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***Abstract*—In this paper we report recent results in [1] concerning local versions of monotonicity for Boolean and pseudo-Boolean functions: say that a pseudo-Boolean (Boolean) function is p-locally monotone if each of its partial derivatives keeps the same sign on tuples which differ on less than p positions. As it turns out, this parameterized notion provides a hierarchy of monotonicities for pseudo-Boolean (Boolean) functions.**

Local monotonicities are tightly related to lattice counterparts of classical partial derivatives via the notion of permutable derivatives. More precisely, p-locally monotone functions have p-permutable lattice derivatives and, in the case of symmetric functions, these two notions coincide. We provide further results relating these two notions, and present a classification of p-locally monotone functions, as well as of functions having p-permutable derivatives, in terms of certain forbidden "sections", i.e., functions which can be obtained by substituting variables for constants. This description is made explicit in the special case when $p = 2$.

I. Introduction

Throughout this paper, let $[n] = \{1, \dots, n\}$ and $\mathbb{B} = \{0, 1\}$. We are interested in the so-called Boolean functions $f\colon \mathbb{B}^n \to \mathbb{B}$ and pseudo-Boolean functions $f\colon \mathbb{B}^n \to \mathbb{R}$, where n denotes the arity of f. The pointwise ordering of functions is denoted by $\leq$, i.e., $f \leq g$ means that $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{B}^n$. The negation of $x \in \mathbb{B}$ is defined by $\overline{x} = x \oplus 1$, where $\oplus$ stands for addition modulo 2. For $x, y \in \mathbb{B}$, we set $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$.

For $k \in [n]$, $\mathbf{x} \in \mathbb{B}^n$, and $a \in \mathbb{B}$, let $\mathbf{x}_k^a$ be the tuple in $\mathbb{B}^n$ whose i-th component is a, if $i = k$, and x_i, otherwise. We use the shorthand notation $\mathbf{x}_{jk}^{ab}$ for $(\mathbf{x}_j^a)_k^b = (\mathbf{x}_k^b)_j^a$. More generally, for $S \subseteq [n]$, $\mathbf{a} \in \mathbb{B}^n$, and $\mathbf{x} \in \mathbb{B}^S$, let $\mathbf{a}_S^{\mathbf{x}}$ be the tuple in $\mathbb{B}^n$ whose i-th component is x_i, if $i \in S$, and a_i, otherwise.

Let $i \in [n]$ and $f\colon \mathbb{B}^n \to \mathbb{R}$. A variable x_i is said to be *essential* in f, or that f *depends* on x_i, if there exists $\mathbf{a} \in \mathbb{B}^n$ such that $f(\mathbf{a}_i^0) \neq f(\mathbf{a}_i^1)$. Otherwise, x_i is said to be *inessential* in f. Let $S \subseteq [n]$ and $f\colon \mathbb{B}^n \to \mathbb{R}$. We say that $g\colon \mathbb{B}^S \to \mathbb{R}$ is an S-*section* of f if there exists $\mathbf{a} \in \mathbb{B}^n$ such that $g(\mathbf{x}) = f(\mathbf{a}_S^{\mathbf{x}})$ for all $\mathbf{x} \in \mathbb{B}^S$ (observe that the components a_i are irrelevant for $i \in S$). By a *section* of f we mean an S-section of f for some $S \subseteq [n]$, i.e., any function which can be obtained from f by replacing some of its variables by constants.

The *(discrete) partial derivative* of $f\colon \mathbb{B}^n \to \mathbb{R}$ with respect to its k-th variable is the function $\Delta_k f\colon \mathbb{B}^n \to \mathbb{R}$ defined by $\Delta_k f(\mathbf{x}) = f(\mathbf{x}_k^1) - f(\mathbf{x}_k^0)$; see [5], [8]. Note that $\Delta_k f$ does not depend on its k-th variable, hence it could be regarded as a function of arity $n - 1$, but for notational convenience we define it as an n-ary function.

A pseudo-Boolean function $f\colon \mathbb{B}^n \to \mathbb{R}$ can always be represented by a multilinear polynomial of degree at most n (see [9]), that is,

$$f(\mathbf{x}) = \sum_{S \subseteq [n]} a_S \prod_{i \in S} x_i \,, \tag{1}$$

where $a_S \in \mathbb{R}$. For instance the multilinear expression for a binary pseudo-Boolean function is given by

$$a_0 + a_1 x_1 + a_2 x_2 + a_{12} x_1 x_2 \,. \tag{2}$$

This representation is very convenient for computing the partial derivatives of f. Indeed, $\Delta_k f$ can be obtained by applying the corresponding formal derivative to the multilinear representation of f. Thus, from (1), we obtain immediately

$$\Delta_k f(\mathbf{x}) = \sum_{S \ni k} a_S \prod_{i \in S \setminus \{k\}} x_i \,. \tag{3}$$

We say that f is *isotone* (resp. *antitone*) *in its k-th variable* if $\Delta_k f(\mathbf{x}) \geq 0$ (resp. $\Delta_k f(\mathbf{x}) \leq 0$) for all $\mathbf{x} \in \mathbb{B}^n$. If f is either isotone or antitone in its k-th variable, then we say that f is *monotone in its k-th variable*. If f is isotone (resp. antitone, monotone) in all of its variables, then f is an *isotone* (resp. *antitone*, *monotone*) *function*.[1] It is clear that any section of an isotone (resp. antitone, monotone) function is also isotone (resp. antitone, monotone).

[1] Note that the terms "positive" and "nondecreasing" (resp. "negative" and "nonincreasing") are often used instead of isotone (resp. antitone), and it is also customary to use the word "monotone" only for isotone functions.

Thus defined, a function $f\colon \mathbb{B}^n \to \mathbb{R}$ is monotone if and only if each of its partial derivatives has the same sign on $\mathbb{B}^n$. In this paper we are interested in some parameterized relaxations of monotonicity: a function $f\colon \mathbb{B}^n \to \mathbb{R}$ is p-locally monotone if each of its partial derivatives has the same sign on tuples which differ on less than p positions. As we will see, these relaxations are tightly related to the following lattice versions of partial derivatives. For $f\colon \mathbb{B}^n \to \mathbb{R}$ and $k \in [n]$, let $\wedge_k f\colon \mathbb{B}^n \to \mathbb{R}$ and $\vee_k f\colon \mathbb{B}^n \to \mathbb{R}$ be the *partial lattice derivatives* defined by

$$\wedge_k f(\mathbf{x}) = f(\mathbf{x}^0_k) \wedge f(\mathbf{x}^1_k) \quad \text{and} \quad \vee_k f(\mathbf{x}) = f(\mathbf{x}^0_k) \vee f(\mathbf{x}^1_k).$$

The latter, known as the k-th *join derivative* of f, was proposed by Fadini [6] while the former, known as the k-th *meet derivative* of f, was introduced by Thayse [12]. In [13] these lattice derivatives were shown to be related to so-called prime implicants and implicates of Boolean functions which play an important role in the consensus method for Boolean and pseudo-Boolean functions. For further background and applications see, e.g., [2], [3], [4], [11], [14].

Observe that, just like in the case of the partial derivative $\Delta_k f$, the k-th variable of each of the lattice derivatives $\wedge_k f$ and $\vee_k f$ is inessential.

The following proposition assembles some basic properties of lattice derivatives.

Proposition 1. *For any pseudo-Boolean functions $f, g\colon \mathbb{B}^n \to \mathbb{R}$ and $j, k \in [n]$, $j \neq k$, the following hold:*

(i) $\wedge_k \wedge_k f = \wedge_k f$ *and* $\vee_k \vee_k f = \vee_k f$;
(ii) *if* $f \leq g$, *then* $\wedge_k f \leq \wedge_k g$ *and* $\vee_k f \leq \vee_k g$;
(iii) $\wedge_j \wedge_k f = \wedge_k \wedge_j f$ *and* $\vee_j \vee_k f = \vee_k \vee_j f$;
(iv) $\vee_k \wedge_j f \leq \wedge_j \vee_k f$.

From equations (1) and (3) it follows that every function is (up to an additive constant) uniquely determined by its partial derivatives. As it turns out, this does not hold when the lattice derivatives are considered. However, as we shall see, there are only two types of such exceptions (see Theorem 22).

Now, if an n-ary pseudo-Boolean function is 2-locally monotone, then for every $j, k \in [n]$, $j \neq k$, we have $\vee_k \wedge_j f = \wedge_j \vee_k f$ (see Lemma 10 below). This motivates the notion of permutable lattice derivatives. As it turns out, p-local monotonicity of f implies permutability of p of its lattice derivatives (see Theorem 21). However the converse does not hold; see Example 24.

The structure of this paper goes as follows. In Section II we formalize the notion of p-local monotonicity. As it turns out, this notion gives rise to a hierarchy of monotonicities whose largest member is the class of all n-ary pseudo-Boolean functions (this is the case when $p = 1$) and whose smallest member is the class of n-ary monotone functions (this is the case when $p = n$). We also provide a characterization of p-locally monotone functions in terms of "forbidden" sections; this characterization is made explicit in the special case when $p = 2$. In Section III we introduce the notion of permutable lattice derivatives. Similarly to local monotonicity, the notion of permutable lattice derivatives gives rise to nested classes, each of which is also described in terms of its sections. In the Boolean case and for $p = 2$, these two parameterized notions coincide; this does not hold for pseudo-Boolean functions even when $p = 2$ (see Example 12). However, in the case of symmetric functions, the notion of being p-locally monotone is equivalent to that of having p-permutable lattice derivatives; see Section IV. In the last section we discuss directions for future research.

II. Local monotonicities

The following definition formulates a local version of monotonicity, where $\Delta_k f(\mathbf{x})$ and $\Delta_k f(\mathbf{y})$ are required to have the same sign only for tuples $\mathbf{x}, \mathbf{y}$ which are close to each other with respect to the Hamming distance. In what follows we assume that $p \in [n]$.

Definition 2. We say that $f\colon \mathbb{B}^n \to \mathbb{R}$ is *p-locally monotone* if, for every $k \in [n]$ and every $\mathbf{x}, \mathbf{y} \in \mathbb{B}^n$, we have

$$\sum_{i \in [n]\setminus\{k\}} |x_i - y_i| < p \quad \Rightarrow \quad \Delta_k f(\mathbf{x})\, \Delta_k f(\mathbf{y}) \geq 0.$$

Any p-locally monotone pseudo-Boolean function is also p'-locally monotone for every $p' \leq p$. Every function $f\colon \mathbb{B}^n \to \mathbb{R}$ is 1-locally monotone, and f is n-locally monotone if and only if it is monotone. Thus p-local monotonicity is a relaxation of monotonicity, and the nested classes of p-locally monotone functions for $p = 1, \ldots, n$ provide a hierarchy of monotonicities for n-ary pseudo-Boolean functions. The weakest nontrivial condition is 2-local monotonicity, therefore we will simply say that f is *locally monotone* whenever f is 2-locally monotone.[2] If f is p-locally monotone for some $p < n$ but not $(p+1)$-locally monotone, then we say that f is *exactly p-locally monotone*, or that the *degree of local monotonicity* of f is p.

If $f\colon \mathbb{B}^n \to \mathbb{B}$ is a Boolean function, then $\Delta_k f(\mathbf{x}) \in \{-1, 0, 1\}$ for all $\mathbf{x} \in \mathbb{B}^n$, hence the condition $\Delta_k f(\mathbf{x})\, \Delta_k f(\mathbf{y}) \geq 0$ in the definition of p-local monotonicity is equivalent to

$$|\Delta_k f(\mathbf{x}) - \Delta_k f(\mathbf{y})| \leq 1. \tag{4}$$

From this it follows that a Boolean function $f\colon \mathbb{B}^n \to \mathbb{B}$ is locally monotone if and only if

$$\big|\Delta_k f(\mathbf{x}) - \Delta_k f(\mathbf{y})\big| \leq \sum_{i \in [n]\setminus\{k\}} |x_i - y_i|. \tag{5}$$

(see [10, Lemma 5.1] for a proof of (5) in a slightly more general framework). In a sense, the latter identity means that $\Delta_k f$ is "1-Lipschitz continuous".

The following proposition is just a reformulation of the definition of p-local monotonicity.

Proposition 3. *A function $f\colon \mathbb{B}^n \to \mathbb{R}$ is p-locally monotone if and only if, for every $k \in [n]$, $S \subseteq [n]\setminus\{k\}$, with $|S| = p-1$, and every $\mathbf{a} \in \mathbb{B}^n$, $\mathbf{x}, \mathbf{y} \in \mathbb{B}^S$, we have*

$$\Delta_k f(\mathbf{a}^{\mathbf{x}}_S)\, \Delta_k f(\mathbf{a}^{\mathbf{y}}_S) \geq 0. \tag{6}$$

[2] In [7], local monotonicity is used to refer to Boolean functions which are monotone (i.e., isotone or antitone in each variable).

As a special case, we have that $f\colon \mathbb{B}^n \to \mathbb{R}$ is locally monotone if and only if, for every $j,k \in [n]$, $j \neq k$, and every $\mathbf{x} \in \mathbb{B}^n$

$$\Delta_k f(\mathbf{x}_j^0)\, \Delta_k f(\mathbf{x}_j^1) \geq 0. \tag{7}$$

By using (4), we see that, for Boolean functions $f\colon \mathbb{B}^n \to \mathbb{B}$, the inequality (7) can be replaced with $|\Delta_{jk} f(\mathbf{x})| \leq 1$, where $\Delta_{jk} f(\mathbf{x}) = \Delta_j \Delta_k f(\mathbf{x}) = \Delta_k \Delta_j f(\mathbf{x})$.

Example 4. The binary Boolean sum

$$f_1(x_1, x_2) = x_1 \oplus x_2 = x_1 + x_2 - 2x_1x_2$$

and the binary Boolean equivalence

$$f_2(x_1, x_2) = \overline{f_1}(x_1, x_2) = x_1 \oplus x_2 \oplus 1 = 1 - x_1 - x_2 + 2x_1x_2$$

are not locally monotone. Indeed, we have $|\Delta_{12} f_1(x_1, x_2)| = |\Delta_{12} f_2(x_1, x_2)| = 2$.

Example 5. Consider the ternary Boolean function $f\colon \mathbb{B}^3 \to \mathbb{B}$ given by

$$f(x_1, x_2, x_3) = x_1 - x_1x_2 + x_2x_3.$$

Since $\Delta_2 f$ may change in sign ($\Delta_2 f(\mathbf{x}) = x_3 - x_1$), the function f is not monotone. However, f is locally monotone since $|\Delta_{12} f(\mathbf{x})| = 1$, $|\Delta_{13} f(\mathbf{x})| = 0$, and $|\Delta_{23} f(\mathbf{x})| = 1$. Thus f is exactly 2-locally monotone. Example 26 in Section IV provides, for each $p \geq 2$, examples of exactly p-locally monotone functions.

Fact 6. *A function $f\colon \mathbb{B}^n \to \mathbb{R}$ is p-locally monotone if and only if so is $\alpha f + \beta$ for every $\alpha, \beta \in \mathbb{R}$, with $\alpha \neq 0$. The same holds for any function obtained from f by negating some of its variables.*

The next theorem gives a characterization of p-locally monotone functions in terms of their sections.

Theorem 7. *A function $f\colon \mathbb{B}^n \to \mathbb{R}$ is p-locally monotone if and only if every p-ary section of f is monotone.*

By combining (7) with Theorem 7, we can easily verify the following corollary.

Corollary 8. *A function $f\colon \mathbb{B}^n \to \mathbb{R}$ is locally monotone if and only if every binary section (2) of f satisfies $a_1(a_1 + a_{12}) \geq 0$ and $a_2(a_2 + a_{12}) \geq 0$.*

Since every binary Boolean function is monotone except for $x \oplus y$ and $x \oplus y \oplus 1$, we also obtain the following corollary.

Corollary 9. *A Boolean function $f\colon \mathbb{B}^n \to \mathbb{B}$ is locally monotone if and only if neither $x \oplus y$ nor $x \oplus y \oplus 1$ is a section of f.*

III. Permutable lattice derivatives

The aim of this section is to relate commutation of lattice derivatives to p-local monotonicity. The starting point is the following result.

Lemma 10. *If $f\colon \mathbb{B}^n \to \mathbb{R}$ is locally monotone, then $\vee_k \wedge_j f = \wedge_j \vee_k f$ for all $j,k \in [n]$, $j \neq k$.*

As we will see in Example 12, Lemma 10 cannot be strengthened to an equivalence. However, the converse of Lemma 10 also holds in the case of Boolean functions.

Theorem 11. *A Boolean function $f\colon \mathbb{B}^n \to \mathbb{B}$ is locally monotone if and only if $\vee_k \wedge_j f = \wedge_j \vee_k f$ holds for all $j,k \in [n]$, $j \neq k$.*

Example 12. Let f be the binary pseudo-Boolean function defined by $f(0,0) = 1$, $f(0,1) = 4$, $f(1,0) = 2$ and $f(1,1) = 3$. Then we have $\vee_2 \wedge_1 f = \wedge_1 \vee_2 f = 3$ and $\vee_1 \wedge_2 f = \wedge_2 \vee_1 f = 2$. However, f is not locally monotone since $\Delta_1 f(\mathbf{x}_2^0) \Delta_1 f(\mathbf{x}_2^1) = -1$.

The above results motivate the following notion of permutability of lattice derivatives, and its relation to local monotonicities.

Definition 13. We say that a pseudo-Boolean function $f\colon \mathbb{B}^n \to \mathbb{R}$ *has p-permutable lattice derivatives* if, for every p-subset $\{k_1, \ldots, k_p\} \subseteq [n]$, every choice of the operators $O_{k_i} \in \{\wedge_{k_i}, \vee_{k_i}\}$ $(i = 1, \ldots, p)$, and every permutation $\pi \in S_p$, the following identity holds:

$$O_{k_1} \cdots O_{k_p} f = O_{k_{\pi(1)}} \cdots O_{k_{\pi(p)}} f.$$

If $f\colon \mathbb{B}^n \to \mathbb{R}$ has n-permutable lattice derivatives, then we simply say that f *has permutable lattice derivatives.*

Every function $f\colon \mathbb{B}^n \to \mathbb{R}$ has 1-permutable lattice derivatives. We will see in Theorem 23 that if a function $f\colon \mathbb{B}^n \to \mathbb{R}$ has p-permutable lattice derivatives, then it also has p'-permutable lattice derivatives for every $p' \leq p$.

Fact 14. *A function $f\colon \mathbb{B}^n \to \mathbb{R}$ has p-permutable lattice derivatives if and only if so has $\alpha f + \beta$ for every $\alpha, \beta \in \mathbb{R}$, with $\alpha \neq 0$. The same holds for any function obtained from f by negating some of its variables.*

Fact 15. *A function $f\colon \mathbb{B}^n \to \mathbb{R}$ has p-permutable lattice derivatives if and only if every p-ary section of f has permutable lattice derivatives.*

In the particular case when $p = 2$, we have the following description of functions having 2-permutable lattice derivatives.

Proposition 16. *A function $f\colon \mathbb{B}^n \to \mathbb{R}$ has 2-permutable lattice derivatives if and only if every binary section (2) of f satisfies $a_1\, a_{12} \geq 0$ or $a_2\, a_{12} \geq 0$ or $|a_{12}| \leq |a_1| \vee |a_2|$.*

Example 12 showed that the class of 2-locally monotone pseudo-Boolean functions is a proper subclass of that of pseudo-Boolean functions which have 2-permutable lattice derivatives. From Corollary 8 and Proposition 16, we obtain the following description of pseudo-Boolean functions which have 2-permutable lattice derivatives but are not 2-locally monotone.

Corollary 17. *A function $f\colon \mathbb{B}^n \to \mathbb{R}$ has 2-permutable lattice derivatives but is not 2-locally monotone if and only if every binary section (2) of f satisfies the following two conditions:*

(i) $a_1\, a_{12} \geq 0$ or $a_2\, a_{12} \geq 0$ or $|a_{12}| \leq |a_1| \vee |a_2|$, and

(*ii*) $a_1(a_1 + a_{12}) < 0$ *or* $a_2(a_2 + a_{12}) < 0$.

Equivalently,

(*i*) $a_1\, a_{12} < 0$ *or* $a_2\, a_{12} < 0$,
(*ii*) $|a_{12}| > |a_1| \wedge |a_2|$,
(*iii*) *if* $a_1\, a_2 > 0$, *then* $|a_{12}| \leq |a_1| \vee |a_2|$,
(*iv*) *if* $a_1\, a_{12} \geq 0$, *then* $|a_{12}| > |a_2|$, *and*
(*v*) *if* $a_2\, a_{12} \geq 0$, *then* $|a_{12}| > |a_1|$.

Now, according to Theorem 11, a Boolean function is 2-locally monotone if and only if it has 2-permutable lattice derivatives. Example 24 below shows that the analogous equivalence does not hold for $p > 2$. However, p-local monotonicity implies p-permutability of lattice derivatives of any pseudo-Boolean function (see Theorem 21 below). To this extent, let us first study how the degree of local monotonicity is affected by taking lattice derivatives.

Lemma 18. *If* $f\colon \mathbb{B}^n \to \mathbb{R}$ *is a monotone function, then* $\wedge_j f$ *and* $\vee_j f$ *are also monotone for all* $j \in [n]$.

Theorem 19. *If* $f\colon \mathbb{B}^n \to \mathbb{R}$ *is a* p*-locally monotone function, then* $\wedge_j f$ *and* $\vee_j f$ *are* $(p-1)$*-locally monotone for all* $j \in [n]$.

Corollary 20. *Let* $0 \leq \ell < p \leq n$. *If* $f\colon \mathbb{B}^n \to \mathbb{R}$ *is* p*-locally monotone, then, for any* ℓ*-subset* $\{k_1, \ldots, k_\ell\} \subseteq [n]$ *and every choice of the operators* $O_{k_i} \in \{\vee_{k_i}, \wedge_{k_i}\}$ $(i = 1, \ldots, \ell)$, *the function* $O_{k_1} \cdots O_{k_\ell} f$ *is* $(p - \ell)$*-locally monotone. In particular, if* $\ell \leq p-2$, *then* $O_{k_1} \cdots O_{k_\ell} f$ *is locally monotone.*

Remark 1. We will see in Example 26 of Section IV that Theorem 19 cannot be sharpened, i.e., the lattice derivatives of a p-locally monotone function are not necessarily p-locally monotone, not even in the case of Boolean functions.

We can now present the relation between p-local monotonicity and p-permutability of lattice derivatives, which generalizes Lemma 10.

Theorem 21. *If* $f\colon \mathbb{B}^n \to \mathbb{R}$ *is* p*-locally monotone, then it has* p*-permutable lattice derivatives.*

A natural question regarding the lattice derivatives is whether a function can be reconstructed from its derivatives. The answer is positive for almost all functions.

Theorem 22. *Let* $f, g\colon \mathbb{B}^n \to \mathbb{R}$ *be pseudo-Boolean functions such that for all* $k \in [n]$ *we have* $\vee_k f = \vee_k g$ *and* $\wedge_k f = \wedge_k g$. *Then either* $f = g$ *or there exists a one-to-one function* $\alpha\colon \mathbb{B} \to \mathbb{R}$ *such that* $f(\mathbf{x}) = \alpha(x_1 \oplus \cdots \oplus x_n)$ *and* $g(\mathbf{x}) = \alpha(x_1 \oplus \cdots \oplus x_n \oplus 1)$ *for all* $\mathbf{x} \in \mathbb{B}^n$.

As in the case of local monotonicity, the classes of functions having permutable lattice derivatives form a chain under inclusion.

Theorem 23. *If* $f\colon \mathbb{B}^n \to \mathbb{R}$ *has* $(p+1)$*-permutable lattice derivatives then* f *has* p*-permutable lattice derivatives.*

If $f\colon \mathbb{B}^n \to \mathbb{B}$ is a Boolean function with p-permutable lattice derivatives for some $p \geq 2$, then f has 2-permutable lattice derivatives by the above theorem, and then Theorem 11 implies that f is 2-locally monotone. Unfortunately, nothing more can be said about the degree of local monotonicity of a Boolean function with p-permutable lattice derivatives. Indeed, the next example shows that there exist n-ary Boolean functions with n-permutable lattice derivatives that are exactly 2-locally monotone.

Example 24. Let $f_n\colon \mathbb{B}^n \to \mathbb{B}$ be the function that takes the value 1 on all tuples of the form

$$\mathbf{x} = (\overbrace{1, \ldots, 1}^{m}, 0, \ldots, 0) \text{ with } 0 \leq m \leq n,$$

and takes the value 0 everywhere else. Using Corollary 9, it is not difficult to verify that f_n is 2-locally monotone. However, if $n \geq 3$, then f_n is not 3-locally monotone, since

$$\begin{aligned} \Delta_2 f(0,0,0,0,\ldots,0) &= -1, \\ \Delta_2 f(1,0,1,0,\ldots,0) &= 1. \end{aligned}$$

Thus f_n is exactly 2-locally monotone.

We will show by induction on n that f_n has n-permutable lattice derivatives. First we compute the meet derivatives:

$$\wedge_k f_n(\mathbf{x}) = \begin{cases} 1, & \text{if } x_1 = \cdots = x_{k-1} = 1 \text{ and} \\ & x_{k+1} = \cdots = x_n = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\wedge_k f$ takes the value 1 only at one tuple, it is monotone. The join derivative $\vee_k f_n$ is essentially the same as the function f_{n-1} (up to the inessential k-th variable of $\vee_k f_n$):

$$\vee_k f_n(\mathbf{x}) = f_{n-1}(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n). \tag{8}$$

Now it follows that if $\{k_1, \ldots, k_n\} = [n]$ and $O_{k_i} \in \{\wedge_{k_i}, \vee_{k_i}\}$ $(i = 1, \ldots, n)$, then

$$O_{k_1} \cdots O_{k_{n-1}} O_{k_n} f = O_{k_{\pi(1)}} \cdots O_{k_{\pi(n-1)}} O_{k_n} f \tag{9}$$

holds for every permutation $\pi \in S_{n-1}$. (If $O_{k_n} = \wedge_{k_n}$, then we use Theorem 21 and the fact that $\wedge_{k_n} f$ is monotone, and if $O_{k_n} = \vee_{k_n}$, then we use (8) and the induction hypothesis.) On the other hand, from the 2-local monotonicity of f we can conclude

$$O_{k_1} \cdots O_{k_{n-2}} O_{k_{n-1}} O_{k_n} f = O_{k_1} \cdots O_{k_{n-2}} O_{k_n} O_{k_{n-1}} f \tag{10}$$

with the help of Theorem 11. Since S_n is generated by S_{n-1} and the transposition $(n-1\ n)$, we see from (9) and (10) that f has n-permutable lattice derivatives.

IV. Symmetric functions

In the previous sections we saw that both notions of local monotonicity and of permutable lattice derivatives lead to two hierarchies of pseudo-Boolean functions which are related by the fact that each p-local monotone class is contained in the corresponding class of functions having p-permutable lattice derivatives. Now, in general this containment is strict. However, under certain assumptions (see Theorem 11), p-local monotonicity is equivalent to p-permutability of lattice derivatives. Hence it is natural to ask for conditions under which these two conditions coincide.

In this section we present a partial answer to this problem by focusing on symmetric pseudo-Boolean functions, i.e., functions $f\colon \mathbb{B}^n \to \mathbb{R}$ that are invariant under all permutations of their variables. Quite surprisingly, in this case the notions of p-local monotonicity and p-permutablity of lattice derivatives become equivalent.

Symmetric functions of arity n are in a one-to-one correspondence with sequences of real numbers of length $n+1$, where the function corresponding to the sequence $\alpha = \alpha_0, \ldots, \alpha_n$ is given by $f(\mathbf{x}) = \alpha_{|\mathbf{x}|}$ ($\mathbf{x} \in \mathbb{B}^n$); here $|\mathbf{x}|$ denotes the Hamming weight of $\mathbf{x}$. Clearly, f is isotone if and only if the corresponding sequence is nondecreasing, i.e., $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$. Similarly, f is antitone if and only if $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n$, and f is monotone if and only if f is either isotone or antitone.[3]

It is easy to see that if f is symmetric, then every section of f is also symmetric; moreover, if f corresponds to the sequence $\alpha = \alpha_0, \ldots, \alpha_n$, then the p-ary sections of f are exactly the symmetric functions corresponding to the subsequences[4] $\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i+p}$ of α of length $p+1$. This observation and Theorem 7 lead to the following description of p-locally monotone pseudo-Boolean functions.

Proposition 25. *Let $f\colon \mathbb{B}^n \to \mathbb{R}$ be a symmetric function corresponding to the sequence $\alpha = \alpha_0, \ldots, \alpha_n$. Then f is p-locally monotone if and only if each subsequence of length $p+1$ of α is either nondecreasing or nonincreasing.*

Unlike in the previous sections, here it will be more convenient to discard the inessential k-th variable of the lattice derivatives $\wedge_k f$ and $\vee_k f$, and regard the latter as $(n-1)$-ary functions. Clearly, if f is symmetric, then so are its lattice derivatives. Moreover, if f corresponds to the sequence $\alpha = \alpha_0, \ldots, \alpha_n$, then $\wedge_k f$ and $\vee_k f$ correspond to the sequences

$$\alpha_0 \wedge \alpha_1, \alpha_1 \wedge \alpha_2, \ldots, \alpha_{n-1} \wedge \alpha_n \text{ and}$$
$$\alpha_0 \vee \alpha_1, \alpha_1 \vee \alpha_2, \ldots, \alpha_{n-1} \vee \alpha_n,$$

respectively, for all $k \in [n]$. Since these sequences do not depend on k, we will write $\wedge f$ and $\vee f$ instead of $\wedge_k f$ and $\vee_k f$, respectively.

The next example shows that Theorem 19 cannot be sharpened.

Example 26. Let $f\colon \mathbb{B}^n \to \mathbb{B}$ be the symmetric function corresponding to the sequence

$$\alpha = 0, 0, \overbrace{1, \ldots, 1}^{p}, \overbrace{0, \ldots, 0}^{p}, 1, 1$$

where $n = 2p+4$ and $p \geq 2$. It follows from Proposition 25 that f is exactly p-locally monotone. To compute $\wedge f$, it is handy to construct a table whose first row contains the sequence α, and in the second row we write $\alpha_i \wedge \alpha_{i+1}$ between α_i and α_{i+1}:

$$\begin{array}{rc} f\colon & 0\ 0\ 1\ 1\ \cdots\ 1\ 1\ 0\ 0\ \cdots\ 0\ 0\ 1\ 1 \\ \wedge f\colon & 0\ 0\ 1\ 1\ \cdots\ 1\ 1\ 0\ 0\ 0\ \cdots\ 0\ 0\ 0\ 1 \end{array}$$

Thus $\wedge f$ corresponds to the sequence

$$0, 0, \overbrace{1, \ldots, 1}^{p-1}, \overbrace{0, \ldots, 0}^{p+1}, 1,$$

and a similar calculation yields that $\vee f$ corresponds to the sequence

$$0, \overbrace{1, \ldots, 1}^{p+1}, \overbrace{0, \ldots, 0}^{p-1}, 1, 1.$$

Now Proposition 25 shows that $\wedge f$ and $\vee f$ are exactly $(p-1)$-locally monotone.

Remark 2. The above example shows that the degree of local monotonicity can decrease, when taking lattice derivatives, and Theorem 19 states that it can decrease by at most one. Other examples can be found to illustrate the cases when this degree stays the same, or even increases. For instance, consider the function $f(\mathbf{x}) = x_1 \oplus \cdots \oplus x_n$, which is not even 2-locally monotone, but its lattice derivatives are constant.

We conclude this section by making explicit the equivalence between the notions of p-local monotonicity and p-permutability of lattice derivatives in the case of symmetric functions.

Theorem 27. *If $f\colon \mathbb{B}^n \to \mathbb{R}$ is a symmetric function, then f is p-locally monotone if and only if f has p-permutable lattice derivatives.*

Remark 3. As a consequence of the above theorem, we can observe that any exactly p-locally monotone symmetric function (for instance, the functions considered in Example 26) has p-permutable but not $(p+1)$-permutable lattice derivatives.

V. Open problems and concluding remarks

We proposed relaxations of monotonicity, namely p-local monotonicity, and we presented characterizations of each in terms of "forbidden" sections. Also, for each p, we observe that p-locally monotone functions have the property that any p of its lattice derivatives permute, and showed that the converse also holds in the special case of symmetric functions. The classes of 2-locally monotone functions, and of functions having 2-permutable lattice derivatives were explicitly described; as a by-product their symmetric difference was obtained. However, similar descriptions elude us for $p \geq 3$. Hence we are left with the following problems.

Problem 1. *For $p \geq 3$, describe the class of p-locally monotone functions and that of functions having p-permutable lattice derivatives.*

Problem 2. *For $p \geq 3$, determine necessary and sufficient conditions on functions for the equivalence between p-local monotonicity and permutability of p lattice derivatives.*

[3] Since if f is isotone (resp. antitone) in one variable, then it is isotone (resp. antitone) in all variables.

[4] Here by a subsequence we mean a sequence of consecutive entries of the original sequence.

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