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GAP vs. PAG

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***Abstract*—We propose a parametrized version of arity gap. The parametrized arity gap $\operatorname{gap}(f, \ell)$ of a function $f\colon A^n \to B$ measures the minimum decrease in the number of essential variables of f when ℓ consecutive identifications of pairs of essential variables are performed. We determine $\operatorname{gap}(f, \ell)$ for an arbitrary function f and a positive integer ℓ. We also propose other variants of arity gap and discuss further problems pertaining to the effect of identification of variables on the number of essential variables of functions.**

I. Introduction

Let A and B be arbitrary nonempty sets. We investigate a variant of the so-called arity gap of functions $f\colon A^n \to B$. The study of arity gap goes back to the 1963 paper by Salomaa [13], where he addressed the question how the number of essential variables of a function is affected by substitution of constants for variables or by identification of variables.

The arity gap of a function $f\colon A^n \to B$ is defined as the minimum decrease in the number of essential variables when any two essential variables of f are identified, and it is denoted by $\operatorname{gap} f$. Concerning the effect of identifying variables on the number of essential variables, Salomaa's main result asserts that in the case when $A = B = \{0, 1\}$, it holds that $\operatorname{gap} f \leq 2$ for every function $f\colon \{0,1\}^n \to \{0,1\}$ with at least two essential variables. (In fact, it is implicit in Salomaa's work that if $|A| = k$ and $|B| \geq 2$, then the arity gap of any function $f\colon A^n \to B$, all variables of which are essential, is at most k, and that there exist functions of arity k for which this upper bound is met.) For background and developments on this topic see, e.g., [4], [5], [6], [7], [8], [14], [16].

In this paper we introduce a parametrized version of arity gap which measures the minimum decrease in the number of essential variables when we make $\ell \geq 1$ successive identifications of pairs of essential variables. For a function $f\colon A^n \to B$, let $\operatorname{gap}(f, \ell)$ denote this minimum decrease. This parametrized notion extends that of arity gap as $\operatorname{gap} f = \operatorname{gap}(f, 1)$. Also, it is tightly related to yet another problem which appears naturally when studying the effect of identification of variables on the number of essential variables of functions: Given a function $f\colon A^n \to B$ and an integer $p \geq 1$, what is the smallest number m such that any m successive identifications of essential variables result in the loss of at least p essential variables? Denoting this smallest number by $\operatorname{pag}(f, p)$, it will become clear that $\operatorname{pag}(f, p)$ is the smallest ℓ such that $\operatorname{gap}(f, \ell) \geq p$.

This paper is organized as follows. In Section II, we recall basic notions and establish preliminary results concerning the simple minor relation and the arity gap, which will be needed in the later sections. Section III is devoted to the study of the parametrized arity gap. In particular, given sets A and B and positive integers n, p, we explicitly determine the possible sequences

$$\operatorname{gap}(f, 1), \operatorname{gap}(f, 2), \dots, \operatorname{gap}(f, \ell), \dots,$$

for functions $f\colon A^n \to B$ depending on all of their variables such that $\operatorname{gap} f = p$. In Section IV, we briefly discuss some further problems related to the effect of several identifications of essential variables on the number of essential variables of a function, as that mentioned above.

The current study was motivated by the questions and remarks made by Dan A. Simovici at the IEEE 41st International Symposium on Multiple-Valued Logic (ISMVL 2011).

II. Preliminaries

A. Functions of several variables and simple minors

For a positive integer n, we will denote $[n] := \{1, \dots, n\}$, and we will assume throughout this paper that A and B are arbitrary sets with at least two elements. A *function of several variables* (or *arguments*) from A to B is a map $f\colon A^n \to B$ for some positive integer n called the *arity* of f. We denote the set of all finitary functions from A to B by

$$\mathcal{F}_{AB} := \bigcup_{n \geq 1} B^{A^n}.$$

We say that the i-th variable is *essential* in $f\colon A^n \to B$, or f *depends* on the i-th variable, if there exist tuples $\mathbf{a} := (a_1, \dots, a_n)$ and $\mathbf{b} := (b_1, \dots, b_n)$ such that $a_j = b_j$ for all $j \neq i$ and $f(\mathbf{a}) \neq f(\mathbf{b})$. A variable that is not essential is called *inessential*. The *essential arity* of f is defined to be the cardinality of the set

$$\operatorname{Ess} f := \{i \in [n] : \text{the } i\text{-th variable is essential in } f\}$$

and is denoted by $\operatorname{ess} f$. If $\operatorname{ess} f = n$, then we say that f is *essentially n-ary*.

Let $f\colon A^n \to B$ and $g\colon A^m \to B$. We say that g is a *simple minor* of f, and we write $g \leq f$, if there exists a map $\alpha\colon [n] \to [m]$ such that $g(a_1, \dots, a_m) = f(a_{\alpha(1)}, \dots, a_{\alpha(n)})$

for all $a_1, \dots, a_m \in A$. (Informally, g is a simple minor of f, if g can be obtained from f by permutation of variables, addition of inessential variables, deletion of inessential variables, or identification of variables.)

Let $f\colon A^n \to B$. For $i, j \in [n]$, $i \neq j$, the simple minor $f_{i \leftarrow j}\colon A^n \to B$ of f given by the rule

$$f_{i \leftarrow j}(a_1, \dots, a_n) = f(a_1, \dots, a_{i-1}, a_j, a_{i+1}, \dots, a_n),$$

for all $a_1, \dots, a_n \in A$, is called an *identification minor* of f, obtained by substituting x_j for x_i. Note that a_j occurs twice on the right-hand side of the above equality while a_i does not appear at all. Thus, the i-th variable is necessarily inessential in $f_{i \leftarrow j}$.

The simple minor relation $\leq$ is a quasiorder on $\mathcal{F}_{AB}$. As for quasiorders, $\leq$ induces an equivalence relation $\equiv$ on $\mathcal{F}_{AB}$. If $f \equiv g$, then we say that f and g are *equivalent.* (Informally, f and g are equivalent, if each of f and g can be obtained from the other by permutation of variables, addition of inessential variables, and deletion of inessential variables.)

Remark 1. If $f \equiv g$, then $\operatorname{ess} f = \operatorname{ess} g$. Every nonconstant function is equivalent to a function that depends on all of its variables.

If $g \leq f$ but $f \not\equiv g$, then we write $g < f$ and say that g is a *strict minor* of f. If $g < f$ but there is no h such that $g < h < f$, then we say that g is a *lower cover* of f and denote this fact by $g \prec f$ or, equivalently, by $f \succ g$.

Remark 2. It was shown in [2] that the lower covers of any function $f\colon A^n \to B$ have the same essential arity when $A = B = \{0, 1\}$. The proof of this fact given in [2] actually shows that this claim is true whenever $|A| = 2$ and $|B| \geq 2$. However, this is not the case when $|A| > 2$.

To this extent, let A be a set with at least three elements, let B be a set with at least two elements, and assume that 0 and 1 are distinct elements of B. Let $\nu\colon A^2 \to B$ be the *inequality predicate*

$$\nu(x, y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y, \end{cases}$$

and let $\wedge\colon B^2 \to B$ and $\vee\colon B^2 \to B$ be arbitrary extensions of the Boolean conjunction and disjunction to B (i.e., arbitrary binary operations on B satisfying $0 \wedge 0 = 0 \wedge 1 = 1 \wedge 0 = 0$, $1 \wedge 1 = 1$, $0 \vee 0 = 0$, $1 \vee 0 = 0 \vee 1 = 1 \vee 1 = 1$). Consider the function $f\colon A^4 \to B$ defined by

$$f(a_1, a_2, a_3, a_4) := \nu(a_1, a_2) \vee \bigwedge_{\substack{1 \leq i < j \leq 4 \\ (i,j) \neq (1,2)}} \nu(a_i, a_j),$$

for all $(a_1, a_2, a_3, a_4) \in A^4$. It is easy to see that $f_{i \leftarrow j}(a_1, a_2, a_3, a_4) = \nu(a_1, a_2)$ for $1 \leq i < j \leq 4$, $(i, j) \neq (1, 2)$, and

$$f_{1 \leftarrow 2}(a_1, a_2, a_3, a_4) = \bigwedge_{2 \leq i < j \leq 4} \nu(a_i, a_j).$$

Furthermore, $\operatorname{ess} f = 4$, $\operatorname{ess} f_{1 \leftarrow 2} = 3$, $\operatorname{ess} f_{i \leftarrow j} = 2$, and $f_{1 \leftarrow 2} \not\leq f_{i \leftarrow j} \not\leq f_{1 \leftarrow 2}$, for every $1 \leq i < j \leq 4$, $(i, j) \neq (1, 2)$. Hence, f has two lower covers of different essential arities.

For background on the simple minor relation and its variants, see [3], [9], [10], [11], [12], [15], [17].

We say that f is *totally symmetric,* if for all permutations π of $[n]$ the identity $f(a_1, \dots, a_n) = f(a_{\pi(1)}, \dots, a_{\pi(n)})$ holds for all $a_1, \dots, a_n \in A$. Observe that a totally symmetric function depends on either all or none of its variables.

Fact 1. *If $f\colon A^n \to B$ is totally symmetric, then for all $i, j, i', j' \in [n]$ ($i \neq j$, $i' \neq j'$), $f_{i \leftarrow j} \equiv f_{i' \leftarrow j'}$. Therefore, if f is nonconstant, then for all distinct $i, j \in [n]$, $f_{i \leftarrow j}$ is, up to equivalence, the unique lower cover of f.*

B. *Functions determined by* supp *and* oddsupp

Following Berman and Kisielewicz [1], we define $\operatorname{supp}\colon \bigcup_{n \geq 1} A^n \to \mathcal{P}(A)$ and $\operatorname{oddsupp}\colon \bigcup_{n \geq 1} A^n \to \mathcal{P}(A)$ as

$$\operatorname{supp}(a_1, \dots, a_n) := \{a_1, \dots, a_n\},$$
$$\operatorname{oddsupp}(a_1, \dots, a_n) := \{a \in A : |\{i \in [n] : a_i = a\}| \text{ is odd}\}.$$

We say that $f\colon A^n \to B$ is *determined by* supp (respectively, *determined by* oddsupp), if there exists a function $\varphi\colon \mathcal{P}(A) \to B$ such that $f = \varphi \circ \operatorname{supp}|_{A^n}$ (respectively, $f = \varphi \circ \operatorname{oddsupp}|_{A^n}$). Note that every function determined by supp or oddsupp is totally symmetric; hence such a function either depends on all of its variables or on none of them. However, not every totally symmetric function is determined by supp or oddsupp.

Remark 3. For any positive integer n, let us define the following subsets of $\mathcal{P}(A)$:

$$\mathcal{P}_{\leq n}(A) := \{S \subseteq A : 1 \leq |S| \leq n\},$$
$$\mathcal{P}'_{\leq n}(A) := \{S \subseteq A : |S| \in \{n, n-2, \dots\}\}.$$

Clearly, $\{\operatorname{supp}(\mathbf{x})\colon \mathbf{x} \in A^n\} = \mathcal{P}_{\leq n}(A)$, therefore the restriction of $\varphi\colon \mathcal{P}(A) \to B$ to $\mathcal{P}_{\leq n}(A)$ uniquely determines the function $\varphi \circ \operatorname{supp}|_{A^n}$, and vice versa. Similarly, we have $\{\operatorname{oddsupp}(\mathbf{x})\colon \mathbf{x} \in A^n\} = \mathcal{P}'_{\leq n}(A)$, and consequently there is a one-to-one correspondence between functions $f\colon A^n \to B$ determined by oddsupp and maps $\varphi\colon \mathcal{P}'_{\leq n}(A) \to B$. In particular, $\varphi \circ \operatorname{supp}|_{A^n}$ (resp. $\varphi \circ \operatorname{oddsupp}|_{A^n}$) is constant if and only if $\varphi|_{\mathcal{P}_{\leq n}(A)}$ (resp. $\varphi|_{\mathcal{P}'_{\leq n}(A)}$) is constant, and has essential arity n otherwise.

Example 2. Every constant function and every unary function is determined by both supp and oddsupp. Furthermore, for each $2 \leq n \leq |A|$, there are nonconstant functions $f\colon A^n \to B$ that are determined by both supp and oddsupp. For instance, let a and b be distinct elements of B and define $f\colon A^n \to B$ by the rule

$$f(a_1, \dots, a_n) := \begin{cases} a & \text{if } a_i \neq a_j \text{ for all } i \neq j, \\ b & \text{otherwise.} \end{cases}$$

It is easy to see that $f = \phi \circ \mathrm{supp}|_{A^n} = \phi \circ \mathrm{oddsupp}|_{A^n}$, where $\phi\colon \mathcal{P}(A) \to B$ is the map

$$\phi(S) := \begin{cases} a & \text{if } |S| = n, \\ b & \text{otherwise.} \end{cases}$$

Our next result shows that, in fact, such nontrivial examples of functions determined by both supp and oddsupp can only be found among functions with small arities.

Proposition 3. *If $n > |A|$, then $f\colon A^n \to B$ is determined by both* supp *and* oddsupp *if and only if f is a constant function.*

As it will become clear from Propositions 4 and 7 below, if a nonconstant function f is determined by oddsupp (supp, respectively) then every simple minor of f is equivalent to a function that is determined by oddsupp (supp, respectively).

Proposition 4. *Let A and B be finite nonempty sets, and let $k := |A|$. If $f\colon A^n \to B$ is a nonconstant function determined by* oddsupp*, then the simple minors of f form a chain*

$$f = f_n \succ f_{n-2} \succ \cdots \succ f_{n-2t-2} \succ f_{n-2t}$$

of length t such that $\mathrm{ess}\, f_{n-2i} = n-2i$ for all $i < t$. Moreover, we either have $\mathrm{ess}\, f_{n-2t} = 1$ and $t = \frac{n-1}{2}$ or $\mathrm{ess}\, f_{n-2t} = 0$ and $\left\lceil \frac{n-k}{2} \right\rceil < t \leq \left\lfloor \frac{n}{2} \right\rfloor$.

In the following two examples we construct for all possible values of k, n and t a function determined by oddsupp whose simple minors form a chain as stated in Proposition 4, thereby showing that this result cannot be sharpened.

Example 5. Let k, n, t be positive integers such that $k \geq 2$ and $\left\lceil \frac{n-k}{2} \right\rceil < t \leq \left\lfloor \frac{n}{2} \right\rfloor$. Then $s := n - 2t + 2$ satisfies the inequalities $2 \leq s \leq k$. Let A be a set with k elements, let B be a set with at least two elements, and let us define $\varphi\colon \mathcal{P}(A) \to B$ by

$$\varphi(S) := \begin{cases} 1 & \text{if } |S| \geq s, \\ 0 & \text{if } |S| < s, \end{cases}$$

where 0 and 1 denote two distinct elements of B. For every $0 \leq i \leq t$, let $f_{n-2i}\colon A^{n-2i} \to B$ be the function determined by oddsupp via φ, i.e., let $f_{n-2i} = \varphi \circ \mathrm{oddsupp}|_{A^{n-2i}}$. It is straightforward to verify with the help of Remark 3 that $f_{n-2t} = f_{s-2}$ is constant, and that $\mathrm{ess}\, f_{n-2i} = n - 2i$ if $i < t$. Moreover, for $i < t$, the unique lower cover of f_{n-2i} is f_{n-2i-2}. Thus the simple minors of f_n form a chain exactly as in (the second case of) Proposition 4.

Example 6. Let k, n, t be positive integers such that $k \geq 2$ and $t = \frac{n-1}{2}$. Let A be a set with k elements, let B be a set with at least two elements, and let $\varphi\colon \mathcal{P}(A) \to B$ be any function that is not constant on singletons, i.e., there exist $a_1, a_2 \in A$ such that $\varphi(\{a_1\}) \neq \varphi(\{a_2\})$. For every odd number r, let $f_r\colon A^r \to B$ be the function determined by oddsupp via φ, i.e., let $f_r = \varphi \circ \mathrm{oddsupp}|_{A^r}$. Then f_r is not constant, hence $\mathrm{ess}\, f_r = r$, and the unique lower cover of f_r is f_{r-2}. Thus the simple minors of f_n form a chain exactly as in (the first case of) Proposition 4.

Reasoning as above, by making use of Remark 3, we have the following analogue of Proposition 4.

Proposition 7. *Let A and B be finite nonempty sets, and let $k := |A|$. If $f\colon A^n \to B$ is a nonconstant function determined by* supp*, then the simple minors of f form a chain*

$$f = f_n \succ f_{n-1} \succ \cdots \succ f_{n-t+1} \succ f_{n-t}$$

of length t such that $\mathrm{ess}\, f_{n-i} = n - i$ for all $i < t$. Moreover, we either have $\mathrm{ess}\, f_{n-t} = 1$ and $t = n - 1$, or $\mathrm{ess}\, f_{n-t} = 0$ and $n - k < t < n$.

As for functions determined by oddsupp, we can find functions which fall into each of the two possible cases provided in Proposition 7.

C. Arity gap

Let $f\colon A^n \to B$ be a function that depends on at least two variables, i.e., $\mathrm{ess}\, f \geq 2$. The *arity gap* of f, denoted $\mathrm{gap}\, f$, is defined as

$$\mathrm{gap}\, f := \min_{\substack{i,j \in \mathrm{Ess}\, f \\ i \neq j}} (\mathrm{ess}\, f - \mathrm{ess}\, f_{i \leftarrow j}).$$

While $f_{i \leftarrow j}$ is not necessarily a lower cover of f in the simple minor quasiorder, every lower cover of f is of the form $f_{i \leftarrow j}$. Therefore, we could define the arity gap of f in an equivalent way as

$$\mathrm{gap}\, f := \min_{g \prec f} (\mathrm{ess}\, f - \mathrm{ess}\, g).$$

Whenever we consider the arity gap of a function $f\colon A^n \to B$, we may assume, without loss of generality, that f depends on all of its variables (see Remark 1).

As made apparent by Willard [16], the notion of arity gap is tightly related to determinability by supp and oddsupp. The following statements are immediate consequences of Propositions 4 and 7.

Corollary 8. *Let $f\colon A^n \to B$, $n > |A|$, be nonconstant.*

(1) *If f is determined by* oddsupp*, then* $\mathrm{gap}\, f = 2$.
(2) *If f is determined by* supp*, then* $\mathrm{gap}\, f = 1$.

We now recall a noteworthy result about the arity gap.

Theorem 9 (Willard [16])**.** *Let A and B be finite nonempty sets, and let $k := |A|$. Suppose that $f\colon A^n \to B$ depends on all of its variables. If $n > k$, then $\mathrm{gap}\, f \leq 2$. Moreover, if $n > \max(k, 3)$, then $\mathrm{gap}\, f = 2$ if and only if f is determined by* oddsupp.

III. Parametrized arity gap

In this section we are interested in a parametrized version of arity gap which measures the minimum decrease in the essential arity when we take $\ell \geq 0$ steps downwards in the simple minor quasiorder. Let $f\colon A^n \to B$, and let $\downarrow^0 f := \{f\}$, and for $\ell \geq 1$, let

$$\downarrow^\ell f := \{g \in \mathcal{F}_{AB} \mid \exists f_1, \ldots, f_{\ell-1} : f \succ f_1 \succ \cdots \succ f_{\ell-1} \succ g\}.$$

If $\downarrow^\ell f \neq \emptyset$, then we define

$$\operatorname{gap}(f,\ell) := \min_{g \in \downarrow^\ell f} (\operatorname{ess} f - \operatorname{ess} g). \tag{1}$$

Note that $\operatorname{gap}(f,\ell)$ is defined only if there exists a chain of length ℓ below f, and in this case $\ell \leq \operatorname{gap}(f,\ell) \leq \operatorname{ess} f$. The arity gap (as defined in Section II-C) corresponds to the case $\ell = 1$, that is, $\operatorname{gap}(f,1) = \operatorname{gap} f$ for every function f. Observe also that $\operatorname{gap}(f,0) = 0$ for every function f.

We saw in Section II-A that taking a strict minor of a function f requires the identification of at least one pair of essential variables of f; otherwise, the minors we obtain are equivalent to f. This means that $\operatorname{gap}(f,\ell)$ can be computed by sequentially identifying a pair of essential variables ℓ times in all possible ways, starting from f, and then determining the sequence of ℓ identifications which results in the minimum loss of essential variables.

Remark 4. It is worth stressing the fact that the identification of variables is performed sequentially, and at each step only one pair of essential variables is identified; otherwise, ambiguities could occur since a priori we do not know which essential variables become inessential after a pair is identified.

We mentioned in Section II-C that not every identification minor $f_{i\leftarrow j}$ is a lower cover of f, and if $f_{i\leftarrow j}$ is not a lower cover of f, then $\operatorname{gap} f < \operatorname{ess} f - \operatorname{ess} f_{i\leftarrow j}$. Moreover, it can be the case that f has two lower covers f_1 and f_2 such that $\operatorname{ess} f_1 < \operatorname{ess} f_2$, and again we would conclude that $\operatorname{gap} f < \operatorname{ess} f - \operatorname{ess} f_1$. Hence, one might be led to thinking that in order to compute $\operatorname{gap}(f,\ell)$ it suffices to choose at each recursion step an identification which results in the minimum loss of essential variables. However, as the following example illustrates, this is not true.

Example 10. Let A be the 5-element field and consider the function $f\colon A^6 \to A$ represented by the polynomial

$$(x_1 - x_2)(x_5 - x_6) + \prod_{\substack{1 \leq i < j \leq 6 \\ (i,j) \neq (5,6)}} (x_i - x_j).$$

It is easy to verify that $\operatorname{ess} f = 6$, $f_{1\leftarrow 2} \equiv 0$, and that f has, up to equivalence, two lower covers, namely,

$$\begin{aligned} f_{1\leftarrow 3} &= (x_1 - x_2) \cdot (x_5 - x_6), \\ f_{5\leftarrow 6} &= \prod_{1 \leq i < j \leq 4} (x_i - x_j) \cdot \prod_{1 \leq i \leq 4} (x_i - x_5)^2. \end{aligned}$$

Figure 1 presents the Hasse diagram of the principal ideal generated by the equivalence class of f in the simple minor poset. The label of each edge $g \prec h$ is the number $\operatorname{ess} h - \operatorname{ess} g$.

Fig. 1. The simple minors of the function f given in Example 10.

We use the following notation for simple minors of f:

$$\begin{aligned} q_1 &= \prod_{1 \leq i < j \leq 4} (x_i - x_j) \cdot \prod_{1 \leq i \leq 4} (x_i - x_5)^2, \\ q_2 &= (x_1 - x_2) \cdot (x_3 - x_4), \\ q_3 &= (x_1 - x_2) \cdot (x_1 - x_3), \\ q_4 &= (x_1 - x_2) \cdot (x_2 - x_3), \\ q_5 &= (x_1 - x_2)^2, \\ q_6 &= -(x_1 - x_2)^2. \end{aligned}$$

Now if we would choose as our first identification the pair $\{5,6\}$, then any other identification of essential variables results in the loss of all the remaining essential variables. In other words, any downward path in Figure 1 which starts from f and passes through q_1 has length 2, and along it we first lose 1 and then 5 essential variables. However, the downward paths that start from f and pass through q_2 have length 4, and along them we lose 2, 1, 1, and then 2 essential variables. This shows that, in order to compute $\operatorname{gap}(f,1)$ as in (1), the minimum value is attained at the lower cover q_1, whereas, for $2 \leq \ell \leq 4$, we need to pass through q_2 for computing $\operatorname{gap}(f,\ell)$. Hence, $\operatorname{gap}(f,0) = 0$, $\operatorname{gap}(f,1) = 1$, $\operatorname{gap}(f,2) = 3$, $\operatorname{gap}(f,3) = 4$, and $\operatorname{gap}(f,4) = 6$.

The following recursion formula is an immediate consequence of the definition:

$$\operatorname{gap}(f,\ell) = \min_{g \prec f} (\operatorname{ess} f - \operatorname{ess} g + \operatorname{gap}(g, \ell - 1)). \tag{2}$$

Theorem 11. *Let A and B be finite nonempty sets, and let $k := |A|$. Let $f\colon A^n \to B$, $\operatorname{ess} f = n$ and $\operatorname{gap} f = 1$. If $1 \leq \ell \leq n - \max(k,3)$, then $\operatorname{gap}(f,\ell) = \ell$.*

Informally speaking, Theorem 11 means that if $\operatorname{gap} f = 1$, then we can walk down from f in the simple minor quasiorder in such a way that in each step we lose only one essential variable, provided that the walk is not too long. The next result asserts that if we consider arbitrarily long walks, then we can lose any number of essential variables. More precisely, for each $2 \leq \ell \leq q \leq n$ we can find a function f with $\operatorname{gap} f = 1$ and $\operatorname{gap}(f,\ell) = q$.

Theorem 12. *For every $2 \leq \ell \leq q \leq n$, there exist sets A and B and a function $f\colon A^n \to B$ such that $\operatorname{ess} f = n$, $\operatorname{gap} f = 1$, $\operatorname{gap}(f,\ell) = q$, $|A| < n$.*

Next we consider the analogue of Theorem 11 for the case $\operatorname{gap} f = 2$.

Theorem 13. *Let A and B be finite nonempty sets, and let $k := |A|$. Let $f\colon A^n \to B$, $\operatorname{ess} f = n$ and $\operatorname{gap} f = 2$. If $1 \leq \ell \leq \lceil \frac{n-k}{2} \rceil$, then $\operatorname{gap}(f,\ell) = 2\ell$.*

The following result makes explicit the fact that, for almost every integer sequence $0 = n_0 < n_1 < n_2 < \cdots < n_r \leq n$, we can find a function $f\colon A^n \to B$ whose parametrized arity gap meets every member of the sequence.

Theorem 14. *Let A be a finite set with k elements and let B be a set with at least two elements. Let $2 \leq n \leq k$, $1 \leq r \leq n-1$, $0 = n_0 < n_1 < n_2 < \cdots < n_r \leq n$ such that $n-1 \leq n_r \leq n$ and $n_{r-1} \neq n-1$. Then there exists a function $f\colon A^n \to B$ such that $\operatorname{gap}(f,\ell) = n_\ell$ for every $0 \leq \ell \leq r$.*

Note that the condition $n_{r-1} \neq n-1$ is necessary, because no function has both an essentially unary function and a constant function as its simple minors.

IV. Concluding remarks

As mentioned in Section I, the parametrized arity gap constitutes a tool for tackling yet another natural problem pertaining to the effect of variable identification on the number of essential variables of a function. Given a function $f\colon A^n \to B$ and an integer $p \geq 1$, what is the smallest number m such that any m successive identifications of essential variables result in the loss of at least p essential variables? Let us denote this smallest number by $\operatorname{pag}(f,p)$. As already mentioned in the introduction, $\operatorname{pag}(f,p)$ is indeed the smallest ℓ such that $\operatorname{gap}(f,\ell) \geq p$.

Example 15. Consider the 6-ary function f of Example 10. We can read off of Figure 1 that

$$\begin{aligned}
&\operatorname{pag}(f,1) = 1, && \operatorname{pag}(f,2) = 2, && \operatorname{pag}(f,3) = 2,\\
&\operatorname{pag}(f,4) = 3, && \operatorname{pag}(f,5) = 4, && \operatorname{pag}(f,6) = 4.
\end{aligned}$$

We may also consider similar problems when we perform several simultaneous identifications of variables. Following the formalism of Willard [16], we view functions of several variables as maps $f\colon A^V \to B$, where $V \subseteq \{x_i : i \in \mathbb{N}\}$. The cardinality of V is called the *arity* of f. In this framework, a function $g\colon A^W \to B$ is a *simple minor* of $f\colon A^V \to B$, if there exists a map $\alpha\colon V \to W$ such that $g(\mathbf{a}) = f(\mathbf{a} \circ \alpha)$ for all $\mathbf{a} \in A^W$.

We denote the set of all equivalence relations on a set V by $\operatorname{Eq}(V)$. Given an equivalence relation $\theta \in \operatorname{Eq}(V)$, denote the canonical surjection by $v_\theta\colon V \to V/\theta$. For a function $f\colon A^V \to B$, we define the function $f^\theta\colon A^{V/\theta} \to B$ by $f^\theta(\mathbf{a}) = f(\mathbf{a} \circ v_\theta)$, and we say that f^θ is obtained from f by *block identification of variables* through θ. We informally identify V/θ with any one of its distinct representatives; in this way f^θ is a simple minor of f, and every simple minor of f is equivalent to f^ψ for some $\psi \in \operatorname{Eq}(V)$. The number of variables identified through θ is

$$e(\theta) := \sum_{X \in V/\theta} (|X| - 1) = |V| - |V/\theta|.$$

Assuming that f depends on all of its variables, i.e., $\operatorname{ess} f = |V|$, we have that $\operatorname{ess} f^\theta \leq |V/\theta| = |V| - e(\theta) = \operatorname{ess} f - e(\theta)$.

Now we can define the analogue of the parametrized arity gap for block identification of variables. For a function $f\colon A^V \to B$ with $\operatorname{ess} f = |V| = n$ and for an integer ℓ such that $0 \leq \ell \leq n-1$, we define

$$\operatorname{b-gap}(f,\ell) := \min_{\substack{\theta \in \operatorname{Eq}(V)\\ e(\theta) = \ell}} (\operatorname{ess} f - \operatorname{ess} f^\theta).$$

Note that if $e(\theta) = 0$, then θ is the trivial equivalence relation $\{(x,x) : x \in V\}$; hence $\operatorname{b-gap}(f,0) = 0$. Note also that $\operatorname{b-gap}(f,1) = \operatorname{gap} f$ for every function f. It is also clear that $\ell \leq \operatorname{b-gap}(f,\ell) \leq n$ for every $0 \leq \ell \leq n-1$, and $\operatorname{b-gap}(f,\ell) \leq \operatorname{gap}(f,\ell)$ for every ℓ for which $\operatorname{gap}(f,\ell)$ is defined.

Let $H(f) := \{\operatorname{ess} f - \operatorname{ess} g : g \leq f\}$. It is clear that $\{\operatorname{b-gap}(f,\ell) : 0 \leq \ell \leq n-1\} \subseteq H(f)$. Moreover, if $f\colon A^V \to B$ is a function such that $\operatorname{ess} f = |V| = n$, then

$$\operatorname{b-gap}(f,\ell) = \min\{m \in H(f) : m \geq \ell\},$$

for every $0 \leq \ell \leq n-1$.

Example 16. Consider the 6-ary function f of Example 10. We can read off of Figure 1 that $H(f) = \{1,2,3,4,6\}$ and

$$\begin{aligned}
&\operatorname{b-gap}(f,1) = 1, && \operatorname{b-gap}(f,2) = 2, && \operatorname{b-gap}(f,3) = 3,\\
&\operatorname{b-gap}(f,4) = 4, && \operatorname{b-gap}(f,5) = 6.
\end{aligned}$$

We can still consider an analogue of the problem stated in the first paragraph of this section. Given a function $f\colon A^V \to B$ that depends on all of its variables and an integer $p \geq 1$, what is the smallest number m such that block identification of variables of f through every equivalence relation θ on V with $e(\theta) = m$ results in the loss of at least p essential variables? Let us denote this smallest number by $\operatorname{b-pag}(f,p)$. It is again clear that $\operatorname{b-pag}(f,p)$ is the smallest ℓ such that $\operatorname{b-gap}(f,\ell) \geq p$.

Example 17. Consider the 6-ary function f of Example 10. We can determine from the values of $\operatorname{b-gap}(f,\ell)$ listed in Example 16, or we can easily read off of Figure 1 that

$$\begin{aligned}
&\operatorname{b-pag}(f,1) = 1, && \operatorname{b-pag}(f,2) = 2, && \operatorname{b-pag}(f,3) = 3,\\
&\operatorname{b-pag}(f,4) = 4, && \operatorname{b-pag}(f,5) = 5, && \operatorname{b-pag}(f,6) = 5.
\end{aligned}$$

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