



**HAL**  
open science

## Optimal speed scaling for multiclass fluid queues

Olivier Brun, Henda Ben Cheikh, Balakrishna Prabhu

► **To cite this version:**

Olivier Brun, Henda Ben Cheikh, Balakrishna Prabhu. Optimal speed scaling for multiclass fluid queues. [Research Report] LAAS-CNRS. 2015. hal-01094506v2

**HAL Id: hal-01094506**

**<https://hal.science/hal-01094506v2>**

Submitted on 27 Jul 2015 (v2), last revised 13 Sep 2016 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Optimal speed scaling for multiclass fluid queues

O. Brun<sup>†</sup>, H. Ben Cheikh<sup>\*</sup>, and B.J. Prabhu<sup>†</sup>

<sup>†</sup>CNRS, LAAS, 7 avenue du colonel Roche, F-31400 Toulouse, France

<sup>†</sup>Université de Toulouse, LAAS, F-31400 Toulouse, France

## Abstract

Inspired by speed-scaling techniques used in servers, we investigate the energy-delay tradeoff in bandwidth-sharing networks in which nodes can regulate their speed according to the load of the system. Assuming that the network is initially congested, we investigate the rate allocation to the classes that drains out the network with minimum total energy and delay cost. We formulate this optimal resource allocation problem as a Markov decision process which proves to be both analytically and computationally challenging. We propose to solve this stochastic problem using a deterministic fluid approximation. For the case of a linear network with two links, we provide numerical evidences in the support of the fluid model as a good approximation to the stochastic control problem. For a single link shared by an arbitrary number of classes, using Pontryagin's Maximum Principle we show that the optimal-fluid solution follows the well-known  $c\mu$  rule and give an explicit expression for the optimal speed.

## 1 Introduction

Optimizing energy consumption has become an important issue in recent years. One of the solutions for reducing energy consumption, called dynamic speed-scaling, involves regulating the speed of the processor as a function of the the load. The optimal speed-scaling policy taking into account the energy-delay trade-off was computed for a single node in [25, 7]. Our main contribution is to generalize the application of this energy-saving technique to bandwidth-sharing networks which were introduced in [17] to model the dynamics of network flows in the Internet. Our model is thus a transposition of speed-scaling to bandwidth-sharing networks in which nodes will be able to regulate their speed in order to minimize a given performance metric.

The main question that we investigate in this paper is the following: assuming that the network is initially in a congested state, how to share the available bandwidth in order to drain out the congestion with minimum total cost which comprises of the mean response times experienced by flows as well as the energy consumption of the network.

The optimal resource-allocation problem in a bandwidth-sharing network can be cast as a Markov decision process with dimension equal to the number of classes of flows in the network. Even without energy costs the computation of closed-form optimal policy has proven to be quite challenging for general networks. Optimal allocations are known only for a few special network topologies such as single node [20, 19], linear networks [23] and star networks [21]. In most cases, the optimal policy is a priority rule that serves classes that do not use competing resources.

When the option of speed-scaling is introduced in a bandwidth-sharing network, a second level of complexity gets added to the computation of the optimal policy. In addition to knowing which classes of traffic to serve, one needs to determine the speed at which each node must operate. This makes the speed-scaling policy for bandwidth-sharing networks even more challenging.

While, in theory, the optimal policy can be computed by value-iteration, in practice, the numerical computation becomes quickly prohibitive due to the exponential relation between the number of states and the number of classes. For example, a network with 5 classes and at most 100 flows per class will result in a Markov chain of with  $101^5$  states.

Our approach to this problem is to analyze an associated fluid model in the spirit of Avram, Bertsimas, and Richard [5] which can be interpreted as a deterministic approximation of the stochastic problem. This

---

<sup>\*</sup>The work of this author was done while she was affiliated with LAAS-CNRS and Universté de Toulouse

technique has been applied in various problems of optimal control of high-dimensional Markov chains [18]. In the context of the bandwidth-sharing networks, which is the focus of this paper, the utility of this technique was demonstrated in [23].

The fluid approach involves approximating the dynamics of the Markov chain using a set of differential equations, and then computing the optimal control policy for this set of differential equations. The utility of this technique lies in its ability to reduce the complexity of the problem to one for which numerical solutions can be efficiently calculated. This feature will be illustrated later when we compare the times required to compute the stochastic as well as the fluid optimal policies.

Other than the relative ease with which it can be computed, it has been shown in [8, 24, 12] that the fluid control is asymptotically optimal when the state of the system is large. These results are especially relevant in the context of this paper. The large state can be interpreted as a congested state of the network caused by a flash-crowd or a momentaneous arrival of a large number of flows. Thus, one can expect the fluid-optimal policy for draining a network from a congested state to be close to the stochastic-optimal policy.

One of the recurrent questions in this type of approximations is how to convert the fluid-optimal policy to the stochastic setting. We shall use a version of the Discrete-review method proposed in [16]. The idea is to periodically apply the fluid-optimal policy with the initial state as the state of the stochastic system sampled at review instants.

In addition to a network, we investigate the canonical case of a single link shared by multiple classes of flows, the so called multi-class queue. The aim is to characterize and compute the policy that minimizes the total cost in delay and energy consumption for the single link case. The single-link case is the basic model to study resource allocation problems and has been widely studied in several related works [19, 4].

Without energy costs, for a single-node, the optimality of the well-known  $c\mu$  rule has been shown in various ways (see, [19] and references therein) Speed-scaling policies for a single server and single class of traffic was analyzed in [2] but for multi-class queues such an analysis has been lacking.

Another related work on energy-aware bandwidth-sharing networks is [3] in which the authors find approximation algorithms for a routing problem. In our model, the routes for the flows are fixed and we use a fluid model to determine the optimal speed profile.

## 1.1 Contributions and Organization of the paper

In the next section, we shall describe the bandwidth-sharing model with speed-scaling and formulate the related fluid optimal control problem. We shall first illustrate the utility of the fluid approximation on a two-node linear network by comparing the costs and the trajectories of the stochastic-optimal and the fluid-optimal policies with discrete-review type updates. It turns out that the analytical computation of the fluid-optimal policy even in the case of the two-node network becomes cumbersome due to the fact that one has to compute the speed of each of the nodes.

In section 3, we shall focus on the single node case. The bulk of our analytical results for the fluid-optimal policies are available only for this scenario. Using Pontryagin's Maximum Principle, we show the optimality of the well-known  $c\mu$  rule, that is, to serve the classes in decreasing order of priority according to the product  $c_i\mu_i$ , where  $c_i$  is the delay cost of class- $i$  and  $\mu_i$  denotes the mean rate of flows of class- $i$ . We also shall give analytical expressions for the optimal server speed as a function of time. The utility and the applicability of the fluid-optimal policies will be shown using several numerical examples.

Finally, in section 4, we present the conclusions and future work in this direction.

## 2 Model description

### 2.1 Stochastic model

We consider a network of a set of  $\mathcal{L}$  links shared by a set  $\mathcal{F}$  of classes of flows with  $N$  being the number of classes. We denote by  $C = \{C_l\}_{l \in \mathcal{L}}$  the vector of links capacity. Class- $i$  flows arrive according to a Poisson process at rate  $\lambda_i$  and have exponentially distributed sizes of mean  $1/\mu_i$ ,  $i \in \mathcal{F}$ . We denote by  $\rho_i = \lambda_i/\mu_i$  the traffic load of class  $i$ , and the total traffic load by  $\rho = \sum_{i=1}^N \rho_i$ . In the following, we let also  $X_i(t)$  be the number of class- $i$  flows in the system at time  $t$ . We define  $\mathbf{X}(t) = (X_1(t), X_2(t), \dots, X_N(t))$  as the state of the system at time  $t$ , and note that the state belongs to  $\mathbb{N}^N$ . A holding cost rate of  $c_i$  is associated with flows

of class- $i$ . Let  $p_i \subset \mathcal{L}$  denote the set of links used by flows of class  $i$ , and let  $P$  be the  $\mathcal{F} \times \mathcal{L}$  incidence matrix in which the entry  $(i, j)$  is 1 if class  $i$  uses link  $j$  and is 0 otherwise.

Link  $l, l \in \mathcal{L}$  can be operated at a speed in the interval  $[0, C_l]$ . We shall assume that the maximum speed  $C_l > \sum_i \rho_i p_{i,l}$ , in order to ensure stability of the system, and that the speed of a link can be changed when the number of flows in the system changes. We let  $\mathbf{u}_i(\mathbf{x})$  be the capacity allocated to class  $i$  when the system is in state  $\mathbf{x}$ . We further assume that all class- $i$  flows share the capacity allocated to the class according to the PS discipline. In the following, we denote by  $\mathbf{u}(\mathbf{x})$  the vector  $(\mathbf{u}_1(\mathbf{x}), \mathbf{u}_2(\mathbf{x}), \dots, \mathbf{u}_N(\mathbf{x}))$ , and note that this vector belongs to the following set

$$\mathcal{U} = \{\mathbf{u} \in \mathbb{R}_+^N : \mathbf{u}P \leq C\} \quad (1)$$

The power required to operate a link at rate  $x$  is assumed to be proportional to  $x^\gamma$  where  $\gamma > 1$ , see for example [2]. The total power consumption of the rate allocation vector  $\mathbf{u}$  is then

$$\sum_{l \in \mathcal{L}} \left( \sum_{j \in \mathcal{F}} u_j P_{j,l} \right)^\gamma. \quad (2)$$

For any given stationary policy  $\mathbf{u}$ , the network state  $\mathbf{X}(t)$  is a Markov process (a multi-dimensional birth-and-death process) with transition rates

$$q(\mathbf{x}, \mathbf{y}) = \begin{cases} \lambda_i & \text{if } \mathbf{y} = \mathbf{x} + \mathbf{e}_i, i \in \mathcal{F} \\ \mu_i u_i(\mathbf{x}) & \text{if } \mathbf{y} = \mathbf{x} - \mathbf{e}_i, i \in \mathcal{F} \end{cases}$$

Let  $\mathcal{S}$  denote the set of states which correspond to the ideal operating point of the network. This could be the set of states within which the network operator would like the number of flows to be so that each flow could receive a given quality of service. We shall assume that at time 0 the network finds itself in a state  $\mathbf{x}(0)$  which is far away from this  $\mathcal{S}$ . This could happen due to unseen circumstances such as flash-crowds or events which attract a large number of users at the same time (a major sports event, or some unexpected news event).

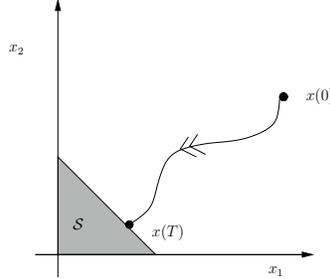


Figure 1: Illustration of the optimal control problem with 2 classes.

The goal of the operator is to find the capacity allocation policy that will bring the network to a state within  $\mathcal{S}$  (or the operating region) while minimizing the total cost incurred (see Figure 1). This cost comprises of two conflicting components: one for the holding the flows in the network and the other for the energy consumption in the network. Intuitively, to lower the holding cost of flows, the operator has to increase the speed of the links which then increases the energy consumption of the network, and vice versa. Formally, we aim to find  $\mathbf{u}^* : \mathbb{N}^N \rightarrow \mathcal{U}$  solving the following problem:

$$\text{Minimize } \mathbb{E}_{\mathbf{x}(0)} \left\{ \int_0^T f(\mathbf{X}(t), \mathbf{u}(\mathbf{X}(t))) dt \right\}, \quad (3)$$

where  $T$  is the first time the process enters the set  $\mathcal{S}$ , that is

$$T = \inf\{t : \mathbf{X}(t) \in \mathcal{S} | \mathbf{X}(0) = \mathbf{x}(0)\},$$

and  $f(\mathbf{x}, \mathbf{u})$  represents the cost rate in state  $\mathbf{x}$  when a control  $\mathbf{u}$  is applied, that is,

$$f(\mathbf{x}, \mathbf{u}) = \sum_{i \in \mathcal{F}} c_i x_i + \kappa \sum_{j \in \mathcal{L}} \left( \sum_{i \in \mathcal{F}} u_i P_{i,j} \right)^\gamma. \quad (4)$$

In (4),  $\kappa$  is a parameter controlling the relative weights of energy consumption and delay, whereas  $\mathbf{c}$  is a vector giving the relative weights of the delays of the classes .

**Remark 1.** *The above problem is equivalent to a stochastic shortest path problem (cf. [9]) in which the cost in the states belonging to  $\mathcal{S}$  is 0 and the rate of transition from these states to any other state is 0 as well. The horizon of the problem now becomes infinite. We shall use this infinite horizon formulation from here on.*

**Remark 2.** *To simplify the analysis, we shall take  $\mathcal{S} = \{\mathbf{0}\}$ , that is the state where all classes have zero flows.*

Problem (3) can be transformed into an equivalent discrete-time problem by uniformizing this CTMC. The sojourn time in state  $\mathbf{x}$  is exponentially distributed with parameter  $\nu(\mathbf{x}) = \sum_i \lambda_i + \sum_i \mu_i u_i(\mathbf{x})$ . We note that  $\nu(\mathbf{x}) \leq \nu$  for all  $\mathbf{x} \in \mathbb{N}^N$ , where

$$\nu = \sum_i \lambda_i + \max_{l \in \mathcal{L}}(C_l) \sum_i \mu_i. \quad (5)$$

The uniformized version of the CTMC is a DTMC with state space  $\mathbb{N}^N$  and transition probabilities

$$p(\mathbf{x}, \mathbf{y}) = \begin{cases} \frac{\lambda_i}{\nu} & \text{if } \mathbf{y} = \mathbf{x} + \mathbf{e}_i, i \in \mathcal{F} \\ \frac{\mu_i}{\nu} u_i(\mathbf{x}) & \text{if } \mathbf{y} = \mathbf{x} - \mathbf{e}_i, i \in \mathcal{F} \\ \frac{1}{\nu} [\nu - \sum_i (\lambda_i + \mu_i u_i(\mathbf{x}))] & \text{if } \mathbf{y} = \mathbf{x}. \end{cases}$$

The equivalent discrete-time problem is as follows:

$$\text{Minimize } J_\phi(\mathbf{x}(0)) = \frac{1}{\nu} \mathbb{E} \left\{ \sum_{k=0}^{\infty} f(\mathbf{x}(k), \mathbf{u}(\mathbf{x}(k))) \right\}. \quad (6)$$

In theory, the optimal policy can then be computed using the value-iteration algorithm [9], that is by iterating over

$$J^{(k+1)}(\mathbf{x}) = \frac{1}{\nu} \min_{\mathbf{u} \in \mathcal{U}} \left[ f(\mathbf{x}, \mathbf{u}) + \sum_{\mathbf{y}} p(\mathbf{x}, \mathbf{y}) J^{(k)}(\mathbf{y}) \right], \quad (7)$$

for all  $\mathbf{x} \in \mathbb{N}^N$  and for  $k = 0, 1, 2, \dots$ . In practice however, this approach, that requires computing at each step  $k$  the optimal cost-to-go  $J^{(k)}(\mathbf{x})$  for each possible initial state  $\mathbf{x}$ , suffers from the curse of dimensionality. If truncation of the state space  $\mathbb{N}^N$  is feasible in light traffic regimes, the above technique is prohibitively expensive when either the number of flow classes or the traffic intensities get large.

## 2.2 Fluid model

As is often done when one is confronted with a computationally hard stochastic control problem (3), we resort to its fluid analog that has deterministic dynamics given by the following set of differential equations:

$$\dot{\mathbf{x}}(t) = \boldsymbol{\lambda} - \boldsymbol{\mu} \cdot \mathbf{u},$$

where  $x_i(t)$  represents the quantity of fluid associated to class  $i$  at time  $t$ ,  $u_i(t)$  is the rate allocated to this class at that time and  $\cdot$  stands for componentwise multiplication. We describe below the associated fluid optimal control problem.

Let  $T$  denote<sup>1</sup> the first time when the total amount of fluid in the network reaches 0, that is,

$$T = \inf \{ t : \mathbf{x}(t) \in \mathcal{S} = \mathbf{0} | \mathbf{X}(0) = \mathbf{x}(0) \},$$

---

<sup>1</sup>With slight abuse of notation, we use  $T$  to denote the first time to enter the set  $\mathcal{S}$  in both the stochastic and fluid models.

We note that the horizon  $T$  is not fixed, and it is also a part of the solution. It is the first time that the process enters the set  $\mathcal{S} = \{\mathbf{0}\}$ .

The fluid control problem is then to find the rate allocation that drains out the queue with minimum total cost. Formally, the problem can be stated as

$$\text{minimize } J(\mathbf{u}; \mathbf{x}_0) = \int_0^T f(\mathbf{x}(t), \mathbf{u}(t)) dt \quad (\text{OPT})$$

subject to

$$\dot{x}_i(t) = \lambda_i - \mu_i u_i(t), \quad i \in \mathcal{F}, \quad (8)$$

$$-x_i(t) \leq 0, \quad i \in \mathcal{F}, \quad (9)$$

$$\mathbf{u}(t) \in \mathcal{U}, \quad (10)$$

$$\mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x}(T) = \mathbf{0}. \quad (11)$$

### 2.3 Implementation of the fluid policy

In this section, we discuss how to use the control policy  $\mathbf{u}(t)$  minimizing (OPT) in order to define an implementable rule for the stochastic network. Roughly speaking, the problem is that the stochastic process can significantly deviate from its fluid limit. Hence, applying the open-loop fluid control law without taking into consideration the state of the network is not an appropriate approach. In recent years, several approaches have been proposed for translating the solution of the associated fluid control problem into an implementable policy for the original stochastic network. In this work, we shall consider the family of so-called discrete-review policies [14, 16]. Those policies step through time in large intervals within which a deterministic control rule is employed. More precisely, network state is reviewed at discrete points in time. At each review instant, the controller formulates a plan for the period of time until the next review instant in order to best track what the fluid control policy would do starting at that point. In [16], Maglaras proposes a discrete-review policy for scheduling stochastic networks that achieves asymptotically optimal performances under fluid scaling, and guarantees stability if the traffic intensity is less than one at each station. See also [10, 6, 22, 13] for related approaches.

In our work, we have used discrete-review policies in their simplest form. Let  $t_k = k \Delta T$  for  $k = 0, 1, 2, \dots$  be the review instants. At time  $t_k$ , we compute the fluid control policy using as initial state the current state of the stochastic network. Dividing the review period in  $m$  steps, we allocate the rate  $u_i(t_k + \frac{n}{m} \Delta T)$  to class  $i$  in the time interval  $[t_k + \frac{n}{m} \Delta T, t_k + \frac{n+1}{m} \Delta T)$ .

In order to illustrate the benefits that can be expected from such an approach, consider the simple 2 links and 3 classes linear network shown in Figure 2. The link capacities are  $C_1 = 10$  and  $C_2 = 20$ . We assume that  $\kappa=1$  and  $\gamma=3$  and consider two scenarios:

- Scenario 1:  $\rho_0 = 0.2$ ,  $\rho_1 = 0.3$ ,  $\rho_2 = 0.1$ ,  $c_0 = 1$ ,  $c_1 = 2$  and  $c_2 = 2$ .
- Scenario 2:  $\rho_0 = 0.4$ ,  $\rho_1 = 0.1$ ,  $\rho_2 = 0.5$ ,  $c_0 = 2$ ,  $c_1 = 1$ ,  $c_2 = 6$ .

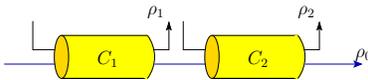


Figure 2: Line Lot network.

Given a sample path for the flow arrival process as well as a sample path of the stochastic process governing flow sizes, one can compute the sample trajectories that would be obtained under the optimal stochastic policy and under the fluid policy reviewed at discrete times. The optimal stochastic policy is easily computed using well-known techniques such as value-iteration or policy-iteration [9]. In our case, we have used the former technique. The fluid control problem can be solved using an optimal control solver. In our case, we have used Bocop, an open source toolbox for optimal control problems [1]. We note that, as clearly demonstrated by Tables 1 and 2, the time required for solving the fluid control problem is much lower than the time required

Scenario	Computing time (s)
1	780s
2	1020s

Table 1: Time required to solve for the optimal stochastic policy using value iteration (relative precision on optimal cost-to-go is  $10^{-3}$ ).

Scenario	Initial State	Computing time (s)
1	(3,5,5)	16 s
2	(4,5,3)	10 s

Table 2: Time required to solve for the optimal fluid policy using Bocop.

to compute the optimal stochastic policy. In other words, whereas computing the optimal stochastic policy is out of reach for large networks, the computation of the optimal fluid control law requires a relatively modest effort.

Figure 3 (left) compares sample trajectories obtained for the 3 classes under the optimal stochastic policy and under the optimal fluid control applied using the discrete review method in the first scenario and for  $\mathbf{x}(0) = (3, 5, 5)$  as initial state. Figure 3 (right) provides a similar comparison in the second scenario for  $\mathbf{x}(0) = (4, 5, 3)$  as initial state. It is apparent from these two examples that the sample trajectories obtained under the fluid policy match very well those obtained under the optimal stochastic policy.

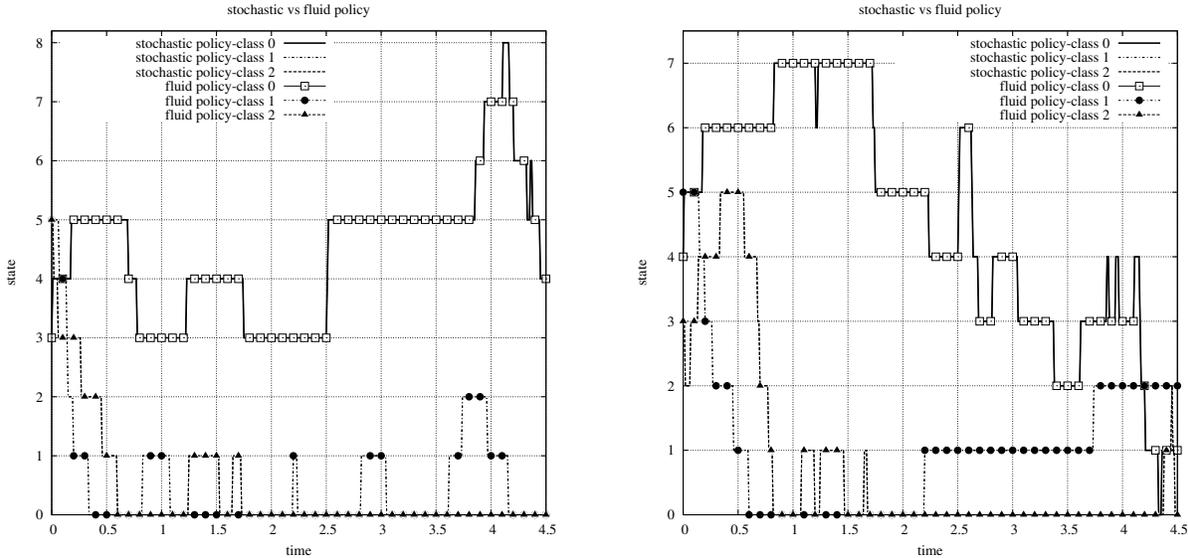


Figure 3: Sample trajectories obtained under the optimal stochastic policy and under the fluid policy for  $x_0(0)=4, x_1(0)=5$  and  $x_2(0)=3$  (left) and for  $x_0(0)=3, x_1(0)=5$  and  $x_2(0)=5$  (right).

### 3 The Single Link Case

In this section, we consider the case of a single node in the network with multiple classes. For this case  $P_{i,j} = 1, \forall i$ . We wish to compute the policy that minimizes the total cost and compute the optimal policy. Without energy costs, this problem was solved in [5]. It was shown that giving full priority to the non-empty

class with the largest  $c\mu$  is optimal. We shall investigate how energy costs affect the priority rule and how the speed varies as a function of the fluid. We first show that in the original stochastic model, the optimal policy is a priority discipline. However given the complexity of the stochastic setting, we were not able to determine the priority criterion. Using the fluid model we show the optimality of the  $c\mu$  rule, that is to give priority to the class  $i$  with the highest  $c_i\mu_i$ . Solving the optimal stochastic control problem numerically we observe the optimal policy coincides with the  $c\mu$  rule.

### 3.1 Stochastic model

In the following, we denote by  $J(\mathbf{x})$  the optimal cost, that is the cost of an optimal policy  $\mathbf{u}^* : \mathbb{N}^N \rightarrow \mathcal{U}$  minimizing (6). In the case of a single server, Bellman's equation takes the form

$$J(\mathbf{x}) = \frac{1}{\nu} \left[ \sum_i c_i x_i + \nu J(\mathbf{x}) + \sum_i \lambda_i \Delta_{\mathbf{x}+\mathbf{e}_i}(i) + \min_{\mathbf{u} \in \mathcal{U}} \left[ \kappa \left( \sum_i u_i \right)^\gamma - \sum_i \mu_i u_i \Delta_{\mathbf{x}}(i) \right] \right], \quad (12)$$

where  $\Delta_{\mathbf{x}}(i) = J(\mathbf{x}) - J(\mathbf{x} - \mathbf{e}_i)$  is the optimal cost differential in state  $\mathbf{x}$ . An optimal policy is to use at state  $\mathbf{x}$  the vector  $\mathbf{u}$  that minimizes

$$\kappa \left( \sum_i u_i \right)^\gamma - \sum_i \mu_i u_i \Delta_{\mathbf{x}}(i)$$

Letting  $s$  be such that  $\sum_i u_i = s$ , the optimal choice is to maximize  $\sum_i \mu_i u_i \Delta_{\mathbf{x}}(i)$ , and the optimal solution is clearly  $u_k = s$  for  $k$  such that for all  $i$  we have  $\mu_k \Delta_{\mathbf{x}}(k) \geq \mu_i \Delta_{\mathbf{x}}(i)$ , whereas  $u_i = 0$  for  $i \neq k$ . We thus conclude that an optimal policy can be obtained by giving full priority to only one class.

It is difficult to obtain more precise results on the optimal speed and the priority-rule for the above described stochastic optimal control problem. Therefore, in the next section we propose to solve the related fluid control model as an approximation to the stochastic-optimal policy.

### 3.2 Fluid model

For the single node case, the problem (OPT) takes the following form

$$\text{minimize } \int_0^T f(\mathbf{y}, \mathbf{u}) dt \quad (\text{OPT-R})$$

subject to

$$\dot{y}_i(t) = \rho_i - u_i(t), \quad i \in \mathcal{F}, \quad (13)$$

$$-y_i(t) \leq 0, \quad i \in \mathcal{F}, \quad (14)$$

$$\mathbf{u}(t) \in \mathcal{U}, \quad (15)$$

$$\mathbf{y}(T) = \mathbf{y}(0), \quad (16)$$

$$\mathbf{y}(T) = \mathbf{0}, \quad (17)$$

where we have made the transformation  $y_i = \frac{x_i}{\mu_i}$  and  $f$  has been redefined as

$$f(\mathbf{y}, \mathbf{u}) = \sum_{i \in \mathcal{F}} \tilde{c}_i y_i + \kappa \sum_{j \in \mathcal{L}} \left( \sum_{i \in \mathcal{F}} u_i P_{i,j} \right)^\gamma, \quad (18)$$

with  $\tilde{c}_i = c_i \mu_i$  for all  $i \in \mathcal{F}$ .

The problem (OPT-R) has pure state constraints which makes the problem more difficult than the ones without state constraints. We will solve using the direct method or the direct adjoining approach [15] which consists of relaxing the pure state constraint and then finding the appropriate value of the Lagrange multiplier.

Ignoring for the moment the constraint on the sum of rates<sup>2</sup>, the Lagrangian (or the augmented Hamiltonian) for this problem is defined as

$$\mathcal{L}(\mathbf{y}, \mathbf{u}, \boldsymbol{\theta}, \boldsymbol{\eta}, \boldsymbol{\beta}) = f(\mathbf{y}, \mathbf{u}) + \sum_i (\theta_i(\rho_i - u_i) - \eta_i y_i - \beta_i u_i),$$

where  $\theta_i$  is the adjoint state for class  $i$ ,  $\eta_i$  is the multiplier for to the pure state constraint  $y_i \geq 0$ , and  $\beta_i$  is multiplier for the constraint  $u_i \geq 0$ .

The necessary conditions for the optimality of  $\mathbf{u}$  are

$$\dot{\theta}_i(t) = -\dot{\mathcal{L}}_{y_i} \quad (19)$$

$$\dot{y}_i(t) = \dot{\mathcal{L}}_{\theta_i} \quad (20)$$

$$\mathbf{u}(t) = \arg \max_{\mathbf{u} \in \mathcal{U}} \mathcal{L} \quad (21)$$

$$\eta_i y_i = 0, \quad \boldsymbol{\eta} \geq 0, \quad (22)$$

$$\beta_i u_i = 0, \quad \boldsymbol{\beta} \geq 0, \quad (23)$$

In addition, due to the pure state constraints there can be jumps in the trajectories of the adjoint variables at points on the boundary of the state space. Let  $t_k$  be the time when one of the state constraints, say of class  $k$ , is satisfied with equality. Then,

$$\theta_k(t_k+) = \theta_k(t_k-) - \nu_k$$

From [15], one infers that the adjoint variables are continuous if the boundary is reached non-tangentially, that is  $\dot{y}_i(t_k-) \neq 0$  when  $y_i$  reaches the boundary. We shall assume that this is indeed the case.

**Assumption 3.1.** *For all  $k$ , the adjoint variable  $\theta_k(t)$  are continuous in  $t$ .*

Using the exact form of the cost rate given in (4), the necessary condition of the optimality of the control (21) (or, equivalently  $\dot{\mathcal{L}}_{\mathbf{u}} = 0$ ) becomes

$$\kappa\gamma \left( \sum_i u_i(t) \right)^{\gamma-1} - (\theta_k(t) + \beta_k(t)) = 0, \quad \forall k \in \mathcal{P}. \quad (24)$$

### 3.2.1 Optimality of the $c\mu$ rule

Let  $\mathcal{P}(t) = \{i : y_i(t) > 0\}$ , i.e., it is the set of non-empty classes at time  $t$ .

**Theorem 3.1.** *At any time  $t > 0$ ,*

1.  $u_i(t) > 0$  iff either  $y_i(t) = 0$  or  $i = \arg \max\{\tilde{c}_j : j \in \mathcal{P}(t)\}$ ;
2. if  $y_i(t) = 0$ , then  $u_i(t) = \rho_i$ .

The result says that amongst the classes with non-zero fluid, it is optimal to only serve the class with the largest  $\tilde{c}$ . This is the same as the  $c\mu$  rule when energy costs are not taken into account.

The proof of this theorem will be based on the following intermediate result which states that only one class in  $\mathcal{P}$  can have a non-zero service rate. The rest of the proof then consists of starting at the end time  $T$  and determining the class that receives service at time  $t$ .

**Proposition 3.1.** *In any interval  $[t_1, t_2]$  such that  $\mathcal{P}(t) = \mathcal{P}(t_1)$ ,  $\forall t \in [t_1, t_2]$ ,  $u_i(t)u_j(t) = 0$ , except on a finite number of time instants.*

*Proof.* For any class  $k \in \mathcal{P}(t)$ , the complementarity conditions (22) imply that  $\eta_k(t) = 0$ . Combining this with (19), one obtains the dynamics of the adjoint variable as

$$\dot{\theta}_k(t) = -\tilde{c}_k, \quad \forall k \in \mathcal{P}(t),$$

---

<sup>2</sup>We shall first give the solution for the infinite capacity case, and later say how to include a finite capacity constraint in the solution.

which upon integration gives

$$\theta_k(t) = -\tilde{c}_k t + d_k, \forall k \in \mathcal{P}(t), \forall t \in [t_1, t_2], \quad (25)$$

where  $d_k$  are constants of integration.

Let  $i$  and  $j$  be two classes in  $\mathcal{P}(t)$  which in addition also satisfy  $u_i(t) > 0$  and  $u_j(t) > 0$ . From the complementarity conditions (23), it then follows that  $\beta_i(t) = \beta_j(t) = 0$ . Substituting this in the necessary conditions for the optimal control (24), we get

$$\theta_i(t) = \theta_j(t). \quad (26)$$

From (25)  $\theta_i$  and  $\theta_j$  are linear functions on  $[t_1, t_2]$  with different slopes, and thus, they can intersect at most in one point within this interval. Therefore, any pair of classes can have non-zero rate simulataneously at most one point in  $[t_1, t_2]$ . Since there are finite number of pairs, the desired result follows.  $\square$

**Corollary 1.** *There exists a  $\tau_N$  such that for all  $t \in [\tau_N, T]$  there is only one class in  $\mathcal{P}(t)$ .*

*Proof.* At  $T$  all classes reach 0, and before  $T$  two classes with non-zero fluid cannot be served simulatenously. Hence, the only way all the classes can be at 0 at  $T$  is that all classes except one were at 0 in some interval before  $T$ .  $\square$

The above result says that in any interval with the same set of non-empty classes, only one class within this set is served at any given time except possibly for a finite number of time instants. Morevoer, the interval  $[0, T]$  can be divided into intervals during which only one non-empty class is served.

*Proof of Theorem 3.1.* Let the class be ordered so that  $\tilde{c}_1 > \tilde{c}_2 > \dots > \tilde{c}_N$ . We will start at time  $T$  and roll the time backwards. The proof will rely on the dynamics of the adjoint variables and the complementarity conditions.

From corollary 1, there exists a  $\tau_N$  such that only one class has non-zero fluid in  $[\tau_N, T]$ . Also, for any  $k \notin \mathcal{P}(t)$ ,  $u_i = \rho_i$ , so that all classes are served at a positive rate in  $[\tau_N, T]$ . From (24) and the complementarity conditions (23),

$$\theta_i(t) - \theta_j(t) = 0, \forall i, j \text{ and } \forall t \in [\tau_N, T], \quad (27)$$

that is

$$\dot{\theta}_i(t) - \dot{\theta}_j(t) = 0, \forall i, j \text{ and } \forall t \in (\tau_N, T), \quad (28)$$

From (19) and (22),

$$\dot{\theta}_i = -\tilde{c}_i + \eta_i(t), \quad i \notin \mathcal{P}(t), \text{ and } \eta_i \geq 0, \quad (29)$$

$$\dot{\theta}_i = -\tilde{c}_i, \quad i \in \mathcal{P}(t). \quad (30)$$

The only way (28), (29), and (30) can be satisfied is if  $\eta_N = 0$ . Thus, class  $N$ , which has the smallest  $\tilde{c}$ , is the last one to finish.

We can take  $\tau_N$  to be the last time one of the other  $N - 1$  was non-zero. Then there is some  $\tau_{N-1}$  such that there are two classes with non-zero fluid in the interval  $[\tau_{N-1}, \tau_N)$ . Let us call one of these two classes  $i$  and the other one  $j$ . One of these is class  $N$ , and we now show that the other is class  $N - 1$ .

Let  $i$  be the class that is served and  $j$  be the class that is idle. From (30), for  $t \in [\tau_{N-1}, \tau_N)$ ,

$$\theta_i(t) = \tilde{c}_i(\tau_N - t) + d_i, \quad (31)$$

$$\theta_j(t) = \tilde{c}_j(\tau_N - t) + d_j. \quad (32)$$

From Assumption 3.1, which gives us the continuity of  $\theta_i$  and  $\theta_j$ , and (27), which tells us that  $\theta_i(\tau_N) = \theta_j(\tau_N)$ , we can conclude that  $d_i = d_j$ . Thus,

$$\theta_i(t) - \theta_j(t) = (\tilde{c}_i - \tilde{c}_j)t. \quad (33)$$

Further, from (23) and (24),

$$\theta_i(t) = \theta_j(t) + \beta_j(t), \quad t \in [\tau_{N-1}, \tau_N).$$

Since  $\beta_j(t) \geq 0$ , the above equality is possible only if  $\tilde{c}_i > \tilde{c}_j$ . This shows that  $j = N$ , that is, the class that is not served is class  $N$ .

In order to determine class  $i$ , we use the argument used in the beginning of this proof to determine the class that is last to be served. Consider all classes except class  $N$ . For these classes, (23) and by consequence (28) will be applicable in the interval  $[\tau_{N-1}, \tau_N)$ . Moreover, (29) and (30) also have to be satisfied. This can occur only if the class with the smallest  $\tilde{c}$  has the corresponding  $\eta = 0$ , which implies that  $\eta_{N-1} = 0$ , and that class  $i$  is class  $N - 1$ .

The above arguments can be recursively applied to show that within  $\mathcal{P}(t)$  the class with the largest  $\tilde{c}$  is the only one that receives non-zero service rate.  $\square$

Theorem 3.1 gives the structure of the optimal policy. It shows that it is optimal to allocate  $u_k(t) = \rho_i$  for those classes  $k$  with  $y_k(t) = 0$ , and to follow the  $c\mu$  rule for the classes with non-zero fluid. That is, for classes with non-zero fluid, it is optimal to serve only the class with the largest value of  $\tilde{c}_i$ , and that this class is served until it reaches 0.

We denote by  $\tau_k$  the first time when class  $k$  starts service. The interval  $[0, T]$  can be divided into sub-intervals  $[\tau_k, \tau_{k+1})$ ,  $k = 1, 2, \dots, N$  with  $\tau_1 = 0$ , and  $\tau_{N+1} = T$ . In the optimal policy, class  $k$  receives a non-zero service rate in the interval from  $[\tau_k, T)$  and a service rate of  $\rho_k$  in the interval  $[\tau_{k+1}, T)$ .

In order to completely characterize the optimal policy, we need to determine the speed at which class  $k$  is served in the interval  $[\tau_k, \tau_{k+1})$ , which we do next.

### 3.2.2 Optimal speed

In the interval  $[\tau_k, \tau_{k+1})$ ,  $\beta_k(t) = 0$  since  $u_k(t) > 0$ , and hence from (24) we have

$$u_k(t) = \left( \frac{\theta_k(t)}{\kappa\gamma} \right)^{\frac{1}{\gamma-1}} - \sum_{i \notin \mathcal{P}(t)} \rho_i, \quad (34)$$

so that, according to Theorem 3.1, we can write the optimal rate allocation as follows

$$u_i(t) = \begin{cases} \left( \frac{\theta_k(t)}{\kappa\gamma} \right)^{\frac{1}{\gamma-1}} - \sum_{j \notin \mathcal{P}(t)} \rho_j & \text{if } i = k \\ 0 & \text{if } i \in \mathcal{P}(t) \setminus \{k\} \\ \rho_i & \text{if } i \notin \mathcal{P}(t) \end{cases} \quad (35)$$

In order to determine an optimal control law  $(\mathbf{u}(t), T)$  we therefore need to solve for the state trajectory  $\mathbf{y}(t)$ , the vector of adjoint variables  $\boldsymbol{\theta}(t)$  and the final time  $T$ . We observe that (19)-(20) provides  $2N$  differential equations, so that the vectors  $\mathbf{y}(t)$  and  $\boldsymbol{\theta}(t)$  are known at each time instant  $t$  up to a constant. Including the final time  $T$ , there are thus  $2N + 1$  constants to be determined. Observe that if we can determine one of the constants, the values of the other  $2N$  constants can be inferred from the  $2N$  boundary conditions  $y_i(0)$  and  $y_i(T) = 0$ ,  $i = 1, \dots, N$ . The remaining equation is given by the transversal condition [11]:

$$\mathcal{L}(\mathbf{y}(T), \mathbf{u}(T), \boldsymbol{\theta}(T), \boldsymbol{\eta}(T), \boldsymbol{\beta}(T)) = 0, \quad (36)$$

from which the value of  $\theta_N(T)$  can be obtained.

Since  $y_i(T) = 0$  for all  $i$  and  $u_i(T) = \rho_i$  for all  $i \neq N$ , (36) yields

$$\kappa \left[ \sum_{i \neq N} \rho_i + u_N(T) \right]^\gamma + \theta_N(T) [\rho_N - u_N(T)] = 0. \quad (37)$$

Using (34) to replace  $\theta_N(T)$  in the above equation, we get

$$\kappa \left[ \sum_{i \neq N} \rho_i + u_N(T) \right]^\gamma + \kappa\gamma \left[ \sum_{i \neq N} \rho_i + u_N(T) \right]^{\gamma-1} [\rho_N - u_N(T)] = 0, \quad (38)$$

from which it follows that

$$u_N(T) = \frac{\sum_{i \neq N} \rho_i + \gamma \rho_N}{\gamma - 1}. \quad (39)$$

We note that  $u_N(T)$  is independent of  $T$  and the initial conditions, and depends only on the traffic intensities of the classes and  $\gamma$ . The next result gives the relations to compute the optimal speeds at other time instants.

From (37) and (39), we obtain the value of  $\theta_N(T)$  as

$$\theta_N(T) = \kappa\gamma \left( \frac{\gamma}{\gamma-1} \sum_i \rho_i \right)^{\gamma-1}. \quad (40)$$

Define for convenience  $\alpha = 1/(\gamma - 1)$ , and note that  $\tau_{N+1} = T$ .

**Proposition 3.2.** *For a fixed  $T$  and with  $\theta_N(\tau_{N+1})$  given in (40),*

1. *provided that  $\tau_{k+1} > 0$ , the service rate of class  $k$ , for  $k = 1, \dots, N$ , is given by*

$$u_k(t) = \left( \frac{\tilde{c}_k(\tau_{k+1} - t) + \theta_k(\tau_{k+1})}{\kappa\gamma} \right)^{\frac{1}{\gamma-1}} - \sum_{i \notin \mathcal{P}(t)} \rho_i, \quad t \in [\tau_k, \tau_{k+1}), \quad (41)$$

where  $\tau_k$  is the unique solution of

$$\begin{aligned} y_k(0) + \rho_k \tau_{k+1} + \sum_{i \notin \mathcal{P}(t)} \rho_i (\tau_{k+1} - \tau_k) &= \frac{\kappa\gamma}{(\alpha+1)\tilde{c}_k} \left( \left( \frac{\tilde{c}_k(\tau_{k+1} - \tau_k) + \theta_k(\tau_{k+1})}{\kappa\gamma} \right)^{\alpha+1} \right. \\ &\quad \left. - \left( \frac{\theta_k(\tau_{k+1})}{\kappa\gamma} \right)^{\alpha+1} \right) \end{aligned} \quad (42)$$

in the interval  $(-\infty, \tau_{k+1})$ .

2. *For  $k = 2, \dots, N$ , it holds that*

$$\theta_{k-1}(\tau_k) = \theta_k(\tau_{k+1}) + \tilde{c}_k(\tau_{k+1} - \tau_k). \quad (43)$$

3. *For  $k = 1, 2, \dots, N$ ,  $\tau_k$  is an increasing function of  $T$ .*

4.  *$T$  is the unique solution of  $\tau_1 = 0$ .*

The proof appears in Appendix A. Here, we explain how to recursively compute the optimal speeds. The recursion moves down from  $N$  to 1. Since  $\theta_N(\tau_{N+1})$  is given, it allows us to compute  $\tau_N$  and  $u_N(t)$ ,  $t \in [\tau_N, \tau_{N+1})$ . Using (43), one then computes  $\theta_{N-1}(\tau_N)$ , which then allows us to compute  $\tau_{N-1}$  and  $u_{N-1}(t)$ ,  $t \in [\tau_{N-1}, \tau_N)$ . This procedure is repeated until we arrive at  $k = 1$ . Finally, we have to compute the value of  $T$  which gives  $\tau_1 = 0$ . This is done by repeatedly computing the  $\tau$  until the right value of  $T$  is found.

**Remark 3** (Properties of the optimal policy). *Define the function  $\Theta(t) = \sum_k \mathbf{1}_{t \in [\tau_k, \tau_{k+1})} \theta_k(t)$ . From (34), the total server speed at time  $t$ ,  $u(t)$ , is given by*

$$u(t) = \left( \frac{\Theta(t)}{\kappa\gamma} \right)^{\frac{1}{\gamma-1}}.$$

*From analysis similar to the one done in Theorem 3.1 (see (31)), it can be shown that  $\Theta(t)$  is a piece-wise linear and decreasing function. Hence, in the optimal policy the total server speed is decreasing as a function of time and scales as  $(T - t)^{\frac{1}{\gamma-1}}$ .*

**Remark 4** (Finite capacity constraint). *The above analysis assumed that there was no constraint on the capacity. In case of a finite capacity constraint, a similar analysis can be done. The same equations as in Proposition 3.2 will hold until the time  $\sum_k u_k(t) = C_l$  or the initial conditions are satisfied. Between this time and 0 the server will be serving at the maximum speed  $C_l$ .*

### 3.3 Numerical results

We now describe the numerical results for the single-link case. In this section, we present one scenario. A second one is available in Appendix B. We shall compare the optimal stochastic policy obtained by solving a stochastic shortest path problem with the fluid policy obtained analytically in Proposition 3.2 as well as with Bocop.

We shall assume that  $\kappa=1$  and  $\gamma=3$ . The parameters of the traffic classes are as follows:  $\rho_0=0.2$ ,  $\rho_1=0.1$ ,  $\rho_2=0.3$ ,  $\mu_0c_0=5$ ,  $\mu_1c_1=6$  and  $\mu_2c_2=4$ . In Figure 4, we compare the optimal fluid rates allocated to each class with those that would be obtained under the optimal stochastic policy when the initial state is  $\mathbf{x}(0) = (10, 11, 12)$ . The fluid rates of each traffic class closely matches its stochastic rates and follows the  $c\mu$  rule. We

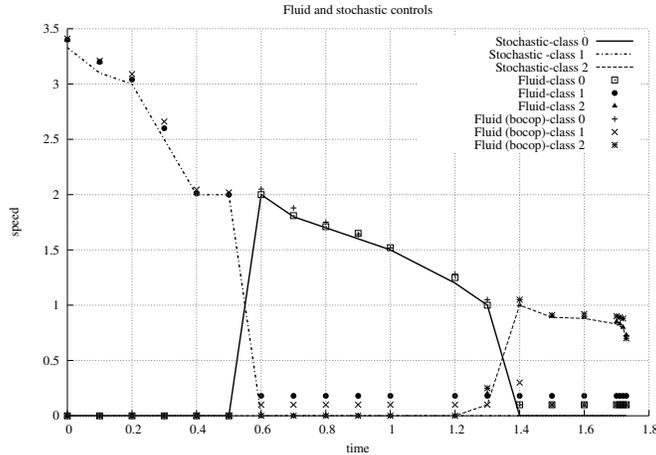


Figure 4: Optimal fluid speed vs. optimal stochastic speed in Scenario 1 for  $x_0(0)=10$ ,  $x_1(0)=11$  and  $x_2(0)=12$ .

compare sample trajectories obtained under the optimal stochastic policy and under the optimal fluid control applied using the discrete-review method in Figure 5 when the initial state is  $\mathbf{x}(0) = (10, 11, 12)$ .

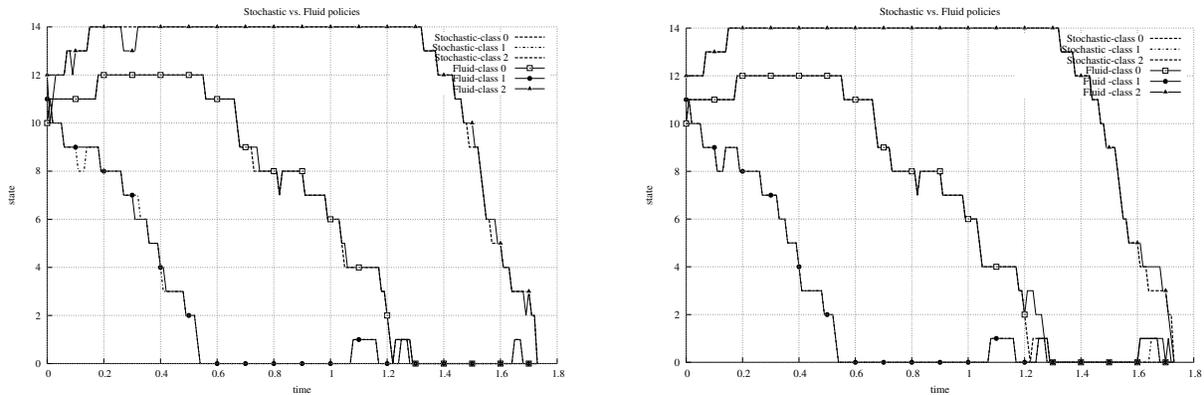


Figure 5: Sample trajectories obtained under the optimal stochastic policy and under the analytical fluid policy (left) and fluid policy from Bocop (right) for  $x_0(0)=10$ ,  $x_1(0)=11$  and  $x_2(0)=12$ .

## 4 Conclusion

In this paper, we modeled the energy-delay tradeoff in bandwidth-sharing networks with speed-scaling. Our objective was to determine the policy minimizing the total cost to drain the network. For the case of a linear network with two links, it was shown via numerical experiments that the fluid model together with

discrete-review provides a good approximation to the stochastic control problem. For a single link shared by an arbitrary number of classes, using Pontryagin's Maximum Principle we completely characterized the fluid optimal control policy. In particular, we established that the optimal fluid policy behaves according to the  $c\mu$  rule. Future work include the extension of the results obtained here for a single link to a linear network topology.

## 5 Acknowledgments

We would like to thank the anonymous referees of the previous version of this report for their comments and suggestions. In particular, we are grateful to the anonymous referee who pointed out a mistake in the proof of Proposition 3.2 that appeared in the previous version of this report.

## References

- [1] Bocop - the optimal control solver, <http://bocop.saclay.inria.fr>.
- [2] L.L.H. Andrew, A. Wierman, and A. Tang. Optimal speed scaling under arbitrary power functions. *SIGMETRICS Perform. Eval. Rev.*, 37(2):39–41, October 2009.
- [3] Matthew Andrews, Antonio Fernández Anta, Lisa Zhang, and Wenbo Zhao. Routing for power minimization in the speed scaling model. *IEEE/ACM Trans. Netw.*, 20(1):285–294, February 2012.
- [4] B. Ata and M. Tongarlak. On scheduling a multiclass queue with abandonments under general delay costs. *Queueing Systems*, pages 1–40, 2012.
- [5] F. Avram, D. Bertsimas, and M. Ricard. An optimal control approach to optimization of multiclass queueing networks. *Proceedings of Workshop on Queueing Networks*, 1994.
- [6] N. Bambos and J. Walrand. Scheduling and stability aspects of a general class of parallel processing systems. *Adv. in Appl. Probab.*, 25:176–202, 1993.
- [7] N. Bansal, H.L. Chan, and K. Pruhs. Speed scaling with an arbitrary power function. In *Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '09, pages 693–701, Philadelphia, PA, USA, 2009. Society for Industrial and Applied Mathematics.
- [8] N. Bäuerle. Asymptotic optimality of tracking policies in stochastic networks. *Ann. Appl. Probab*, 10:1065–1083, 2000.
- [9] D.P. Bertsekas. *Dynamic Programming and Optimal Control I*. Athena Scientific, Belmont, Massachusetts, 1995.
- [10] D. Bertsimas and G. Van Ryzin. A stochastic and dynamic vehicle routing problem in the euclidean plane. *Oper. Res.*, 39:601–615, 1991.
- [11] B. Chachuat. Nonlinear and dynamic optimization - from theory to practice. Technical report, Ecole Polytechnique Federale de Lausanne, 2006.
- [12] A. Gajrat and A. Hordijk. Fluid approximation of a controlled multiclass tandem network. *Queueing Systems*, 35:349–380, 2000.
- [13] N. Gans and G. Van Ryzin. Optimal control of a muticlass, flexible queueing system. *Oper. Res.*, 45:677–693, 1997.
- [14] J.M. Harrison. The bigstep approach to flow management in stochastic processing networks. *Stochastic Networks: Theory and Applications (F. Kelly, S. Zachary and I. Ziedins eds)*, pages 57–90, 1996.
- [15] R.F. Hartl, S.P. Sethi, and R.G. Vickson. A survey of the maximum principles for optimal control problems with state constraints. *SIAM Review*, 37(2):181–218, June 1995.

- [16] C. Maglaras. Discrete-review policies for scheduling stochastic networks: trajectory tracking and fluid-scale asymptotic optimality. *The Annals of Applied Probability*, 10(3):897–929, 2000.
- [17] Laurent Massoulié and James Roberts. Bandwidth sharing: Objectives and algorithms. *IEEE/ACM Trans. Netw.*, 10(3):320–328, June 2002.
- [18] Sean Meyn. *Control Techniques for Complex Networks*. Cambridge University Press, 2007. Cambridge Books Online.
- [19] P. Nain and D. Towsley. Optimal scheduling in a machine with stochastic varying processing rate. *IEEE Transactions on Automatic Control*, 39:1853–1855, 1994.
- [20] R. Righter and J. G. Shanthikumar. Scheduling multiclass single server queueing systems to stochastically maximize the number of successful departures. *Probability in the Engineering and Informational Sciences*, pages 323–333, 1989.
- [21] B. Tan, L. Ying, and R. Srikant. Short-term fairness and long-term qos in the internet. *Performance Evaluation*, 67:406–414, 2010.
- [22] L. Tassiulas and S. Papavassiliou. Optimal anticipative scheduling with asynchronous transmission opportunities. *IEEE Trans. Automat. Control*, 40:2052–2062, 1995.
- [23] I.M Verloop and S.C. Borst. Heavy-traffic delay minimization in bandwidth-sharing networks. *In: Proc. IEEE Infocom*, 2007.
- [24] I.M. Verloop and R. Núñez-Queija. Asymptotically optimal parallel resource assignment with interference. *Queueing Systems*, 65:43–92, 2010.
- [25] Adam Wierman, Lachlan L. H. Andrew, and Ao Tang. Power-aware speed scaling in processor sharing systems: Optimality and robustness. *Perform. Eval.*, 69(12):601–622, December 2012.

## A Proof of Proposition 3.2

*Proof.* 1. Let  $k < N$ . For  $t \in [\tau_k, \tau_{k+1})$  the arguments used to obtain (31) and (32) are applicable not only to classes  $N - 1$  and  $N$  but also to any classes  $k$  and  $k + 1$ , where  $k$  is the highest priority non-zero class. Similarly, the continuity assumption and (27) can be applied to obtain  $d_k = d_{k+1} = \theta_{k+1}(\tau_{k+1})$ . Thus,

$$\theta_k(t) = \tilde{c}_k(\tau_{k+1} - t) + \theta_{k+1}(\tau_{k+1}), \quad t \in [\tau_k, \tau_{k+1}). \quad (44)$$

The relation (41) now follows from (34) and the above equation.

For  $k = N$ , from (30) and the given value of  $\theta_N(\tau_{N+1})$ , we get

$$\theta_N(t) = \tilde{c}_N(\tau_{N+1} - t) + \theta_N(\tau_{N+1}), \quad t \in [\tau_N, \tau_{N+1}).$$

For (42), first we use the dynamics (13) of  $y_k$  and the boundary condition  $y_k(\tau_{k+1}) = 0$ , that is,

$$y_k(\tau_{k+1}) = y_k(0) + \rho_k \tau_{k+1} - \int_{\tau_k}^{\tau_{k+1}} u_k(t) dt, = 0.$$

which when combined with (41) gives (42).

We now show that  $\tau_k$  is the unique solution of (42). For a fixed  $\tau_{k+1}$ , let  $\Delta_k = \tau_{k+1} - \tau_k$ . We can rewrite (42) as

$$g(\Delta) = y_k(0) + \rho_k \tau_{k+1},$$

where  $g$  is defined using the remaining terms in (42). Note that the right-hand side of the above equation is strictly positive whereas  $g(0) = 0$ . Also,  $g'(\Delta) = u_k(\tau_k) > 0$  for any  $\Delta > 0$ . Therefore,  $g$  is an increasing function in the interval  $[0, \infty)$  and there is a unique value of  $\Delta$  satisfying the above equation. This shows that there is a unique  $\tau_k \in (-\infty, \tau_{k+1})$ .

2. Integrating (30), we get  $\theta_k(\tau_k) = \tilde{c}_k(\tau_{k+1} - \tau_k) + \theta_k(\tau_{k+1})$ , where  $\theta_k(\tau_{k+1})$  is well defined by Assumption 3.1. Using  $k-1$  instead of  $k$  in (44) and combining with the previous equality, we get

$$\theta_{k-1}(\tau_k) = \theta_k(\tau_{k+1}) + \tilde{c}_k(\tau_{k+1} - \tau_k).$$

3. We shall show this property using recursion. Consider a  $k < N$  and assume that  $\tau_i$  is increasing for  $i > k$ . For convenience, define  $\rho_{-i} = \sum_{j < i} \rho_j$ . Our starting point is (42) which defines  $\tau_k$ . Note that  $\tau_i$  and  $\theta_i, \forall i$  are function of  $T$ . We differentiate (42) with respect to  $T$  to obtain

$$\rho_k \tau'_{k+1} + \rho_{-k}(\tau'_{k+1} - \tau'_k) = u(\tau_k)(\tau'_{k+1} - \tau'_k) + \frac{u(\tau_k) - u(\tau_{k+1})}{\tilde{c}_k} \theta'_k,$$

where  $u(t) = \sum_i u_i(t)$  is the total server speed at time  $t$ . Upon rearranging the above equation we get

$$\tau'_k = \tau'_{k+1} + (u(\tau_k) - \rho_{-k})^{-1} \left[ -\rho_k \tau'_{k+1} + \frac{u(\tau_k) - u(\tau_{k+1})}{\tilde{c}_k} \theta'_k \right]. \quad (45)$$

Our base case is  $k = N$ . Note that  $\theta_N(T)$  is a constant independent of  $T$ , and that  $\tau_{k+1} = T$ . Hence, (45) simplifies to

$$\tau'_k = 1 - (u(\tau_k) - \rho_{-k})^{-1} \rho_k.$$

Now,  $u(\tau_k) - \rho_{-k}$  is the speed at which class  $k$  is served in the interval  $[\tau_k, \tau_{k+1})$ . This speed is larger than  $\rho_k$ , and hence the right-hand side of the above equation is positive. This concludes the base case.

For  $k < N$ , we first investigate the term  $\theta'_k$  in (45). Developing the recursion (43), we obtain

$$\theta_k(\tau_{k+1}) = \tilde{c}_N T - \tilde{c}_{k+1} \tau_{k+1} + \sum_{j=k+2}^N (\tilde{c}_{j-1} - \tilde{c}_j) \tau_j,$$

which upon differentiation gives

$$\theta_k(\tau_{k+1})' = \tilde{c}_N - \tilde{c}_k \tau'_{k+1} + \sum_{j=k+2}^N (\tilde{c}_{j-1} - \tilde{c}_j) \tau'_j,$$

Since  $\tilde{c}_{j-1} - \tilde{c}_j > 0$  and  $\tau'_j$  are also positive (by induction hypothesis),  $\theta'_k > -\tilde{c}_{k+1} \tau'_{k+1} > -\tilde{c}_k \tau'_{k+1}$ . Substituting this inequality in (45), we get

$$\begin{aligned} \tau'_k &> \tau'_{k+1} + (u(\tau_k) - \rho_{-k})^{-1} [-\rho_k - u(\tau_k) + u(\tau_{k+1})] \tau_{k+1}' \\ &= \tau'_{k+1} + (u(\tau_k) - \rho_{-k})^{-1} [-(u(\tau_k) - \rho_{-k}) + u(\tau_{k+1}) - \rho_{-(k+1)}] \tau_{k+1}' \\ &= (u(\tau_k) - \rho_{-k})^{-1} [u(\tau_{k+1}) - \rho_{-(k+1)}] \tau_{k+1}' \\ &> 0 \quad (\text{since } u(\tau_{k+1}) - \rho_{-(k+1)} = u_{k+1}(\tau_{k+1}) > 0). \end{aligned}$$

4. Since  $\tau_1$  is an increasing function of  $T$ , there is a unique  $T$  such that  $\tau_1 = 0$ . □

## B A second numerical experiment

As a second scenario, we again consider the case of three classes, (the value of  $\kappa$  (resp  $\gamma$ ) is still 0.5 (1 resp)). The parameters of the traffic classes are as follows:  $\rho_0=0.5, \rho_1=0.6, \rho_2=0.6, \mu_0 c_0=4, \mu_1 c_1=2$  and  $\mu_2 c_2=8$ .

In Figure 6, we compare the optimal fluid rates allocated to each class with those that would be obtained under the optimal stochastic policy when the initial state is  $\mathbf{x}(0) = (11, 12, 13)$ . The fluid rates of each traffic class closely matches its stochastic rates and follows the  $c\mu$  rule.

Figure 7, 8 show a sample path of class 1, 2 and 3 under the optimal fluid control when the initial state is  $\mathbf{x}(0) = (11, 12, 13)$ .

In figure 9, 10, we compare a sample path of the three classes for  $x_0(0)=9, x_1(0)=6$  and  $x_2(0) = 5$ .

Again, we note that these trajectories closely match those obtained under the optimal stochastic policy, and that both policies follow the  $c\mu$  rule. Indeed, they both give priority to class-2 which has a higher value of  $c_i \mu_i$  ( $c_2 \mu_2 = 8 > 4 = c_0 \mu_0 > 2 = c_1 \mu_1$ ).

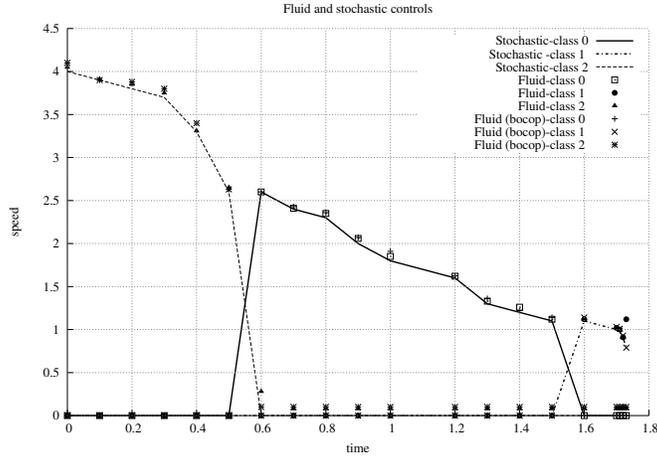


Figure 6: Optimal fluid speed vs. optimal stochastic speed in Scenario 2 for  $x_0(0)=11$ ,  $x_1(0)=12$  and  $x_1(0)=13$ .

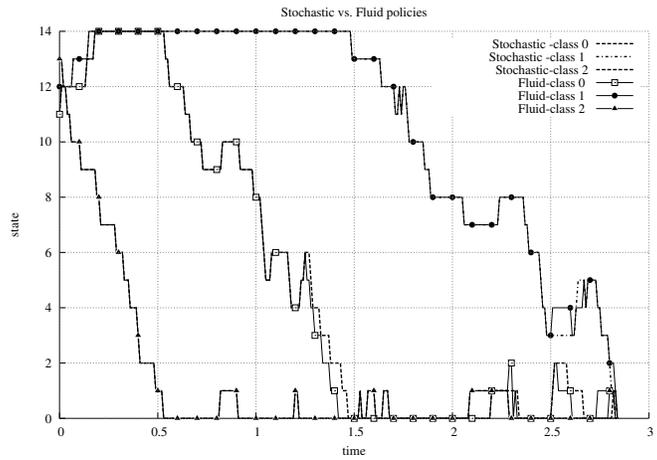


Figure 7: Sample trajectories obtained under the optimal stochastic policy and under the fluid policy (obtained in section 3) for  $x_0(0)=11$ ,  $x_1(0)=12$  and  $x_2(0) = 13$ .

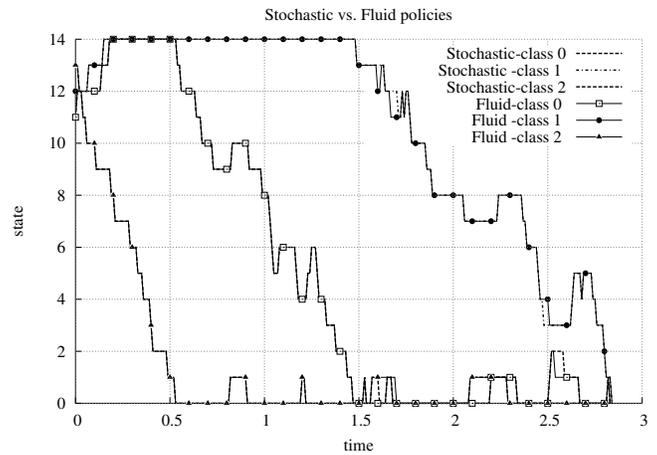


Figure 8: Sample trajectories obtained under the optimal stochastic policy and under the fluid policy (bocop) for  $x_0(0)=11$ ,  $x_1(0)=12$  and  $x_2(0) = 13$ .

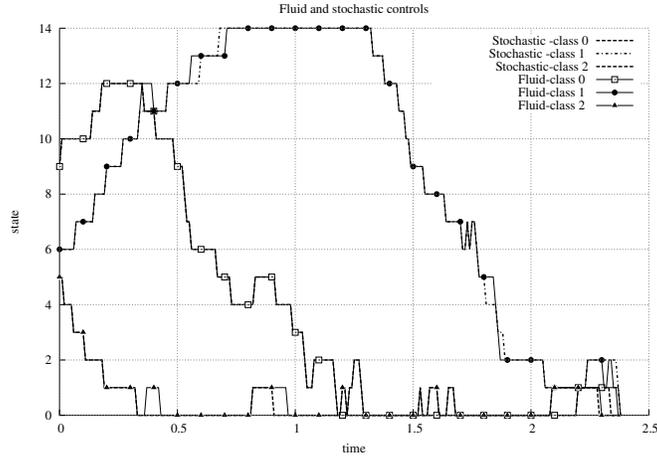


Figure 9: Sample trajectories obtained under the optimal stochastic policy and under the fluid policy (obtained in section 3) for  $x_0(0)=9$ ,  $x_1(0)=6$  and  $x_2(0) = 5$ .

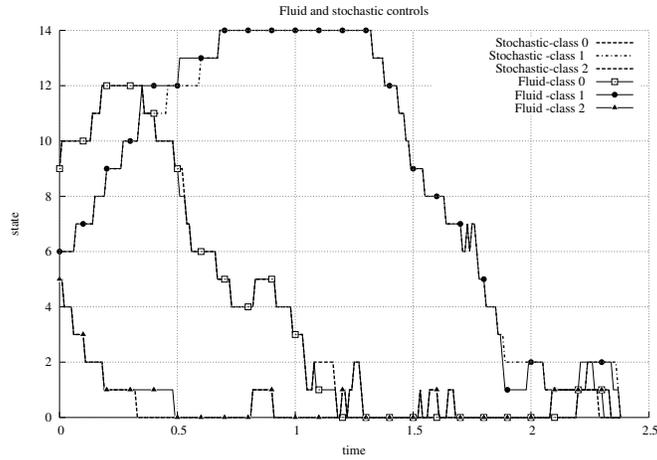


Figure 10: Sample trajectories obtained under the optimal stochastic policy and under the fluid policy (bocop) for  $x_0(0)=9$ ,  $x_1(0)=6$  and  $x_2(0) = 5$ .