



# An improved result for the full justification of asymptotic models for the propagation of internal waves

Samer Israwi, Ralph Lteif, Raafat Talhouk

## ► To cite this version:

Samer Israwi, Ralph Lteif, Raafat Talhouk. An improved result for the full justification of asymptotic models for the propagation of internal waves. 2014. hal-01094516

**HAL Id: hal-01094516**

**<https://hal.science/hal-01094516>**

Preprint submitted on 12 Dec 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# An improved result for the full justification of asymptotic models for the propagation of internal waves

Samer Israwi \*

Ralph Lteif \*<sup>†</sup>

Raafat Talhouk \*

December 12, 2014

## Abstract

We consider here asymptotic models that describe the propagation of one-dimensional internal waves at the interface between two layers of immiscible fluids of different densities, under the rigid lid assumption and with uneven bottoms. The aim of this paper is to show that the full justification result of the model obtained by Duchêne, Israwi and Talhouk [to appear in *SIAM J. Math. Anal.*, (arXiv:1304.4554v2)], in the sense that it is consistent, well-posed, and that its solutions remain close to exact solutions of the full Euler system with corresponding initial data, can be improved in two directions. The first direction is taking into account medium amplitude topography variations and the second direction is allowing strong nonlinearity using a new pseudo-symmetrizer, thus canceling out the smallness assumption of the Camassa-Holm regime for the existence and uniqueness results.

## 1 Introduction

### 1.1 Presentation of the problem

In this work, we are interested in the propagation of internal waves in a two-fluid system, which consists in two layers of immiscible, homogeneous, ideal, incompressible fluids of different densities, under the only influence of gravity. The domain of the two layers is infinite in the horizontal space variable (assumed to be of dimension  $d = 1$ ). We assume medium amplitude topography variations (non-flat bottom) and that the surface is confined by a flat rigid lid.

The derivation of the governing equations of such a system is not new: see [2], [5] and [16]. Under the aforementioned configuration, the governing equations describing the evolution of the flow may be reduced to a system of two coupled evolution equations located at the interface between the two layers (see [12, 32] for the water-wave configuration, and [5] for the bi-fluidic case), named *full Euler system*. In particular, the well-posedness in Sobolev spaces of the Cauchy problem for bi-fluidic full Euler system has been answered satisfactorily in the presence of a small amount of surface tension, see [22] (that is, with an existence of solutions on a time scale consistent with physical observations). However, the theoretical study of this system is extremely challenging.

Because of the complexity of these equations their solutions are very difficult to describe, this explains why a great deal of interests has been drawn to *asymptotic models*, in order to predict accurately the main behavior of the system, provided some parameters describing the domain and nature of the flow are small. Parameters of interests include

$$\mu = \frac{d_1^2}{\lambda^2}, \quad \epsilon = \frac{a}{d_1}, \quad \beta = \frac{a_b}{d_1}, \quad \delta = \frac{d_1}{d_2}, \quad \gamma = \frac{\rho_1}{\rho_2}, \quad \text{Bo} = \frac{g(\rho_2 - \rho_1)\lambda^2}{\sigma},$$

where  $a$ (resp.  $a_b$ ) is the maximal vertical deformation of the interface(resp. bottom) with respect to its rest position;  $\lambda$  is a characteristic horizontal length;  $d_1$  (resp.  $d_2$ ) is the depth of the upper (resp. lower)

---

\*Laboratory of Mathematics-EDST and Faculty of Sciences I, Lebanese University, Beirut, Lebanon. (s\_israwi83@hotmail.com)(rtalhouk@ul.edu.lb)

<sup>†</sup>LAMA, UMR 5127 CNRS, Université de Savoie, 73376 Le Bourget du lac cedex, France. (ralphlteif\_90@hotmail.com)

layer; and  $\rho_1$  (resp.  $\rho_2$ ) is the density of the upper (resp. lower) layer,  $g$  the gravitational acceleration,  $\sigma$  the interfacial tension coefficient and  $Bo$  the classical Bond number, which measures the ratio of gravity forces over capillary forces. In the following we use  $bo = \mu Bo$  instead of the classical Bond number,  $Bo$ . Mathematically speaking,  $\mu$  and  $\epsilon$  measure respectively the amount of dispersion and nonlinearity which will contribute to the evolution of internal waves. Let us introduce some earlier results directly related to the present paper.

Bona, Lannes and Saut [5] followed a strategy initiated in [3, 4] in the water-wave setting (one layer of fluid, with free surface) to derive a large class of models for different regimes, under the rigid-lid assumption, neglecting surface tension effects and with flat bottom, (see also [2] where a topography and surface tension is added to the system, and [14] where the rigid-lid assumption is removed). Shallow water ( $\mu \ll 1$ ) asymptotic models for uni-dimensional internal waves have been derived and studied in the pioneer works of [25, 26, 27]. More recently, weakly ( $\epsilon = \mathcal{O}(\mu)$ ) and strongly ( $\epsilon \sim 1$ ) nonlinear models in two-dimensions have been derived by Camassa and Choi in [9, 10]. They obtain bi-fluidic extensions of the classical shallow water (or Saint-Venant [29]), Boussinesq [6, 7] and Green-Naghdi [19, 30] models. Similar systems have been derived in [28] (with the additional assumption of  $\gamma \approx 1$ ) and in [11] (using a different approach, *i.e.* making use of the Hamiltonian structure of the full Euler equations). The models derived in these papers are systematically justified by a consistency result: roughly speaking, sufficiently smooth solutions of the full Euler system satisfy the equations of the asymptotic model, up to a small remainder.

Contrarily to the water-wave case, large amplitude internal waves in a bi-fluidic system are known to generate Kelvin-Helmholtz instabilities that appear at high frequencies, so that surface tension is necessary in order to regularize the flow. However, when adding a small amount of surface tension, Lannes [22] proved that, thanks to a stability criterion, the problem becomes well-posed with a time of existence that does not vanish as the surface tension goes to zero and thus is consistent with the observations. Therefore, adding a small amount of surface tension at the interface in the Euler system guarantees the well-posedness of the system and does not change our asymptotic models. The study of Lannes focuses on the two-layer fluid system with a flat bottom ( $\beta = 0$ ). However, we believe that the theory in the uneven bottom case does not differ much from the one in the flat bottom configuration. In [20], the well-posedness and stability results have been proved for bi-fluidic shallow-water system, and in [15] for a class of Boussinesq-type systems, neglecting surface tension and under reasonable assumptions on the flow (typically, the shear velocity must be sufficiently small). However, the well-posedness of the Green-Naghdi model in the bi-fluidic case is not clear, and similar systems have been proved to be ill-posed in [24], which has led to various propositions in order to overcome this difficulty; see [8, 13] and references therein. Green-Naghdi models consist in higher order extensions, which has since been widely used in coastal oceanography, as it takes into account the dispersive effects neglected by the shallow-water model, thus are consistent with precision  $\mathcal{O}(\mu^2)$  instead of  $\mathcal{O}(\mu)$ , and allows waves of greater amplitude (whereas Boussinesq models are limited to the long wave regime:  $\epsilon = \mathcal{O}(\mu)$ ). Finally, we mention the recent work of Xu [31], which studies and rigorously justify the so-called intermediate long wave system, obtained in a regime similar to ours:  $\epsilon \sim \sqrt{\mu}$ , but  $\delta \sim \sqrt{\mu}$ .

*In this work, we present a Green-Naghdi type model in the Camassa-Holm (or medium amplitude) regime  $\epsilon = \mathcal{O}(\sqrt{\mu})$  and we assume medium amplitude topography variations. More precisely, we assume that there exists  $\beta_{max} < \infty$  such that*

$$\beta = \mathcal{O}(\sqrt{\mu}) \quad \text{with} \quad \beta \in [0, \beta_{max}].$$

We improve in this paper the existence and uniqueness results obtained in [17] using a new pseudo-symmetrizer, thus canceling out the smallness assumption  $\epsilon = \mathcal{O}(\sqrt{\mu})$  and we prove that our Green-Naghdi model is fully justified as an asymptotic model for a set of dimensionless parameters limited to the so-called *Camassa-Holm regime*, that we describe precisely below.

We first consider the so-called *shallow water regime* for two layers of comparable depths:

$$\mathcal{P}_{\text{SW}} \equiv \left\{ (\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) : 0 < \mu \leq \mu_{\max}, 0 \leq \epsilon \leq 1, \delta \in (\delta_{\min}, \delta_{\max}), \right. \\ \left. 0 \leq \gamma < 1, 0 \leq \beta \leq \beta_{\max}, \text{bo}_{\min} \leq \text{bo} \leq \infty \right\}, \quad (1.1)$$

with given  $0 \leq \mu_{\max}, \delta_{\min}^{-1}, \delta_{\max}, \text{bo}_{\min}^{-1}, \beta_{\max} < \infty$ .

The two additional key restrictions for the validity of the model (3.9) are as follows:

$$\mathcal{P}_{\text{CH}} \equiv \left\{ (\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{SW}} : \epsilon \leq M\sqrt{\mu}, \quad \beta \leq M\sqrt{\mu} \quad \text{and} \quad \nu \equiv \frac{1 + \gamma\delta}{3\delta(\gamma + \delta)} - \frac{1}{\text{bo}} \geq \nu_0 \right\}, \quad (1.2)$$

with given  $0 \leq M, \nu_0^{-1} < \infty$ .

We denote for convenience

$$M_{\text{SW}} \equiv \max \{ \mu_{\max}, \delta_{\min}^{-1}, \delta_{\max}, \text{bo}_{\min}^{-1}, \beta_{\max} \}, \quad M_{\text{CH}} \equiv \max \{ M_{\text{SW}}, M, \nu_0^{-1} \}.$$

We prove that the full Euler system is consistent with our model, and that our system is well-posed (in the sense of Hadamard) in Sobolev spaces, and stable with respect to perturbations of the equations.

## 1.2 Organization of the paper

We start by introducing in Section 2 the non-dimensionalized full Euler system and the Green-Naghdi model.

In Section 3, we present our new model where the asymptotic model is precisely derived and motivated. Sections 4 and 5 contain the necessary ingredients for the proof of our results.

In Section 6, we explain the full justification of asymptotic models and we state its main ingredients.

We conclude this section with an inventory of the notations used throughout the present paper.

**Notations** In the following,  $C_0$  denotes any nonnegative constant whose exact expression is of no importance.

The notation  $a \lesssim b$  means that  $a \leq C_0 b$  and we write  $A = \mathcal{O}(B)$  if  $A \leq C_0 B$ .

We denote by  $C(\lambda_1, \lambda_2, \dots)$  a nonnegative constant depending on the parameters  $\lambda_1, \lambda_2, \dots$  and whose dependence on the  $\lambda_j$  is always assumed to be nondecreasing.

We use the condensed notation

$$A_s = B_s + \langle C_s \rangle_{s > \underline{s}},$$

to express that  $A_s = B_s$  if  $s \leq \underline{s}$  and  $A_s = B_s + C_s$  if  $s > \underline{s}$ .

Let  $p$  be any constant with  $1 \leq p < \infty$  and denote  $L^p = L^p(\mathbb{R})$  the space of all Lebesgue-measurable functions  $f$  with the standard norm

$$\|f\|_{L^p} = \left( \int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p} < \infty.$$

The real inner product of any functions  $f_1$  and  $f_2$  in the Hilbert space  $L^2(\mathbb{R})$  is denoted by

$$(f_1, f_2) = \int_{\mathbb{R}} f_1(x) f_2(x) dx.$$

The space  $L^\infty = L^\infty(\mathbb{R})$  consists of all essentially bounded, Lebesgue-measurable functions  $f$  with the norm

$$\|f\|_{L^\infty} = \text{ess sup } |f(x)| < \infty.$$

Let  $k \in \mathbb{N}$ , we denote by  $W^{k,\infty} = W^{k,\infty}(\mathbb{R}) = \{f \in L^\infty, |f|_{W^{k,\infty}} < \infty\}$ , where  $|f|_{W^{k,\infty}} = \sum_{\alpha \in \mathbb{N}, \alpha \leq k} |\partial_x^\alpha f|_{L^\infty}$ .

For any real constant  $s \geq 0$ ,  $H^s = H^s(\mathbb{R})$  denotes the Sobolev space of all tempered distributions  $f$  with the norm  $|f|_{H^s} = |\Lambda^s f|_{L^2} < \infty$ , where  $\Lambda$  is the pseudo-differential operator  $\Lambda = (1 - \partial_x^2)^{1/2}$ .

For a given  $\mu > 0$ , we denote by  $H_\mu^{s+1}(\mathbb{R})$  the space  $H^{s+1}(\mathbb{R})$  endowed with the norm

$$|\cdot|_{H_\mu^{s+1}}^2 \equiv |\cdot|_{H^s}^2 + \mu |\cdot|_{H^{s+1}}^2.$$

For any function  $u = u(t, x)$  and  $v(t, x)$  defined on  $[0, T) \times \mathbb{R}$  with  $T > 0$ , we denote the inner product, the  $L^p$ -norm and especially the  $L^2$ -norm, as well as the Sobolev norm, with respect to the spatial variable  $x$ , by  $(u, v) = (u(t, \cdot), v(t, \cdot))$ ,  $|u|_{L^p} = |u(t, \cdot)|_{L^p}$ ,  $|u|_{L^2} = |u(t, \cdot)|_{L^2}$ , and  $|u|_{H^s} = |u(t, \cdot)|_{H^s}$ , respectively. We denote  $L^\infty([0, T); H^s(\mathbb{R}))$  the space of functions such that  $u(t, \cdot)$  is controlled in  $H^s$ , uniformly for  $t \in [0, T)$ :

$$\|u\|_{L^\infty([0, T); H^s(\mathbb{R}))} = \operatorname{ess\,sup}_{t \in [0, T)} |u(t, \cdot)|_{H^s} < \infty.$$

Finally,  $C^k(\mathbb{R})$  denote the space of  $k$ -times continuously differentiable functions.

For any closed operator  $T$  defined on a Banach space  $X$  of functions, the commutator  $[T, f]$  is defined by  $[T, f]g = T(fg) - fT(g)$  with  $f, g$  and  $fg$  belonging to the domain of  $T$ . The same notation is used for  $f$  an operator mapping the domain of  $T$  into itself.

## 2 Previously obtained models

### 2.1 The full Euler system

The equations governing the evolution of the aforescribed system in the introduction read (using non-dimensionalized variables and the Zakharov/Craig-Sulem formulation) [12, 32]:

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G^\mu \psi = 0, \\ \partial_t (H^{\mu, \delta} \psi - \gamma \partial_x \psi) + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x (|H^{\mu, \delta} \psi|^2 - \gamma |\partial_x \psi|^2) \\ \quad = \mu \epsilon \partial_x \mathcal{N}^{\mu, \delta} - \mu \frac{\gamma + \delta}{b_0} \frac{\partial_x (k(\epsilon \sqrt{\mu} \zeta))}{\epsilon \sqrt{\mu}}, \end{cases} \quad (2.1)$$

where we denote

$$\mathcal{N}^{\mu, \delta} \equiv \frac{\left( \frac{1}{\mu} G^\mu \psi + \epsilon (\partial_x \zeta) H^{\mu, \delta} \psi \right)^2 - \gamma \left( \frac{1}{\mu} G^\mu \psi + \epsilon (\partial_x \zeta) (\partial_x \psi) \right)^2}{2(1 + \mu |\epsilon \partial_x \zeta|^2)}.$$

$\zeta(t, x)$  represent the deformation of the interface between the two layers and  $b(x)$  represent the deformation of the bottom,  $\psi$  is the trace of the velocity potential of the upper-fluid at the interface.

The function  $k(\zeta) = -\partial_x \left( \frac{1}{\sqrt{1 + |\partial_x \zeta|^2}} \partial_x \zeta \right)$  denotes the mean curvature of the interface and  $\sigma$  the surface (or interfacial) tension coefficient.

We will refer to (2.1) as the *full Euler system*, and solutions of this system will be exact solutions of the problem.

Finally,  $G^\mu$  and  $H^{\mu, \delta}$  are the so-called Dirichlet-Neumann operators, defined as follows:

**Definition 2.2 (Dirichlet-Neumann operators)** Let  $\zeta \in H^{t_0+1}(\mathbb{R})$ ,  $t_0 > 1/2$ , such that there exists  $h_0 > 0$  with  $h_1 \equiv 1 - \epsilon \zeta \geq h_0 > 0$  and  $h_2 \equiv \frac{1}{\delta} + \epsilon \zeta - \beta b \geq h_0 > 0$ , and let  $\psi \in L_{\text{loc}}^2(\mathbb{R})$ ,  $\partial_x \psi \in H^{1/2}(\mathbb{R})$ . Then we define

$$\begin{aligned} G^\mu \psi &\equiv G^\mu[\epsilon \zeta] \psi \equiv \sqrt{1 + \mu |\epsilon \partial_x \zeta|^2} (\partial_n \phi_1)|_{z=\epsilon \zeta} = -\mu \epsilon (\partial_x \zeta) (\partial_x \phi_1)|_{z=\epsilon \zeta} + (\partial_z \phi_1)|_{z=\epsilon \zeta}, \\ H^{\mu, \delta} \psi &\equiv H^{\mu, \delta}[\epsilon \zeta, \beta b] \psi \equiv \partial_x (\phi_2|_{z=\epsilon \zeta}) = (\partial_x \phi_2)|_{z=\epsilon \zeta} + \epsilon (\partial_x \zeta) (\partial_z \phi_2)|_{z=\epsilon \zeta}, \end{aligned}$$

where  $\phi_1$  and  $\phi_2$  are uniquely defined (up to a constant for  $\phi_2$ ) as the solutions in  $H^2(\mathbb{R})$  of the Laplace's problems:

$$\begin{cases} (\mu \partial_x^2 + \partial_z^2) \phi_1 = 0 & \text{in } \Omega_1 \equiv \{(x, z) \in \mathbb{R}^2, \epsilon \zeta(x) < z < 1\}, \\ \partial_z \phi_1 = 0 & \text{on } \Gamma_t \equiv \{(x, z) \in \mathbb{R}^2, z = 1\}, \\ \phi_1 = \psi & \text{on } \Gamma \equiv \{(x, z) \in \mathbb{R}^2, z = \epsilon \zeta\}, \end{cases} \quad (2.3)$$

$$\begin{cases} (\mu \partial_x^2 + \partial_z^2) \phi_2 = 0 & \text{in } \Omega_2 \equiv \{(x, z) \in \mathbb{R}^2, -\frac{1}{\delta} + \beta b(x) < z < \epsilon \zeta\}, \\ \partial_n \phi_2 = \partial_n \phi_1 & \text{on } \Gamma, \\ \partial_n \phi_2 = 0 & \text{on } \Gamma_b \equiv \{(x, z) \in \mathbb{R}^2, z = -\frac{1}{\delta} + \beta b(x)\}. \end{cases} \quad (2.4)$$

The existence and uniqueness of a solution to (2.3)-(2.4), and therefore the well-posedness of the Dirichlet-Neumann operators follow from classical arguments detailed, for example, in [23].

## 2.2 The Green-Naghdi model

The key ingredient for constructing shallow water asymptotic models lies in the expansion given in [14, 18] of the Dirichlet-Neumann operators, with respect to the shallowness parameter,  $\mu$ . Thanks to such an expansion, one is able to obtain the so-called Green-Naghdi model (for internal waves). This model has been introduced in [16] (with a flat bottom) and generalized in [18]. It is also convenient to introduce a new velocity variable, namely the shear mean velocity  $v$  is equivalently defined as

$$v \equiv u_2 - \gamma u_1 \quad (2.5)$$

where  $u_1$  and  $u_2$  are the horizontal velocities integrated across each layer:

$u_1(t, x) = \frac{1}{h_1(t, x)} \int_{\epsilon \zeta(t, x)}^1 \partial_x \phi_1(t, x, z) dz$  and  $u_2(t, x) = \frac{1}{h_2(t, x)} \int_{-\frac{1}{\delta} + \beta b(x)}^{\epsilon \zeta(t, x)} \partial_x \phi_2(t, x, z) dz$ , where  $\phi_1$  and  $\phi_2$  are the solutions to the Laplace's problems (2.3)-(2.4).

The expansions of the Dirichlet-Neumann operators may be written in terms of the new variable  $v$ .

Plugging the expansions given in [18, Proposition 7] into the full Euler system (2.1), and withdrawing all  $\mathcal{O}(\mu^2)$  terms yields (in the unidimensional case  $d = 1$ ),

$$\begin{cases} \partial_t \zeta + \partial_x \left( \frac{h_1 h_2}{h_1 + \gamma h_2} v \right) = 0, \\ \partial_t \left( v + \mu \overline{\mathcal{Q}}[h_1, h_2] v \right) + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \left( \frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2} v^2 \right) = \\ \mu \epsilon \partial_x (\overline{\mathcal{R}}[h_1, h_2] v) + \mu \frac{\gamma + \delta}{\text{bo}} \partial_x^3 \zeta, \end{cases} \quad (2.6)$$

where we denote  $h_1 = 1 - \epsilon \zeta$  and  $h_2 = \delta^{-1} + \epsilon \zeta - \beta b$ , as well as

$$\begin{aligned} \overline{\mathcal{Q}}[h_1, h_2] v &= \mathcal{T}[h_2, \beta b] \left( \frac{h_1 v}{h_1 + \gamma h_2} \right) - \gamma \mathcal{T}[h_1, 0] \left( \frac{-h_2 v}{h_1 + \gamma h_2} \right), \\ &= -\frac{1}{3h_2} \partial_x \left( h_2^3 \partial_x \left( \frac{h_1 v}{h_1 + \gamma h_2} \right) \right) + \frac{1}{2h_2} \beta \left[ \partial_x \left( h_2^2 (\partial_x b) \frac{h_1 v}{h_1 + \gamma h_2} \right) - h_2^2 (\partial_x b) \partial_x \left( \frac{h_1 v}{h_1 + \gamma h_2} \right) \right] \\ &\quad + \beta^2 (\partial_x b)^2 \left( \frac{h_1 v}{h_1 + \gamma h_2} \right) - \gamma \left[ \frac{1}{3h_1} \partial_x \left( h_1^3 \partial_x \left( \frac{h_2 v}{h_1 + \gamma h_2} \right) \right) \right]. \end{aligned}$$

$$\begin{aligned} \overline{\mathcal{R}}[h_1, h_2] v &= \frac{1}{2} \left( -h_2 \partial_x \left( \frac{h_1 v}{h_1 + \gamma h_2} \right) + \beta (\partial_x b) \left( \frac{h_1 v}{h_1 + \gamma h_2} \right) \right)^2 - \frac{\gamma}{2} \left( h_1 \partial_x \left( \frac{-h_2 v}{h_1 + \gamma h_2} \right) \right)^2 \\ &\quad - \left( \frac{h_1 v}{h_1 + \gamma h_2} \right) \mathcal{T}[h_2, \beta b] \left( \frac{h_1 v}{h_1 + \gamma h_2} \right) + \gamma \left( \frac{-h_2 v}{h_1 + \gamma h_2} \right) \mathcal{T}[h_1, 0] \left( \frac{-h_2 v}{h_1 + \gamma h_2} \right), \end{aligned}$$

with,

$$\mathcal{T}[h, b]V \equiv \frac{-1}{3h} \partial_x (h^3 \partial_x V) + \frac{1}{2h} [\partial_x (h^2 (\partial_x b) V) - h^2 (\partial_x b) (\partial_x V)] + (\partial_x b)^2 V.$$

If additionally, one assume medium amplitude topography variations, then the above system may be simplified. More precisely, we assume that there exists  $\beta_{max} < \infty$  such that

$$\beta = \mathcal{O}(\sqrt{\mu}) \quad \text{with} \quad \beta \in [0, \beta_{max}]$$

Withdrawing again  $\mathcal{O}(\mu^2)$  in (2.6) terms, one obtains

$$\begin{cases} \partial_t \zeta + \partial_x \left( \frac{h_1 h_2}{h_1 + \gamma h_2} v \right) = 0, \\ \partial_t \left( v + \mu \left( \overline{\mathcal{Q}}[h_1, h_{2f}] v + \beta \overline{\mathcal{P}}[h_1, h_{2f}] v \right) \right) + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \left( \frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2} v^2 \right) = \\ \mu \epsilon \partial_x \left( \overline{\mathcal{R}}[h_1, h_{2f}] v + \beta \overline{\mathcal{S}}[h_1, h_{2f}] v \right) + \mu \frac{\gamma + \delta}{\text{bo}} \partial_x^3 \zeta, \end{cases} \quad (2.7)$$

where we denote  $h_{2f} = \delta^{-1} + \epsilon \zeta$  ( $f$  corresponds to flat topography) with

$$\overline{\mathcal{Q}}[h_1, h_{2f}] v = \mathcal{T}[h_{2f}, 0] \left( \frac{h_1 v}{h_1 + \gamma h_{2f}} \right) - \gamma \mathcal{T}[h_1, 0] \left( \frac{-h_{2f} v}{h_1 + \gamma h_{2f}} \right),$$

and

$$\begin{aligned} \overline{\mathcal{P}}[h_1, h_{2f}] v &= \frac{1}{3h_{2f}} \partial_x \left( 3(h_{2f})^2 b \partial_x \left( \frac{h_1 v}{h_1 + \gamma h_{2f}} \right) \right) - \frac{1}{3h_{2f}} \partial_x \left( (h_{2f})^3 \partial_x \left( \frac{h_1 (\gamma b) v}{(h_1 + \gamma h_{2f})^2} \right) \right) \\ &- \frac{b}{3(h_{2f})^2} \partial_x \left( (h_{2f})^3 \partial_x \left( \frac{h_1 v}{h_1 + \gamma h_{2f}} \right) \right) \\ &+ \frac{1}{2h_{2f}} \left[ \partial_x \left( (h_{2f})^2 (\partial_x b) \frac{h_1 v}{h_1 + \gamma h_{2f}} \right) - h_{2f}^2 (\partial_x b) \partial_x \left( \frac{h_1 v}{h_1 + \gamma h_{2f}} \right) \right] \\ &- \gamma \left[ \frac{1}{3h_1} \partial_x \left( h_1^3 \partial_x \left( \frac{\gamma b h_{2f} v}{(h_1 + \gamma h_{2f})^2} \right) \right) \right] - \gamma \left[ \frac{1}{3h_1} \partial_x \left( h_1^3 \partial_x \left( \frac{-b v}{h_1 + \gamma h_{2f}} \right) \right) \right], \end{aligned}$$

and

$$\begin{aligned} \overline{\mathcal{R}}[h_1, h_{2f}] v &= \frac{1}{2} \left( h_{2f} \partial_x \left( \frac{h_1 v}{h_1 + \gamma h_{2f}} \right) \right)^2 - \frac{\gamma}{2} \left( h_1 \partial_x \left( \frac{-h_{2f} v}{h_1 + \gamma h_{2f}} \right) \right)^2 \\ &- \left( \frac{h_1 v}{h_1 + \gamma h_{2f}} \right) \left[ -\frac{1}{3h_{2f}} \partial_x \left( h_{2f}^3 \partial_x \left( \frac{h_1 v}{h_1 + \gamma h_{2f}} \right) \right) \right] \\ &+ \gamma \left( \frac{-h_{2f} v}{h_1 + \gamma h_{2f}} \right) \left[ \frac{1}{3h_1} \partial_x \left( h_1^3 \partial_x \left( \frac{h_{2f} v}{h_1 + \gamma h_{2f}} \right) \right) \right]. \end{aligned}$$

and

$$\begin{aligned}
\overline{S}[h_1, h_{2f}]v &= h_{2f}\partial_x\left(\frac{h_1v}{h_1 + \gamma h_{2f}}\right)\left[h_{2f}\partial_x\left(\frac{h_1\gamma bv}{(h_1 + \gamma h_{2f})^2}\right) - b\partial_x\left(\frac{h_1v}{h_1 + \gamma h_{2f}}\right)\right] \\
&- h_{2f}(\partial_x b)\left(\frac{h_1v}{h_1 + \gamma h_{2f}}\right)\partial_x\left(\frac{h_1v}{h_1 + \gamma h_{2f}}\right) \\
&+ \gamma\left(h_1\partial_x\left(\frac{-h_{2f}v}{h_1 + \gamma h_{2f}}\right)\right)\left[h_1\partial_x\left(\frac{h_{2f}\gamma bv}{(h_1 + \gamma h_{2f})^2}\right) - h_1\partial_x\left(\frac{bv}{h_1 + \gamma h_{2f}}\right)\right] \\
&- \frac{h_1v}{h_1 + \gamma h_{2f}}\left[\frac{-1}{3h_{2f}}\partial_x\left(h_{2f}^3\partial_x\left(\frac{h_1\gamma bv}{(h_1 + \gamma h_{2f})^2}\right)\right) + \frac{1}{3h_{2f}}\partial_x\left(3h_{2f}^2b\partial_x\left(\frac{h_1v}{h_1 + \gamma h_{2f}}\right)\right)\right] \\
&- \frac{b}{3h_{2f}^2}\partial_x\left(h_{2f}^3b\partial_x\left(\frac{h_1v}{h_1 + \gamma h_{2f}}\right)\right) \\
&+ \frac{1}{2h_{2f}}\left[\partial_x\left(h_{2f}^2(\partial_x b)\frac{h_1v}{h_1 + \gamma h_{2f}}\right) - h_{2f}^2(\partial_x b)\partial_x\left(\frac{h_1v}{h_1 + \gamma h_{2f}}\right)\right] \\
&- \frac{\gamma h_1 b}{(h_1 + \gamma h_{2f})^2}\left[\frac{-1}{3h_{2f}}\partial_x\left(h_{2f}^3\partial_x\left(\frac{h_1v}{h_1 + \gamma h_{2f}}\right)\right)\right] \\
&+ \gamma\left(\frac{-h_{2f}v}{h_1 + \gamma h_{2f}}\right)\left[\frac{1}{3h_1}\partial_x\left(h_1^3\partial_x\left(\frac{\gamma b h_{2f}v}{(h_1 + \gamma h_{2f})^2} - \frac{bv}{h_1 + \gamma h_{2f}}\right)\right)\right], \\
&+ \gamma\left(\frac{-\gamma b h_{2f}v}{(h_1 + \gamma h_{2f})^2} + \frac{bv}{h_1 + \gamma h_{2f}}\right)\left[\frac{1}{3h_1}\partial_x\left(h_1^3\partial_x\left(\frac{h_{2f}v}{h_1 + \gamma h_{2f}}\right)\right)\right]
\end{aligned}$$

**Remark 2.8** The following approximation formally hold using that:

$$\frac{1}{1-X} = 1 + X + X^2 + \mathcal{O}(X^3) \text{ with } X \ll 1,$$

$$\begin{aligned}
\frac{1}{h_1 + \gamma h_2} &= \frac{1}{h_1 + \gamma h_{2f} - \gamma \beta b} \\
&= \frac{1}{(h_1 + \gamma h_{2f})\left(1 - \frac{\gamma \beta b}{h_1 + \gamma h_{2f}}\right)} \\
&= \frac{1}{(h_1 + \gamma h_{2f})}\left(1 + \frac{\gamma \beta b}{h_1 + \gamma h_{2f}} + \frac{(\gamma \beta b)^2}{(h_1 + \gamma h_{2f})^2} + \mathcal{O}(\beta^3)\right).
\end{aligned}$$

**Proposition 2.9 (Consistency)** For  $\mathbf{p} = (\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{SW}}$ , let  $U^{\mathbf{p}} = (\zeta^{\mathbf{p}}, \psi^{\mathbf{p}})^{\top}$  be a family of solutions of the full Euler system (2.1) such that there exists  $T > 0$ ,  $s \geq s_0 + 1$ ,  $s_0 > 1/2$  for which  $(\zeta^{\mathbf{p}}, \partial_x \psi^{\mathbf{p}})^{\top}$  and  $(\partial_t \zeta^{\mathbf{p}}, \partial_t \partial_x \psi^{\mathbf{p}})^{\top}$  are bounded in  $L^\infty([0, T]; H^{s+N})^2$  ( $N$  sufficiently large), uniformly with respect to  $\mathbf{p} \in \mathcal{P}_{\text{SW}}$ . Moreover, assume that  $b \in H^{s+N}$  and there exists  $h_{01} > 0$  such that

$$h_1 = 1 - \epsilon \zeta^{\mathbf{p}} \geq h_{01} > 0, \quad h_{2f} = 1/\delta + \epsilon \zeta^{\mathbf{p}} \geq h_{01} > 0, \quad h_2 = 1/\delta + \epsilon \zeta^{\mathbf{p}} - \beta b \geq h_{01} > 0.$$

Define  $v^{\mathbf{p}}$  as in (2.5). Then  $(\zeta^{\mathbf{p}}, v^{\mathbf{p}})^{\top}$  satisfies (2.7) up to a remainder term,  $R = (0, r)^{\top}$ , bounded by

$$\|r\|_{L^\infty([0, T]; H^s)} \leq \mu^2 C_1,$$

with  $C_1 = C(h_{01}^{-1}, M_{\text{SW}}, |b|_{H^{s+N}}, \|(\zeta^{\mathbf{p}}, \partial_x \psi^{\mathbf{p}})^{\top}\|_{L^\infty([0, T]; H^{s+N})^2}, \|(\partial_t \zeta^{\mathbf{p}}, \partial_t \partial_x \psi^{\mathbf{p}})^{\top}\|_{L^\infty([0, T]; H^{s+N})^2})$ , uniformly with respect to  $\mathbf{p} \in \mathcal{P}_{\text{SW}}$ .

*Proof.*

This results has been proved in [18, Proposition 8] for the system (2.6). So the proof is straightforwardly adapted to the simplified system (2.7), using in particular the following estimates, valid for  $s > 1/2$ :

$$\begin{aligned}
\left|\frac{1}{h_1 + \gamma h_2} - \frac{1}{h_1 + \gamma h_{2f}}\right|_{H^s} &\leq \beta C(h_{01}^{-1}, |\zeta^{\mathbf{p}}|_{H^s})|b|_{H^s}; \\
\left|\frac{1}{h_1 + \gamma h_2} - \frac{1}{h_1 + \gamma h_{2f}} - \frac{\gamma \beta b}{(h_1 + \gamma h_{2f})^2}\right|_{H^s} &\leq \beta^2 C(h_{01}^{-1}, |\zeta^{\mathbf{p}}|_{H^s})|b^2|_{H^s};
\end{aligned}$$

see, for example, [23, Proposition B.4]. □



### 3 Construction of the new model

The present work is limited to the so-called Camassa-Holm regime, that is using additional assumption  $\epsilon = \mathcal{O}(\sqrt{\mu})$ . In this section, we manipulate the Green-Naghdi system (2.7), systematically withdrawing  $\mathcal{O}(\mu^2, \mu\epsilon^2, \mu\epsilon\beta)$  terms, in order to recover our model.

One can check that the following approximations formally hold:

$$\begin{aligned}\overline{\mathcal{Q}}[h_1, h_{2f}]v + \beta\overline{\mathcal{P}}[h_1, h_{2f}]v &= -\lambda\partial_x^2 v - \epsilon\frac{\gamma+\delta}{3}((\theta-\alpha)v\partial_x^2\zeta + (\alpha+2\theta)\partial_x(\zeta\partial_x v) - \theta\zeta\partial_x^2 v) \\ &\quad + \beta\frac{\gamma+\delta}{3}\left((\frac{\alpha_1}{2} + \theta_1)v\partial_x^2 b + (\alpha_1 + 2\theta_1)\partial_x(b\partial_x v) - \theta_1 b\partial_x^2 v\right) \\ &\quad + \mathcal{O}(\epsilon^2, \epsilon\beta), \\ \overline{\mathcal{R}}[h_1, h_{2f}]v + \beta\overline{\mathcal{S}}[h_1, h_{2f}]v &= \alpha\left(\frac{1}{2}(\partial_x v)^2 + \frac{1}{3}v\partial_x^2 v\right) + \mathcal{O}(\epsilon, \beta).\end{aligned}$$

with

$$\lambda = \frac{1+\gamma\delta}{3\delta(\gamma+\delta)}, \quad \alpha = \frac{1-\gamma}{(\gamma+\delta)^2} \quad \text{and} \quad \theta = \frac{(1+\gamma\delta)(\delta^2-\gamma)}{\delta(\gamma+\delta)^3}, \quad (3.1)$$

and

$$\alpha_1 = \frac{1}{(\gamma+\delta)^2} \quad \text{and} \quad \theta_1 = \frac{\delta(1+\gamma\delta)}{(\gamma+\delta)^3}. \quad (3.2)$$

Plugging these expansions into system (2.7) yields a simplified model, with the same order of precision of the original model (that is  $\mathcal{O}(\mu^2)$ ) in the Camassa-Holm regime. However, we will use several additional transformations, in order to produce an equivalent model (again, in the sense of consistency), which possess a structure similar to symmetrizable quasilinear systems, and allows the study of the subsequent sections.

Using the same techniques as in [17, Section 4.2] but with a different symmetric operator  $\mathfrak{T}[\epsilon\zeta, \beta b]$  defined below, we obtain the following equation:

$$\begin{aligned}\mathfrak{T}[\epsilon\zeta, \beta b](\partial_t v + \epsilon\varsigma v\partial_x v) - q_1(\epsilon\zeta, \beta b)\partial_t \left(v + \mu(\overline{\mathcal{Q}}[h_1, h_{2f}]v + \beta\overline{\mathcal{P}}[h_1, h_{2f}]v)\right) \\ + q_1(\epsilon\zeta, \beta b)\mu\frac{\gamma+\delta}{\text{bo}}\partial_x^3\zeta + \mu\epsilon q_1(\epsilon\zeta, \beta b)\partial_x \left(\overline{\mathcal{R}}[h_1, h_{2f}]v + \beta\overline{\mathcal{S}}[h_1, h_{2f}]v\right) \\ = \epsilon\varsigma q_1(\epsilon\zeta, \beta b)v\partial_x v - \mu\epsilon\frac{2\alpha}{3}\partial_x((\partial_x v)^2) + \mu\beta\omega(\partial_x\zeta)(\partial_x^2 b) + \mathcal{O}(\mu^2, \mu\epsilon^2, \mu\epsilon\beta)\end{aligned} \quad (3.3)$$

where we denote  $\omega = \frac{(\gamma+\delta)^2}{3}\left(\frac{\alpha_1}{2} + \theta_1\right)$  and

$$\mathfrak{T}[\epsilon\zeta, \beta b]V = q_1(\epsilon\zeta, \beta b)V - \mu\nu\partial_x\left(q_2(\epsilon\zeta, \beta b)\partial_x V\right), \quad (3.4)$$

with  $q_i(X, Y) \equiv 1 + \kappa_i X + \omega_i Y$  ( $i = 1, 2$ ) and  $\nu, \kappa_1, \kappa_2, \omega_1, \omega_2, \varsigma$  are defined as follow:

$$\nu = \lambda - \frac{1}{\text{bo}} = \frac{1+\gamma\delta}{3\delta(\gamma+\delta)} - \frac{1}{\text{bo}}, \quad (3.5)$$

$$\left(\lambda - \frac{1}{\text{bo}}\right)\kappa_1 = \frac{\gamma+\delta}{3}(2\theta-\alpha), \quad \left(\lambda - \frac{1}{\text{bo}}\right)\kappa_2 = (\gamma+\delta)\theta, \quad (3.6)$$

$$\left(\lambda - \frac{1}{\text{bo}}\right)\omega_1 = -\theta_1\frac{\gamma+\delta}{3}, \quad \left(\lambda - \frac{1}{\text{bo}}\right)\omega_2 = -\frac{(\gamma+\delta)}{3}(\alpha_1 + 2\theta_1), \quad (3.7)$$

$$\left(\lambda - \frac{1}{\text{bo}}\right)\varsigma = \frac{2\alpha-\theta}{3} - \frac{1}{\text{bo}}\frac{\delta^2-\gamma}{(\delta+\gamma)^2}. \quad (3.8)$$

When plugging the estimate (3.3) in (2.7), and after multiplying the second equation by  $q_1(\epsilon\zeta, \beta b)$ , we obtain the following system of equations:

$$\begin{cases} \partial_t \zeta + \partial_x \left( \frac{h_1 h_2}{h_1 + \gamma h_2} v \right) = 0, \\ \mathfrak{T}[\epsilon\zeta, \beta b] (\partial_t v + \epsilon \zeta v \partial_x v) + (\gamma + \delta) q_1(\epsilon\zeta, \beta b) \partial_x \zeta + \frac{\epsilon}{2} q_1(\epsilon\zeta, \beta b) \partial_x \left( \frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2} |v|^2 - \varsigma |v|^2 \right) \\ = -\mu \epsilon \frac{2}{3} \alpha \partial_x ((\partial_x v)^2) + \mu \beta \omega (\partial_x \zeta) (\partial_x^2 b), \end{cases} \quad (3.9)$$

**Proposition 3.10 (Consistency)** *For  $\mathbf{p} = (\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{SW}}$ , let  $U^{\mathbf{p}} = (\zeta^{\mathbf{p}}, \psi^{\mathbf{p}})^{\top}$  be a family of solutions of the full Euler system (2.1) such that there exists  $T > 0$ ,  $s \geq s_0 + 1$ ,  $s_0 > 1/2$  for which  $(\zeta^{\mathbf{p}}, \partial_x \psi^{\mathbf{p}})^{\top}$  and  $(\partial_t \zeta^{\mathbf{p}}, \partial_t \partial_x \psi^{\mathbf{p}})^{\top}$  are bounded in  $L^\infty([0, T]; H^{s+N})^2$  with sufficiently large  $N$ , and uniformly with respect to  $\mathbf{p} \in \mathcal{P}_{\text{SW}}$ . Moreover assume that  $b \in H^{s+N}$  and there exists  $h_{01} > 0$  such that*

$$h_1 = 1 - \epsilon \zeta^{\mathbf{p}} \geq h_{01} > 0, \quad h_{2f} = 1/\delta + \epsilon \zeta^{\mathbf{p}} \geq h_{01} > 0, \quad h_2 = 1/\delta + \epsilon \zeta^{\mathbf{p}} - \beta b \geq h_{01} > 0. \quad (\text{H1})$$

*Define  $v^{\mathbf{p}}$  as in (2.5). Then  $(\zeta^{\mathbf{p}}, v^{\mathbf{p}})^{\top}$  satisfies (3.9) up to a remainder term,  $R = (0, r)^{\top}$ , bounded by*

$$\|r\|_{L^\infty([0, T]; H^s)} \leq (\mu^2 + \mu \epsilon^2 + \mu \epsilon \beta) C,$$

*with  $C = C(h_{01}^{-1}, M_{\text{SW}}, |b|_{H^{s+N}}, \|(\zeta^{\mathbf{p}}, \partial_x \psi^{\mathbf{p}})^{\top}\|_{L^\infty([0, T]; H^{s+N})^2}, \|(\partial_t \zeta^{\mathbf{p}}, \partial_t \partial_x \psi^{\mathbf{p}})^{\top}\|_{L^\infty([0, T]; H^{s+N})^2})$ .*

*Proof.*

Let  $U = (\zeta, \psi)^{\top}$  satisfy the hypothesis above (withdrawing the explicit dependence with respect to parameters  $\mathbf{p}$  for the sake of readability). We know from Proposition 2.9 that  $(\zeta, v)^{\top}$  satisfies the system (2.7) up to a remainder  $R_0 = (0, r_0)^{\top}$  bounded by,

$$\|r_0\|_{L^\infty([0, T]; H^s)} \leq \mu^2 C_1,$$

with  $C_1 = C(h_{01}^{-1}, M_{\text{SW}}, |b|_{H^{s+N}}, \|(\zeta^{\mathbf{p}}, \partial_x \psi^{\mathbf{p}})^{\top}\|_{L^\infty([0, T]; H^{s+N})^2}, \|(\partial_t \zeta^{\mathbf{p}}, \partial_t \partial_x \psi^{\mathbf{p}})^{\top}\|_{L^\infty([0, T]; H^{s+N})^2})$ , uniformly with respect to  $\mathbf{p} \in \mathcal{P}_{\text{SW}}$ .

The proof now consists in checking that all terms neglected in the above calculations can be rigorously estimated in the same way. We do not develop each estimate, but rather provide the precise bound on the various remainder terms. One has

$$\begin{aligned} & \left| \partial_t (\overline{Q}[h_1, h_{2f}]v + \beta \overline{P}[h_1, h_{2f}]v) - \left[ -\lambda \partial_x^2 \partial_t v - \epsilon \frac{\gamma + \delta}{3} \partial_t ((\beta - \alpha)v \partial_x^2 \zeta + (\alpha + 2\beta) \partial_x (\zeta \partial_x v) - \beta \zeta \partial_x^2 v) \right. \right. \\ & \quad \left. \left. + \beta \frac{\gamma + \delta}{3} \partial_t \left( \left( \frac{\alpha_1}{2} + \theta_1 \right) v \partial_x^2 b + (\alpha_1 + 2\theta_1) \partial_x (b \partial_x v) - \theta_1 b \partial_x^2 v \right) \right] \right|_{H^s} \\ & \leq (\epsilon^2 + \epsilon \beta) C(s+3), \end{aligned}$$

with  $C(s+3) \equiv C(M_{\text{SW}}, h_{01}^{-1}, |\zeta|_{H^{s+3}}, |\partial_t \zeta|_{H^{s+2}}, |v|_{H^{s+3}}, |\partial_t v|_{H^{s+2}}, |b|_{H^{s+3}})$ , and

$$\left| \partial_x (\overline{R}[h_1, h_{2f}]v + \beta \overline{S}[h_1, h_{2f}]v) - \partial_x \left[ \alpha \left( \frac{1}{2} (\partial_x v)^2 + \frac{1}{3} v \partial_x^2 v \right) \right] \right|_{H^s} \leq (\epsilon + \beta) C(s+3).$$

It follows that (3.3) is valid up to a remainder  $R_1$ , bounded by

$$|R_1|_{H^s} \leq (\mu^2 + \mu \epsilon^2 + \mu \epsilon \beta) C(s+3)$$

Finally,  $(\zeta, v)$  satisfies (3.9) up to a remainder  $R$ , bounded by

$$|R|_{H^s} \leq |R_1|_{H^s} + |R_0|_{H^s} \leq (\mu^2 + \mu \epsilon^2 + \mu \epsilon \beta) C.$$

where we use that

$$|v|_{H^{s+3}} + |\partial_t v|_{H^{s+2}} \leq C.$$

The estimate on  $v$  follows directly from the identity  $\partial_x \left( \frac{h_1 h_2}{h_1 + \gamma h_2} v \right) = -\frac{1}{\mu} G^{\mu, \epsilon} \psi = \partial_t \zeta$ . The estimate on  $\partial_t v$  can be proved, for example, following [14, Prop. 2.12]. This concludes the proof of Proposition 3.10.  $\square$

## 4 Preliminary results

In this section, we recall the operator  $\mathfrak{T}[\epsilon\zeta, \beta]$ , defined in (3.4):

$$\mathfrak{T}[\epsilon\zeta, \beta b]V = q_1(\epsilon\zeta, \beta b)V - \mu\nu\partial_x(q_2(\epsilon\zeta, \beta b)\partial_x V). \quad (4.1)$$

with  $\nu, \kappa_1, \kappa_2, \omega_1, \omega_2$  are constants and  $\nu = \frac{1+\gamma\delta}{3\delta(\gamma+\delta)} - \frac{1}{\text{bo}} \geq \nu_0 > 0$ .

The operator  $\mathfrak{T}[\epsilon\zeta, \beta b]$ , has exactly the same structure as the one introduced in [17]. In the following, we seek sufficient conditions to ensure the strong ellipticity of the operator  $\mathfrak{T}$  which will yield to the well-posedness and continuity of the inverse  $\mathfrak{T}^{-1}$ .

As a matter of fact, this condition, namely (H2) (and similarly the classical non-zero depth condition, (H1)) simply consists in assuming that the deformation of the interface is not too large as given in [17] but here we have to take into account the topographic variation that plays a role in (H1) and in (H2). For fixed  $\zeta \in L^\infty$  and  $b \in L^\infty$ , the restriction reduces to an estimate on  $\epsilon_{\max}|\zeta|_{L^\infty} + \beta_{\max}|b|_{L^\infty}$  with  $\epsilon_{\max} = \min(M\sqrt{\mu_{\max}}, 1)$ , and (H1)-(H2) hold uniformly with respect to  $(\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{CH}}$ .

Let us shortly detail the argument. Recall the non-zero depth condition

$$\exists h_{01} > 0, \quad \text{such that} \quad \inf_{x \in \mathbb{R}} h_1 \geq h_{01} > 0, \quad \inf_{x \in \mathbb{R}} h_{2f} \geq h_{01} > 0, \quad \inf_{x \in \mathbb{R}} h_2 \geq h_{01} > 0. \quad (\text{H1})$$

where  $h_1 = 1 - \epsilon\zeta$  and  $h_2 = 1/\delta + \epsilon\zeta - \beta b$  are the depth of respectively the upper and the lower layer of the fluid and  $h_{2f} = 1/\delta + \epsilon\zeta$  the depth of the lower layer of the fluid when the bottom is flat.

It is straightforward to check that, since for all  $(\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{CH}}$ , the following condition

$$\epsilon_{\max}|\zeta|_{L^\infty} + \beta_{\max}|b|_{L^\infty} < \min(1, \frac{1}{\delta_{\max}})$$

is sufficient to define  $h_{01} > 0$  such that (H1) is valid independently of  $(\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{CH}}$ .

In the same way, we introduce the condition

$$\exists h_{02} > 0, \quad \text{such that} \quad \inf_{x \in \mathbb{R}} (1 + \epsilon\kappa_1\zeta + \beta\omega_1 b) \geq h_{02} > 0; \quad \inf_{x \in \mathbb{R}} (1 + \epsilon\kappa_2\zeta + \beta\omega_2 b) \geq h_{02} > 0. \quad (\text{H2})$$

*In what follows, we will always assume that (H1) and (H2) are satisfied.* It is a consequence of our work that such assumption may be imposed only on the initial data, and then is automatically satisfied over the relevant time scale.

Now the preliminary results proved in [17, Section 5] remain true for the operator  $\mathfrak{T}[\epsilon\zeta, \beta b]$ . Let us recall these results,

**Lemma 4.2** *Let  $\zeta \in L^\infty$ ,  $b \in L^\infty$  and  $\epsilon_{\max} = \min(M\sqrt{\mu_{\max}}, 1)$  be such that there exists  $h_0 > 0$  with*

$$\max(|\kappa_1|, |\kappa_2|, 1, \delta_{\max})\epsilon_{\max}|\zeta|_{L^\infty} + \max(|\omega_1|, |\omega_2|, \delta_{\max})\beta_{\max}|b|_{L^\infty} \leq 1 - h_0 < 1.$$

*Then there exists  $h_{01}, h_{02} > 0$  such that (H1)-(H2) hold for any  $(\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{CH}}$ .*

Before asserting the strong ellipticity of the operator  $\mathfrak{T}$ , let us first recall the quantity  $|\cdot|_{H_\mu^1}$ , which is defined as

$$\forall v \in H^1(\mathbb{R}), \quad |v|_{H_\mu^1}^2 = |v|_{L^2}^2 + \mu |\partial_x v|_{L^2}^2,$$

and is equivalent to the  $H^1(\mathbb{R})$ -norm but not uniformly with respect to  $\mu$ . We define by  $H_\mu^1(\mathbb{R})$  the space  $H^1(\mathbb{R})$  endowed with this norm.

**Lemma 4.3** Let  $(\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{CH}}$  and  $\zeta \in L^\infty(\mathbb{R})$ ,  $b \in L^\infty(\mathbb{R})$  such that (H2) is satisfied. Then the operator

$$\mathfrak{T}[\epsilon\zeta, \beta b] : H_\mu^1(\mathbb{R}) \longrightarrow (H_\mu^1(\mathbb{R}))^*$$

is uniformly continuous and coercive. More precisely, there exists  $c_0 > 0$  such that

$$(\mathfrak{T}u, v) \leq c_0 |u|_{H_\mu^1} |v|_{H_\mu^1}; \quad (4.4)$$

$$(\mathfrak{T}u, u) \geq \frac{1}{c_0} |u|_{H_\mu^1}^2 \quad (4.5)$$

with  $c_0 = C(M_{\text{CH}}, h_{02}^{-1}, \epsilon|\zeta|_{L^\infty}, \beta|b|_{L^\infty})$ .

Moreover, the following estimates hold:

(i) Let  $s_0 > \frac{1}{2}$  and  $s \geq 0$ . If  $\zeta, b \in H^{s_0}(\mathbb{R}) \cap H^s(\mathbb{R})$  and  $u \in H^{s+1}(\mathbb{R})$  and  $v \in H^1(\mathbb{R})$ , then:

$$|(\Lambda^s \mathfrak{T}[\epsilon\zeta, \beta b]u, v)| \leq C_0 \left( (1 + \epsilon|\zeta|_{H^{s_0}} + \beta|b|_{H^{s_0}}) |u|_{H_\mu^{s+1}} + \langle (\epsilon|\zeta|_{H^s} + \beta|b|_{H^s}) |u|_{H_\mu^{s_0+1}} \rangle_{s>s_0} \right) |v|_{H_\mu^1}, \quad (4.6)$$

(ii) Let  $s_0 > \frac{1}{2}$  and  $s \geq 0$ . If  $\zeta, b \in H^{s_0+1}(\mathbb{R}) \cap H^s(\mathbb{R})$ ,  $u \in H^s(\mathbb{R})$  and  $v \in H^1(\mathbb{R})$ , then:

$$\begin{aligned} |(\Lambda^s, \mathfrak{T}[\epsilon\zeta, \beta b]u, v)| &\leq C_0 \left( (\epsilon|\zeta|_{H^{s_0+1}} + \beta|b|_{H^{s_0+1}}) |u|_{H_\mu^s} + \langle (\epsilon|\zeta|_{H^s} + \beta|b|_{H^s}) |u|_{H_\mu^{s_0+1}} \rangle_{s>s_0+1} \right) |v|_{H_\mu^1} \\ &\leq \max(\epsilon, \beta) C_0 \left( (|\zeta|_{H^{s_0+1}} + |b|_{H^{s_0+1}}) |u|_{H_\mu^s} + \langle (|\zeta|_{H^s} + |b|_{H^s}) |u|_{H_\mu^{s_0+1}} \rangle_{s>s_0+1} \right) |v|_{H_\mu^1} \end{aligned} \quad (4.7)$$

where  $C_0 = C(M_{\text{CH}}, h_{02}^{-1})$ .

The following lemma offers an important invertibility result on  $\mathfrak{T}$ .

**Lemma 4.8** Let  $(\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{CH}}$  and  $\zeta \in L^\infty(\mathbb{R})$ ,  $b \in L^\infty(\mathbb{R})$  such that (H2) is satisfied. Then the operator

$$\mathfrak{T}[\epsilon\zeta, \beta b] : H^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

is one-to-one and onto. Moreover, one has the following estimates:

(i)  $(\mathfrak{T}[\epsilon\zeta, \beta b])^{-1} : L^2 \rightarrow H_\mu^1(\mathbb{R})$  is continuous. More precisely, one has

$$\|\mathfrak{T}^{-1}\|_{L^2(\mathbb{R}) \rightarrow H_\mu^1(\mathbb{R})} \leq c_0,$$

with  $c_0 = C(M_{\text{CH}}, h_{02}^{-1}, \epsilon|\zeta|_{L^\infty}, \beta|b|_{L^\infty})$ .

(ii) Additionally, if  $\zeta, b \in H^{s_0+1}(\mathbb{R})$  with  $s_0 > \frac{1}{2}$ , then one has for any  $0 < s \leq s_0 + 1$ ,

$$\|\mathfrak{T}^{-1}\|_{H^s(\mathbb{R}) \rightarrow H_\mu^{s+1}(\mathbb{R})} \leq c_{s_0+1}.$$

(iii) If  $\zeta, b \in H^s(\mathbb{R})$  with  $s \geq s_0 + 1$ ,  $s_0 > \frac{1}{2}$ , then one has

$$\|\mathfrak{T}^{-1}\|_{H^s(\mathbb{R}) \rightarrow H_\mu^{s+1}(\mathbb{R})} \leq c_s$$

where  $c_s = C(M_{\text{CH}}, h_{02}^{-1}, \epsilon|\zeta|_{H^s}, \beta|b|_{H^s})$ , thus uniform with respect to  $(\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{CH}}$ .

Finally, let us introduce the following technical estimate, which is used several times in the subsequent sections.

**Corollary 4.9** Let  $(\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{CH}}$  and  $\zeta, b \in H^s(\mathbb{R})$  with  $s \geq s_0 + 1$ ,  $s_0 > \frac{1}{2}$ , such that (H2) is satisfied. Assume moreover that  $u \in H^{s-1}(\mathbb{R})$  and that  $v \in H^1(\mathbb{R})$ . Then one has

$$\begin{aligned} |(\Lambda^s, \mathfrak{T}^{-1}[\epsilon\zeta, \beta b]u, \mathfrak{T}[\epsilon\zeta, \beta b]v)| &= |(\Lambda^s, \mathfrak{T}[\epsilon\zeta, \beta b]\mathfrak{T}^{-1}u, v)| \\ &\leq \max(\epsilon, \beta) C(M_{\text{CH}}, h_{02}^{-1}, |\zeta|_{H^s}, |b|_{H^s}) |u|_{H^{s-1}} |v|_{H_\mu^1} \end{aligned} \quad (4.10)$$

## 5 Linear analysis

Let us recall the system (3.9).

$$\begin{cases} \partial_t \zeta + \partial_x \left( \frac{h_1 h_2}{h_1 + \gamma h_2} v \right) = 0, \\ \mathfrak{T}[\epsilon \zeta, \beta b] (\partial_t v + \epsilon \zeta v \partial_x v) + (\gamma + \delta) q_1(\epsilon \zeta, \beta b) \partial_x \zeta + \frac{\epsilon}{2} q_1(\epsilon \zeta, \beta b) \partial_x \left( \frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2} |v|^2 - \varsigma |v|^2 \right) \\ = -\mu \epsilon^{\frac{2}{3}} \alpha \partial_x ((\partial_x v)^2) + \mu \beta \omega (\partial_x \zeta) (\partial_x^2 b), \end{cases} \quad (5.1)$$

with  $h_1 = 1 - \epsilon \zeta$ ,  $h_2 = 1/\delta + \epsilon \zeta - \beta b$ ,  $q_i(X, Y) = 1 + \kappa_i X + \omega_i Y$  ( $i = 1, 2$ ),  $\kappa_i, \omega_i, \varsigma$  defined in (3.6), (3.7), (3.8), and

$$\mathfrak{T}[\epsilon \zeta, \beta b] V = q_1(\epsilon \zeta, \beta b) V - \mu \nu \partial_x (q_2(\epsilon \zeta, \beta b) \partial_x V).$$

In order to ease the reading, we define the function

$$f : X \rightarrow \frac{(1 - X)(\delta^{-1} + X - \beta b)}{1 - X + \gamma(\delta^{-1} + X - \beta b)},$$

and

$$g : X \rightarrow \left( \frac{(1 - X)}{1 - X + \gamma(\delta^{-1} + X - \beta b)} \right)^2.$$

One can easily check that

$$f(\epsilon \zeta) = \frac{h_1 h_2}{h_1 + \gamma h_2}, \quad f'(\epsilon \zeta) = \frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2} \quad \text{and} \quad g(\epsilon \zeta) = \left( \frac{h_1}{h_1 + \gamma h_2} \right)^2.$$

Additionally, let us denote

$$\kappa = \frac{2}{3} \alpha = \frac{2}{3} \frac{1 - \gamma}{(\delta + \gamma)^2} \quad \text{and} \quad q_3(\epsilon \zeta) = \frac{1}{2} \left( \frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2} - \varsigma \right),$$

so that one can rewrite,

$$\begin{cases} \partial_t \zeta + f(\epsilon \zeta) \partial_x v + \epsilon \partial_x \zeta f'(\epsilon \zeta) v - \beta \partial_x b g(\epsilon \zeta) v = 0, \\ \mathfrak{T} \left( \partial_t v + \frac{\epsilon}{2} \varsigma \partial_x (v^2) \right) + (\gamma + \delta) q_1(\epsilon \zeta, \beta b) \partial_x \zeta + \epsilon q_1(\epsilon \zeta, \beta b) \partial_x (q_3(\epsilon \zeta) v^2) = -\mu \epsilon \kappa \partial_x ((\partial_x v)^2) + \mu \beta \omega (\partial_x \zeta) (\partial_x^2 b). \end{cases} \quad (5.2)$$

$$\text{with } \partial_x (q_3(\epsilon \zeta)) = \frac{-\gamma \epsilon \partial_x \zeta (h_1 + h_2)^2 + \gamma \beta \partial_x b h_1 (h_1 + h_2)}{(h_1 + \gamma h_2)^3}.$$

The equations can be written after applying  $\mathfrak{T}^{-1}$  to the second equation in (5.2) as

$$\partial_t U + A_0[U] \partial_x U + A_1[U] \partial_x U + B[U] = 0, \quad (5.3)$$

with

$$A_0[U] = \begin{pmatrix} \epsilon f'(\epsilon \zeta) v & f(\epsilon \zeta) \\ \mathfrak{T}^{-1}(Q_0(\epsilon \zeta, \beta b) \cdot) & \epsilon \mathfrak{T}^{-1}(\mathfrak{Q}[\epsilon \zeta, \beta b, v] \cdot) \end{pmatrix}, \quad A_1[U] = \begin{pmatrix} 0 & 0 \\ \epsilon^2 \mathfrak{T}^{-1}(Q_1(\epsilon \zeta, \beta b, v) \cdot) & \epsilon \varsigma v \end{pmatrix}, \quad (5.4)$$

$$B[U] = \begin{pmatrix} -\beta \partial_x b g(\epsilon \zeta) v \\ \epsilon \mathfrak{T}^{-1} \left( \frac{\gamma \beta q_1(\epsilon \zeta, \beta b) h_1 (h_1 + h_2) v^2}{(h_1 + \gamma h_2)^3} \partial_x b \right) \end{pmatrix} \quad (5.5)$$

where  $Q_0(\epsilon \zeta, \beta b)$ ,  $Q_1(\epsilon \zeta, \beta b, v)$  are defined as

$$Q_0(\epsilon \zeta, \beta b) = (\gamma + \delta) q_1(\epsilon \zeta, \beta b) - \mu \beta \omega \partial_x^2 b, \quad Q_1(\epsilon \zeta, \beta b, v) = -\gamma q_1(\epsilon \zeta, \beta b) \frac{(h_1 + h_2)^2}{(h_1 + \gamma h_2)^3} v^2 \quad (5.6)$$

and the operator  $\mathfrak{Q}[\epsilon, \beta b, v]$  defined by

$$\mathfrak{Q}[\epsilon \zeta, \beta b, v]f \equiv 2q_1(\epsilon \zeta, \beta b)q_3(\epsilon \zeta)v f + \mu \kappa \partial_x(f \partial_x v). \quad (5.7)$$

Following the classical theory of quasilinear hyperbolic systems, the well-posedness of the initial value problem of the above system will rely on a precise study of the properties, and in particular energy estimates, for the linearized system around some reference state  $\underline{U} = (\underline{\zeta}, \underline{v})^\top$ :

$$\begin{cases} \partial_t U + A_0[\underline{U}] \partial_x U + A_1[\underline{U}] \partial_x U + B[\underline{U}] = 0; \\ U|_{t=0} = U_0. \end{cases} \quad (5.8)$$

## 5.1 Energy space

Let us first remark that by construction, one has a pseudo-symmetrizer of the system that allows to cancel the smallness assumption of the Camassa-Holm regime  $\epsilon = \mathcal{O}(\sqrt{\mu})$  for the existence and uniqueness results, given by

$$Z[\underline{U}] = \begin{pmatrix} \frac{Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v})}{f(\epsilon \underline{\zeta})} & 0 \\ 0 & \mathfrak{T}[\epsilon \underline{\zeta}, \beta b] \end{pmatrix} \quad (5.9)$$

One should add an additional assumption in order to ensure that our pseudo-symmetrizer is defined and positive which is:

$$\exists h_{03} > 0 \text{ such that } Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v}) \geq h_{03} > 0. \quad (\text{H3})$$

Let us now define our energy space.

**Definition 5.10** For given  $s \geq 0$  and  $\mu, T > 0$ , we denote by  $X^s$  the vector space  $H^s(\mathbb{R}) \times H_\mu^{s+1}(\mathbb{R})$  endowed with the norm

$$\forall U = (\zeta, v) \in X^s, \quad |U|_{X^s}^2 \equiv |\zeta|_{H^s}^2 + |v|_{H^s}^2 + \mu |\partial_x v|_{H^s}^2,$$

while  $X_T^s$  stands for the space of  $U = (\zeta, v)$  such that  $U \in C^0([0, \frac{T}{\max(\epsilon, \beta)}]; X^s)$  and  $\partial_t U \in L^\infty([0, \frac{T}{\max(\epsilon, \beta)}] \times \mathbb{R})$ , endowed with the canonical norm

$$\|U\|_{X_T^s} \equiv \sup_{t \in [0, T/\max(\epsilon, \beta)]} |U(t, \cdot)|_{X^s} + \text{ess sup}_{t \in [0, T/\max(\epsilon, \beta)], x \in \mathbb{R}} |\partial_t U(t, x)|.$$

A natural energy for the initial value problem (5.8) is now given by:

$$E^s(U)^2 = (\Lambda^s U, Z[\underline{U}] \Lambda^s U) = (\Lambda^s \zeta, \frac{Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v})}{f(\epsilon \underline{\zeta})} \Lambda^s \zeta) + (\Lambda^s v, \mathfrak{T}[\epsilon \underline{\zeta}, \beta b] \Lambda^s v). \quad (5.11)$$

In order to ensure the equivalency of  $X^s$  with the energy of the pseudo-symmetrizer it requires to add the additional assumption given in (H3).

**Lemma 5.12** Let  $\mathbf{p} = (\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{CH}}$ ,  $s \geq 0$ ,  $\underline{U} \in L^\infty(\mathbb{R})$  and  $b \in W^{2,\infty}(\mathbb{R})$ , satisfying (H1), (H2), and (H3). Then  $E^s(U)$  is equivalent to the  $|\cdot|_{X^s}$ -norm.

More precisely, there exists  $c_0 = C(M_{\text{CH}}, h_{01}^{-1}, h_{02}^{-1}, h_{03}^{-1}, \epsilon |\underline{U}|_{L^\infty}, \beta |b|_{W^{2,\infty}}) > 0$  such that

$$\frac{1}{c_0} E^s(U) \leq |U|_{X^s} \leq c_0 E^s(U).$$

*Proof.*

This is a straightforward application of Lemma 4.3, and that for  $Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v}) \geq h_{03} > 0$  and  $f(\epsilon \underline{\zeta}) > 0$ ,

$$\inf_{x \in \mathbb{R}} \frac{Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v})}{f(\epsilon \underline{\zeta})} \geq \inf_{x \in \mathbb{R}} \left( Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v}) \right) \left( \sup_{x \in \mathbb{R}} f(\epsilon \underline{\zeta}) \right)^{-1},$$

$$\sup_{x \in \mathbb{R}} \frac{Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v})}{f(\epsilon \underline{\zeta})} \leq \sup_{x \in \mathbb{R}} \left( Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v}) \right) \left( \inf_{x \in \mathbb{R}} f(\epsilon \underline{\zeta}) \right)^{-1}.$$

where we recall that if (H1) is satisfied then,  $\underline{h}_1 = 1 - \epsilon \underline{\zeta}$ ,  $\underline{h}_2 = 1/\delta + \epsilon \underline{\zeta} - \beta b$  satisfy

$$\inf_{x \in \mathbb{R}} \underline{h}_1 \geq h_{01}, \quad \sup_{x \in \mathbb{R}} |\underline{h}_1| \leq 1 + 1/\delta, \quad \inf_{x \in \mathbb{R}} \underline{h}_2 \geq h_{01}, \quad \sup_{x \in \mathbb{R}} |\underline{h}_2| \leq 1 + 1/\delta.$$

□

**Lemma 5.13** *Let  $\mathbf{p} = (\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{CH}}$ , and let  $U = (\zeta_u, u)^\top \in L^\infty$ ,  $b \in W^{2,\infty}$  satisfies (H1), (H2) and (H3). Then for any  $V, W \in X^0$ , one has*

$$\left| \left( Z[U]V, W \right) \right| \leq C |V|_{X^0} |W|_{X^0}, \quad (5.14)$$

with  $C = C(M_{\text{CH}}, h_{01}^{-1}, h_{02}^{-1}, \epsilon |U|_{L^\infty}, \beta |b|_{W^{2,\infty}})$ .

Moreover, if  $U \in X^s$ ,  $b \in H^{s+2}$ ,  $V \in X^{s-1}$  with  $s \geq s_0 + 1$ ,  $s_0 > 1/2$ , then one has

$$\left| \left( [\Lambda^s, Z[U]]V, W \right) \right| \leq C |V|_{X^{s-1}} |W|_{X^0} \quad (5.15)$$

$$\left| \left( [\Lambda^s, Z^{-1}[U]]V, Z[U]W \right) \right| \leq C |V|_{H^{s-1} \times H^{s-1}} |W|_{X^0} \quad (5.16)$$

with  $C = C(M_{\text{CH}}, h_{01}^{-1}, h_{02}^{-1}, \epsilon |U|_{X^s}, \beta |b|_{H^{s+2}})$ .

*Proof.*

The proof of the Lemma 5.13 is the same as in [17, Lemma 6.4] adapted to our pseudo-symmetrizer. □

## 5.2 Energy estimates

Our aim is to establish *a priori* energy estimates concerning our linear system. In order to be able to use the linear analysis to both the well-posedness and stability of the nonlinear system, we consider the following modified system

$$\begin{cases} \partial_t U + A_0[\underline{U}] \partial_x U + A_1[\underline{U}] \partial_x U + B[\underline{U}] = F; \\ U|_{t=0} = U_0. \end{cases} \quad (5.17)$$

where we added a right-hand-side  $F$ , whose properties will be precised in the following Lemmas.

We begin by asserting a basic  $X^0$  energy estimate, that we extend to  $X^s$  space ( $s > 3/2$ ) later on.

In the analysis below, the additional assumption  $\epsilon \leq M\sqrt{\mu}$  in  $\mathbf{p} \in \mathcal{P}_{\text{CH}}$  is not used anymore (apart from the simplifications it offers when constructing system (5.1)).

**Lemma 5.18 ( $X^0$  energy estimate)** *Set  $(\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{CH}}$ . Let  $T > 0$  and  $U \in L^\infty([0, T/\max(\epsilon, \beta)]; X^0)$  and  $\underline{U}, \partial_x \underline{U} \in L^\infty([0, T/\max(\epsilon, \beta)] \times \mathbb{R})$  and  $b \in W^{3,\infty}$  such that  $\partial_t \underline{U} \in L^\infty([0, T/\max(\epsilon, \beta)] \times \mathbb{R})$  and  $\underline{U}, b$  satisfies (H1), (H2), and (H3) and  $U, \underline{U}$  satisfy system (5.17), with a right hand side,  $F$ , such that*

$$(F, Z[\underline{U}]U) \leq C_F \max(\epsilon, \beta) |U|_{X^0}^2 + f(t) |U|_{X^0},$$

with  $C_F$  a constant and  $f$  a positive integrable function on  $[0, T/\max(\epsilon, \beta)]$ .

Then there exists  $\lambda, C_1 \equiv C(\|\partial_t \underline{U}\|_{L^\infty}, \|\underline{U}\|_{L^\infty}, \|\partial_x \underline{U}\|_{L^\infty}, \|b\|_{W^{3,\infty}}, C_F)$  such that

$$\forall t \in [0, \frac{T}{\max(\epsilon, \beta)}], \quad E^0(U)(t) \leq e^{\max(\epsilon, \beta)\lambda t} E^0(U_0) + \int_0^t e^{\max(\epsilon, \beta)\lambda(t-t')} (f(t') + \max(\epsilon, \beta) C_1) dt', \quad (5.19)$$

The constants  $\lambda$  and  $C_1$  are independent of  $\mathbf{p} = (\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{CH}}$ , but depend on  $M_{\text{CH}}, h_{01}^{-1}, h_{02}^{-1}$ , and  $h_{03}^{-1}$ .

*Proof.*

Let us take the inner product of (5.17) by  $Z[\underline{U}]U$ :

$$(\partial_t U, Z[\underline{U}]U) + (A_0[\underline{U}]\partial_x U, Z[\underline{U}]U) + (A_1[\underline{U}]\partial_x U, Z[\underline{U}]U) + (B[\underline{U}], Z[\underline{U}]U) = (F, Z[\underline{U}]U),$$

From the symmetry property of  $Z[\underline{U}]$ , and using the definition of  $E^s(U)$ , one deduces

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E^0(U)^2 &= \frac{1}{2} (U, [\partial_t, Z[\underline{U}]]U) - (Z[\underline{U}]A_0[\underline{U}]\partial_x U, U) - (Z[\underline{U}]A_1[\underline{U}]\partial_x U, U) \\ &\quad - (B[\underline{U}], Z[\underline{U}]U) + (F, Z[\underline{U}]U). \end{aligned} \quad (5.20)$$

Let us first estimate  $(B[\underline{U}], Z[\underline{U}]U)$ . One has

$$\begin{aligned} (B[\underline{U}], Z[\underline{U}]U) &= \left( -g(\epsilon \underline{\zeta}) \underline{v} \beta \partial_x b, \frac{Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v})}{f(\epsilon \underline{\zeta})} \zeta \right) \\ &\quad + \left( \epsilon \mathfrak{T}^{-1} \left( \frac{\gamma \beta q_1(\epsilon \underline{\zeta}, \beta b) h_1 (h_1 + h_2) \underline{v}^2 \partial_x b}{(h_1 + \gamma h_2)^3} \right), \mathfrak{T}[\epsilon \underline{\zeta}, \beta b] v \right) \\ &\equiv A_1 + A_2. \end{aligned}$$

$$|A_1| \leq \beta C(\|\underline{U}\|_{L^\infty}, \|b\|_{W^{2,\infty}}) |U|_{X^0}.$$

In order to control  $A_2$ , using the symmetry property of  $\mathfrak{T}[\epsilon \underline{\zeta}]$ , we write

$$\left( \epsilon \mathfrak{T}^{-1} \left( \frac{\gamma \beta q_1(\epsilon \underline{\zeta}, \beta b) h_1 (h_1 + h_2) \underline{v}^2 \partial_x b}{(h_1 + \gamma h_2)^3} \right), \mathfrak{T}[\epsilon \underline{\zeta}, \beta b] v \right) = \epsilon \left( \frac{\gamma \beta q_1(\epsilon \underline{\zeta}, \beta b) h_1 (h_1 + h_2) \underline{v}^2 \partial_x b}{(h_1 + \gamma h_2)^3}, v \right).$$

Using the Cauchy-Schwartz inequality, one deduces

$$|A_2| \leq \beta C(\|\underline{U}\|_{L^\infty}, \|b\|_{W^{1,\infty}}) |U|_{X^0}.$$

Altogether, one has

$$(B[\underline{U}], Z[\underline{U}]U) \leq \beta C_1 |U|_{X^0} \leq \max(\epsilon, \beta) C_1 |U|_{X^0}. \quad (5.21)$$

Now we have,

$$Z[\underline{U}]A_0[\underline{U}] = \begin{pmatrix} \epsilon \frac{Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v})}{f(\epsilon \underline{\zeta})} f'(\epsilon \underline{\zeta}) \underline{v} & Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v}) \\ Q_0(\epsilon \underline{\zeta}, \beta b) & \epsilon \mathfrak{N}[\epsilon \underline{\zeta}, \beta b, \underline{v}] \end{pmatrix}$$

and

$$Z[\underline{U}]A_1[\underline{U}] = \begin{pmatrix} 0 & 0 \\ \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v}) & \epsilon \varsigma \mathfrak{T}[\epsilon \underline{\zeta}, \beta b](\underline{v}.) \end{pmatrix}.$$

One has,

$$\begin{aligned} (Z[\underline{U}]A_0[\underline{U}]\partial_x U, U) &= \left( \epsilon \frac{(Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v}))}{f(\epsilon \underline{\zeta})} f'(\epsilon \underline{\zeta}) \underline{v} \partial_x \zeta, \zeta \right) \\ &\quad + \left( Q_0(\epsilon \underline{\zeta}, \beta b) \partial_x v, \zeta \right) + \left( \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v}) \partial_x v, \zeta \right) \\ &\quad + \left( Q_0(\epsilon \underline{\zeta}, \beta b) \partial_x \zeta, v \right) + \left( \epsilon \mathfrak{N}[\epsilon \underline{\zeta}, \beta b, \underline{v}] \partial_x v, v \right) \end{aligned}$$

and

$$(Z[\underline{U}]A_1[\underline{U}]\partial_x U, U) = \left( \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v}) \partial_x \zeta, v \right) + \epsilon \varsigma \left( \mathfrak{T}[\epsilon \underline{\zeta}, \beta b](\underline{v} \partial_x v), v \right).$$



So that,

$$\begin{aligned}
(Z[\underline{U}]A_0[\underline{U}]\partial_x U, U) + (Z[\underline{U}]A_1[\underline{U}]\partial_x U, U) &= \left( \epsilon \frac{Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v})}{f(\epsilon \underline{\zeta})} f'(\epsilon \underline{\zeta}) \underline{v} \partial_x \zeta, \zeta \right) \\
&+ \left( Q_0(\epsilon \underline{\zeta}, \beta b) \partial_x v, \zeta \right) + \left( \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v}) \partial_x v, \zeta \right) \\
&+ \left( Q_0(\epsilon \underline{\zeta}, \beta b) \partial_x \zeta, v \right) + \left( \epsilon \mathfrak{Q}[\epsilon \underline{\zeta}, \beta b, \underline{v}] \partial_x v, v \right) \\
&+ \left( \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v}) \partial_x \zeta, v \right) + \epsilon \varsigma \left( \mathfrak{T}[\epsilon \underline{\zeta}, \beta b](\underline{v} \partial_x v), v \right).
\end{aligned}$$

One deduces that,

$$\begin{aligned}
(Z[\underline{U}]A_0[\underline{U}]\partial_x U, U) + (Z[\underline{U}]A_1[\underline{U}]\partial_x U, U) &= -\frac{1}{2} \left( \epsilon \partial_x \left( \frac{Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v})}{f(\epsilon \underline{\zeta})} f'(\epsilon \underline{\zeta}) \underline{v} \right) \zeta, \zeta \right) \\
&- \left( \partial_x (Q_0(\epsilon \underline{\zeta}, \beta b)) \zeta, v \right) - \epsilon^2 \left( \partial_x (Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v})) \zeta, v \right) \\
&+ \left( \epsilon \mathfrak{Q}[\epsilon \underline{\zeta}, \beta b, \underline{v}] \partial_x v, v \right) + \epsilon \varsigma \left( \mathfrak{T}[\epsilon \underline{\zeta}, \beta b](\underline{v} \partial_x v), v \right).
\end{aligned}$$

One can easily remark that we didn't use the smallness assumption of the Camassa-Holm regime  $\epsilon = \mathcal{O}(\sqrt{\mu})$  since we do not have anymore  $\partial_x v$  in the third term of the above identity.

One make use of the identity below,

$$\begin{aligned}
(\mathfrak{T}[\epsilon \underline{\zeta}, \beta b](\underline{v} \partial_x V), V) &= (q_1(\epsilon \underline{\zeta}, \beta b) \underline{v} \partial_x V - \mu \nu \partial_x (q_2(\epsilon \underline{\zeta}, \beta b) \partial_x (\underline{v} \partial_x V)), V) \\
&= -\frac{1}{2} \left( \partial_x (q_1(\epsilon \underline{\zeta}, \beta b) \underline{v}) V, V \right) + \mu \nu \left( q_2(\epsilon \underline{\zeta}, \beta b) \partial_x (\underline{v} \partial_x V), \partial_x V \right) \\
&= -\frac{1}{2} \left( \partial_x (q_1(\epsilon \underline{\zeta}, \beta b) \underline{v}) V, V \right) + \mu \nu \left( q_2(\epsilon \underline{\zeta}, \beta b) (\partial_x \underline{v}) \partial_x V, \partial_x V \right) \\
&\quad - \mu \nu \frac{1}{2} \left( \partial_x (q_2(\epsilon \underline{\zeta}, \beta b) \underline{v}) \partial_x V, \partial_x V \right).
\end{aligned}$$

Using the same techniques as in [17, Lemma 6.5] we obtains,

$$\left| (Z[\underline{U}]A_0[\underline{U}]\partial_x U, U) + (Z[\underline{U}]A_1[\underline{U}]\partial_x U, U) \right| \leq \max(\epsilon, \beta) C \left( \|\underline{U}\|_{L^\infty} + \|\partial_x \underline{U}\|_{L^\infty} + \|b\|_{W^{3,\infty}} \right) |U|_{X^0}^2. \quad (5.22)$$

The last term to estimate is  $(U, [\partial_t, Z[\underline{U}]]U)$ .

One has

$$\begin{aligned}
(U, [\partial_t, Z[\underline{U}]]U) &\equiv (v, [\partial_t, \mathfrak{T}]v) + \left( \zeta, \left[ \partial_t, \frac{Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v})}{f(\epsilon \underline{\zeta})} \right] \zeta \right) \\
&= \left( v, (\partial_t q_1(\epsilon \underline{\zeta}, \beta b)) v \right) - \mu \nu \left( v, \partial_x ((\partial_t q_2(\epsilon \underline{\zeta}, \beta b)) (\partial_x v)) \right) \\
&\quad + \left( \zeta, \partial_t \left( \frac{Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v})}{f(\epsilon \underline{\zeta})} \right) \zeta \right).
\end{aligned}$$

From Cauchy-Schwarz inequality and since  $\underline{\zeta}$  and  $b$  satisfies (H1), one deduces

$$\begin{aligned}
\left| \frac{1}{2} (U, [\partial_t, Z[\underline{U}]]U) \right| &\leq \epsilon C (\|\partial_t \underline{U}\|_{L^\infty}, \|\underline{U}\|_{L^\infty}) |U|_{X^0}^2 \\
&\leq \max(\epsilon, \beta) C (\|\partial_t \underline{U}\|_{L^\infty}, \|\underline{U}\|_{L^\infty}) |U|_{X^0}^2.
\end{aligned} \quad (5.23)$$

One can now conclude with the proof of the  $X^0$  energy estimate. Plugging (5.21), (5.22) and (5.23) into (5.20), and making use of the assumption of the Lemma on F. This yields

$$\frac{1}{2} \frac{d}{dt} E^0(U)^2 \leq \max(\epsilon, \beta) C_1 E^0(U)^2 + \left( f(t) + \max(\epsilon, \beta) C_1 \right) E^0(U),$$

where  $C_1 \equiv C(\|\partial_t \underline{U}\|_{L^\infty}, \|\underline{U}\|_{L^\infty}, \|\partial_x \underline{U}\|_{L^\infty}, \|b\|_{W^{3,\infty}}, C_F)$ . Consequently

$$\frac{d}{dt} E^0(U) \leq \max(\epsilon, \beta) C_1 E^0(U) + \left( f(t) + \max(\epsilon, \beta) C_1 \right).$$

Making use of the usual trick, we compute for any  $\lambda \in \mathbb{R}$ ,

$$e^{\max(\epsilon, \beta)\lambda t} \partial_t (e^{-\max(\epsilon, \beta)\lambda t} E^0(U)) = -\max(\epsilon, \beta)\lambda E^0(U) + \frac{d}{dt} E^0(U).$$

Thanks to the above inequality, one can choose  $\lambda = C_1$ , so that for all  $t \in [0, \frac{T}{\max(\epsilon, \beta)}]$ , one deduces

$$\frac{d}{dt} (e^{-\max(\epsilon, \beta)\lambda t} E^0(U)) \leq \left( f(t) + \max(\epsilon, \beta) C_1 \right) e^{-\max(\epsilon, \beta)\lambda t}.$$

Integrating this differential inequality yields

$$\forall t \in [0, \frac{T}{\max(\epsilon, \beta)}], \quad E^0(U)(t) \leq e^{\max(\epsilon, \beta)\lambda t} E^0(U_0) + \int_0^t e^{\max(\epsilon, \beta)\lambda(t-t')} \left( f(t') + \max(\epsilon, \beta) C_1 \right) dt'. \quad (5.24)$$

This proves the energy estimate (5.19).  $\square$

Let us now turn to the *a priori* energy estimate in “large”  $X^s$  norm.

**Lemma 5.25 (  $X^s$  energy estimate )** *Set  $(\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{CH}}$ , and  $s \geq s_0 + 1$ ,  $s_0 > 1/2$ . Let  $U = (\zeta, v)^\top$  and  $\underline{U} = (\underline{\zeta}, \underline{v})^\top$  be such that  $U, \underline{U} \in L^\infty([0, T/\max(\epsilon, \beta)]; X^s)$ ,  $\partial_t \underline{U} \in L^\infty([0, T/\max(\epsilon, \beta)] \times \mathbb{R})$ ,  $b \in H^{s+2}$  and  $\underline{U}$  satisfies (H1),(H2), and (H3) uniformly on  $[0, T/\max(\epsilon, \beta)]$ , and such that system (5.17) holds with a right hand side,  $F$ , with*

$$(\Lambda^s F, Z[\underline{U}] \Lambda^s U) \leq C_F \max(\epsilon, \beta) |U|_{X^s}^2 + f(t) |U|_{X^s},$$

where  $C_F$  is a constant and  $f$  is an integrable function on  $[0, T/\max(\epsilon, \beta)]$ .

Then there exists  $\lambda, C_2 = C(\|\underline{U}\|_{X_T^s}, \|b\|_{H^{s+2}}, C_F)$  such that the following energy estimate holds:

$$E^s(U)(t) \leq e^{\max(\epsilon, \beta)\lambda t} E^s(U_0) + \int_0^t e^{\max(\epsilon, \beta)\lambda(t-t')} (f(t') + \max(\epsilon, \beta) C_2) dt', \quad (5.26)$$

The constants  $\lambda$  and  $C_2$  are independent of  $\mathbf{p} = (\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{CH}}$ , but depend on  $M_{\text{CH}}, h_{01}^{-1}, h_{02}^{-1}$ , and  $h_{03}^{-1}$ .

**Remark 5.27** *In this Lemma, and in the proof below, the norm  $\|\underline{U}\|_{X_T^s}$  is to be understood as essential sup:*

$$\|U\|_{X_T^s} \equiv \text{ess sup}_{t \in [0, T/\epsilon]} |U(t, \cdot)|_{X^s} + \text{ess sup}_{t \in [0, T/\epsilon], x \in \mathbb{R}} |\partial_t U(t, x)|.$$

*Proof.*

Let us multiply the system (5.17) on the right by  $\Lambda^s Z[\underline{U}] \Lambda^s U$ , and integrate by parts. One obtains

$$\begin{aligned} (\Lambda^s \partial_t U, Z[\underline{U}] \Lambda^s U) + (\Lambda^s A_0[\underline{U}] \partial_x U, Z[\underline{U}] \Lambda^s U) + (\Lambda^s A_1[\underline{U}] \partial_x U, Z[\underline{U}] \Lambda^s U) + (\Lambda^s B[\underline{U}], Z[\underline{U}] \Lambda^s U) \\ = (\Lambda^s F, Z[\underline{U}] \Lambda^s U), \end{aligned} \quad (5.28)$$

from which we deduce, using the symmetry property of  $Z[\underline{U}]$ , as well as the definition of  $E^s(U)$ :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E^s(U)^2 &= \frac{1}{2} (\Lambda^s U, [\partial_t, Z[\underline{U}]] \Lambda^s U) - (Z[\underline{U}] A_0[\underline{U}] \partial_x \Lambda^s U, \Lambda^s U) \\ &\quad - (Z[\underline{U}] A_1[\underline{U}] \partial_x \Lambda^s U, \Lambda^s U) - ([\Lambda^s, A_0[\underline{U}] + A_1[\underline{U}]] \partial_x U, Z[\underline{U}] \Lambda^s U) \\ &\quad - (\Lambda^s B[\underline{U}], Z[\underline{U}] \Lambda^s U) + (\Lambda^s F, Z[\underline{U}] \Lambda^s U). \end{aligned} \quad (5.29)$$

We now estimate each of the different components of the r.h.s of the above identity.

- *Estimate of  $(\Lambda^s B[\underline{U}], Z[\underline{U}] \Lambda^s U)$ ,*

$$\begin{aligned} (\Lambda^s B[\underline{U}], Z[\underline{U}] \Lambda^s U) &= \left( \Lambda^s \left( -g(\epsilon \underline{\zeta}) \underline{v} \beta \partial_x b \right), \frac{Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v})}{f(\epsilon \underline{\zeta})} \Lambda^s \zeta \right) \\ &\quad + \epsilon \left( \Lambda^s \underline{\mathfrak{T}}^{-1} \left( \frac{\gamma \beta q_1(\epsilon \underline{\zeta}, \beta b) h_1 (h_1 + h_2) \underline{v}^2}{(h_1 + \gamma h_2)^3} \partial_x b \right), \mathfrak{T}[\epsilon \underline{\zeta}, \beta b] \Lambda^s v \right). \end{aligned}$$

Using Cauchy-Schwartz inequality, Lemma (4.3) and Lemma (4.8) one has,

$$|(\Lambda^s B[\underline{U}], Z[\underline{U}] \Lambda^s U)| \leq \beta C(\|\underline{U}\|_{X_T^s}, \|b\|_{H^{s+1}}) |U|_{X^s} \leq \max(\epsilon, \beta) C_2 |U|_{X^s}. \quad (5.30)$$

- *Estimate of  $(Z[\underline{U}] A_0[\underline{U}] \partial_x \Lambda^s U, \Lambda^s U) + (Z[\underline{U}] A_1[\underline{U}] \partial_x \Lambda^s U, \Lambda^s U)$ .*

Thanks to Sobolev embedding, one has for  $s > s_0 + 1$ ,  $s_0 > 1/2$

$$C(\|\underline{U}\|_{L^\infty} + \|\partial_x \underline{U}\|_{L^\infty}) \leq C(\|\underline{U}\|_{X_T^s}),$$

One can use the  $L^2$  estimate derived in (5.22), applied to  $\Lambda^s U$ . One deduces

$$\left| (Z[\underline{U}] A_0[\underline{U}] \partial_x \Lambda^s U, \Lambda^s U) + (Z[\underline{U}] A_1[\underline{U}] \partial_x \Lambda^s U, \Lambda^s U) \right| \leq \max(\epsilon, \beta) C(\|\underline{U}\|_{X_T^s} + \|b\|_{W^{3,\infty}}) |U|_{X^s}^2. \quad (5.31)$$

- *Estimate of  $([\Lambda^s, A[\underline{U}]] \partial_x U, Z[\underline{U}] \Lambda^s U)$ , where  $A[\underline{U}] = A_0[\underline{U}] + A_1[\underline{U}]$ . Using the definition of  $A[\cdot]$  and  $Z[\cdot]$  in (5.4) and (5.9), one has*

$$\begin{aligned} ([\Lambda^s, A[\underline{U}]] \partial_x U, Z[\underline{U}] \Lambda^s U) &= \left( [\Lambda^s, \epsilon f'(\epsilon \underline{\zeta}) \underline{v}] \partial_x \zeta + [\Lambda^s, f(\epsilon \underline{\zeta})] \partial_x v, \frac{Q(\epsilon \underline{\zeta}, \beta b, \underline{v})}{f(\epsilon \underline{\zeta})} \Lambda^s \zeta \right) \\ &\quad + \left( [\Lambda^s, \underline{\mathfrak{T}}^{-1}(Q(\epsilon \underline{\zeta}, \beta b, \underline{v}))] \partial_x \zeta, \underline{\mathfrak{T}} \Lambda^s v \right) + \epsilon \left( [\Lambda^s, \underline{\mathfrak{T}}^{-1} \mathfrak{Q}[\epsilon \underline{\zeta}, \beta b, \underline{v}] + \varsigma \underline{v}] \partial_x v, \underline{\mathfrak{T}} \Lambda^s v \right). \end{aligned}$$

Here and in the following, we denote  $\underline{\mathfrak{T}} \equiv \mathfrak{T}[\epsilon \underline{\zeta}, \beta b]$  and  $Q(\epsilon \underline{\zeta}, \beta b, \underline{v}) = Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v})$ .

Using the same techniques as in [17, Lemma 6.6] and since  $\underline{\zeta}$  and  $b$  satisfies (H1), we proved

$$\left| ([\Lambda^s, A[\underline{U}]] \partial_x U, Z[\underline{U}] \Lambda^s U) \right| \leq \max(\epsilon, \beta) C(\|\underline{U}\|_{X_T^s} + \|b\|_{H^{s+2}}) |U|_{X^s}^2. \quad (5.32)$$

- *Estimate of  $\frac{1}{2}(\Lambda^s U, [\partial_t, Z[\underline{U}]] \Lambda^s U)$ .*

One has

$$\begin{aligned} (\Lambda^s U, [\partial_t, Z[\underline{U}]] \Lambda^s U) &\equiv (\Lambda^s v, [\partial_t, \underline{\mathfrak{T}}] \Lambda^s v) + (\Lambda^s \zeta, [\partial_t, \frac{Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v})}{f(\epsilon \underline{\zeta})}] \Lambda^s \zeta) \\ &= \left( \Lambda^s v, (\partial_t q_1(\epsilon \underline{\zeta}, \beta b)) \Lambda^s v \right) - \mu \nu \left( \Lambda^s v, \partial_x ((\partial_t q_2(\epsilon \underline{\zeta}, \beta b)) (\partial_x \Lambda^s v)) \right) \\ &\quad + \left( \Lambda^s \zeta, \partial_t \left( \frac{Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v})}{f(\epsilon \underline{\zeta})} \right) \Lambda^s \zeta \right) \\ &= \epsilon \kappa_1 \left( \Lambda^s v, (\partial_t \underline{\zeta}) \Lambda^s v \right) + \mu \nu \epsilon \kappa_2 \left( \Lambda^s \partial_x v, (\partial_t \underline{\zeta}) \Lambda^s \partial_x v \right) \\ &\quad + \left( \Lambda^s \zeta, \partial_t \left( \frac{Q_0(\epsilon \underline{\zeta}, \beta b) + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v})}{f(\epsilon \underline{\zeta})} \right) \Lambda^s \zeta \right). \end{aligned}$$

From Cauchy-Schwarz inequality and since  $\zeta$  and  $b$  satisfies (H1), one deduces

$$\begin{aligned} \left| \frac{1}{2}(\Lambda^s U, [\partial_t, Z[\underline{U}]] \Lambda^s U) \right| &\leq \epsilon C(\|\partial_t \underline{U}\|_{L^\infty}, \|\underline{U}\|_{L^\infty}) |U|_{X^s}^2 \\ &\leq \max(\epsilon, \beta) C(\|\partial_t \underline{U}\|_{L^\infty}, \|\underline{U}\|_{L^\infty}) |U|_{X^s}^2. \end{aligned}$$

and continuous Sobolev embedding yields,

$$\left| \frac{1}{2}(\Lambda^s U, [\partial_t, Z[\underline{U}]] \Lambda^s U) \right| \leq \max(\epsilon, \beta) C(\|\underline{U}\|_{X_T^s}) |U|_{X^s}^2. \quad (5.33)$$

One can now conclude the proof of the  $X^s$  energy estimate. Plugging (5.30), (5.31), (5.32) and (5.33) into (5.29), and making use of the assumption of the Lemma on F.

$$\frac{1}{2} \frac{d}{dt} E^s(U)^2 \leq \max(\epsilon, \beta) C_2 E^s(U)^2 + E^s(U) (f(t) + \max(\epsilon, \beta) C_2),$$

with  $C_2 = C(\|\underline{U}\|_{X_T^s}, \|b\|_{H^{s+2}}, C_F)$ , and consequently

$$\frac{d}{dt} E^s(U) \leq \max(\epsilon, \beta) C_2 E^s(U) + (f(t) + \max(\epsilon, \beta) C_2).$$

Making use of the usual trick, we compute for any  $\lambda \in \mathbb{R}$ ,

$$e^{\max(\epsilon, \beta) \lambda t} \partial_t (e^{-\max(\epsilon, \beta) \lambda t} E^s(U)) = -\max(\epsilon, \beta) \lambda E^s(U) + \frac{d}{dt} E^s(U).$$

Thus with  $\lambda = C_2$ , one has for all  $t \in [0, \frac{T}{\max(\epsilon, \beta)}]$ ,

$$\frac{d}{dt} (e^{-\max(\epsilon, \beta) \lambda t} E^s(U)) \leq (f(t) + \max(\epsilon, \beta) C_2) e^{-\max(\epsilon, \beta) \lambda t}.$$

Integrating this differential inequality yields,

$$E^s(U)(t) \leq e^{\max(\epsilon, \beta) \lambda t} E^s(U_0) + \int_0^t e^{\max(\epsilon, \beta) \lambda (t-t')} (f(t') + \max(\epsilon, \beta) C_2) dt'.$$

□

### 5.3 Well-posedness of the linear system

**Proposition 5.34** *Let  $\mathbf{p} = (\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{CH}}$  and  $s \geq s_0 + 1$  with  $s_0 > 1/2$ , and let  $\underline{U} = (\zeta, v)^\top \in X_T^s$  (see Definition 5.10),  $b \in H^{s+2}$  be such that (H1), (H2), and (H3) are satisfied for  $t \in [0, T/\max(\epsilon, \beta)]$ , uniformly with respect to  $\mathbf{p} \in \mathcal{P}_{\text{CH}}$ . For any  $U_0 \in X^s$ , there exists a unique solution to (5.8),  $U^{\mathbf{p}} \in C^0([0, T/\max(\epsilon, \beta)]; X^s) \cap C^1([0, T/\max(\epsilon, \beta)]; X^{s-1}) \subset X_T^s$ , with  $\lambda_T, C_0 = C(\|\underline{U}\|_{X_T^s}, T, M_{\text{CH}}, h_{01}^{-1}, h_{02}^{-1}, h_{03}^{-1}, \|b\|_{H^{s+2}})$ , independent of  $\mathbf{p} \in \mathcal{P}_{\text{CH}}$ , such that one has the energy estimates*

$$\forall 0 \leq t \leq \frac{T}{\max(\epsilon, \beta)}, \quad E^s(U^{\mathbf{p}})(t) \leq e^{\max(\epsilon, \beta) \lambda_T t} E^s(U_0) + \max(\epsilon, \beta) C_0 \int_0^t e^{\max(\epsilon, \beta) \lambda_T (t-t')} dt'$$

$$\text{and } E^{s-1}(\partial_t U^{\mathbf{p}}) \leq C_0 e^{\max(\epsilon, \beta) \lambda_T t} E^s(U_0) + \max(\epsilon, \beta) C_0^2 \int_0^t e^{\max(\epsilon, \beta) \lambda_T (t-t')} dt' + \max(\epsilon, \beta) C_0.$$

*Proof.*

Existence and uniqueness of a solution to the initial value problem (5.8) follows, by standard techniques, from the estimate (5.26) in Lemma 5.25:

$$E^s(U)(t) \leq e^{\max(\epsilon, \beta) \lambda_T t} E^s(U_0) + \max(\epsilon, \beta) C_0 \int_0^t e^{\max(\epsilon, \beta) \lambda_T (t-t')} dt', \quad (5.35)$$

(since  $F \equiv 0$ , and omitting the index  $\mathbf{p}$  for the sake of simplicity).

First, let us notice that using the system of equation (5.8), one can deduce an energy estimate on the time-derivative of the solution. Indeed, one has

$$\begin{aligned}
|\partial_t U|_{X^{s-1}} &= | -A_0[\underline{U}]\partial_x U - A_1[\underline{U}]\partial_x U - B[\underline{U}] |_{X^{s-1}} \\
&\leq | -\epsilon f'(\epsilon \underline{\zeta}) \underline{v} \partial_x \zeta - f(\epsilon \underline{\zeta}) \partial_x v + \beta \partial_x b g(\epsilon \underline{\zeta}) \underline{v} |_{H^{s-1}} \\
&\quad + |\mathfrak{T}[\epsilon \underline{\zeta}, \beta b]^{-1} \left( Q_0(\epsilon \underline{\zeta}, \beta b) \partial_x \zeta + \epsilon \mathfrak{Q}[\epsilon \underline{\zeta}, \beta b, \underline{v}] \partial_x v + \epsilon^2 Q_1(\epsilon \underline{\zeta}, \beta b, \underline{v}) \partial_x \zeta \right. \\
&\quad \left. + \epsilon \frac{\gamma \beta q_1(\epsilon \underline{\zeta}, \beta b) h_1 (h_1 + h_2) \underline{v}^2 \partial_x b}{(h_1 + \gamma h_2)^3} \right) + \epsilon \zeta \underline{v} \partial_x v |_{H_\mu^s} \\
&\leq C(|\underline{U}|_{X^s}, |b|_{H^{s+1}}) |U|_{X^s} + \beta C_0 \\
&\leq C_0 E^s(U)(t) + \beta C_0 \\
&\leq C_0 e^{\max(\epsilon, \beta) \lambda_T t} E^s(U_0) + \max(\epsilon, \beta) C_0^2 \int_0^t e^{\max(\epsilon, \beta) \lambda_T (t-t')} dt' + \max(\epsilon, \beta) C_0.
\end{aligned} \tag{5.36}$$

The completion of the proof is as follows. In order to construct a solution to (5.8), we use a sequence of Friedrichs mollifiers, defined by  $J_\nu \equiv (1 - \nu \partial_x^2)^{-1/2}$  ( $\nu > 0$ ), in order to reduce our system to ordinary differential equation systems on  $X^s$ , which are solved uniquely by Cauchy-Lipschitz theorem. Estimates (5.35), (5.36) hold for each  $U_\nu \in C^0([0, T/\max(\epsilon, \beta)]; X^s)$ , uniformly in  $\nu > 0$ . One deduces that a subsequence converges towards  $U \in L^2([0, T/\max(\epsilon, \beta)]; X^s)$ , a (weak) solution of the Cauchy problem (5.8). By regularizing the initial data as well, one can show that the system induces a smoothing effect in time, and that the solution  $U \in C^0([0, T/\max(\epsilon, \beta)]; X^s) \cap C^1([0, T/\max(\epsilon, \beta)]; X^{s-1})$  is actually a strong solution. The uniqueness is a straightforward consequence of (5.35) (with  $U_0 \equiv 0$ ) applied to the difference of two solutions.  $\square$

## 5.4 Priori estimate

In this subsection, we control the difference of two solutions of the nonlinear system, with different initial data and right-hand sides.

**Proposition 5.37** *Let  $(\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{\text{CH}}$  and  $s \geq s_0 + 1$ ,  $s_0 > 1/2$ , and assume that there exists  $U_i$  for  $i \in \{1, 2\}$ , such that  $U_i = (\zeta_i, v_i)^\top \in X_T^s$ ,  $U_2 \in L^\infty([0, T/\max(\epsilon, \beta)]; X^{s+1})$ ,  $b \in H^{s+2}$ ,  $U_1$  satisfy (H1), (H2) and (H3) on  $[0, T/\max(\epsilon, \beta)]$ , with  $h_{01}, h_{02}, h_{03} > 0$ , and  $U_i$  satisfy*

$$\begin{aligned}
\partial_t U_1 + A_0[U_1] \partial_x U_1 + A_1[U_1] \partial_x U_1 + B[U_1] &= F_1, \\
\partial_t U_2 + A_0[U_2] \partial_x U_2 + A_1[U_2] \partial_x U_2 + B[U_2] &= F_2,
\end{aligned}$$

with  $F_i \in L^1([0, T/\max(\epsilon, \beta)]; X^s)$ .

Then there exists constants  $C_0 = C(M_{\text{CH}}, h_{01}^{-1}, h_{02}^{-1}, h_{03}^{-1}, \max(\epsilon, \beta) |U_1|_{X^s}, \max(\epsilon, \beta) |U_2|_{X^s}, |b|_{H^{s+2}})$  and  $\lambda_T = (C_0 \times C(|U_2|_{L^\infty([0, T/\max(\epsilon, \beta)]; X^{s+1})}) + C_0)$  such that for all  $t \in [0, \frac{T}{\max(\epsilon, \beta)}]$ ,

$$E^s(U_1 - U_2)(t) \leq e^{\max(\epsilon, \beta) \lambda_T t} E^s(U_1|_{t=0} - U_2|_{t=0}) + C_0 \int_0^t e^{\max(\epsilon, \beta) \lambda_T (t-t')} E^s(F_1 - F_2)(t') dt'.$$

*Proof.*

When multiplying the equations satisfied by  $U_i$  on the left by  $Z[U_i]$ , one obtains

$$\begin{aligned}
Z[U_1] \partial_t U_1 + \Sigma_0[U_1] \partial_x U_1 + \Sigma_1[U_1] \partial_x U_1 + Z[U_1] B[U_1] &= Z[U_1] F_1 \\
Z[U_2] \partial_t U_2 + \Sigma_0[U_2] \partial_x U_2 + \Sigma_1[U_2] \partial_x U_2 + Z[U_2] B[U_2] &= Z[U_2] F_2;
\end{aligned}$$

with  $\Sigma_0[U] = Z[U]A_0[U]$  and  $\Sigma_1[U] = Z[U]A_1[U]$ . Subtracting the two equations above, and defining  $V = U_1 - U_2 \equiv (\zeta, v)^\top$  one obtains

$$\begin{aligned} Z[U_1]\partial_t V + \Sigma_0[U_1]\partial_x V + \Sigma_1[U_1]\partial_x V + (Z[U_1]B[U_1] - Z[U_2]B[U_2]) = \\ Z[U_1](F_1 - F_2) - (\Sigma_0[U_1] + \Sigma_1[U_1] - \Sigma_0[U_2] - \Sigma_1[U_2])\partial_x U_2 \\ - (Z[U_1] - Z[U_2])(\partial_t U_2 - F_2). \end{aligned}$$

We then apply  $Z^{-1}[U_1]$  and deduce the following system satisfied by  $V$ :

$$\begin{cases} \partial_t V + A_0[U_1]\partial_x V + A_1[U_1]\partial_x V + Z^{-1}[U_1](Z[U_1]B[U_1] - Z[U_2]B[U_2]) = F \\ V(0) = (U_1 - U_2)|_{t=0}, \end{cases} \quad (5.38)$$

$$\begin{aligned} \text{where, } F &\equiv F_1 - F_2 \\ &- Z^{-1}[U_1](\Sigma_0[U_1] + \Sigma_1[U_1] - \Sigma_0[U_2] - \Sigma_1[U_2])\partial_x U_2 \\ &- Z^{-1}[U_1](Z[U_1] - Z[U_2])(\partial_t U_2 - F_2). \end{aligned} \quad (5.39)$$

We wish to use the energy estimate of Lemma 5.25 to the linear system (5.38).

The additional term now is  $Z^{-1}[U_1](Z[U_1]B[U_1] - Z[U_2]B[U_2])$ .

So we have to control,

$$(\Lambda^s Z^{-1}[U_1](Z[U_1]B[U_1] - Z[U_2]B[U_2]), Z[U_1]\Lambda^s V) = B.$$

One has,

$$B = (\Lambda^s(Z[U_1]B[U_1] - Z[U_2]B[U_2]), \Lambda^s V) + ([\Lambda^s, Z^{-1}[U_1]]Z[U_1]B[U_1] - Z[U_2]B[U_2], Z[U_1]\Lambda^s V)$$

$$B = B_1 + B_2.$$

Now we have to estimate the terms  $(B_1)$  and  $(B_2)$ .

$$\begin{aligned} (B_1) &= \left( \Lambda^s \left( \frac{-Q(\epsilon\zeta_1, \beta b, v_1)\beta\partial_x b g(\epsilon\zeta_1)v_1}{f(\epsilon\zeta_1)} + \frac{Q(\epsilon\zeta_2, \beta b, v_2)\beta\partial_x b g(\epsilon\zeta_2)v_2}{f(\epsilon\zeta_2)} \right), \Lambda^s \zeta_v \right) \\ &+ \left( \Lambda^s \left( \frac{\epsilon\gamma\beta q_1(\epsilon\zeta_1, \beta b)h_1(h_1 + h_2)v_1^2\partial_x b}{(h_1 + \gamma h_2)^3} - \frac{\epsilon\gamma\beta q_1(\epsilon\zeta_2, \beta b)h_1(h_1 + h_2)v_2^2\partial_x b}{(h_1 + \gamma h_2)^3} \right), \Lambda^s v \right) \end{aligned}$$

With  $Q(\epsilon\zeta_i, \beta b, v_i) = Q_0(\epsilon\zeta_i, \beta b) + \epsilon^2 Q_1(\epsilon\zeta_i, \beta b, v_i)$  for  $i = 1, 2$ .

In order to control  $(B_1)$  we use the following decompositions,

$$\begin{aligned} \bullet &\left( \frac{-Q_0(\epsilon\zeta_1, \beta b)\beta\partial_x b g(\epsilon\zeta_1)v_1}{f(\epsilon\zeta_1)} + \frac{Q_0(\epsilon\zeta_2, \beta b)\beta\partial_x b g(\epsilon\zeta_2)v_2}{f(\epsilon\zeta_2)} \right) \\ &= \left( \frac{-Q_0(\epsilon\zeta_1, \beta b)g(\epsilon\zeta_1)}{f(\epsilon\zeta_1)} + \frac{Q_0(\epsilon\zeta_2, \beta b)g(\epsilon\zeta_2)}{f(\epsilon\zeta_2)} \right) (\beta\partial_x b v_1) \\ &\quad - \beta(v_1 - v_2) \frac{Q_0(\epsilon\zeta_2, \beta b)g(\epsilon\zeta_2)\partial_x b}{f(\epsilon\zeta_2)}. \\ \bullet &\beta \left( \frac{-\epsilon^2 Q_1(\epsilon\zeta_1, \beta b, v_1)\partial_x b g(\epsilon\zeta_1)v_1}{f(\epsilon\zeta_1)} + \frac{\epsilon^2 Q_1(\epsilon\zeta_2, \beta b, v_2)\partial_x b g(\epsilon\zeta_2)v_2}{f(\epsilon\zeta_2)} \right) \\ &= \left( \frac{-\epsilon^2 Q_1(\epsilon\zeta_1, \beta b, v_1)g(\epsilon\zeta_1)}{f(\epsilon\zeta_1)} + \frac{\epsilon^2 Q_1(\epsilon\zeta_2, \beta b, v_2)g(\epsilon\zeta_2)}{f(\epsilon\zeta_2)} \right) (\beta\partial_x b v_1) \\ &\quad - \beta(v_1 - v_2) \frac{\epsilon^2 Q_1(\epsilon\zeta_2, \beta b, v_2)g(\epsilon\zeta_2)\partial_x b}{f(\epsilon\zeta_2)}. \end{aligned}$$

$$\begin{aligned}
& \bullet \left( \frac{\epsilon \gamma \beta q_1(\epsilon \zeta_1, \beta b) h_1 (h_1 + h_2) v_1^2 \partial_x b}{(h_1 + \gamma h_2)^3} - \frac{\epsilon \gamma \beta q_1(\epsilon \zeta_2, \beta b) h_1 (h_1 + h_2) v_2^2 \partial_x b}{(h_1 + \gamma h_2)^3} \right) \\
&= \left( \frac{\epsilon v_1 \gamma q_1(\epsilon \zeta_1, \beta b) h_1 (h_1 + h_2)}{(h_1 + \gamma h_2)^3} - \frac{\epsilon v_1 \gamma q_1(\epsilon \zeta_2, \beta b) h_1 (h_1 + h_2)}{(h_1 + \gamma h_2)^3} \right) (\beta \partial_x b v_1) \\
&\quad + \left( \frac{\epsilon \gamma q_1(\epsilon \zeta_2, \beta b) h_1 (h_1 + h_2) \partial_x b}{(h_1 + \gamma h_2)^3} \right) \beta (v_1^2 - v_2^2).
\end{aligned}$$

Using the fact that,  $\epsilon^2 Q_1(\epsilon \zeta_i, \beta b, v_i) = Q_1(\epsilon \zeta_i, \beta b, \epsilon v_i)$  is a polynomial, one deduces,

$$\begin{aligned}
|B_1| &\leq C(\beta |v_1|_{H^s}, |b|_{H^{s+2}}) \epsilon |\zeta_1 - \zeta_2|_{H^s} |\zeta_v|_{H^s} + C(\epsilon |\zeta_2|_{H^s}, |b|_{H^{s+2}}) \beta |v_1 - v_2|_{H^s} |\zeta_v|_{H^s} \\
&+ C(\beta |v_1|_{H^s}, |b|_{H^{s+1}}) \epsilon |\zeta_1 - \zeta_2|_{H^s} |\zeta_v|_{H^s} + C(\epsilon |\zeta_2|_{H^s}, \epsilon |v_2|_{H^s}, |b|_{H^{s+1}}) \beta |v_1 - v_2|_{H^s} |\zeta_v|_{H^s} \\
&+ C(\beta |v_1|_{H^s}, \epsilon |v_1|_{H^s}, |b|_{H^{s+1}}) \epsilon |\zeta_1 - \zeta_2|_{H^s} |v|_{H^s} + C(\epsilon |\zeta_2|_{H^s}, \epsilon |v_1|_{H^s}, \epsilon |v_2|_{H^s}, |b|_{H^{s+1}}) \beta |v_1 - v_2|_{H^s} |v|_{H^s}. \\
&\leq \max(\epsilon, \beta) C_0 E^s(U_1 - U_2) E^s(V). \\
&\leq \max(\epsilon, \beta) C_0 E^s(V)^2.
\end{aligned}$$

with  $C_0 = C(M_{\text{CH}}, h^{-1}, h_{03}^{-1}, \max(\epsilon, \beta) |U_1|_{X^s}, \max(\epsilon, \beta) |U_2|_{X^s}, |b|_{H^{s+2}})$ .

The contribution of  $(B_2)$  is immediately bounded using Lemma 5.13:

$$\begin{aligned}
|B_2| &= \left( [\Lambda^s, Z^{-1}[U_1]] (Z[U_1]B[U_1] - Z[U_2]B[U_2]) , Z[U_1]\Lambda^s V \right) \\
&\leq C |Z[U_1]B[U_1] - Z[U_2]B[U_2]|_{H^{s-1} \times H^{s-1}} |V|_{X^s} \\
&\leq C \left( \left| \frac{-Q(\epsilon \zeta_1, \beta b, v_1) \beta \partial_x b g(\epsilon \zeta_1) v_1}{f(\epsilon \zeta_1)} + \frac{Q(\epsilon \zeta_2, \beta b, v_2) \beta \partial_x b g(\epsilon \zeta_2) v_2}{f(\epsilon \zeta_2)} \right|_{H^{s-1}} \right. \\
&\quad \left. + \left| \frac{\epsilon \gamma \beta q_1(\epsilon \zeta_1, \beta b) h_1 (h_1 + h_2) v_1 \partial_x b}{(h_1 + \gamma h_2)^3} - \frac{\epsilon \gamma \beta q_1(\epsilon \zeta_2, \beta b) h_1 (h_1 + h_2) v_2^2 \partial_x b}{(h_1 + \gamma h_2)^3} \right|_{H^{s-1}} \right) |V|_{X^s} \\
&\leq \max(\epsilon, \beta) C_0 E^s(U_1 - U_2) E^s(V). \\
&\leq \max(\epsilon, \beta) C_0 E^s(V)^2.
\end{aligned}$$

So we have,

$$|B| \leq C_0 \max(\epsilon, \beta) E^s(V)^2.$$

Now one needs to control accordingly the right hand side  $F$ .

In order to do so, we take advantage of the following Lemma.

**Lemma 5.40** *Let  $(\mu, \epsilon, \delta, \gamma, \beta, b_0) \in \mathcal{P}_{\text{CH}}$  and  $s \geq s_0 > 1/2$ . Let  $V = (\zeta_v, v)^\top$ ,  $W = (\zeta_w, w)^\top \in X^s$  and  $U_1 = (\zeta_1, v_1)^\top$ ,  $U_2 = (\zeta_2, v_2)^\top \in X^s$ ,  $b \in H^{s+1}$  such that there exists  $h > 0$  with*

$$1 - \epsilon \zeta_1 \geq h > 0, \quad 1 - \epsilon \zeta_2 \geq h > 0, \quad \frac{1}{\delta} + \epsilon \zeta_1 - \beta b \geq h > 0, \quad \frac{1}{\delta} + \epsilon \zeta_2 - \beta b \geq h > 0.$$

*Then one has*

$$\begin{aligned}
& \left| \left( \Lambda^s (Z[U_1] - Z[U_2]) V , W \right) \right| \leq \epsilon C |U_1 - U_2|_{X^s} |V|_{X^s} |W|_{X^0} \\
& \left( \Lambda^s (Z[U_1]A[U_1] - Z[U_2]A[U_2]) V , W \right) \leq \epsilon C |U_1 - U_2|_{X^s} |V|_{X^s} |W|_{X^0}
\end{aligned}$$

with  $C = C(M_{\text{CH}}, h^{-1}, \epsilon |U_1|_{X^s}, \epsilon |U_2|_{X^s}, |b|_{H^{s+2}})$ , and denoting  $A[\cdot] \equiv A_0[\cdot] + A_1[\cdot]$ .

*Proof.*

We prove the Lemma 5.40 using the same techniques as in the Proof of [17, Lemma 7.2], adapted to our

pseudo-symmetrizer as  $\epsilon^2 Q_1(\epsilon \zeta_i, \beta b, v_i) = Q_1(\epsilon \zeta_i, \beta b, \epsilon v_i)$  is a polynomial.  $\square$

Let us continue the proof of Proposition 5.37, by estimating  $F$  defined in (5.39).

More precisely we want to estimate

$$\begin{aligned} (\Lambda^s F, Z[U_1] \Lambda^s V) &= (\Lambda^s F_1 - \Lambda^s F_2, Z[U_1] \Lambda^s V) \\ &- (\Lambda^s (\Sigma_0[U_1] + \Sigma_1[U_1] - \Sigma_0[U_2] - \Sigma_1[U_2]) \partial_x U_2, \Lambda^s V) \\ &- (\Lambda^s, Z^{-1}[U_1]) (\Sigma_0[U_1] + \Sigma_1[U_1] - \Sigma_0[U_2] - \Sigma_1[U_2]) \partial_x U_2, Z[U_1] \Lambda^s V) \\ &- (\Lambda^s (Z[U_1] - Z[U_2]) (\partial_t U_2 - F_2), \Lambda^s V) \\ &- (\Lambda^s, Z^{-1}[U_1]) (Z[U_1] - Z[U_2]) (\partial_t U_2 - F_2), Z[U_1] \Lambda^s V). \end{aligned}$$

We adapt the proof given in [17, Proposition 7.1] to our pseudo-symmetrizer, we deduce that,

$$\begin{aligned} |(\Lambda^s F, Z[U_1] \Lambda^s V)| &\leq \epsilon C \times (|\partial_x U_2|_{X^s} + |\partial_t U_2 - F_2|_{X^s}) E^s(V)^2 + C E^s(V) E^s(F_1 - F_2) \\ &\leq \max(\epsilon, \beta) C \times (|\partial_x U_2|_{X^s} + |\partial_t U_2 - F_2|_{X^s}) E^s(V)^2 + C E^s(V) E^s(F_1 - F_2). \end{aligned}$$

with  $C = C(M_{CH}, h^{-1}, h_{03}^{-1}, \epsilon |U_1|_{X^s}, \epsilon |U_2|_{X^s}, |b|_{H^{s+2}})$ .

Then one has

$$|(\Lambda^s F, Z[U_1] \Lambda^s V)| \leq \max(\epsilon, \beta) C_0 \times (|\partial_x U_2|_{X^s} + |\partial_t U_2 - F_2|_{X^s}) E^s(V)^2 + C_0 E^s(V) E^s(F_1 - F_2).$$

We can now conclude by Lemma 5.25, and the proof of Proposition 5.37 is complete.  $\square$

## 6 Full justification of the asymptotic model

A model is said to be *fully justified* (using the terminology of [23]) if the Cauchy problem for both the full Euler system and the asymptotic model is well-posed for a given class of initial data, and over the relevant time scale; and if the solutions with corresponding initial data remain close. We conclude our work by stating all the ingredients for the full justification of our model.

**Theorem 6.1 (Existence and uniqueness)** *Let  $\mathbf{p} = (\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{CH} \setminus (\epsilon \leq M\sqrt{\mu})$  and  $s \geq s_0 + 1$ ,  $s_0 > 1/2$ , and assume  $U_0 = (\zeta_0, v_0)^\top \in X^s$ ,  $b \in H^{s+1}$  satisfies (H1), (H2), and (H3). Then there exists a maximal time  $T_{\max} > 0$ , uniformly bounded from below with respect to  $\mathbf{p} \in \mathcal{P}_{CH}$ , such that the system of equations (3.9) admits a unique strong solution  $U = (\zeta, v)^\top \in C^0([0, T_{\max}]; X^s) \cap C^1([0, T_{\max}]; X^{s-1})$  with the initial value  $(\zeta, v)|_{t=0} = (\zeta_0, v_0)$ , and preserving the conditions (H1), (H2) and (H3) (with different lower bounds) for any  $t \in [0, T_{\max})$ .*

*Moreover, there exists  $\lambda, C_0 = C(h_{01}^{-1}, h_{02}^{-1}, h_{03}^{-1}, M_{CH}, T, |U_0|_{X^s}, |b|_{H^{s+2}})$ , independent of  $\mathbf{p} \in \mathcal{P}_{CH}$ , such that  $T_{\max} \geq T / \max(\epsilon, \beta)$ , and one has the energy estimates*

$$\forall 0 \leq t \leq \frac{T}{\max(\epsilon, \beta)},$$

$$|U(t, \cdot)|_{X^s} + |\partial_t U(t, \cdot)|_{X^{s-1}} \leq C_0 e^{\max(\epsilon, \beta) \lambda t} + \max(\epsilon, \beta) C_0^2 \int_0^t e^{\max(\epsilon, \beta) \lambda (t-t')} dt' + \max(\epsilon, \beta) C_0$$

*If  $T_{\max} < \infty$ , one has*

$$|U(t, \cdot)|_{X^s} \longrightarrow \infty \quad \text{as } t \longrightarrow T_{\max},$$

*or one of the conditions (H1), (H2), (H3) ceases to be true as  $t \longrightarrow T_{\max}$ .*

*Proof.*

We construct a sequence of approximate solution  $(U^n = (\zeta^n, v^n))_{n \geq 0}$  through the induction relation

$$U^0 = U_0, \quad \text{and} \quad \forall n \in \mathbb{N}, \quad \begin{cases} \partial_t U^{n+1} + A[U^n] \partial_x U^{n+1} + B[U^n] = 0; \\ U^{n+1}|_{t=0} = U_0. \end{cases} \quad (6.2)$$



By Proposition 5.34, there exists  $U^{n+1} \in C^0([0, \frac{T_{n+1}}{\max(\epsilon, \beta)}]; X^s) \cap C^1([0, \frac{T_{n+1}}{\max(\epsilon, \beta)}]; X^{s-1})$  unique solution to (6.2) if  $U^n \in C^0([0, \frac{T_n}{\max(\epsilon, \beta)}]; X^s) \cap C^1([0, \frac{T_n}{\max(\epsilon, \beta)}]; X^{s-1}) \subset X_T^s$ , and satisfies (H1), (H2) and (H3).

*Existence and uniform control of the sequence  $U^n$ .*

The existence of  $T' > 0$  such that the sequence  $U^n$  is uniquely defined, controlled in  $X_{T'}^s$ , and satisfies (H1), (H2) and (H3), uniformly with respect to  $n \in \mathbb{N}$ , is obtained by induction, as follows.

Proposition 5.34 yields

$$E^s(U^{n+1})(t) \leq e^{\max(\epsilon, \beta)\lambda_n t} E^s(U_0) + \max(\epsilon, \beta) C_n \int_0^t e^{\max(\epsilon, \beta)\lambda_n(t-t')} dt'.$$

$$\begin{aligned} |\partial_t U^{n+1}(t, \cdot)|_{X^{s-1}} &\leq C_n E^s(U^{n+1})(t) + \max(\epsilon, \beta) C_n \\ &\leq C_n e^{\max(\epsilon, \beta)\lambda_n t} E^s(U_0) + \max(\epsilon, \beta) C_n^2 \int_0^t e^{\max(\epsilon, \beta)\lambda_n(t-t')} dt' + \max(\epsilon, \beta) C_n, \end{aligned}$$

with  $C_n, \lambda_n = C(M_{CH}, h_{01,n}^{-1}, h_{02,n}^{-1}, T_n, \|U^n\|_{X_{T_n}^s}, |b|_{H^{s+2}})$ , provided  $U^n \in X_{T_n}^s$  satisfies (H1), (H2) and (H3) with positive constants  $h_{01,n}, h_{02,n}$ , and  $h_{03,n}$  on  $[0, T_n/\max(\epsilon, \beta)]$ .

It is a consequence of the work [17, Theorem 7.3] and by taking into account the topographic variation that the assumptions (H1) and (H2) may be imposed only on the initial data and then is automatically satisfied over the relevant time scale. Let us prove it now for (H3),

Since  $U^n = (\zeta^n, v^n)^\top$  satisfies (6.2), one has

$$\partial_t \zeta^{n+1} = -f(\epsilon \zeta^n) \partial_x v^{n+1} - \epsilon f'(\epsilon \zeta^n) v^n \partial_x \zeta^{n+1} + \beta \partial_x b g(\epsilon \zeta^n) v^n,$$

and

$$\begin{aligned} \partial_t v^{n+1} &= -\mathfrak{T}[\epsilon \zeta^n]^{-1} \left( Q_0(\epsilon \zeta^n, \beta b) \partial_x \zeta^{n+1} + \epsilon \mathfrak{Q}[\epsilon \zeta^n, \beta b, v^n] \partial_x v^{n+1} + \epsilon^2 Q_1(\epsilon \zeta^n, \beta b, v^n) \partial_x \zeta^{n+1} \right. \\ &\quad \left. + \epsilon \frac{\gamma \beta q_1(\epsilon \zeta^n, \beta b) h_1 (h_1 + h_2) v^{n2} \partial_x b}{(h_1 + \gamma h_2)^3} \right) - \epsilon \zeta v^n \partial_x v^{n+1}. \end{aligned}$$

Using continuous Sobolev embedding of  $H^{s-1}$  into  $L^\infty$  ( $s-1 > 1/2$ ), and since  $U^n$  satisfies (H1), (H2) with  $h_{01,n}, h_{02,n}$  on  $[0, T_n/\max(\epsilon, \beta)]$ , one deduces that

$$|\partial_t \zeta^{n+1}|_{L^\infty} \leq C(M_{CH}, h_{01,n}^{-1}, h_{02,n}^{-1}, \beta |b|_{H^s}) \|U^n\|_{X_{T_n}^s}, \quad (6.3)$$

and

$$|\partial_t v^{n+1}|_{L^\infty} \leq C(M_{CH}, h_{01,n}^{-1}, h_{02,n}^{-1}, \beta |b|_{H^{s+1}}) \|U^n\|_{X_{T_n}^s}. \quad (6.4)$$

Let  $g^{n+1} = a_1(\epsilon \zeta^{n+1}, \beta b) + a_2(\epsilon \zeta^{n+1}, \beta b) \epsilon^2 (v^{n+1})^2$ ,

where  $(a_1(\epsilon \zeta^{n+1}, \beta b), a_2(\epsilon \zeta^{n+1}, \beta b)) = \left( (\gamma + \delta) q_1(\epsilon \zeta^{n+1}, \beta b) - \mu \beta \omega \partial_x^2 b, -\gamma q_1(\epsilon \zeta^{n+1}, \beta b) \frac{(h_1 + h_2)^2}{(h_1 + \gamma h_2)^3} \right)$

One has,

$$\begin{aligned} g^{n+1} &= g^{n+1}|_{t=0} + \int_0^t \partial_t a_1(\epsilon \zeta^{n+1}, \beta b) + \epsilon^2 \int_0^t \partial_t a_2(\epsilon \zeta^{n+1}, \beta b) (v^{n+1})^2 + 2\epsilon^2 \int_0^t a_2(\epsilon \zeta^{n+1}, \beta b) v^{n+1} \partial_t v^{n+1} \\ &= g^{n+1}|_{t=0} + (\gamma + \delta) \epsilon \kappa_1 \int_0^t \partial_t \zeta^{n+1} + \epsilon^3 \int_0^t a'_2(\epsilon \zeta^{n+1}, \beta b) \partial_t \zeta^{n+1} (v^{n+1})^2 + 2\epsilon^2 \int_0^t a_2(\epsilon \zeta^{n+1}, \beta b) v^{n+1} \partial_t v^{n+1} \end{aligned}$$

so that (6.3) and (6.4) yields

$$\|g^{n+1} - g^{n+1}|_{t=0}\|_{L^\infty} \leq C(M_{CH}, h_{01,n}^{-1}, h_{02,n}^{-1}, |b|_{H^{s+2}}) \|U^n\|_{X_{T_n}^s}.$$

Now, one has  $g^{n+1}|_{t=0} \equiv g^0|_{t=0} \geq h_{03,0} > 0$ , independent of  $n$ . Thus one can easily prove (by induction) that it is possible to chose  $T' > 0$  such that  $g^{n+1} > \alpha/2$  holds on  $[0, T'/\max(\epsilon, \beta)]$ , and the above energy estimates hold uniformly with respect to  $n$ , on  $[0, T'/\max(\epsilon, \beta)]$ .

More precisely, one has that  $U^n$  satisfies (H3) with  $\frac{h_{03}}{2} > 0$  and the estimates

$$\begin{aligned} E^s(U^n)(t) &\leq e^{\max(\epsilon, \beta)\lambda t} E^s(U_0) + \max(\epsilon, \beta) C_0 \int_0^t e^{\max(\epsilon, \beta)\lambda(t-t')} dt' \\ |\partial_t U^n(t, \cdot)|_{X^{s-1}} &\leq C_0 E^s(U^n)(t) + \max(\epsilon, \beta) C_0 \\ &\leq C_0 e^{\max(\epsilon, \beta)\lambda t} E^s(U_0) + \max(\epsilon, \beta) C_0^2 \int_0^t e^{\max(\epsilon, \beta)\lambda(t-t')} dt' + \max(\epsilon, \beta) C_0. \end{aligned} \tag{6.5}$$

on  $[0, T'/\max(\epsilon, \beta)]$ , where  $C_0, \lambda = C(M_{CH}, h_{01}^{-1}, h_{02}^{-1}, h_{03}^{-1}, T', |U_0|_{X^s}, |b|_{H^{s+2}})$  are uniform with respect to  $n$ .

For the completion of the proof (*Convergence of  $U^n$  towards a solution of the nonlinear problem*) we use the same techniques as in the proof of [17, Theorem 7.3]  $\square$

**Theorem 6.6 (Stability)** *Let  $\mathbf{p} = (\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{CH}$  and  $s \geq s_0 + 1$  with  $s_0 > 1/2$ , and assume  $U_{0,1} = (\zeta_{0,1}, v_{0,1})^\top \in X^s$ ,  $U_{0,2} = (\zeta_{0,2}, v_{0,2})^\top \in X^{s+1}$ , and  $b \in H^{s+2}$  satisfies (H1), (H2), and (H3). Denote  $U_j$  the solution to (3.9) with  $U_j|_{t=0} = U_{0,j}$ . Then there exists  $T, \lambda, C_0 = C(M_{CH}, h_{01}^{-1}, h_{02}^{-1}, h_{03}^{-1}, |U_{0,1}|_{X^s}, |U_{0,2}|_{X^{s+1}}, |b|_{H^{s+2}})$  such that  $\forall t \in [0, \frac{T}{\max(\epsilon, \beta)}]$ ,*

$$\|(U_1 - U_2)(t, \cdot)\|_{X^s} \leq C_0 e^{\max(\epsilon, \beta)\lambda t} \|U_{1,0} - U_{2,0}\|_{X^s}.$$

*Proof.*

The existence and uniform control of the solution  $U_1$  (resp.  $U_2$ ) in  $L^\infty([0, T/\max(\epsilon, \beta)]; X^s)$  (resp.  $L^\infty([0, T/\max(\epsilon, \beta)]; X^{s+1})$ ) is provided by Theorem 6.1. The proposition is then a direct consequence of the *a priori* estimate of Proposition 5.37, with  $F_1 = F_2 = 0$ , and Lemma 5.12.  $\square$

**Theorem 6.7 (Convergence)** *Let  $\mathbf{p} = (\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{CH}$  (see (1.2)) and  $s \geq s_0 + 1$  with  $s_0 > 1/2$ , and let  $U^0 \equiv (\zeta^0, \psi^0)^\top \in H^{s+N}(\mathbb{R})^2$ ,  $b \in H^{s+N}$  with  $N$  sufficiently large, (satisfy the hypotheses of Theorem 5 in [22] adapted to non flat topography) as well as (H1), (H2), and (H3). Then there exists  $C, T > 0$ , independent of  $\mathbf{p}$ , such that*

- *There exists a unique solution  $U \equiv (\zeta, \psi)^\top$  to the full Euler system (2.1), defined on  $[0, T]$  and with initial data  $(\zeta^0, \psi^0)^\top$  (provided by Theorem 5 in [22] adapted to non flat topography<sup>1</sup>);*
- *There exists a unique solution  $U_a \equiv (\zeta_a, v_a)^\top$  to our new model (3.9), defined on  $[0, T]$  and with initial data  $(\zeta^0, v^0)^\top$  (provided by Theorem 6.1);*
- *With  $v$ , defined as in (2.5), one has*

$$\|(\zeta, v) - (\zeta_a, v_a)\|_{L^\infty([0, T]; X^s)} \leq C \mu^2 t.$$

<sup>1</sup>The study of Lannes focuses on the two-layer fluid system with a flat bottom ( $\beta = 0$ ). However, we believe that the theory in the uneven bottom case does not differ much from the one in the flat bottom configuration.

*Proof.*

As stated above, the existence of  $U$  is provided by Theorem 5 in [22], and the existence of  $U_a$  is given by our Theorem 6.1 (we choose  $T$  as the minimum of the existence time of both solutions; it is bounded from below, independently of  $\mathbf{p} \in \mathcal{P}_{\text{CH}}$ ). If  $N$  is large enough, then  $U \equiv (\zeta, \psi)^\top$  satisfies the assumptions of our consistency result, Proposition 3.10, and therefore  $(\zeta, v)^\top$  solves (3.9) up to a residual  $R = (r_1, r_2)^\top$ , with  $|R|_{L^\infty([0,T];H^s)} \leq C(M_{\text{CH}}, h_{01}^{-1}, |b|_{H^{s+N}}, |U^0|_{H^{s+N}})(\mu^2 + \mu\epsilon^2)$ . The result follows from the stability Proposition 5.37, with  $F_1 = (r_1, \mathfrak{T}[\epsilon\zeta]^{-1}r_2)^\top$  and  $F_2 = 0$ .  $\square$

## References

- [1] S. ALINHAC AND P. GÉRARD, *Opérateurs pseudo-différentiels et théorème de Nash-Moser*. Savoirs Actuels. InterEditions et Editions du CNRS, Paris, 1991.
- [2] C. T. ANH, Influence of surface tension and bottom topography on internal waves. *Math. Models Methods Appl. Sci.*, 19(12):2145–2175, 2009.
- [3] J. L. BONA, M. CHEN, AND J.-C. SAUT, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I. Derivation and linear theory. *J. Nonlinear Sci.*, 12(4):283–318, 2002.
- [4] J. L. BONA, M. CHEN, AND J.-C. SAUT, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II. The nonlinear theory. *Nonlinearity*, 17(3):925–952, 2004.
- [5] J. L. BONA, D. LANNES, AND J.-C. SAUT, Asymptotic models for internal waves. *J. Math. Pures Appl. (9)*, 89(6):538–566, 2008.
- [6] J. BOUSSINESQ, Théorie de l’intumescence liquide appelée onde solitaire ou de translation se propageant dans un canal rectangulaire. *C.R. Acad. Sci. Paris Sér. A-B*, 72:755–759, 1871.
- [7] J. BOUSSINESQ, Théorie des ondes et des remous qui se propagent le long d’un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. *J. Math. Pures Appl.*, 17(2):55–108, 1872.
- [8] W. CHOI, R. BARROS, AND T.-C. JO, A regularized model for strongly nonlinear internal solitary waves. *J. Fluid Mech.*, 629:73–85, 2009.
- [9] W. CHOI AND R. CAMASSA, Weakly nonlinear internal waves in a two-fluid system. *J. Fluid Mech.*, 313:83–103, 1996.
- [10] W. CHOI AND R. CAMASSA, Fully nonlinear internal waves in a two-fluid system. *J. Fluid Mech.*, 396:1–36, 1999.
- [11] W. CRAIG, P. GUYENNE, AND H. KALISCH, Hamiltonian long-wave expansions for free surfaces and interfaces. *Comm. Pure Appl. Math.*, 58(12):1587–1641, 2005.
- [12] W. CRAIG AND C. SULEM, Numerical simulation of gravity waves. *J. Comput. Phys.*, 108(1):73–83, 1993.
- [13] C.J. COTTER, D.D. HOLM AND J.R. PERCIVAL, The square root depth wave equations *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 466:3621–3633, 2010.
- [14] V. DUCHÊNE, Asymptotic shallow water models for internal waves in a two-fluid system with a free surface. *SIAM J. Math. Anal.*, 42(5):2229–2260, 2010.
- [15] V. DUCHÊNE, Boussinesq/Boussinesq systems for internal waves with a free surface, and the KdV approximation. *M2AN Math. Model. Numer. Anal.*, 46:145–185, 2011.

- [16] V. DUCHÊNE, Decoupled and unidirectional asymptotic models for the propagation of internal waves. *M3AS:Math. Models Methods Appl. Sci.*, 24(01):1–65, 2014.
- [17] V. DUCHÊNE, S. ISRAWI AND R. TALHOUK, A new fully justified asymptotic model for the propagation of internal waves in the Camassa-Holm regime. *To appear in SIAM J. Math. Anal.* (arXiv:1304.4554v2)
- [18] V. DUCHÊNE, S. ISRAWI AND R. TALHOUK, Shallow water asymptotic models for the propagation of internal waves. *Discrete Contin. Dyn. Syst. Ser. S*, 7(2):239–269, 2014.
- [19] A. E. GREEN AND P. M. NAGHDI, A derivation of equations for wave propagation in water of variable depth. *J. Fluid Mech.*, 78(02):237–246, 1976.
- [20] P. GUYENNE, D. LANNES, AND J.-C. SAUT, Well-posedness of the Cauchy problem for models of large amplitude internal waves. *Nonlinearity*, 23(2):237–275, 2010.
- [21] S. ISRAWI, Large time existence for 1d Green-Naghdi equations. *Nonlinear analysis: Theory, Methods & Applications*, 74(1):81–93, 2011.
- [22] D. LANNES, A stability criterion for two-fluid interfaces and applications. *Arch. Ration. Mech. Anal.*, 208(2):481–567, 2013.
- [23] D. LANNES, *Water waves: mathematical analysis and asymptotics*, volume 188 of Mathematical Surveys and Monographs, AMS, 2013.
- [24] R. LISKA, L. MARGOLIN AND B. WENDROFF, Nonhydrostatic two-layer models of incompressible flow *Comput. Math. Appl.*, 29:25–37, 1995.
- [25] Z. L. MAL’TSEVA, Unsteady long waves in a two-layer fluid. *Dinamika Sploshn. Sredy*, (93-94):96–110, 1989.
- [26] Y. MATSUNO, A unified theory of nonlinear wave propagation in two-layer fluid systems. *J. Phys. Soc. Japan*, 62(6):1902–1916, 1993.
- [27] M. MIYATA, An internal solitary wave of large amplitude. *La mer*, 23(2):43–48, 1985.
- [28] L. A. OSTROVSKY AND J. GRUE, Evolution equations for strongly nonlinear internal waves. *Phys. Fluids*, 15(10):2934–2948, 2003.
- [29] B. DE SAINT-VENANT, Théorie du mouvement non-permanent des eaux, avec application aux crues des rivières et à l’introduction des marées dans leur lit. *C.R. Acad. Sci. Paris*, 73:147–154, 1871.
- [30] F. SERRE, Contribution à l’étude des écoulements permanents et variables dans les canaux. *La Houille Blanche*, (6):830–872, 1953.
- [31] L. XU, Intermediate long wave systems for internal waves. *Nonlinearity*, 25(3):597–640, 2012.
- [32] V. E. ZAKHAROV, Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys.*, 9:190–194, 1968.