



HAL
open science

Sign changes in short intervals of coefficients of spinor Zeta function of a Siegel cusp form of genus 2

Emmanuel Royer, Jyoti Sengupta, Jie Wu

► **To cite this version:**

Emmanuel Royer, Jyoti Sengupta, Jie Wu. Sign changes in short intervals of coefficients of spinor Zeta function of a Siegel cusp form of genus 2. *International Journal of Number Theory*, 2014, 10 (2), pp.327 - 339. 10.1142/S1793042113500966 . hal-01093998

HAL Id: hal-01093998

<https://hal.science/hal-01093998>

Submitted on 11 Dec 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

SIGN CHANGES IN SHORT INTERVALS OF COEFFICIENTS OF SPINOR ZETA FUNCTION OF A SIEGEL CUSP FORM OF GENUS 2

EMMANUEL ROYER, JYOTI SENGUPTA, AND JIE WU

ABSTRACT. In this paper, we establish a Voronoi formula for the spinor zeta function of a Siegel cusp form of genus 2. We deduce from this formula quantitative results on the number of its positive (resp. negative) coefficients in some short intervals.

CONTENTS

1. Introduction	1
2. Truncated Voronoi formula	3
3. Proof of the Theorem	6
References	10

1. INTRODUCTION

Let S_k be the space of Siegel cusp forms of integral weight k on the group $\mathrm{Sp}_4(\mathbb{Z}) \subset \mathrm{GL}_4(\mathbb{Q})$ and let $F \in S_k$ be an eigenfunction of all the Hecke operators. Let

$$Z_F(s) := \prod_{p \in \mathcal{P}} Z_{F,p}(p^{-s}) \quad (\mathrm{Re} s > 1)$$

be the spinor zeta function of F . Here \mathcal{P} is the set of prime numbers and if $\alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p}$ are the Satake p -parameters attached to F then

$$Z_{F,p}(t)^{-1} := (1 - \alpha_{0,p}t)(1 - \alpha_{0,p}\alpha_{1,p}t)(1 - \alpha_{0,p}\alpha_{2,p}t)(1 - \alpha_{0,p}\alpha_{1,p}\alpha_{2,p}t).$$

They satisfy

$$\alpha_{0,p}^2 \alpha_{1,p} \alpha_{2,p} = 1$$

for all p . A Siegel form is in the Maass subspace S_k^M of S_k if it is a linear combination of Siegel forms F that are eigenvectors of all the Hecke operators and for which there

2010 *Mathematics Subject Classification.* 11F46, 11F30, 11M41, 11N37, 11N56.

Key words and phrases. Spinor zeta function, Siegel form, Fourier coefficients, Voronoi formula.

The work was supported by a grant from the Indo-French Centre for the Promotion of Advanced Research (CEFIPRA Project No. 4601-2). Part of this work has been done during the visit of the second author at l'Institut Élie Cartan de Lorraine and finished during the visit of the first and third authors at Tata Institute of Fundamental Research. They would like to thank these institutes for hospitality.

exists a primitive modular form, f , of weight $2k - 2$ such that

$$Z_F(s) = \zeta\left(s - \frac{1}{2}\right)\zeta\left(s + \frac{1}{2}\right)L(f, s).$$

Here $L(f, s)$ is the L-function of f . This happens only if k is even. The bijective linear application between S_k^M and the space of modular forms of weight $2k - 2$ is called the Saito-Kurokawa lifting [16]. The Ramanujan-Petersson conjecture says that

$$(1) \quad |\alpha_{j,p}| = 1 \text{ for } j = 0, 1, 2 \text{ and all primes } p.$$

It is not true for Siegel Hecke-eigenforms in S_k^M . But, if k is odd or, if k is even and in the orthogonal complement of S_k^M , then it has been established by Weissauer [15]. We denote by H_k^* the set of Siegel cuspidal Hecke-eigenforms of weight k and genus 2 that, if k is even, are in the orthogonal complement of S_k^M . The forms we consider in this paper all belong to H_k^* . According to Breulmann [2], a Siegel Hecke-eigenform is in S_k^M if and only if all its Hecke eigenvalues are positive.

According to [1, 6], the function

$$\Lambda_F(s) := (2\pi)^{-s}\Gamma\left(s + k - \frac{3}{2}\right)\Gamma\left(s + \frac{1}{2}\right)Z_F(s)$$

has an entire continuation to \mathbb{C} since $F \in H_k^*$. Further it satisfies the functional equation

$$(2) \quad \Lambda_F(s) = (-1)^k \Lambda_F(1 - s)$$

on \mathbb{C} . The spinor zeta function of F has the Dirichlet expansion:

$$Z_F(s) = \sum_{n \geq 1} a_F(n) n^{-s}$$

for $\text{Re } s > 1$. By using (1), one sees that

$$(3) \quad |a_F(n)| \leq d_4(n)$$

for all $n \geq 1$, where $d_4(n)$ is the number of solutions in positive integers a, b, c, d of $n = abcd$.

In this paper, we investigate the problem of sign changes for the sequence $(a_F(n))_{n \geq 1}$ in short intervals. Define

$$\mathcal{N}_F^\pm(x) := \sum_{\substack{n \leq x \\ a_F(n) \geq 0}} 1.$$

We apply a method due to Lau & Tsang [11] to establish the following Theorem. Convergence issues however appear and we have to deal with them.

Theorem— *Let F be in H_k^* and $\varepsilon > 0$. There are constants $c > 0$ absolute and $x_0(F)$ depending only on F such that for all $x \geq x_0(F)$, we have*

$$\mathcal{N}_F^\pm(x + cx^{3/4}) - \mathcal{N}_F^\pm(x) \gg x^{3/8-\varepsilon},$$

where the implied constant in \gg depends only on ε .

Remark - An ingredient of our proof is the inequality

$$(4) \quad \sum_{n \leq x} a_F(n) \ll_{F, \varepsilon} x^{3/5+\varepsilon} \quad (x \geq 2).$$

(see Lemma 1). We also prove, and use an Omega-result:

$$\sum_{n \leq x} a_F(n) = \Omega_{\pm}(x^{3/8})$$

(see Lemma 2).

Two related problems have already been studied. Denote by $\lambda_F(n)$ the n -th normalised Hecke eigenvalue of F . Then we have

$$(5) \quad \sum_{n=1}^{\infty} \frac{\lambda_F(n)}{n^s} = \frac{Z_F(s)}{\zeta(2s+1)} \quad (\text{Res} > 1).$$

In [8], Kohnen proved that

$$\#\{n \leq x: \lambda_F(n) \geq 0\} \rightarrow \infty \quad (x \rightarrow \infty).$$

Then, Das [5] proved that, as x tends to $+\infty$, the quantities

$$\frac{1}{\#\{p \in \mathcal{P}: p \leq x\}} \#\{p \in \mathcal{P} \cap [1, x]: \lambda_F(p) \geq 0\}$$

are bounded from below (and naturally also bounded from above). In [9], Kohnen & Sengupta proved that under the same assumption there is an integer $n \ll k^2(\log k)^{20}$ such that $\lambda_F(n) < 0$. Their result has been generalised to higher levels by Brown [3].

Remark - Das's result is on the counting function of the Hecke eigenvalues. It implies however the result on the coefficients of the spinor zeta function since (5) implies

$$a_F(n) = \sum_{\substack{(d,m) \in \mathbb{N}^2 \\ d^2 m = n}} \frac{\lambda_F(m)}{d}.$$

Moreover, the proof of Kohnen & Sengupta can be adapted to prove that there is an integer $n \ll k^2(\log k)^{20}$ such that $a_F(n) < 0$.

2. TRUNCATED VORONOI FORMULA

The aim of this section is to establish the following truncated Voronoi formula, which will be needed in the proof of the Theorem.

Lemma 1 – Let F be in H_k^* . Then for any $A > 0$ and $\varepsilon > 0$, we have

$$(6) \quad \sum_{n \leq x} a_F(n) = \frac{x^{3/8}}{(2\pi)^{3/4}} \sum_{n \leq M} \frac{a_F(n)}{n^{5/8}} \cos\left(4\sqrt{2\pi}(nx)^{1/4} + \frac{\pi}{4}\right) + O_{A,F,\varepsilon}\left((x^3 M^{-1})^{1/4+\varepsilon} + (xM)^{1/4+\varepsilon}\right)$$

uniformly for $x \geq 2$ and $1 \leq M \leq x^A$, where the implied constant depends on A , F and ε only. In particular

$$(7) \quad \sum_{n \leq x} a_F(n) \ll_{F,\varepsilon} x^{3/5+\varepsilon} \quad (x \geq 2).$$

Proof. Without loss of generality, we assume that $M \in \mathbb{N}$. Let $\kappa := 1 + \varepsilon$ and

$$(8) \quad T^4 = 4\pi^2(M + \frac{1}{2})x.$$

By the Perron formula (see [14, Corollary II.2.4]) we have

$$(9) \quad \sum_{n \leq x} a_F(n) = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} Z_F(s) \frac{x^s}{s} ds + O_{F,\varepsilon}(x^{3/4+\varepsilon}M^{-1/4} + x^\varepsilon).$$

We shift the line of integration horizontally to $\text{Re } s = -\varepsilon$, the main term gives

$$\frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} Z_F(s) \frac{x^s}{s} ds = Z_F(0) + \frac{1}{2\pi i} \int_{\mathcal{L}} Z_F(s) \frac{x^s}{s} ds,$$

where \mathcal{L} is the contour joining the points $\kappa \pm iT$ and $-\varepsilon \pm iT$. Using the convexity bound [12, §1.3]

$$Z_F(\sigma + it) \ll_{F,\varepsilon} (|t| + 1)^{\max\{2(1-\sigma), 0\} + \varepsilon} \quad (-\varepsilon \leq \sigma \leq \kappa),$$

the integrals over the horizontal segments and the term $Z_F(0)$ can be absorbed in

$$O_{F,\varepsilon}\left((Tx)^\varepsilon(T + T^{-1}x)\right) = O_{F,\varepsilon}\left(x^{1/4+\varepsilon}M^{1/4} + x^{3/4+\varepsilon}M^{-1/4}\right).$$

To handle the integral over the vertical segment $\mathcal{L}_v := [-\varepsilon - iT, -\varepsilon + iT]$, we invoke the functional equation (2). We deduce that

$$(10) \quad \frac{1}{2\pi i} \int_{\mathcal{L}_v} Z_F(s) \frac{x^s}{s} ds = (-1)^k \sum_{n \geq 1} \frac{a_F(n)}{n} I_{\mathcal{L}_v}(nx),$$

where

$$I_{\mathcal{L}_v}(y) := \frac{1}{2\pi i} \int_{\mathcal{L}_v} (2\pi)^{2s-1} \frac{\Gamma(k - \frac{1}{2} - s)\Gamma(\frac{3}{2} - s)}{\Gamma(s + k - \frac{3}{2})\Gamma(s + \frac{1}{2})} \frac{y^s}{s} ds.$$

By using the Stirling formula

$$\Gamma(\sigma + it) = \sqrt{2\pi}|t|^{\sigma-1/2} e^{-\pi|t|/2 + i(t \log|t| - t) + i \text{sgn}(t)(\pi/2)(\sigma-1/2)} \{1 + O(t^{-1})\}$$

uniformly for $\sigma_1 \leq \sigma \leq \sigma_2$ and $|t| \geq 1$, the quotient of the four gamma factors is

$$(11) \quad |t|^{2-4\sigma} e^{-4i(t \log|t| - t) + i \text{sgn}(t)\pi(1-k)} \{1 + O(t^{-1})\}$$

for bounded σ and any $|t| \geq 1$, where the implied constant depends on σ and k . Together with the second mean value theorem for integrals [14, Theorem I.0.3], we obtain

$$(12) \quad \begin{aligned} I_{\mathcal{L}_v}(nx) &\ll (nx)^{-\varepsilon} \left(\left| \int_1^T t^{1+4\varepsilon} e^{-ig(t)} dt \right| + T^{1+4\varepsilon} \right) \\ &\ll T \left(\frac{T^4}{nx} \right)^\varepsilon \left(\left| \int_a^T e^{-ig(t)} dt \right| + 1 \right) \end{aligned}$$

for some $1 \leq a \leq T$, where $g(t) := t \log(t^4/(4\pi^2 nx)) - 4t$. In view of (8), we have

$$g'(t) = -\log(4\pi^2 nx/t^4) < 0 \quad \text{and} \quad |g'(t)| \geq |\log(n/(M + \frac{1}{2}))|$$

for $n \geq M + 1$ and $1 \leq t \leq T$. Using (3) and [14, Theorem I.6.2], we infer that

$$\begin{aligned} \sum_{n>M} \frac{a_F(n)}{n} I_{\mathcal{L}_V}(nx) &\ll T \left(\frac{T^4}{x} \right)^\varepsilon \sum_{n>M} \frac{d_4(n)}{n^{1+\varepsilon}} \left(\left| \log \frac{n}{M + \frac{1}{2}} \right|^{-1} + 1 \right) \\ (13) \quad &\ll T \left(\frac{T^4}{x} \right)^\varepsilon \left\{ \sum_{M < n \leq 2M} \frac{d_4(n)(M + \frac{1}{2})}{n^{1+\varepsilon}|n - M - \frac{1}{2}|} + \frac{1}{M^{\varepsilon/2}} \right\} \\ &\ll T \left(\frac{T^4}{\sqrt{M}x} \right)^\varepsilon \\ &\ll Tx^\varepsilon. \end{aligned}$$

For $n \leq M$, we extend the segment of integration \mathcal{L}_V to an infinite line \mathcal{L}_V^* in order to apply Lemma 1 in [4]. Write

$$\mathcal{L}_V^\pm := [\frac{1}{2} + \varepsilon \pm iT, \frac{1}{2} + \varepsilon \pm i\infty), \quad \mathcal{L}_h^\pm := [-\varepsilon \pm iT, \frac{1}{2} + \varepsilon \pm iT]$$

and define \mathcal{L}_V^* to be the positively oriented contour consisting of \mathcal{L}_V , \mathcal{L}_V^\pm and \mathcal{L}_h^\pm . In view of (11), the contribution over the horizontal segments \mathcal{L}_h^\pm is

$$\begin{aligned} I_{\mathcal{L}_h^\pm}(nx) &\ll \int_{-\varepsilon}^{1/2-\varepsilon} (2\pi)^{2\sigma-1} T^{2-4\sigma} \frac{(nx)^\sigma}{T} d\sigma \\ &\ll T \int_{-\varepsilon}^{1/2-\varepsilon} \left(\frac{nx}{T^4} \right)^\sigma d\sigma \\ &\ll Tx^\varepsilon. \end{aligned}$$

As in (12), for $n \leq M$ we get that

$$\begin{aligned} I_{\mathcal{L}_V^\pm}(nx) &\ll (nx)^{1/2+\varepsilon} \left(\int_T^\infty t^{-1-4\varepsilon} e^{-ig(t)} dt + \frac{1}{T^{1+4\varepsilon}} \right) \\ &\ll T \left(\frac{nx}{T^4} \right)^{1/2+\varepsilon} \left(\left| \log \frac{M + \frac{1}{2}}{n} \right|^{-1} + 1 \right) \\ &\ll T \left(\left| \log \frac{M + \frac{1}{2}}{n} \right|^{-1} + 1 \right). \end{aligned}$$

So

$$\begin{aligned} \sum_{n \leq M} \frac{a_F(n)}{n} (I_{\mathcal{L}_V^\pm}(nx) + I_{\mathcal{L}_h^\pm}(nx)) &\ll Tx^{\varepsilon/2} \sum_{n \leq M} \frac{d_4(n)}{n} \left(\left| \log \frac{M + \frac{1}{2}}{n} \right|^{-1} + 1 \right) \\ (14) \quad &\ll Tx^{\varepsilon/2} \sum_{n \leq M} \frac{d_4(n)(M + \frac{1}{2})}{n|n - M - \frac{1}{2}|} + Tx^\varepsilon \\ &\ll Tx^\varepsilon. \end{aligned}$$

Define

$$I_{\mathcal{L}_v^*}(y) = \frac{1}{4\pi^2 i} \int_{\mathcal{L}_v^*} \frac{\Gamma(k - \frac{1}{2} - s)\Gamma(\frac{3}{2} - s)\Gamma(s)}{\Gamma(s + k - \frac{3}{2})\Gamma(s + \frac{1}{2})\Gamma(1 + s)} (4\pi^2 y)^s ds.$$

After a change of variable s into $1 - s$, we see that

$$I_{\mathcal{L}_v^*}(y) = \frac{I_0(4\pi^2 y)}{2\pi},$$

with

$$I_0(t) := \frac{1}{2\pi i} \int_{\mathcal{L}_\varepsilon} \frac{\Gamma(s + k - \frac{3}{2})\Gamma(s + \frac{1}{2})\Gamma(1 - s)}{\Gamma(k - \frac{1}{2} - s)\Gamma(\frac{3}{2} - s)\Gamma(2 - s)} t^{1-s} ds.$$

Here \mathcal{L}_ε consists of the line $s = \frac{1}{2} - \varepsilon + i\tau$ with $|\tau| \geq T$, together with three sides of the rectangle whose vertices are $\frac{1}{2} - \varepsilon - iT$, $1 + \varepsilon - iT$, $1 + \varepsilon + iT$ and $\frac{1}{2} - \varepsilon + iT$. Note that all the poles of the integrand in $I_0(t)$ lie on the left of the line \mathcal{L}_ε .

Using a result due to Chandrasekharan and Narasimhan [4, Lemma 1] generalised by Lau & Tsang [11, Lemma 2.2] we obtain (note that a factor $\sqrt{2}$ is missing for the definition of e_0 in both references)

$$I_0(t) = \frac{(-1)^k}{\sqrt{2\pi}} t^{3/8} \cos\left(4t^{1/4} + \frac{\pi}{4}\right) + O(t^{1/8}).$$

It hence follows that

$$(15) \quad I_{\mathcal{L}_v^*}(nx) = (-1)^k \frac{(nx)^{3/8}}{(2i)^{3/4}} \cos\left(4\sqrt{2\pi}(nx)^{1/4} + \frac{\pi}{4}\right) + O((nx)^{1/8}).$$

We conclude

$$(16) \quad \sum_{n \leq M} \frac{a_F(n)}{n} I_{\mathcal{L}_v^*}(nx) = \frac{(-1)^k}{(2\pi)^{3/4}} x^{3/8} \sum_{n \leq M} \frac{a_F(n)}{n^{5/8}} \cos\left(4\sqrt{2\pi}(nx)^{1/4} + \frac{\pi}{4}\right) + O(x^{1/4+\varepsilon} M^{1/4})$$

from (14) and (15). Finally the asymptotic formula (6) by (9)-(10), (13) and (16).

Since

$$x^{3/8} \sum_{n \leq M} \frac{a_F(n)}{n^{5/8}} \cos\left(4\sqrt{2\pi}(nx)^{1/4} + \frac{\pi}{4}\right) \ll (xM)^{3/8+\varepsilon},$$

the choice of $M = x^{3/5}$ in (6) gives (7). □

3. PROOF OF THE THEOREM

We establish a lemma that has a similar statement as a one due to Lau & Wu [10, Lemma 3.2]. However, due to convergence issue, the proof is more delicate.

Lemma 2– Let F be in H_k^* . Define

$$S_F(x) := \sum_{n \leq x} a_F(n).$$

There exist positive absolute constants C, c_1, c_2 and $X_0(F)$ depending only on F such that for all $X \geq X_0(F)$, we can find $x_1, x_2 \in [X, X + CX^{3/4}]$ for which

$$S_F(x_1) > c_1 X^{3/8} \quad \text{and} \quad S_F(x_2) < -c_2 X^{3/8}.$$

Proof. We begin the proof with Theorem C of Hafner [7]. In order to use this result, it is more convenient to introduce the notion of (C, ℓ) -summability and to present related simple facts (see [13] for more details). Let $\{g_n(t)\}_{n \geq 0}$ be a sequence of functions. We write

$$s(g; n) := \sum_{0 \leq \nu \leq n} g_\nu(t), \quad \sigma(g; n) := \frac{1}{C_n^{(\ell+1)}} \sum_{\nu=0}^n C_{n-\nu}^{(\ell)} s(g; \nu),$$

where $C_n^{(\ell)} := \binom{\ell+n-1}{n}$. We say that the series of general term $g_n(t)$ is uniformly (C, ℓ) -summable to the sum $G(t)$ if $\sigma(g; n)$ converges uniformly to $G(t)$ as $n \rightarrow \infty$. We have $C_0^{(\ell)} + \dots + C_n^{(\ell)} = C_n^{(\ell+1)}$ and if the series $\sum_n \int g_n(t) dt$ converges then the series of general term $\int g_n(t) dt$ is also (C, ℓ) -summable and their limits are the same.

As in [7, page 151], for $\rho > -1$ and $x \notin 2\pi\mathbb{N}$, define

$$A_\rho(x) := \frac{1}{\Gamma(\rho+1)} \sum_{2\pi n \leq x} a_F(n) (x - 2\pi n)^\rho.$$

Now let \mathcal{C} be the rectangle with vertices $c \pm iR$ and $1 - b \pm iR$ (taken in the counter-clockwise direction), where $b > c > \max\{1, |k - \frac{3}{2}|\}$ and $R > |k - \frac{3}{2}|$ are real numbers. Let

$$Q_\rho(x) := \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(s)(2\pi)^{-s} Z_F(s)}{\Gamma(s+\rho+1)} x^{\rho+s} ds.$$

Denote by $\mathcal{C}_{0,b}$ the oriented polygonal path with vertices $-i\infty, -iR, b - iR, b + iR, iR$ and $+i\infty$. Let

$$f_\rho(x) := \frac{1}{2\pi i} \int_{\mathcal{C}_{0,b}} \frac{\Gamma(1-s)\Delta(s)}{\Gamma(2+\rho-s)\Delta(1-s)} x^{1+\rho-s} ds$$

where

$$\Delta(s) = \Gamma(s + k - \frac{3}{2})\Gamma(s + \frac{1}{2}).$$

By [7, Theorem C], the series of general term $(-1)^k (2\pi n)^{-1-\rho} a_F(n) f_\rho(2\pi n x)$ is uniformly (C, ℓ) -summable for $\ell > \max\{\frac{1}{2} - \rho, 0\}$ on any finite closed interval in $(0, \infty)$ only under the condition $\rho > -1$ and the sum is $A_\rho(x) - Q_\rho(x)$. In particular, we can fix $\ell = 1$ and $\rho = 0$. We shall say C -summable for $(C, 1)$ -summable.

The only pole of the integrand of $Q_0(x)$ is 0, it is encircled by \mathcal{C} hence

$$Q_0(x) \ll_F 1 \quad (x \geq 1).$$

To estimate $f_0(x)$, we use again the result by Lau & Tsang [11, Lemma 2.2] already used to establish Voronoi formula. We get

$$(17) \quad f_0(y) = \frac{(-1)^k}{\sqrt{2\pi}} y^{3/8} \cos\left(4y^{1/4} + \frac{\pi}{4}\right) + (-1)^k e_1 y^{1/8} \cos\left(4y^{1/4} + \frac{3\pi}{4}\right) + O\left(\frac{1}{y^{1/8}}\right),$$

where e_1 is a absolute constant.

Let

$$\begin{aligned}\Phi(v) &:= (2\pi)^{3/4} \frac{A_0(2\pi v^4)}{v^{3/2}}, \\ g_n(v) &:= \frac{a_F(n)}{n^{5/8}} \cos\left(4\sqrt{2\pi}n^{1/4}v + \frac{\pi}{4}\right), \\ g_n^*(v) &:= \frac{e_1}{v} \frac{a_F(n)}{n^{7/8}} \sin\left(4\sqrt{2\pi}n^{1/4}v + \frac{\pi}{4}\right).\end{aligned}$$

Then the series of general term $g_n(v) - g_n^*(v)$ is uniformly C-summable on any finite closed interval in $(0, \infty)$ and the sum is $\Phi(v) + O(v^{-3/2})$ (here the term $O(v^{-3/2})$ comes from $Q_0(2\pi v^4)$ and the O-term of (17)). In view of (4), a simple partial integration shows that the series of general term $g_n^*(v)$ converges to the sum $\sum_n g_n^*(v)$ uniformly on any finite closed interval in $(0, \infty)$. Thus the series of general term $g_n(v)$ is uniformly C-summable on any finite closed interval in $(0, \infty)$ and the sum is $\Phi(v) + \sum_n g_n^*(v) + O(v^{-3/2})$.

Let t be any large natural number, $\kappa > 1$ a large parameter that will be fixed later. Write

$$K_\tau(u) = (1 - |u|)(1 + \tau \cos(4\sqrt{2\pi}\kappa u))$$

with $\tau = \pm 1$. We consider the integral

$$J_\tau = \int_{-1}^1 \Phi(t + \kappa u) K_\tau(u) du.$$

We have

$$\begin{aligned}\int_{-1}^1 g_n(t + \kappa u) K_\tau(u) du &= r_\beta \frac{a_F(n)}{n^{5/8}}, \\ \int_{-1}^1 g_n^*(t + \kappa u) K_\tau(u) du &= s_\beta e_1 \frac{a_F(n)}{n^{7/8}},\end{aligned}$$

where

$$\begin{aligned}r_\beta &:= \int_{-1}^1 K_\tau(u) \cos\left(4\sqrt{2\pi}\beta(t + \kappa u) + \frac{\pi}{4}\right) du, \\ s_\beta &:= \int_{-1}^1 \frac{K_\tau(u)}{t + \kappa u} \sin\left(4\sqrt{2\pi}\beta(t + \kappa u) + \frac{\pi}{4}\right) du.\end{aligned}$$

As in [10, (3.13)], we have

$$r_\beta = \delta_{\beta=1} \frac{\tau}{2} + O\left(\frac{1}{\kappa^2 \beta^2} + \delta_{\beta \neq 1} \frac{1}{\kappa^2 (\beta - 1)^2}\right)$$

and

$$s_\beta \ll (t\beta\kappa)^{-1}.$$

It follows that

$$\begin{aligned} \int_{-1}^1 g_1(t + \kappa u) K_\tau(u) du &= \frac{\tau}{2} + O\left(\frac{1}{\kappa^2}\right), \\ \int_{-1}^1 g_n(t + \kappa u) K_\tau(u) du &\ll \frac{d_4(n)}{\kappa^2 n^{9/8}} \quad (n \geq 2), \\ \int_{-1}^1 g_n^*(t + \kappa u) K_\tau(u) du &\ll \frac{d_4(n)}{\kappa t n^{9/8}}, \end{aligned}$$

where all the implied constants are absolute. These estimates show that

$$\begin{aligned} \sum_{n \geq 1} \int_{-1}^1 g_n(t + \kappa u) K_\tau(u) du &= \frac{\tau}{2} + O\left(\frac{1}{\kappa^2}\right), \\ \sum_{n \geq 1} \int_{-1}^1 g_n^*(t + \kappa u) K_\tau(u) du &\ll \frac{1}{\kappa t}. \end{aligned}$$

In view of the remark about C-summability, we obtain

$$J_\tau = \frac{\tau}{2} + O\left(\frac{1}{\kappa t} + \frac{1}{t^{3/2}}\right).$$

We fix κ large enough. When $X \geq \kappa^4$, we take $t = \lfloor X^{1/4} \rfloor$. So $t > 2\kappa$ and the O-term in J_τ is $\ll \kappa^{-2}$, so the main term dominates if κ has been chosen sufficiently large. Therefore

$$J_{-1} < -\frac{1}{4} \quad \text{and} \quad J_1 > \frac{1}{4}.$$

Since $S_F(x) = A_0(2\pi x)$, we rewrite this as

$$\int_{-1}^1 \frac{S_F(t + \kappa u)}{(t + \kappa u)^{3/2}} K_{-1}(u) du < -\frac{1}{4(2\pi)^{3/4}} \quad \text{and} \quad \int_{-1}^1 \frac{S_F(t + \kappa u)}{(t + \kappa u)^{3/2}} K_1(u) du > \frac{1}{4(2\pi)^{3/4}}.$$

The kernel function $K_\tau(u)$ is nonnegative and satisfies

$$1 - (3\pi\kappa)^{-2} \leq \int_{-1}^1 K_\tau(u) du \leq 2 \quad (\tau = \pm 1).$$

As a consequence, we have

$$\frac{S_F((t + \kappa\eta_+)^4)}{(t + \kappa\eta_+)^{3/2}} \geq \frac{1}{2(2\pi)^{3/4}}$$

and

$$\frac{S_F((t + \kappa\eta_-)^4)}{(t + \kappa\eta_-)^{3/2}} \leq -\frac{1}{4(1 - (3\pi\kappa)^{-2})(2\pi)^{3/4}}$$

for some $\eta_\pm \in [-1, 1]$. These two points deviate from X by a distance $\ll X^{3/4}$, since the difference between $(t \pm \kappa)^4$ is $\ll \kappa t^3 \asymp X^{3/4}$.

This implies the result of Lemma 2. \square

Now we are ready to prove the Theorem.

By Lemma 2, for any $x \geq X_0(\mathbb{F})$ we can pick three points $x < x_1 < x_2 < x_3 < x + 3Cx^{3/4}$ such that $S_{\mathbb{F}}(x_i) < -cx^{3/8}$ ($i = 1, 3$) and $S_{\mathbb{F}}(x_2) > cx^{3/8}$ for some absolute constant $c > 0$. (Note that $y + Cy^{3/4} \leq x + 3Cx^{3/4}$ for $y = x + Cx^{3/4}$.) Hence we deduce that

$$\sum_{\substack{x_1 < n < x_2 \\ a_{\mathbb{F}}(n) > 0}} a_{\mathbb{F}}(n) \geq S_{\mathbb{F}}(x_2) - S_{\mathbb{F}}(x_1) > 2cx^{3/8}$$

and

$$\sum_{\substack{x_2 < n < x_3 \\ a_{\mathbb{F}}(n) < 0}} (-a_{\mathbb{F}}(n)) \geq -(S_{\mathbb{F}}(x_3) - S_{\mathbb{F}}(x_2)) > 2cx^{3/8}.$$

Thus, the Theorem follows as each term in the two sums are positive and $\ll_{\varepsilon} n^{\varepsilon}$.

REFERENCES

- [1] A. N. Andrianov, *Euler products that correspond to Siegel's modular forms of genus 2*, Uspehi Mat. Nauk **29** (1974), no. 3 (177), 43–110. MR 0432552 (55 #5540)
- [2] Stefan Breulmann, *On Hecke eigenforms in the Maaß space*, Math. Z. **232** (1999), no. 3, 527–530. MR 1719682 (2000j:11066)
- [3] Jim Brown, *The first negative Hecke eigenvalue of genus 2 Siegel cuspforms with level $N \geq 1$* , Int. J. Number Theory **6** (2010), no. 4, 857–867. MR 2661286 (2012b:11073)
- [4] K. Chandrasekharan and Raghavan Narasimhan, *The approximate functional equation for a class of zeta-functions*, Math. Ann. **152** (1963), 30–64. MR 0153643 (27 #3605)
- [5] Soumya Das, *On the natural densities of eigenvalues of a siegel cusp form of degree 2*, Int. J. Number Theory **9** (2013), no. 1, 9–15.
- [6] S. A. Evdokimov, *Characterization of the Maass space of Siegel modular cusp forms of genus 2*, Mat. Sb. (N.S.) **112(154)** (1980), no. 1(5), 133–142, 144. MR 575936 (82a:10028)
- [7] James Lee Hafner, *On the representation of the summatory functions of a class of arithmetical functions*, Analytic number theory (Philadelphia, Pa., 1980), Lecture Notes in Math., vol. 899, Springer, Berlin, 1981, pp. 148–165. MR 654524 (83g:10030)
- [8] Winfried Kohnen, *Sign changes of Hecke eigenvalues of Siegel cusp forms of genus two*, Proc. Amer. Math. Soc. **135** (2007), no. 4, 997–999 (electronic). MR 2262899 (2007j:11057)
- [9] Winfried Kohnen and Jyoti Sengupta, *The first negative Hecke eigenvalue of a Siegel cusp form of genus two*, Acta Arith. **129** (2007), no. 1, 53–62. MR 2326486 (2009a:11112)
- [10] Y.-K. Lau and J. Wu, *The number of Hecke eigenvalues of same signs*, Math. Z. **263** (2009), no. 4, 959–970. MR 2551607 (2010j:11072)
- [11] Yuk-Kam Lau and Kai-Man Tsang, *Large values of error terms of a class of arithmetical functions*, J. Reine Angew. Math. **544** (2002), 25–38. MR 1887887 (2003c:11119)
- [12] Philippe Michel, *Analytic number theory and families of automorphic L-functions*, Automorphic forms and applications, IAS/Park City Math. Ser., vol. 12, Amer. Math. Soc., Providence, RI, 2007, pp. 181–295. MR 2331346 (2008m:11104)
- [13] Charles N. Moore, *Summable series and convergence factors*, Dover Publications Inc., New York, 1966. MR 0201863 (34 #1743)
- [14] Gérald Tenenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge Studies in Advanced Mathematics, vol. 46, Cambridge University Press, Cambridge, 1995, Translated from the second French edition (1995) by C. B. Thomas. MR 1342300 (97e:11005b)
- [15] Rainer Weissauer, *Endoscopy for $\mathrm{GSp}(4)$ and the cohomology of Siegel modular threefolds*, Lecture Notes in Mathematics, vol. 1968, Springer-Verlag, Berlin, 2009. MR 2498783 (2010h:11086)
- [16] D. Zagier, *Sur la conjecture de Saito-Kurokawa (d'après H. Maass)*, Seminar on Number Theory, Paris 1979–80, Progr. Math., vol. 12, Birkhäuser Boston, Mass., 1981, pp. 371–394. MR 633910 (83b:10031)

EMMANUEL ROYER, CLERMONT UNIVERSITÉ, UNIVERSITÉ BLAISE PASCAL, LABORATOIRE DE MATHÉMATIQUES,
BP 10448, F-63000 CLERMONT-FERRAND, FRANCE

Current address: Emmanuel Royer, Université Blaise Pascal, Laboratoire de mathématiques, Les Cézeaux,
BP 80026, F-63171 Aubière Cedex, France

E-mail address: emmanuel.royer@math.univ-bpclermont.fr

JYOTI SENGUPTA, SCHOOL OF MATHEMATICS, T.I.F.R., HOMI BHABHA ROAD, 400 005 MUMBAI, INDIA

E-mail address: sengupta@math.tifr.res.in

JIE WU, INSTITUT ÉLIE CARTAN, CNRS UMR 7502, UNIVERSITÉ DE LORRAINE, INRIA, F-54506 VANDŒUVRE-
LÈS-NANCY, FRANCE

E-mail address: Jie.Wu@univ-lorraine.fr