A Survey on Intersections of Maximal Partial Clones of Boolean Partial Functions

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Abstract—We survey known results and present some new ones about intersections of maximal partial clones on a 2-element set.

I. Preliminaries

Let A be a finite non-singleton set. Without loss of generality we assume that $A = \mathbf{k} := \{0, \ldots, k-1\}$. For a positive integer n, an n-ary partial function on $\mathbf{k}$ is a map $f : \mathrm{dom}\,(f) \to \mathbf{k}$ where $\mathrm{dom}\,(f)$ is a subset of $\mathbf{k}^n$ called the *domain* of f. Let $\mathrm{Par}^{(n)}(\mathbf{k})$ denote the set of all n-ary partial functions on $\mathbf{k}$ and let $\mathrm{Par}(\mathbf{k}) := \bigcup_{n \geq 1} \mathrm{Par}^{(n)}(\mathbf{k})$. An n-ary partial function g is said to be a *total function* if $\mathrm{dom}\,(g) = \mathbf{k}^n$. Let $\mathrm{Op}(\mathbf{k})$ be the set of all total functions on $\mathbf{k}$.

For $n, m \geq 1$, $f \in \mathrm{Par}^{(n)}(\mathbf{k})$ and $g_1, \ldots, g_n \in \mathrm{Par}^{(m)}(\mathbf{k})$, the *composition* of f and $g_1, \ldots, g_n$, denoted by $f[g_1, \ldots, g_n] \in \mathrm{Par}^{(m)}(\mathbf{k})$, is defined by

$$\mathrm{dom}\,(f[g_1, \ldots, g_n]) :=$$
$$\{\vec{a} \in \mathbf{k}^m \mid \vec{a} \in \bigcap_{i=1}^{m} \mathrm{dom}\,(g_i), (g_1(\vec{a}), \ldots, g_m(\vec{a})) \in \mathrm{dom}\,(f)\},$$

and $f[g_1, \ldots, g_n](\vec{a}) := f(g_1(\vec{a}), \ldots, g_n(\vec{a}))$, for all $\vec{a} \in \mathrm{dom}\,(f[g_1, \ldots, g_n])$.

For every positive integer n and each $1 \leq i \leq n$, let e_i^n denote the *n-ary i-th projection function* defined by $e_i^n(a_1, \ldots, a_n) = a_i$ for all $(a_1, \ldots, a_n) \in \mathbf{k}^n$. Furthermore, let $J_{\mathbf{k}} := \{e_i^n : 1 \leq i \leq n\}$ be the set of all (total) projections.

Definition 1. A *partial clone* on $\mathbf{k}$ is a composition closed subset of $\mathrm{Par}(\mathbf{k})$ containing $J_{\mathbf{k}}$. If a partial clone is contained in the set of all total functions $\mathrm{Op}(\mathbf{k})$, then it is called a *clone* on $\mathbf{k}$.

Remark 1. There are two other equivalent definitions for partial clones. One definition uses Mal'tsev's formalism and the other uses the concept of one point extension. These definitions can be found in chapter 20 of [12].

The partial clones on $\mathbf{k}$ (clones on $\mathbf{k}$), ordered by inclusion, form a lattice $\mathcal{L}_{P_{\mathbf{k}}}$ ($\mathcal{L}_{O_{\mathbf{k}}}$, respectively) in which the infimum is the set-theoretical intersection. That means that the intersection of an arbitrary family of partial clones (clones) on $\mathbf{k}$ is

also a partial clone on $\mathbf{k}$ (clone on $\mathbf{k}$, respectively). A *maximal partial clone on* $\mathbf{k}$ is a coatom of the lattice $\mathcal{L}_{P_{\mathbf{k}}}$. Therefore a partial clone M is maximal if there is no partial clone C over $\mathbf{k}$ such that $M \subset C \subset \mathrm{Par}(\mathbf{k})$. Similarly a clone M is called a *maximal clone* if there is no clone C on $\mathbf{k}$ such that $M \subset C \subset \mathrm{Op}(\mathbf{k})$. We say that a partial clone C_0 on A is *covered* by a partial clone C_1 on A if there is no partial clone C such that $C_0 \subset C \subset C_1$. Therefore a maximal partial clone is a partial clone covered by $\mathrm{Par}(\mathbf{k})$.

Example 2. Let

$$\Omega_k :=$$
$$\bigcup_{n \geq 1} \{f \in \mathrm{Par}^{(n)}(\mathbf{k}) \mid \mathrm{dom}\,(f) \neq \emptyset \Longrightarrow \mathrm{dom}\,(f) = \mathbf{k}^n\} =$$
$$\bigcup_{n \geq 1} \{f \in \mathrm{Par}^{(n)}(\mathbf{k}) \mid \mathrm{dom}\,(f) \neq \emptyset \Longrightarrow f \in \mathrm{Op}(\mathbf{k})\}.$$

Then Ω_k is a maximal partial clone on $\mathbf{k}$.

An interesting and somehow difficult problem in clone theory is to study intersections of maximal clones and maximal partial clones. The lattice $\mathcal{L}_{O_2}$ is known and was completely described by E. Post in [14]. In chapter 14 of [12] are listed all submaximal elements of $\mathcal{L}_{O_3}$, i.e., all clones on $\mathbf{3}$ covered by the maximal elements of $\mathcal{L}_{O_3}$. Several results dealing with intersection of maximal clones can be found in the literature, we refer the reader to the list of reference in [12] for details.

In this paper we focus on $\mathcal{L}_{P_2}$, the lattice of partial clones of Boolean functions. We survey the known results and present some new results concerning the intersections of the maximal elements of $\mathcal{L}_{P_2}$.

With one exception, every maximal partial clone is the polymorphism of a relation. We have:

Definition 3. For $h \geq 1$, let ρ be an h-ary relation on $\mathbf{k}$ and f be an n-ary partial function on $\mathbf{k}$. We say that f *preserves* ρ if for every $h \times n$ matrix $M = [M_{ij}]$ whose columns $M_{*j} \in \rho$, $(j = 1, \ldots n)$ and whose rows $M_{i*} \in \mathrm{dom}\,(f)$ $(i = 1, \ldots, h)$, the h-tuple $(f(M_{1*}), \ldots, f(M_{h*})) \in \rho$. Define

$$\mathrm{pPol}\,\rho := \{f \in \mathrm{Par}(\mathbf{k}) \mid f \text{ preserves } \rho\}.$$

It is well known that $\mathrm{pPol}\,\rho$ is a partial clone called the *partial clone determined by the relation ρ*.

Note that if there is no $h \times n$ matrix $M = [M_{ij}]$ whose columns $M_{*j} \in \rho$ and whose rows $M_{i*} \in \mathrm{dom}\,(f)$, then $f \in \mathrm{pPol}\,\rho$.

Example 4. Let $\mathbf{2} := \{0,1\}$, let $\{(0,0),(0,1),(1,1)\}$ be the natural order on $\mathbf{2}$ and consider the binary relation $\{(0,1),(1,0)\}$ on $\mathbf{2}$.

Then $\mathrm{pPol}\,\{(0,0),(0,1),(1,1)\}$ is the set of all *monotone* partial functions and $\mathrm{pPol}\,\{(0,1),(1,0)\}$ is the set of all *self-dual* partial functions on $\mathbf{2}$. For simplicity we will write $\mathrm{pPol}\,(\leq)$ and $\mathrm{pPol}\,(\neq)$ for $\mathrm{pPol}\,(\{(0,0),(0,1),(1,1)\})$ and $\mathrm{pPol}\,(\{(0,1),(1,0)\})$, respectively.

It is easy to see that if ρ is an h-ary relation on $\mathbf{2}$, then $\mathrm{pPol}\,\rho = \mathrm{pPol}\,(\rho \otimes \mathbf{2})$. This fact motivates the concept of irredundant relation.

Let $h \geq 1$ and let ρ be an h-ary relation on $\mathbf{k}$. We say that ρ is *repetition-free* if for all $1 \leq i < j \leq h$, there exists $(a_1, \ldots, a_h) \in \rho$ with $a_i \neq a_j$. Moreover, ρ is said to be *irredundant* if it is repetition-free and has no fictitious components, i.e., there is no $i \in \{1, \ldots, h\}$ such that $(a_1, \ldots, a_h) \in \rho$ implies $(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_h) \in \rho$ for all $x \in \mathbf{k}$.

It can be shown that if μ is a nonempty relation, then we can find an irredundant relation ρ such that $\mathrm{pPol}\,\mu = \mathrm{pPol}\,\rho$ (see, e. g. [4]) for details).

The following result, known as the *Definability Lemma*, was first established by B. Romov in [15] (see Lemma 20.3.4 in [12]) and has been widely used to handle maximal partial clones via the relational approach.

Lemma 5. *(The Definability Lemma)* Let $h_1, \ldots, h_n, t \geq 1$ be integers, ρ_i be an h_i-ary relation on $\mathbf{k}$, $i = 1, \ldots, n$, and β be a t-ary irredundant relation on $\mathbf{k}$. Then

$$\bigcap_{1 \leq i \leq n} \mathrm{pPol}\,\rho_i \subseteq \mathrm{pPol}\,\beta$$

if and only if there exists a family of h_i-ary auxiliary relations $\{\varrho_1, \ldots, \varrho_n\}$ whose vertex sets is $\{1, \ldots, t\}$, and such that

$$\beta = \{(x_1, \ldots, x_t) \in \mathbf{k}^t \mid (i_1^j, \ldots, i_{h_j}^j) \in \varrho_j \Rightarrow$$
$$(x_{i_1^j}, \ldots, x_{i_{h_j}^j}) \in \rho_j, \quad \text{for} \quad j = 1, \ldots, n\}.$$

Example 6. Let ρ_1 be a binary and ρ_2 be a ternary relation on $\mathbf{k}$. Let β be the 4-ary relation defined by

$$\beta := \{(x_1, \ldots, x_4) \in \mathbf{k}^4 \mid (x_1, x_2) \in \rho_1, \ (x_3, x_2) \in \rho_1,$$
$$\text{and } (x_1, x_4, x_3) \in \rho_2\}.$$

Then $\mathrm{pPol}\,\rho_1 \cap \mathrm{pPol}\,\rho_2 \subseteq \mathrm{pPol}\,\beta$. (Here $n = 2$, $\varrho_1 = \{(1,2),(3,2)\}$ and $\varrho_2 = \{(1,4,3)\}$.)

As mentioned earlier, Freivald showed that there are exactly eight maximal partial clones on $\mathbf{2}$. The following two relations determine maximal partial clones. Set

$R_1 := \{(x,x,y,y) \mid x,y \in \mathbf{2}\} \cup \{(x,y,y,x) \mid x,y \in \mathbf{2}\}$ and
$R_2 := R_1 \cup \{(x,y,x,y) \mid x,y \in \mathbf{2}\}$.

Theorem 7. ([3]) *There are exactly 8 maximal partial clones on $\mathbf{2}$, namely:* $\mathrm{pPol}\,\{0\}$, $\mathrm{pPol}\,\{1\}$, $\mathrm{pPol}\,\{(0,1)\})$, $\mathrm{pPol}\,(\leq)$, $\mathrm{pPol}\,(\neq)$, $\mathrm{pPol}\,(R_1)$, $\mathrm{pPol}\,(R_2)$ *and* Ω_2 .

The three maximal partial clones $\mathrm{pPol}\,R_1$, $\mathrm{pPol}\,R_2$ and Ω_2 contain the unary functions $\mathrm{Op}^{(1)}(\mathbf{2})$ (i.e., maps) on $\mathbf{2}$. Such partial clones are called *Słupecki type* partial clones in [8], [16]. They are the only three maximal partial clones of Słupecki type on $\mathbf{2}$.

It is known that $\mathrm{pPol}\,R_2 \cap \mathrm{Op}(\mathbf{2})$ is the maximal clone of all (total) linear functions over $\mathbf{2}$ (see, e.g., section 5.2.4 of [12]). Alekzeev and Voronenko studied the classes of partial clones of Boolean functions that contain $\mathrm{pPol}\,R_2 \cap \mathrm{Op}(\mathbf{2})$ on $\mathbf{2}$. From the main result of [1] we have:

Theorem 8. *The interval of partial clones* $[\mathrm{pPol}\,R_2 \cap \mathrm{Op}(\mathbf{2}), \mathrm{Par}(\mathbf{2})]$ *is of continuum cardinality on $\mathbf{2}$.*

The proof of this result is quite complicated and is given in ([1]) (see also Theorem 20.8.1 of [12]). We refer the reader to Theorem 19 of [8] and Theorem 20.7.13 of [12] for the generalization of Theorem 8 to partial clones on $\mathbf{k}$ with $k \geq 3$. A consequence of Theorem 8 is that the interval of partial clones $[\mathrm{pPol}\,(R_2) \cap \Omega_2, \mathrm{Par}(\mathbf{2})]$ is of continuum cardinality on $\mathbf{2}$.

On the other hand it is shown in [5] that $\mathrm{pPol}\,R_1 \cap \mathrm{Op}(\mathbf{k})$ is the clone over $\mathbf{k}$ generated by $\mathrm{Op}^{(1)}(\mathbf{k})$ for every $k \geq 2$. We present a result similar to Theorem 8 that is established for $\mathrm{pPol}\,R_1$ in [5].

For $n \geq 3$ define the $2n$-ary relation τ_{2n} on $\mathbf{2}$ by setting: $(x_1, \ldots, x_{2n}) \in \tau_n$ if and only if either $x_1 = \cdots = x_{2n}$, or each of 0 and 1 appears exactly n times in $(x_1, \ldots, x_{2n})$.

It is shown in [5] that $\mathrm{Op}^{(1)}(\mathbf{2}) \subseteq \mathrm{pPol}\,\tau_{2n}$ for all $n \geq 3$. Since

$$R_1 = \{(x_1, x_2, x_3, x_4) \in \mathbf{2}^4 \mid$$
$$(x_1, \ldots, x_1, x_2, x_3, \ldots, x_3, x_4) \in \tau_{2n}\}$$

holds for all $n \geq 3$, it follows from Lemma 5 that $\mathrm{pPol}\,\tau_{2n} \subseteq \mathrm{pPol}\,R_1$ for all $n \geq 3$.

Let $\mathbf{P} := \{3, 5, 7, \ldots\}$ be the set of all odd prime numbers and $\mathcal{P}(\mathbf{P})$ be its power set. It is shown in [5] that the map

$$\chi : \mathcal{P}(\mathbf{P}) \to [\mathrm{pPol}\,R_1 \cap \Omega_2, \mathrm{pPol}\,R_1]$$

defined by $X \longmapsto \chi(X) := \bigcap_{t \in \mathbf{P} \setminus X} \mathrm{pPol}\,\tau_{2t}$ is one-to-one. Hence we have the following result:

Theorem 9. *The interval of partial clones* $[\mathrm{pPol}\,(R_1) \cap \Omega_2, \mathrm{Par}(\mathbf{2})]$ *is of continuum cardinality on $\mathbf{2}$.*

Together with D. Lau, the second author studied several intersections of Słupecki type partial clones on a non-singleton finite set. The following result comes from [8]:

Theorem 10. *The partial clone* $\mathrm{pPol}\,R_1 \cap \mathrm{pPol}\,R_2$ *is covered by the maximal partial clone* $\mathrm{pPol}\,R_2$.

The dual of the above result does not hold for $\mathrm{pPol}\,R_1$. It is shown in [8] that there is at least one partial clone that strictly

lies between $\mathrm{pPol}\,R_1 \cap \mathrm{pPol}\,R_2$ and $\mathrm{pPol}\,R_1$. Indeed, let

$$\lambda := \{(x_1,\ldots,x_7) \in \mathbf{2}^7 \mid (x_1,x_2,x_5,x_6) \in R_1,$$
$$(x_2,x_4,x_6,x_7) \in R_1, \text{ and } (x_1,x_2,x_3,x_4) \in R_2\}.$$

Then it is shown in [8] that $\mathrm{pPol}\,R_1 \cap \mathrm{pPol}\,R_2 \subsetneq \mathrm{pPol}\,\lambda \subsetneq \mathrm{pPol}\,R_1$. Therefore the partial clone $\mathrm{pPol}\,R_1 \cap \mathrm{pPol}\,R_2$ is not covered by the maximal partial clone $\mathrm{pPol}\,R_1$. To our knowledge, little seems to be known about the interval of partial clones $[\mathrm{pPol}\,R_1 \cap \mathrm{pPol}\,R_2, \mathrm{pPol}\,R_1]$.

Intersections of maximal partial clones that are not of Słupecki type have been studied as well. Intersections of the form $\mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta$ where $\rho, \theta \in \{\{0\}, \{1\}, \{(0,1)\}, \neq, \leq\}$, with the exception of $\{\rho, \theta\} = \{\leq, \neq\}$, have been studied in [6], [7]. Almost all proofs given in [6], [7] are based on the composition of partial functions. In the same direction, deeper results were established in [10] and [9] where partial clones are handled via relations, and all proofs are based on the Lemma 5. Let

$$C_M := \mathrm{pPol}\,\{0\} \cap \mathrm{pPol}\,\{1\} \cap \mathrm{pPol}\,\{(0,1)\} \cap \mathrm{pPol}\,(\leq) \quad \text{and}$$
$$C_D := \mathrm{pPol}\,\{0\} \cap \mathrm{pPol}\,\{1\} \cap \mathrm{pPol}\,\{(0,1)\} \cap \mathrm{pPol}\,(\neq).$$

Then C_M (respectively C_D) is the set of all idempotent monotonic partial functions on $\mathbf{2}$ (idempotent self-dual partial functions on $\mathbf{2}$). We have:

Theorem 11. *The interval $[C_M, \mathrm{Par}(\mathbf{2})]$ contains exactly 25 partial clones and the interval $[C_D, \mathrm{Par}(\mathbf{2})]$ contains exactly 33 partial clones on $\mathbf{2}$.*

The intervals of partial clones $[C_M, \mathrm{Par}(\mathbf{2})]$ and $[C_D, \mathrm{Par}(\mathbf{2})]$ are completely described in [10] (see also [9]). D. Lau informed the second author that some of the results in [10] exist in the unpublished manuscript [18] by B. Strauch.

In view of results from [1], [5], [10], [18], [19], it was thought that if $2 \leq i \leq 5$ and $M_1, \ldots, M_i$ are non-Słupecki maximal partial clones on $\mathbf{2}$, then the interval $[M_1 \cap \cdots \cap M_i, \mathrm{Par}(\mathbf{2})]$ is either finite or countably infinite. It is shown in [11] that the interval of partial clones $[\mathrm{pPol}\,(\leq) \cap \mathrm{pPol}\,(\neq), \mathrm{Par}(\mathbf{2})]$ is infinite. This result is mentioned in Theorem 20.8 of [12] (with an independent proof given in [13]) and in chapter 8 of the PhD thesis [17]. However, it remained an open problem to determine whether $[\mathrm{pPol}\,(\leq) \cap \mathrm{pPol}\,(\neq), \mathrm{Par}(\mathbf{2})]$ is countably or uncountably infinite.

The following relations, introduced in [2], are needed to settle this question. For $n \geq 5$ and $n > k \geq 1$ we denote by $\sigma_k^n \subseteq \mathbf{2}^{2n}$ the $(2n)$-ary relation defined by

$$\sigma_k^n := \{(x_1,\ldots,x_n,y_1,\ldots,y_n) \in \mathbf{2}^{2n} \mid$$
$$\forall\, i = 1,\ldots,n,\ x_i \neq y_i, \text{ and } \forall\, k = 1,\ldots,n,\ y_{i+k} \leq x_i\},$$

where the subscripts $i + j$ in the above definition are taken modulo n.

Now for $n \geq 5$ and $n > k \geq 1$, we denote by $\rho_k^n \subseteq \mathbf{2}^{4n}$ the $(4n)$-ary relation defined by

$$\rho_k^n := \{(x_1,\ldots,x_{2n},y_1,\ldots,y_{2n}) \in \mathbf{2}^{4n} \mid$$
$$(x_1,\ldots,x_n,y_1,\ldots,y_n) \in \sigma_1^n, \text{ and }$$
$$(x_{n+1},\ldots,x_{2n},y_{n+1},\ldots,y_{2n}) \in \sigma_k^n\}.$$

By Lemma 5 we have that for all $n \geq 5$ and $n > k \geq 1$

$$\mathrm{pPol}\,(\leq) \cap \mathrm{pPol}\,(\neq) \subseteq \mathrm{pPol}\,\sigma_k^n \cap \mathrm{pPol}\,\sigma_1^n \subseteq \mathrm{pPol}\,\rho_k^n.$$

Denote by $\mathbf{E}_{\geq 4} := \{4, 6, 8, \ldots\}$ the set of all even integers greater or equal to 4 and denote by $\mathcal{P}(\mathbf{E}_{\geq 4})$ the power set of $\mathbf{E}_{\geq 4}$. Furthermore, for every **even** integer $k \geq 4$, let $n(k) := k(k+1)+1$. Since $\mathrm{pPol}\,(\leq) \cap \mathrm{pPol}\,(\neq) \subseteq \mathrm{pPol}\,(\rho_k^n)$ for every $n \geq 5$ and every $n > k \geq 1$, we have that

$$\mathrm{pPol}\,(\leq) \cap \mathrm{pPol}\,(\neq) \subseteq \bigcap_{t \in \mathbf{E}_{\geq 4} \setminus X} \mathrm{pPol}\,\rho^{n(t)}$$

for every subset X of $\mathbf{E}_{\geq 4}$. It was shown in [2] that the map

$$\chi \colon \mathcal{P}(\mathbf{E}_{\geq 4}) \to [\mathrm{pPol}\,(\leq) \cap \mathrm{pPol}\,(\neq), \mathrm{Par}(\mathbf{2})]$$

defined by $X \longmapsto \chi(X) := \bigcap_{t \in \mathbf{E}_{\geq 4} \setminus X} \mathrm{pPol}\,\rho^{n(t)}$ is one-to-one. Hence we have the following result:

Theorem 12. *The interval of partial clones $[\mathrm{pPol}\,(\leq) \cap \mathrm{pPol}\,(\neq), \mathrm{Par}(\mathbf{2})]$ is of continuum cardinality on $\mathbf{2}$.*

We conclude this survey with the following new result, which provides several examples of finite intervals of the form above.

Theorem 13. *Let $\rho \in \{\{0\}, \{1\}, \{(0,1)\}, \neq, \leq\}$ and $\theta \in \{R_1, R_2\}$. Then the partial clone $\mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta$ is covered by the maximal partial clone $\mathrm{pPol}\,\theta$ over $\mathbf{2}$. In particular, each interval of partial clones of the form $[\mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta, \mathrm{Par}(\mathbf{2})]$ has size 2.*

The proof is based on the following fact established after Lemma 3 in [8].

Fact 14. *Let $\mathrm{pPol}\,\rho$ and $\mathrm{pPol}\,\theta$ be two distinct maximal partial clones on $\mathbf{k}$. Suppose that*

$$[\mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta \subseteq \mathrm{pPol}\,\lambda] \Longrightarrow$$
$$[\mathrm{pPol}\,\lambda \subseteq \mathrm{pPol}\,\rho \quad or \quad \mathrm{pPol}\,\lambda = \mathrm{pPol}\,\theta]$$

holds for every irredundant relation λ. Then the partial clone $\mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta$ is covered by the maximal partial clone $\mathrm{pPol}\,\theta$ on $\mathbf{k}$.

We need the following notation. For $v = (v_1, \ldots, v_t) \in \mathbf{2}^t$, we define $\ker(v) := \{(i,j) \in \{1,\ldots,t\}^2 \mid v_i = v_j\}$. Note that $\ker(v)$ is a binary equivalence relation on the set $\{1, \ldots, t\}$ with at most two blocks.

Proof of Theorem 13. Let $\theta \in \{R_1, R_2\}$. We consider three cases:

a) $\mathrm{pPol}\,\{(0,1)\} \cap \mathrm{pPol}\,\theta$ is covered by $\mathrm{pPol}\,\theta$. Let $t \geq 1$ and λ be a t-ary irredundant relation such that

$$\mathrm{pPol}\,\{(0,1)\} \cap \mathrm{pPol}\,\theta \subseteq \mathrm{pPol}\,\lambda.$$

By Lemma 5, there is a binary relation ϱ_1 and a 4-ary relation ϱ_2, with $\{\varrho_1,\varrho_2\}$ covering the set $\{1,\ldots,t\}$ and such that

$$\lambda = \{(x_1,\ldots,x_t) \in \mathbf{2}^t \mid$$
$$\forall\,(j_1,j_2) \in \varrho_1,\ (x_{j_1},\ x_{j_2}) \in \{(0,1)\},\ \text{and}$$
$$\forall\,(i_1,\ldots,i_4) \in \varrho_2,\ (x_{i_1},\ldots,x_{i_4}) \in \theta\}.$$

Note that if $\varrho_1 = \emptyset$, then λ can be defined from θ and by Lemma 5 $\mathrm{pPol}\,\theta \subseteq \mathrm{pPol}\,\lambda$, thus $\mathrm{pPol}\,\theta = \mathrm{pPol}\,\lambda$ by the maximality of $\mathrm{pPol}\,\theta$. So assume $\varrho_1 \neq \emptyset$. Without loss of generality, let $(1,2) \in \varrho_1$, i.e., $(x_1,x_2) = (0,1)$ for every tuple $(x_1,\ldots,x_t) \in \lambda$. Fix $\underset{\sim}{v} = (0,1,v_3,\ldots,v_t) \in \lambda$ and set

$$\mu := \{(x_1,x_2) \in \mathbf{2}^2 \mid (x_1,x_2,x_{i_3},\ldots,x_{i_t}) \in \lambda\}$$

where, for $j = 3,\ldots,t$, $i_j = 1$ if $(1,i_j) \in \ker\,(\underset{\sim}{v})$ and $i_j = 2$ if $(2,i_j) \in \ker\,(\underset{\sim}{v})$. Then we have that $\mathrm{pPol}\,\lambda \subseteq \mathrm{pPol}\,\mu$ by Lemma 5. As $\underset{\sim}{v} = (0,1,v_3,\ldots,v_t) \in \lambda$ we have $(0,1) \in \mu$ and since every $(x_1,\ldots,x_t) \in \lambda$ satisfies $x_1 = 0, x_2 = 1$, we have that $\mu = \{(0,1)\}$. By Fact 14, $\mathrm{pPol}\,\{(0,1)\} \cap \mathrm{pPol}\,\theta$ is covered by $\mathrm{pPol}\,\theta$. The proof of the claim that $\mathrm{pPol}\,\{0\} \cap \mathrm{pPol}\,\theta$ and $\mathrm{pPol}\,\{1\} \cap \mathrm{pPol}\,\theta$ are covered by $\mathrm{pPol}\,\theta$ follows similarly.

b) $\mathrm{pPol}\,\{\neq\} \cap \mathrm{pPol}\,\theta$ is covered by $\mathrm{pPol}\,\theta$. We proceed as in case a), and choose $\underset{\sim}{v} = (v_1,v_2,\ldots,v_t) \in \lambda$. Then either $(v_1,v_2) = (0,1)$ or $(v_1,v_2) = (1,0)$. Suppose that $(v_1,v_2) = (0,1)$ and consider the map $\neg(0) = 1, \neg(1) = 0$. Since $\neg \in \mathrm{pPol}\,(\neq) \cap \mathrm{pPol}\,\theta$, we have $\neg \in \mathrm{pPol}\,\lambda$ and so $\neg(\underset{\sim}{v}) := (\neg(v_1),\ldots,\neg(v_t)) \in \lambda$. Again consider the relation μ defined in a). It is easy to see that μ is the binary relation $\neq$ and the rest of the proof is as above. The case $(v_1,v_2) = (1,0)$ follows similarly.

c) $\mathrm{pPol} \leq \cap\, \mathrm{pPol}\,\theta$ is covered by $\mathrm{pPol}\,\theta$. Again proceed as in case a) with the assumption that $(1,2) \in \varrho_1$. Note that in this case we have $(0,\ldots,0),(1,\ldots,1) \in \lambda$. Moreover since λ is irredundant, there is $\underset{\sim}{v} = (v_1,v_2,\ldots,v_t) \in \lambda$ such that $v_1 \neq v_2$. As $v_1 \leq v_2$ we get $(v_1,v_2) = (0,1)$. Again consider the relation μ as defined in case a). From $\underset{\sim}{v} \in \lambda$ we obtain $(0,1) \in \mu$ and as $(i,\ldots,i) \in \lambda$ we have $(i,i) \in \mu$, for $i = 0,1$. Note that $(1,0) \notin \mu$ since for every $(x_1,x_2,\ldots,x_t) \in \lambda, x_1 \leq x_2$. So μ is the binary relation $\leq$. The rest of the proof is as in case a). $\square$

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