + 1$  is the conductor of  $\Gamma(F) = \text{o}(R)$ .

**Example 4.2.** Let  $f = x^7$  and  $g = x^4 + x^2$ . The above resultant is then  $F = y^7 - 7x^2y^3 - x^4 - 14x^2y^2 - 7x^2y - x^2$ . Then  $\Gamma(F) = \text{o}(R) = \langle 2, 7 \rangle$ .

```
gap> Resultant(x-t^7, y-t^4-t^2,t);
y^7-7*x^2*y^3-x^4-14*x^2*y^2-7*x^2*y-x^2
gap> s:=SemigroupOfValuesOfCurve_Local([t^7,t^4+t^2]);
<Modular numerical semigroup satisfying 7x mod 14 <= x >
gap> MinimalGeneratingSystem(last);
[ 2, 7 ]
gap> IsFreeNumericalSemigroup(s);
true
```

Let the notations be as above. For all  $k \geq 2$ , let  $G_k(X, Y) \in \mathbb{K}[[X, Y]]$  such that  $\text{o}(G_k(x^n, g(x))) = r_k$ . It follows from [1] that  $\deg_Y G_k = \frac{n}{d_k}$ . If  $g_k(x) = G_k(x^n, g(x))$ , then we have the following.

**Proposition 4.3.** *The set  $\{x^n, g, g_2, \dots, g_h\}$  is a basis of  $R$ , that is,  $R = \mathbb{K}[[x^n, g, g_2, \dots, g_h]]$  and  $M_o(R) = \mathbb{K}[[x^n, x^m, x^{r_2}, \dots, x^{r_h}]]$ .*



Note that, by a similar argument as in Section 2, we may assume that  $f = x^n, g = x^m + \sum_{i \in G(\Gamma(F))} c_i^1 x^i$ , and for all  $k \geq 2, g_k = x^{r_k} + \sum_{i \in G(\Gamma(F))} c_i^k x^i$ , where  $G(\Gamma(F)) = \{j \in \mathbb{N} \mid j \notin \Gamma(F)\}$  is the set of gaps of  $\Gamma(F)$ .

Let the notations be as in Section 3. The morphism

$$D : \mathbb{K}[[u]] \longrightarrow T = \mathbb{K}[[u]][[H_f, H_g, H_{g_2}, \dots, H_{g_h}]]$$

gives us a deformation of  $T|_{u=1} = R = \mathbb{K}[[f(x), g(x), g_2(x), \dots, g_h(x)]]$  to  $T|_{u=0} = \mathbb{K}[[x^n, x^m, x^{r_2}, \dots, x^{r_h}]]$ . Note that, since  $\langle n, m, r_2, \dots, r_h \rangle$  is free with respect to the given arrangement, then it is a complete intersection (see for instance [14]). For all  $k \in \{1, \dots, h\}$ , write  $e_k r_k = \sum_{i=0}^{k-1} \theta_i^k r_i$  with  $0 \leq \theta_i^k < e_i$  for all  $i \in \{1, \dots, k-1\}$ . If  $B$  is the ideal of  $\mathbb{K}[X_0, X_1, \dots, X_h]$  generated by

$$\{X_1^{e_2} - X_0^{\frac{m}{d_2}}, X_2^{e_2} - X_0^{\theta_0^2} X_1^{\theta_1^2}, \dots, X_h^{e_h} - X_0^{\theta_0^h} X_1^{\theta_1^h} \dots X_{h-1}^{\theta_{h-1}^h}\}$$

then

$$\mathbb{K}[[x^n, x^m, x^{r_2}, \dots, x^{r_h}]] \simeq \mathbb{K}[[X_0, X_1, \dots, X_h]]/B.$$

Let  $\bar{F}(X, Y)$  be the  $x$ -resultant of  $X - x^n, y - g(x)$ . By hypothesis,  $\bar{F}(X, Y)$  is a polynomial. Furthermore,  $\bar{F}(X, Y) = Y^n + a_1(X)Y^{n-1} + \dots + a_n(X)$  with  $\text{o}(a_i(X)) > i$  for all  $2 \leq i \leq n$ . Set  $G_{h+1} = \bar{F}$  and for all  $k \geq 1$ , let

$$G_{k+1} = G_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} G_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k X_0^{\alpha_0^k} G_1^{\alpha_1^k} \dots G_k^{\alpha_k^k},$$

where the following conditions hold:

- (1) for all  $i \in \{0, \dots, k-1\}$ ,  $0 \leq \theta_i^k < e_i$ ;
- (2) for all  $\underline{\alpha}^k$ , if  $c_{\underline{\alpha}^k}^k \neq 0$ , then for all  $i \in \{1, \dots, k\}$ ,  $0 \leq \alpha_i^k < e_i$ ;
- (3) for all  $\underline{\alpha}^k$ , if  $c_{\underline{\alpha}^k}^k \neq 0$ , then  $\alpha_0^k n + \sum_{i=1}^k \alpha_i^k r_i = D_i^k > e_k r_k = \theta_0^k r_0 + \sum_{i=1}^{k-1} \theta_i^k r_i$ .

It follows from Section 3. that if  $I$  (respectively  $J$ ) is the ideal generated by

$$(X_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} X_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k X_0^{\alpha_0^k} X_1^{\alpha_1^k} \dots X_k^{\alpha_k^k})_{1 \leq k \leq h}$$

(respectively  $(X_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} X_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k u^{D_i^k - e_k r_k} X_0^{\alpha_0^k} X_1^{\alpha_1^k} \dots X_k^{\alpha_k^k})_{1 \leq k \leq h}$ ) in  $\mathbb{K}[[X_0, \dots, X_h]]$  (respectively  $\mathbb{K}[[u]][[X_0, \dots, X_h]]$ ), then

$$R = \mathbb{K}[[x^n, g(x), g_2(x), \dots, g_h(x)]] \simeq \mathbb{K}[[X_0, X_1, \dots, X_h]]/I$$

and

$$\mathbb{K}[[u]][[x^n, H_g, H_{g_2}, \dots, H_{g_h}]] \simeq \mathbb{K}[[u]][[X_0, X_1, \dots, X_h]]/J.$$

Furthermore,  $\mathbb{K}[[u]][[X_0, X_1, \dots, X_h]]/J$  is a flat  $\mathbb{K}[[u]]$ -module. This gives us a family of formal space curves parametrized by  $u$  which is a deformation from  $\mathbb{K}[[X_0, X_1, \dots, X_h]]/I$  to the formal toric variety  $\mathbb{K}[[X_0, X_1, \dots, X_h]]/B$ . The later being a complete intersection, we get the following.

**Theorem 4.4.** *Every irreducible singularity of a plane curve  $X = f(x), Y = g(x)$  of  $\mathbb{K}^2$  has a deformation into a formal monomial complete intersection curve of  $\mathbb{K}^{h+1}$  for some  $h \geq 1$ .*

**Example 4.5.** Let  $f(x) = x^4, g(x) = x^6 + x^7$ . The minimal polynomial of  $(f(x), g(x))$  is given by:

$$F(X, Y) = Y^4 - 2X^3Y^2 + X^6 - 4X^5Y - X^7 = (Y^2 - X^3)^2 - 4X^5Y - X^7$$

Let  $r_0 = 4 = d_1, r_1 = 6 = m_1$  and  $G_1 = Y$ . We have  $d_2 = \text{gcd}(6, 4) = 2$ , hence  $m_2 = 7$ . It follows that  $r_2 = 13$ . Note that if  $G_2 = Y^2 - X^3$ , then  $g_2(x) = G_2(f(x), g(x)) = 2x^{13} + x^{14}$ . Hence  $\Gamma(F) = \text{o}(R) = \langle 4, 6, 13 \rangle$  and  $\{f(x), g(x), g_2(x)\}$  is a basis of  $R$ . Let us double check it.

```
gap> SemigroupOfValuesOfCurve_Local([x^4, x^6+x^7], "basis");
[ x^4, x^7+x^6, -1/2*x^15+x^13 ]
```

(Observe that the output is different, since this is a reduced basis: we change  $2x^{13} + x^4$  with  $x^{13} + \frac{1}{2}x^{14}$ , and then using that  $14 = 2 \times 4 + 6$ , we replace this last polynomial with  $x^{13} - \frac{1}{2}x^{15}$ .)

Consequently,  $H_R = \mathbb{K}[x^4, x^6 + ux^7, 2x^{13} + ux^{14}, u^4, u^6, u^{13}]$ . With the notations above,  $e_1 = 3, e_2 = 2$ , hence  $\mathbb{K}[x^4, x^6, x^{13}] \simeq T = \mathbb{K}[X_0, X_1, X_2]/(X_1^2 - X_0^3, X_2^2 - X_0^5 X_1)$ , and

$$\mathbb{K}[u] \longrightarrow \mathbb{K}[u][X_0, X_1, X_2]/(X_1^2 - X_0^3, X_2^2 - 4X_0^5 X_1 - u^2 X_0^7)$$

gives us a deformation from  $R$  to  $T$  (we can also change  $X_2$  with  $\frac{1}{2}X_2$ , and then  $B = (X_1^2 - X_0^3, X_2^2 - 4X_0^4 X_1)$ ).

## 5. SEMIGROUP OF A POLYNOMIAL CURVE

Let  $\mathbb{K}$  be a field and let  $f_1(x), \dots, f_s(x)$  be  $s$  polynomials of  $\mathbb{K}[x]$ . Let  $A = \mathbb{K}[f_1, \dots, f_s]$  be a subalgebra of  $\mathbb{K}[x]$ , and assume, without loss of generality, that  $f_i$  is monic for all  $i \in \{1, \dots, s\}$ . Given  $f(x) = \sum_{i=0}^p c_i x^i \in A$ , with  $c_p \neq 0$ , we set  $d(f) = p$  and  $M(f) = c_p x^p$ , the *degree* and *leading monomial*, respectively. We also define  $\text{supp}(f) = \{i \mid c_i \neq 0\}$ . The set  $d(A) = \{d(f) \mid f \in A\}$  is a submonoid of  $\mathbb{N}$ . We shall assume that  $\lambda_A(\mathbb{K}[x]/A) < \infty$ . In particular  $d(A)$  is a numerical semigroup. We say that  $\{f_1, \dots, f_s\}$  is a *basis* of  $A$  if  $\{d(f_1), \dots, d(f_s)\}$  generates  $d(A)$ . Clearly,  $\{f_1, \dots, f_s\}$  is a basis of  $A$  if and only if  $\mathbb{K}[M(f), f \in A] = \mathbb{K}[M(f_1), \dots, M(f_s)]$ . For several variables, these basis are known in the literature as SAGBI basis ([15, 4]).

The following result is the analogue to Proposition 2.6. Actually, in this section we follow a similar argument to the one used in Section 2, with the advantage that sequences of degrees decrease, and thus the finiteness conditions are easier to deduce.

**Proposition 5.1.** *Given  $f(x) \in \mathbb{K}[x]$ , there exist  $g(x) \in A$  and  $r(x) \in \mathbb{K}[x]$  such that the following conditions hold:*

- (1)  $f(x) = g(x) + r(x) = \sum_{\alpha} c_{\alpha} f_1^{\alpha_1} \cdots f_s^{\alpha_s} + r(x)$ ,
- (2) if  $g(x) \neq 0$  (respectively  $r(x) \neq 0$ ), then  $d(g) \leq d(f)$  (respectively  $d(r) \leq d(f)$ ),
- (3)  $\text{supp}(r(x)) \subseteq \mathbb{N} \setminus \langle d(f_1), \dots, d(f_s) \rangle$ .

*Proof.* The assertion is clear if  $f \in \mathbb{K}$ . Suppose that  $f \notin \mathbb{K}$  and let  $f(x) = \sum_{i=0}^p c_i x^i$  with  $p = d(f) > 0$ .

- (1) If  $p \notin \langle d(f_1), \dots, d(f_s) \rangle$ , then we set  $g^1 = 0, r^1 = c_p x^p$  and  $f^1 = f - c_p x^p$ .
- (2) If  $p \in \langle d(f_1), \dots, d(f_s) \rangle$ , then  $c_p x^p = c_{\underline{\theta}} M(f_1)^{\theta_1} \cdots M(f_s)^{\theta_s}$ . We set  $g^1 = c_{\underline{\theta}} f_1^{\theta_1} \cdots f_s^{\theta_s}, r^1 = 0$  and  $f^1 = f - g^1$ .

In this way,  $f = f^1 + g^1 + r^1$ ,  $g^1 \in \mathbf{A}$ ,  $\text{supp}(r^1) \subseteq \mathbb{N} \setminus \langle d(f_1), \dots, d(f_s) \rangle$ , and if  $f^1 \notin \mathbb{K}$ , then  $d(f^1) < d(f) = p$ . Then we restart with  $f^1$ . Clearly there is  $k \geq 1$  such that  $f^k \in \mathbb{K}$ . We set  $g = g^1 + \dots + g^k + f^k$  and  $r = r^1 + \dots + r^k$ .  $\square$

We call the polynomial  $r(x)$  of the theorem above the *remainder* of  $f$  with respect to  $\{f_1, \dots, f_s\}$  and we denote it by  $R(f, \{f_1, \dots, f_s\})$ . This is an adaptation of the procedure called subduction in [15], with the difference that we are adding  $f^k$  to  $g$ , and thus whenever the subduction process produces a constant, the remainder is zero.

The following characterization of basis is analogous to that given in Proposition 2.7 for formal space curves; we omit the proof since it is roughly the same.

**Proposition 5.2.** *The set  $\{f_1, \dots, f_s\}$  is a basis of  $A$  if and only if  $R(f, \{f_1, \dots, f_s\}) = 0$  for all  $f \in A$ .*

Let the notations be as above and let

$$\phi : \mathbb{K}[X_1, \dots, X_s] \longrightarrow \mathbb{K}[x], \quad \phi(X_i) = M(f_i), \quad \text{for all } i \in \{1, \dots, s\}.$$

Let  $\{F_1, \dots, F_r\}$  be a generating system of the kernel of  $\phi$ . Clearly for all  $i = 1, \dots, r$ ,  $F_i$  is a binomial. If  $F_i = X_1^{\alpha_i} \cdots X_s^{\alpha_i} - X_1^{\beta_i} \cdots X_s^{\beta_i}$ , we set  $S_i = f_1^{\alpha_i} \cdots f_s^{\alpha_i} - f_1^{\beta_i} \cdots f_s^{\beta_i}$ . Observe that if  $d = \sum_{k=1}^s \alpha_k^i d(f_k) = \sum_{k=1}^s \beta_k^i d(f_k)$ , then  $d(S_i) < d$ . With this we obtain the following particular case of [16, Corollary 11.5] (taking into account that the subduction process is not exactly our remainder).

**Theorem 5.3.** *The set  $\{f_1, \dots, f_s\}$  is a basis of  $A$  if and only if  $R(S_i, \{f_1, \dots, f_s\}) = 0$  for all  $i \in \{1, \dots, r\}$ .*

*Proof.* The proof goes as in Theorem 2.8, but now the inequalities are reversed and  $p$  is defined as a maximum instead of a minimum.  $\square$

**Algorithm 5.4.** Let the notations be as above.

- (1) If  $R(S_k(f_1, \dots, f_s), \{f_1, \dots, f_s\}) = 0$  for all  $k = 1, \dots, r$ , then  $\{f_1, \dots, f_s\}$  is a basis of  $A$ .
- (2) If  $r(x) = R(S_k(f_1, \dots, f_s), \{f_1, \dots, f_s\}) \neq 0$  for some  $1 \leq k \leq r$ , and if  $M(r) = ax^q$ , then we set  $f_{s+1} = \frac{1}{a}r(x)$ , and we restart with  $\{f_1, \dots, f_{s+1}\}$ . Note that in this case,  $\langle d(f_1), \dots, d(f_s) \rangle \subsetneq \langle d(f_1), \dots, d(f_s), d(f_{s+1}) \rangle$ .

This process will stop, giving a basis of  $A$ .

Suppose that  $\{f_1, \dots, f_s\}$  is a basis of  $A$ . Also suppose that  $f_i$  is monic for all  $i \in \{1, \dots, s\}$ . We say that  $\{f_1, \dots, f_r\}$  is a *minimal basis* of  $A$  if  $\{d(f_1), \dots, d(f_s)\}$  minimally generates the semigroup  $d(A)$ . We say that  $\{f_1, \dots, f_r\}$  is a *reduced basis* of  $A$  if  $\text{supp}(f_i(x) - M(f_i)) \in \mathbb{N} \setminus d(A)$ .

Let  $i \in \{1, \dots, s\}$ . If  $d(f_i) \in \langle d(f_1), \dots, d(f_{i-1}), d(f_{i+1}), \dots, d(f_s) \rangle$ , then  $\{f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_s\}$  is also a basis of  $A$ . Furthermore, by applying the division process of Proposition 5.1 to  $f_i - M(f_i)$ , we can always construct a reduced basis of  $A$ . Minimal reduced basis are not unique as in the formal space curve case, but “almost” unique.

**Corollary 5.5.** *Up to constants, the algebra  $A$  has a unique minimal reduced basis.*

*Proof.* Let  $\{f_1, \dots, f_s\}$  and  $\{g_1, \dots, g_{s'}\}$  be two minimal reduced bases of  $A$ . Clearly  $s = s'$ . Let  $i = 1$ . There exists  $j_1$  such that  $d(f_1) = d(g_{j_1})$ . If  $f_1 - g_{j_1} \notin \mathbb{K}$ , then  $d(f_1 - g_{j_1}) \notin \Gamma(\mathbf{A})$ , which is a contradiction because  $f_1 - g_{j_1} \in A$ . The same argument shows that for all  $i \geq 2$ , there exists  $j_i$  such that  $f_i - g_{j_i} \in \mathbb{K}$ .  $\square$

**Corollary 5.6.** *Let  $\{f_1, \dots, f_s\}$  be a reduced basis of  $A$ . For all  $i \in \{1, \dots, s\}$ ,  $\text{supp}(f_i - M(f_i)) \subseteq G(\Gamma A)$ .*

**Example 5.7.** We compute  $d(R)$  for  $R = \mathbb{K}[x^6 + x^3, x^4 + x^2]$ ;  $f_1 = x^6 + x^3$  and  $f_2 = x^4 + x^2$ . We start by computing the kernel of  $\phi : \mathbb{K}[X_1, X_2] \rightarrow \mathbb{K}[x]$ , with  $\phi(X_1) = x^6$  and  $\phi(X_2) = x^4$ . This kernel is generated by  $F_1 = X_1^2 - X_2^3$ . Hence  $S_1 = -3x^{10} + 2x^9 - 3x^8$ . Since  $10 \in \langle 4, 6 \rangle$ ,  $R(S_1, \{f_1, f_2\}) = \frac{-2}{3}x^9 - x^7 - x^5$ . We add  $f_3 = x^9 + \frac{3}{2}x^7 + \frac{3}{2}x^5$  to our generating set.

In the next step  $\phi : \mathbb{K}[X_1, X_2, X_3] \rightarrow \mathbb{K}[x]$ , with  $\phi(X_1) = x^6$ ,  $\phi(X_2) = x^4$  and  $\phi(X_3) = x^9$ ;  $\ker \phi = (X_1^2 - X_2^3, X_1^3 - X_3^2)$ , whence  $S_1 = x^{16} - x^{15} + \frac{7}{4}x^{14} + \frac{1}{2}x^{12} + \frac{3}{4}x^{10} - \frac{1}{3}x^9$  and  $S_2 = x^{10} - \frac{2}{3}x^9 + x^8$ . It turns out that  $R(S_1, \{f_1, f_2, f_3\}) = 0 = R(S_2, \{f_1, f_2, f_3\})$ , and consequently  $\{f_1, f_2, f_3\}$  is a (reduced minimal) basis of  $R$ . Also  $d(R) = \langle 4, 6, 9 \rangle$ .

**Example 5.8.** Let us compute  $K[x^7, x^9 + x^{10}, x^{17}, x^{19}]$  with our implementation.

```
gap> MinimalGeneratingSystem(SemigroupOfValuesOfCurve_Global(
[x^7, x^9+x^10, x^17, x^19]);
[ 7, 10, 16, 19, 25 ]
```

## 6. DEFORMATION TO A TORIC IDEAL

Let the notations be as in Section 5. Given  $f(x) = \sum_{i=0}^p c_i x^i \in \mathbb{K}[x]$ , we set  $h_f(u, x) = \sum_{i=0}^p c_i u^{p-i} x^i$ . We set  $h_A = \mathbb{K}[h_f \mid f \in A]$ . With these notations we have the following result, and its proof is similar to that of Proposition 3.1.

**Proposition 6.1.** *The set  $\{f_1, \dots, f_r\}$  is a basis of  $A$  if and only if  $h_A = \mathbb{K}[h_{f_1}, \dots, h_{f_s}, u^{d(f_1)}, \dots, x^{d(f_s)}]$ .*

**Remark 6.2.** Suppose that  $\{f_1, \dots, f_s\}$  is a basis of  $A$ . By the inclusion morphism of rings  $D : \mathbb{K}[u] \rightarrow B = \mathbb{K}[u, h_{f_1}, \dots, h_{f_s}]$ ,  $B$  is a  $\mathbb{K}[u]$ -module. When  $u = 1$  (respectively  $u = 0$ ), we get  $B|_{u=1} = A$  (respectively  $B|_{u=0} = \mathbb{K}[M(f_1), \dots, M(f_s)]$ ). Hence we get a deformation from  $A$  to  $\mathbb{K}[M(f_1), \dots, M(f_s)]$ . More precisely let

$$\psi : \mathbb{K}[X_1, \dots, X_s] \rightarrow \mathbb{K}[f_1, \dots, f_s]$$

and

$$h_\psi : \mathbb{K}[u][X_1, \dots, X_s] \rightarrow \mathbb{K}[u][h_{f_1}, \dots, h_{f_s}]$$

be the morphisms of rings such that  $h_\psi(u) = u$ ,  $\psi(X_i) = f_i$  and  $h_\psi(X_i) = h_{f_i}$  for all  $i = 1, \dots, s$ . For all  $i = 1, \dots, r$ , let

$$S_i = f_1^{\alpha_i} \cdots f_s^{\alpha_i} - f_1^{\beta_i} \cdots f_s^{\beta_i} = \sum_{\underline{\theta}^i} c_{\underline{\theta}^i}^i f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}$$

with  $d(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}) = D_{\underline{\theta}^i}^i < \sum_{k=1}^s \alpha_k^i d(f_k) = \sum_{k=1}^s \beta_k^i d(f_k) = p_i$ . Let  $I$  (respectively  $J$ ) be the ideal generated by  $(X_1^{\alpha_i} \cdots X_s^{\alpha_i} - X_1^{\beta_i} \cdots X_s^{\beta_i} - \sum_{\underline{\theta}^i} c_{\underline{\theta}^i}^i X_1^{\theta_1^i} \cdots X_s^{\theta_s^i})_{1 \leq i \leq r}$  (respectively  $(X_1^{\alpha_i} \cdots X_s^{\alpha_i} - X_1^{\beta_i} \cdots X_s^{\beta_i} - \sum_{\underline{\theta}^i} u^{p_i - D_{\underline{\theta}^i}^i} c_{\underline{\theta}^i}^i X_1^{\theta_1^i} \cdots X_s^{\theta_s^i})_{1 \leq i \leq r}$ ) in  $\mathbb{K}[X_1, \dots, X_s]$  (respectively  $\mathbb{K}[u][X_1, \dots, X_s]$ ).

Since  $\{f_1, \dots, f_s\}$  is a basis of  $A$ , then the kernel of  $\psi$  (respectively  $h_\psi$ ) is generated by  $I$  (respectively  $J$ ). Now the morphism

$$\mathbb{K}[u] \longrightarrow \mathbb{K}[u][X_1, \dots, X_r]/J$$

is flat (because  $p(u)$  is not a zero divisor for all  $p(u) \in \mathbb{K}[u]$ ). Hence we get a family of polynomial space curves parametrized by  $u$  which gives us a deformation from  $\mathbb{K}[X_1, \dots, X_r]/I$  to  $\mathbb{K}[X_1, \dots, X_r]/(F_1, \dots, F_r)$ .

In particular we get the following analogue to Theorem 3.3.

**Theorem 6.3.** *Every polynomial space curve of  $\mathbb{K}^l$ , parametrized by  $Y_1 = g_1(x), \dots, Y_l = g_l(x)$  has a deformation into a monomial curve of  $\mathbb{K}^r$  for some positive integer  $r$ .*

## 7. BASIS OF $K[f(x), g(x)]$

Let  $f(x) = \sum_{i=1}^n a_i x^i$  and  $g(x) = \sum_{j=1}^m b_j x^j$  be two polynomials of  $\mathbb{K}[x]$  and suppose, without loss of generality, that the following conditions hold:

- (1)  $a_n = b_m = 1$ .
- (2)  $n \geq m$ ,
- (3) for all  $d > 1$ ,  $f(x), g(x) \notin \mathbb{K}[x^d]$  (that is, the gcd of  $\text{supp}(f(x)) \cup \text{supp}(g(x)) = 1$ ).

Let the notations be as in Section 5, in particular  $A = \mathbb{K}[f, g]$ . Let also  $F(X, Y)$  be the  $x$ -resultant of  $X - f(x), Y - g(x)$ , that is,  $F(X, Y)$  is the generator of the kernel of the map  $\psi : \mathbb{K}[X, Y] \longrightarrow \mathbb{K}[x], \psi(X) = f(x)$  and  $\psi(Y) = g(x)$ . Since  $\mathbb{K}[f, g] = \mathbb{K}[f, g - f]$ , then we shall assume that  $n > m$ . Write  $F(X, Y) = Y^n + c_1(X)Y^{n-1} + \cdots + c_n(X)$ . Given a polynomial  $G(X, Y) \notin (F(X, Y))\mathbb{K}[X, Y]$ , we set  $\text{int}(F, G) = \deg_x G(f(x), g(x))$ . Assume that  $\mathbb{K}$  is algebraically closed with characteristic zero. Let  $d$  be a divisor of  $n$ , and let  $G$  be a monic polynomial in  $\mathbb{K}[X][Y]$  of degree  $\frac{n}{d}$  in  $Y$ . Write  $F = G^d + \alpha_1(X, Y)G^{d-1} + \cdots + \alpha_d(X, Y)$  where for all  $k \in \{1, \dots, d\}$ , if  $\alpha_k \neq 0$ , then  $\deg_Y \alpha_k < \frac{n}{d}$ . We say that  $G$  is a  $d$ th approximate root of  $F$  if  $\alpha_1 = 0$ . There is a unique  $d$ th approximate root of  $F$ . We denote it by  $\text{App}(F, d)$ . The following results can be found in [1].

**Theorem 7.1.** *Under the standing hypothesis.*

- (1)  $F(X, Y)$  has one place at infinity, that is, for all  $k \in \{1, \dots, n\}$ , if  $c_k(X) \neq 0$ , then  $\deg_X c_k(X) < k$ .
- (2)  $\{\text{int}(F, G) \mid G \in \mathbb{K}[X, Y] \setminus (F)\}$  is a numerical semigroup.
- (3) Let  $D(n)$  be the set of divisors of  $n$ . The set  $\{\text{int}(F, \text{App}(F, d)) \mid d \in D(n)\}$  generates  $\Gamma(F)$ .

We call  $\{\text{int}(F, G) \mid G \in \mathbb{K}[X, Y] \setminus (F)\}$  the semigroup of  $F$ , and we denote it by  $\Gamma(F)$ .

**Corollary 7.2.** *Let the notations be as above. We have  $d(A) = \Gamma(F)$ .*

*Proof.* In fact,  $h(x) \in A$  if and only if  $h(x) = P(f(x), g(x))$  for some  $P(X, Y) \in \mathbb{K}[X, Y]$ . Hence  $a \in d(A)$  if and only if  $a = \text{int}(F, P), P \in \mathbb{K}[X, Y]$  which means that  $a \in \Gamma(F)$ .  $\square$

Let  $F(X, Y) = Y^n + c_1(X)Y^{n-1} + \cdots + c_n(X)$  be as above, and assume, after a possible change of variables  $X' = X, Y' = Y + \frac{c_1}{n}$ , that  $c_1(X, Y) = 0$  (note that this does not change  $A$ ). In particular  $\text{App}(F, n) = Y$ . A system of generators of  $\Gamma(F)$  can be found algorithmically in the following way.

Let  $r_0 = d_1 = n = \text{int}(F, X), r_1 = \deg_X a_n(X) = \text{int}(F, \text{App}(F, n))$ , and  $d_2 = \text{gcd}(r_0, r_1)$ . We set  $G_2 = \text{App}(F, d_2), r_2 = \text{int}(F, G_2) = \deg_x G_2(f(x), g(x))$ , and  $d_3 = \text{gcd}(r_3, d_2)$ , and so on... With these notations we have the following:

- (1)  $d_1 > d_2 > \dots$  and there exists  $h \geq 1$  such that  $d_{h+1} = 1$ ;
- (2)  $\Gamma(F) = \mathfrak{d}(A)$  is generated by  $\{r_0, r_1, \dots, r_h\}$ ;
- (3)  $r_k d_k > r_{k-1} d_{k-1}$  for all  $k \in \{1, \dots, h\}$ ;
- (4)  $\Gamma(F) = \mathfrak{d}(A)$  is free with respect to the arrangement  $(r_0, \dots, r_h)$ . More precisely, let  $e_k = \frac{d_k}{d_{k+1}}$  for all  $k \in \{1, \dots, h\}$ . Then  $e_k r_k \in \langle r_0, \dots, r_{k-1} \rangle$ ;
- (5)  $C = \sum_{k=1}^h (e_k - 1)r_k - n + 1$  is the conductor of  $\Gamma(F) = \mathfrak{d}(A)$ .

**Lemma 7.3.** *If  $A = \mathbb{K}[x]$ , then  $r_k = d_{k+1}$  for all  $k = 1, \dots, h$ . In particular  $\deg_x G_h(f(x), g(x)) = 1$  and  $m$  divides  $n$ .*

*Proof.* If  $A = \mathbb{K}[x]$  then  $C = 0$ , hence  $\sum_{k=1}^h (e_k - 1)r_k = n - 1$ . Since  $r_k \geq d_{k+1}$ , then  $\sum_{k=1}^h (e_k - 1)r_k \geq n - 1$  with equality if and only if  $r_k = d_{k+1}$  for all  $k = 1, \dots, h$ . Since  $m = r_1 = d_2 = \gcd(n, m)$ , then  $m$  divides  $n$ .  $\square$

**Lemma 7.4.** [18, Theorem 2] *If  $\gcd(n, m) = 1$ , then  $\{f(x), g(x)\}$  is a basis of  $A$ .*

*Proof.* If  $\gcd(n, m) = 1$ , then  $\Gamma(F) = \mathfrak{d}(A) = \langle n, m \rangle$ . Hence  $\{f(x), g(x)\}$  is a basis of  $A$ .  $\square$

**Lemma 7.5.** *Suppose that  $\gcd(n, m) = p_1 \cdots p_l$  where  $p_i$  is a positive prime number for all  $i \in \{1, \dots, l\}$  (and the  $p_i$ 's are not necessarily distinct). The set  $\{f(x), g(x)\}$  is not a basis of  $A$ . Furthermore, if  $c$  is the cardinality of a basis of  $A$ , then  $2 \leq c \leq l + 2$ . In particular, if  $\gcd(n, m)$  is a prime number  $p > 1$ , then a basis of  $A$  has either two or three elements.*

*Proof.* Since  $\gcd(n, m) > 1$ , then the first assertion is clear. On the other hand, since  $d_2 = \gcd(n, m) = p_1 \cdots p_r$ , we have  $A \neq \mathbb{K}[x]$ , and  $h \leq l + 1$ . Hence  $\Gamma(F) = \mathfrak{d}(A)$  has at most  $l + 2$  generators. The result now follows.  $\square$

**Remark 7.6.** Let  $\underline{r} = (r_0 = n, r_1 = m, r_2, \dots, r_h)$  be a sequence of integers and for all  $k \geq 1$ , let  $d_k = \gcd(r_0, \dots, r_{k-1})$  and  $e_k = \frac{d_k}{d_{k+1}}$ . Assume that the following conditions hold:

- (1)  $d_1 > d_2 > \dots > d_{h+1} = 1$ ;
- (2)  $r_k d_k > r_{k-1} d_{k-1}$  for all  $k \in \{1, \dots, h\}$ ;
- (3)  $e_k r_k \in \langle r_0, \dots, r_{k-1} \rangle$  for all  $k = 1, \dots, h$ .

Such a sequence is called a  $\delta$ -sequence and it is well known (see [1]) that there exists a polynomial  $\tilde{F}(X, Y)$  with one place at infinity such that the semigroup  $\{\text{rank}_{\mathbb{K}} \mathbb{K}[X, Y]/(\tilde{F}, G), G \notin (F)\}$  is generated by  $\underline{r}$ .

Abhyankar asked whether every semigroup generated by a  $\delta$ -sequence is the semigroup of a polynomial curve (for example, the  $\delta$ -sequence  $(10, 4, 5)$  generates the semigroup  $\langle 4, 5 \rangle$  which is the semigroup of the polynomial curve  $A = \mathbb{K}[x^4, x^5]$ ). It has been proved recently that the answer is no ([8]). It would be nice to see which supplementary conditions a  $\delta$ -sequence should satisfy in order to generate the semigroup of a polynomial curve.

**Remark 7.7.** Let  $f(x)$  and  $g(x)$  be as above, and let  $A = \mathbb{K}[f(x), g(x)]$ . Let also  $F(X, Y)$  be the  $x$ -resultant of  $X - f(x)$  and  $Y - g(x)$ . Let  $r_0 = n, r_1 = m, r_2, \dots, r_h$  be the generators of  $\Gamma(F)$  calculated as above. Let  $1 \leq k \leq h$  and let  $G_k(X, Y) = \text{App}(F, d_k)$ . We have  $\mathfrak{d}(G_k(f(x), g(x))) = r_k$ , but  $G_k$  is not the unique polynomial with this condition (for example,  $\mathfrak{d}((G_k + \lambda)(f(x), g(x))) = r_k$  for all  $\lambda \neq 0$ ). Hence it is natural to ask the following: is there a polynomial  $G(X, Y)$  (of degree  $< n$  in  $Y$ ) such that  $G$  is parametrized by polynomials in  $x$ ? Such a polynomial, if it exists, should be of degree  $\frac{n}{d_k}$  and should have the contact with  $F$  at a characteristic exponent of  $F$  (see [1] for the definition of the characteristic exponents of a curve with one place at infinity). Hence the existence of such a polynomial implies that a polynomial curve can be approximated by polynomial curves.

Let the notations be as above, in particular  $F(X, Y) = Y^n + c_1(X)Y^{n-1} + \dots + c_n(X)$  is the  $x$ -resultant of  $(X - f(x), Y - g(x))$ . Let  $G_1 = Y, G_2, \dots, G_h$  be the set of approximate roots of  $F(X, Y)$  constructed algorithmically as above. In particular  $r_0 = n, r_1 = m, r_2 = \text{int}(F, G_2), \dots, r_h = \text{int}(F, G_h)$  generate  $\mathfrak{d}(A)$ . For all  $k = 2, \dots, h$ , let  $g_k(x) = G_k(f(x), g(x))$  and let  $M(g_k) = b_{r_k} x^{r_k}$ . We have  $A = \mathbb{K}[f(x), g(x), g_2(x), \dots, g_h(x)]$ . Furthermore, the map

$$D : \mathbb{K}[u] \longrightarrow B = \mathbb{K}[u][h_f, h_g, h_{g_2}, \dots, h_{g_h}]$$

introduced in Remark 6.2 gives us a deformation of the polynomial curve  $B|_{u=1} = A$  into  $B|_{u=0} = \mathbb{K}[t^n, t^m, t^{r_2}, \dots, t^{r_h}]$ . Note that, since  $\langle n, m, r_2, \dots, r_h \rangle$  is free with respect to the given arrangement, then it is a complete intersection. For all  $k \in \{1, \dots, h\}$ , write  $e_k r_k = \sum_{i=0}^{k-1} \theta_i^k r_i$  with  $0 \leq \theta_i^k < e_i$  for every  $i \in \{1, \dots, k-1\}$ . With the notations above, if  $T$  is the ideal of  $\mathbb{K}[X_0, X_1, \dots, X_h]$  generated by

$$\{X_1^{e_2} - X_0^{\frac{m}{d_2}}, X_2^{e_2} - X_0^{\theta_0^2} X_1^{\theta_1^2}, \dots, X_h^{e_h} - X_0^{\theta_0^h} X_1^{\theta_1^h} \dots X_{h-1}^{\theta_{h-1}^h}\},$$

then

$$\mathbb{K}[x^n, x^m, x^{r_2}, \dots, x^{r_h}] \simeq \mathbb{K}[X_0, X_1, \dots, X_h]/T.$$

Set  $G_{h+1} = F$  and for all  $k \geq 1$ , let

$$G_{k+1} = G_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} G_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k X_0^{\alpha_0^k} \cdot G_1^{\alpha_1^k} \dots G_k^{\alpha_k^k},$$

where the following conditions hold:

- (1) for all  $i \in \{0, \dots, k-1\}$ ,  $0 \leq \theta_i^k < e_i$ ;
- (2) for all  $\underline{\alpha}^k$ , if  $c_{\underline{\alpha}^k}^k \neq 0$ , then for all  $i \in \{1, \dots, k\}$ ,  $0 \leq \alpha_i^k < e_i$ ,
- (3) for all  $\underline{\alpha}^k$ , if  $c_{\underline{\alpha}^k}^k \neq 0$ , then  $\alpha_0^k n + \sum_{i=1}^k \alpha_i^k r_i = D_i^k < e_k r_k = \theta_0^k r_0 + \sum_{i=1}^k \theta_i^k r_i$ .

It follows from Section 6 that if  $I$  (respectively  $J$ ) is the ideal generated by  $(X_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} X_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k X_0^{\alpha_0^k} \cdot X_1^{\alpha_1^k} \dots X_k^{\alpha_k^k})_{1 \leq k \leq h}$  (respectively  $(X_k^{e_k} - X_0^{\theta_0^k} \prod_{i=1}^{k-1} X_i^{\theta_i^k} + \sum_{\underline{\alpha}^k} c_{\underline{\alpha}^k}^k u^{e_k r_k - D_i^k} X_0^{\alpha_0^k} \cdot X_1^{\alpha_1^k} \dots X_k^{\alpha_k^k})_{1 \leq k \leq h}$ ) in  $\mathbb{K}[X_0, \dots, X_h]$  (respectively  $\mathbb{K}[u][X_0, \dots, X_h]$ ), then

$$A = \mathbb{K}[x^n, g(x), g_2(x), \dots, g_h(x)] \simeq \mathbb{K}[X_0, X_1, \dots, X_h]/I$$

and

$$\mathbb{K}[u][x^n, h_{g(x)}, h_{g_2(x)}, \dots, h_{g_h(x)}] \simeq \mathbb{K}[u][X_0, X_1, \dots, X_h]/J.$$

Furthermore,  $\mathbb{K}[u][X_0, X_1, \dots, X_h]/J$  is a flat  $\mathbb{K}[u]$ -module. This gives us a family of space curve parametrized by  $u$  which is a deformation from  $\mathbb{K}[X_0, X_1, \dots, X_h]/I$  to the toric variety  $\mathbb{K}[X_0, X_1, \dots, X_h]/T$ . The later being a complete intersection, we get the following result.

**Theorem 7.8.** *Every polynomial curve  $X = f(x), Y = g(x)$  of  $\mathbb{K}^2$  has a deformation into a monomial complete intersection curve of  $\mathbb{K}^{h+1}$  for some positive integer  $h$ .*

**Example 7.9.** Let  $f(x) = x^6 + x^3, g(x) = x^4$ . The minimal polynomial of  $(f(x), g(x))$  is given by:

$$F(X, Y) = Y^6 - 2X^2Y^3 - 4XY^3 - Y^3 + X^4.$$

Let  $r_0 = 6 = d_1, r_1 = 4$  and  $G_1 = Y$ . We have  $d_2 = \gcd(6, 4) = 2$ , and  $G_2 = \text{App}(F, 2) = Y^3 - X^2 - 2X - \frac{1}{2}$ . Since  $g_2(x) = G_2(f(x), g(x)) = -2x^9 - 3x^6 - 2x^3 - \frac{1}{2}$ , then  $r_2 = 9$  and  $d_3 = 1$ , hence  $\Gamma(F) = d(A) = \langle 6, 4, 9 \rangle$  and  $\{f(x), g(x), g_2(x)\}$  is a basis of  $A$ . Consequently,  $h_A = \mathbb{K}[x^6 + u^3x^3, x^4, 2x^9 + 3u^3x^6 + 2u^6x^3 + \frac{1}{2}u^9, u^6, u^4, u^9]$ . Note that, with the notations above,  $e_1 = 3, e_2 = 2$ , hence  $\mathbb{K}[x^6, x^4, 2x^9] \simeq \mathbb{K}[X_0, X_1, X_2]/(X_1^3 - X_0^2, X_2^2 - 4X_0^3) = \mathbb{K}[X_0, X_1, X_2]/T$ ,  $\mathbb{K}[x^6 + x^3, x^4, 2x^9 - 3x^6 - 2x^3 - \frac{1}{2}] \simeq \mathbb{K}[X_0, X_1, X_2]/(X_1^3 - X_0^2 - 2X_0 - \frac{1}{2}, X_2^2 - 4X_0^3 - 5X_0^2 - 2X_0 - \frac{1}{4})$ , and

$$\mathbb{K}[u] \longrightarrow \mathbb{K}[u][X_0, X_1, X_2]/(X_1^3 - X_0^2 - 2u^6X_0 - \frac{1}{2}u^9, X_2^2 - 4X_0^3 - 5u^6X_0^2 - 2u^{12}X_0 - \frac{1}{4}u^{18})$$

gives us a deformation from  $A$  to  $\mathbb{K}[X_0, X_1, X_2]/T$ .

The computation of the approximate roots and of  $\Gamma(F)$  can be performed with the algorithm presented in [2].

```
gap> f:=y^6-2*x^2*y^3-4*x*y^3-y^3+x^4;;
gap> SemigroupOfValuesOfPlaneCurveWithSinglePlaceAtInfinity(f);
<Numerical semigroup with 3 generators>
gap> MinimalGeneratingSystem(last);
[ 4, 6, 9 ]
gap> SemigroupOfValuesOfPlaneCurveWithSinglePlaceAtInfinity(f);
```

[ [ 6, 4, 9 ], [ y, y^3-x^2-2\*x-1/2 ] ]

**Example 7.10.** Let  $f(x) = x^6 + x, g(x) = x^4$ . The minimal polynomial of  $(f(x), g(x))$  is given by:

$$F(X, Y) = Y^6 - 2X^2Y^3 - 4XY^2 - Y + X^4.$$

Let  $r_0 = 6 = d_1, r_1 = 4$  and  $G_1 = Y$ . We have  $d_2 = \gcd(6, 4) = 2$ , and  $G_2 = \text{App}(F, 2) = Y^3 - X^2$ . Since  $g_2(x) = G_2(f(x), g(x)) = -2x^7 - x^2$ , then  $r_2 = 7$  and  $d_3 = 1$ , hence  $\Gamma(F) = \text{d}(A) = \langle 6, 4, 7 \rangle$  and  $\{f(x), g(x), g_2(x)\}$  is a basis of  $A$ . Consequently,  $h_A = \mathbb{K}[x^6 + u^5x, x^4, 2x^7 + u^5x^2, u^6, u^4, u^7]$ . Note that, with the notations above,  $e_1 = 3, e_2 = 2$ , hence  $\mathbb{K}[x^6, x^4, x^7] \simeq \mathbb{K}[X_0, X_1, X_2]/(X_1^3 - X_0^2, X_2^2 - X_0X_1^2) = \mathbb{K}[X_0, X_1, X_2]/T$ , and

$$\mathbb{K}[u] \longrightarrow \mathbb{K}[u][X_0, X_1, X_2]/(X_1^3 - X_0^2, X_2^2 - 4X_0X_1^2 - u^{10}X_1)$$

gives us a deformation from  $A$  to  $\mathbb{K}[X_0, X_1, X_2]/T$  (we can also change  $X_2$  with  $\frac{1}{2}X_2$ , and then  $T = (X_1^3 - X_0^2, X_2^2 - 4X_0X_1^2)$ ).

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