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Defining the space in a general spacetime

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Abstract

A global vector field v on a “spacetime” differentiable manifold V , of dimension $N + 1$, defines a congruence of world lines: the maximal integral curves of v , or orbits. The associated global space N_v is the set of these orbits. A “ v -adapted” chart on V is one for which the \mathbb{R}^N vector $\mathbf{x} \equiv (x^j)$ ($j = 1, \dots, N$) of the “spatial” coordinates remains constant on any orbit l . We consider non-vanishing vector fields v that have non-periodic orbits, each of which is a closed set. We prove transversality theorems relevant to such vector fields. Due to these results, it can be considered plausible that, for such a vector field, there exists in the neighborhood of any point $X \in V$ a chart χ that is v -adapted and “nice”, i.e., such that the mapping $\bar{\chi} : l \mapsto \mathbf{x}$ is injective — unless v has some “pathological” character. This leads us to define a notion of “normal” vector field. For any such vector field, the mappings $\bar{\chi}$ build an atlas of charts, thus providing N_v with a canonical structure of differentiable manifold (when the topology defined on N_v is Hausdorff, for which we give a sufficient condition met in important physical situations). Previously, a local space manifold M_F had been associated with any “reference frame” F , defined as an equivalence class of charts. We show that, if F is made of nice v -adapted charts, M_F is naturally identified with an open subset of the global space manifold N_v .

Keywords: Physical space; global vector field; reference fluid; orbit space; adapted chart; differentiable manifold; Kruskal-Szekeres coordinates.

1 Introduction

1.1 Physical motivation

The theory of relativity says that space and time merge into “a kind of union of the two” (in Minkowski’s words): the spacetime. However, the notion of a *physical space* should be useful also in relativistic physics. In our opinion it is even needed, for the following two reasons. (i) In experimental/observational work, one of course needs to define the spatial position of the experimental apparatus and/or of the observed system, and this is true also if the relativistic effects have to be considered. (ii) In quantum mechanics, the “state” of a quantum-mechanical particle is a function ψ of the position x belonging to some 3-D “physical space” M , and taking values in \mathbb{C} (or in a complex vector bundle). Note that defining such a space as a spacelike 3-D *submanifold of the spacetime manifold* (e.g. [1]) can work to define an initial condition for a field in space-time, but does not allow one to define a spatial position in the way that is needed in the two foregoing examples: in those, one needs to identify spatial points that exist at least for some open interval of time — e.g. to state that some objects maintain a fixed spatial position in some reference frame, or to define the stationary states of a quantum particle. In practice, the spatial position is taken to be the triplet of the spatial coordinates, $\mathbf{x} \equiv (x^j)$ ($j = 1, 2, 3$). However, *a priori*, \mathbf{x} does not have a precise geometric meaning in a theory starting from a spacetime structure. Only a notion of “spatial tensors” has been defined for a general spacetime of relativistic gravity. This definition was based on the concept of “reference fluid” [2, 3, 4], also named “reference body” [5] — i.e., a three-dimensional congruence of world lines, whose the tangent vector is assumed to be a time-like vector field. The latter can be normed to become a four-velocity field v , i.e. $\mathbf{g}(v, v) = 1$ where \mathbf{g} is the spacetime metric. The data of the four-velocity field v allows one to define the spatial projection operator Π_X (depending on the point X in the spacetime manifold V) [2, 3, 4, 5, 6, 7]. A “spatial vector” is then defined as a *spacetime vector* which is equal to its spatial projection. A full algebra of “spatial tensors” can be defined in the same way, and also, once a relevant connection has been defined, a spatial tensor analysis [2, 3, 4, 5, 6].

However, it is possible in a general spacetime manifold V to define a relevant physical space as a *3-D differentiable manifold*, at least *locally* in V .

To see this, consider a coordinate system or chart:

$$\chi : U \rightarrow \mathbb{R}^4, \quad X \mapsto \chi(X) = \mathbf{X} \equiv (x^\mu) \quad (\mu = 0, \dots, 3), \quad (1)$$

where U is an open subset of V : the domain of the chart. Then one may define a set of world lines, each of which, l , has constant spatial coordinates a^j in the chart χ :¹

$$l_{\mathbf{a}} = \{X \in U; \chi(X) = (x^\mu) \text{ is such that } x^j = a^j \text{ for } j = 1, 2, 3\}. \quad (3)$$

Let us suppose for a moment that the chart χ is in fact a Cartesian coordinate system on the Minkowski spacetime. Then that chart defines an inertial reference frame. In that case, it is clear that, for any event X , with $\chi(X) = (x^\mu)$, the triplet $\mathbf{x} \equiv (x^j)$ ($j = 1, 2, 3$) defines the spatial position associated in that chart with the event X . Note that the data of \mathbf{x} is equivalent to specifying a unique world line in the “congruence (3)”. [By this we mean the set of the world lines (3), when $\mathbf{a} \equiv (a^j)$ takes any value in \mathbb{R}^3 such that the corresponding world line (3) is not empty.] That world line is thus uniquely determined by the event X and may be noted $l(X)$. Events X' that have different values of the time coordinate x^0 , but that have the same values of the spatial coordinates x^j ($j = 1, 2, 3$), can be said to occur at the same spatial position in the inertial frame as does X . Thus, the whole of $l(X)$ is needed. However, each world line in the congruence (3) stays invariant if we change the coordinate system by a purely spatial coordinate change:

$$x'^0 = x^0, \quad x'^k = \phi^k((x^j)). \quad (4)$$

It is clear that this transformation leaves us in the same inertial reference frame. With the new chart χ' , the new triplet $\mathbf{x}' \equiv (x'^k) \equiv \phi(\mathbf{x})$ corresponds to the same spatial position in the inertial frame as does \mathbf{x} with the first chart χ . And indeed, that world line of the congruence which is defined in the chart χ' by the data of \mathbf{x}' is just the same as that world line of the congruence which is defined in the chart χ by the data of \mathbf{x} . The spatial position

¹ Note that, if we assume that V is endowed with a Lorentzian metric \mathbf{g} whose component g_{00} in the chart χ verifies $g_{00} > 0$ in U , then each among the world lines l is time-like, because in the chart χ the tangent vector to l has components $\propto (1, 0, 0, 0)$, which may be normed to

$$v^0 \equiv \frac{1}{\sqrt{g_{00}}}, \quad v^j = 0. \quad (2)$$

of the event X in the inertial frame is therefore most precisely defined by *the world line $l(X)$ of the congruence which passes at X* .

Now note that very little in the foregoing paragraph actually depends on whether or not the chart χ is a Cartesian coordinate system on the Minkowski spacetime: only the qualification of the reference frame as being inertial depends on that. It is just that we are accustomed to consider a spatial position in an inertial frame in a flat spacetime, and special relativity makes it natural to accept that it is actually the world line $l(X)$ which best represents that spatial position. Hence, consider a general spacetime, and define a congruence of world lines from the data of a coordinate system as in Eq. (3). In the domain of the chart, we may then define the spatial position of an event X as the unique world line $l(X)$ of the congruence (3) which passes at X — i.e., $l(X)$ is the unique world line of the congruence (3), such that $X \in l(X)$. Thus, *the data of a coordinate system on the spacetime defines a three-dimensional space M , of which the points (the elements of M) are the world lines of the congruence (3) associated with that coordinate system.*

1.2 Local reference frame and local space manifold

The foregoing approach can be used to define precise notions of a reference frame and its unique associated space manifold [8]. First, the invariance of the congruence (3) under the purely spatial coordinate changes (4) allows one to define a *reference frame* as being an equivalence class of charts related by a change (4). More exactly, the following is an equivalence relation between charts which are defined on a given open subspace U of the spacetime manifold V :

$$\chi \mathcal{R}_U \chi' \iff [\forall \mathbf{X} \in \chi(U), \quad \phi^0(\mathbf{X}) = x^0 \text{ and } \frac{\partial \phi^k}{\partial x^0}(\mathbf{X}) = 0 \ (k = 1, \dots, 3)], \quad (5)$$

where $f \equiv \chi' \circ \chi^{-1} \equiv (\phi^\mu)$ is the transition map, which is defined on $\chi(U)$. Thus a reference frame F is a set of charts defined on the same open domain U and exchanging by a purely spatial coordinate change (4). Then the *space manifold* M or M_F associated with the reference frame F is defined as the set of the world lines (3). In detail: let $P_S : \mathbb{R}^4 \rightarrow \mathbb{R}^3$, $\mathbf{X} \equiv (x^\mu) \mapsto \mathbf{x} \equiv (x^j)$, be the spatial projection. A world line l is an element of M_F iff there is a chart $\chi \in F$ and a triplet $\mathbf{x} \in P_S(\chi(U))$, such that l is the set of *all* points X in

the domain U , whose spatial coordinates are \mathbf{x} :

$$l \equiv \{ X \in U; P_S(\chi(X)) = \mathbf{x} \}. \quad (6)$$

(Thus, l is not necessarily a connected set.) It results easily from (4) that (6) holds true then in any chart $\chi' \in F$, of course with the transformed spatial projection triplet $\mathbf{x}' = \phi(\mathbf{x}) \equiv (\phi^j(\mathbf{x}))$ [8]. For a chart $\chi \in F$, one defines the “associated chart” as the mapping which associates, with a world line $l \in M$, the constant triplet of the spatial coordinates of the points $X \in l$:

$$\tilde{\chi} : M \rightarrow \mathbb{R}^3, \quad l \mapsto \mathbf{x} \text{ such that } \forall X \in l, P_S(\chi(X)) = \mathbf{x}. \quad (7)$$

One shows then that the set \mathcal{T} of the subsets $\Omega \subset M$ such that,

$$\forall \chi \in F, \quad \tilde{\chi}(\Omega) \text{ is an open set in } \mathbb{R}^3 \quad (8)$$

is a topology on M . Finally one shows that the set of the associated charts: $\tilde{F} \equiv \{\tilde{\chi}; \chi \in F\}$, is an atlas on the topological space (M, \mathcal{T}) , hence defines a structure of *differentiable manifold* on M [8]. *Thus the space manifold M_F is browsed by precisely the triplet $\mathbf{x} \equiv (x^j)$ made with the spatial projection of the spacetime coordinates $\mathbf{X} \equiv (x^\mu) \equiv \chi(X)$ of a chart $\chi \in F$, see Eq. (7).*

Using these results, one may define the space of quantum-mechanical states, for a given reference frame F in a given spacetime (V, \mathbf{g}) , as being the set \mathcal{H} of the square-integrable functions defined on the corresponding space manifold M [9]. One may also define the full algebra of spatial tensors: the pointwise algebra is defined simply as the tensor algebra of the tangent vector space TM_x to the space manifold M at some arbitrary point $x \in M$ [10].

1.3 Goal and summary

Thus, by defining a reference frame as a set F of charts that all have the same open domain U and that exchange by a purely spatial coordinate change (4), one can then define the associated space manifold M_F as the set of the world lines (6) [8]. These definitions are relevant to physical applications [9, 10]. However, they apply to a *parametrizable* domain U of the spacetime manifold V , i.e., to an open set U , such that at least one regular chart can be defined over the whole of U . Since the manifold V itself as a whole is in general not

parametrizable, a reference frame is in general only a local one, and so the associated space manifold does not look “maximal”. *The aim of the present work* is to define *global* reference fluids, to associate with any of them a *global physical space*, and to *link* these concepts with the formerly defined *local concepts*. As the “spacetime”, we consider a differentiable manifold V having dimension $N + 1$, thus N is the dimension of the “space” manifold to be defined. We define a global reference fluid by the data of a *non-vanishing* global vector field v on V . We do not need that $N = 3$, nor that V be endowed with a Lorentzian metric \mathbf{g} for which v be a time-like vector field. This was already true for the former “local” work [8]. Note, however, that a time-like vector field on a Lorentzian manifold (V, \mathbf{g}) is non-vanishing; and that, if a Lorentzian manifold (V, \mathbf{g}) is time-oriented, which indeed is usually required for a spacetime, then by definition there exists at least one global time-like vector field on V . We define the “global space” associated with v as the set N_v of the maximal integral curves (or “orbits”) of v . To reach our goal, we take the following steps:

(a) Section 2 studies when a given vector field v on a differentiable manifold V is such that there locally exists charts of V which are *adapted* [2] to the congruence associated with v ; i.e., charts in which the “spatial” position $\mathbf{x} \equiv (x^j)$ ($j = 1, \dots, N$) is constant on any orbit l of v ; see Definition 1. We need also that the mapping $l \mapsto \mathbf{x}$ be injective. The desired situation is defined by Proposition 2. According to a transversality argument, this situation should be attainable, in general, if v does not vanish and each of its orbits is non-periodic and is closed in V . In Subsect. 2.4, two theorems of transversality and another theorem pertaining to differential topology allow us to formalize that argument in Theorem 4. This justifies us in introducing a notion of “normal” vector field by Definition 2, which ensures the local existence of v -adapted charts through Theorem 5.

(b) For any v -adapted chart χ , one may define the mapping $\bar{\chi}$ which associates with any orbit l the constant spatial position \mathbf{x} . We show in Section 3 that, using the set \mathcal{A} of the injective mappings $\bar{\chi}$, one can endow the global orbit set N_v with a topology \mathcal{T}' for which this set is an atlas of charts. See Theorem 6. Thus, when \mathcal{T}' is metrizable and separable, we do have a canonical structure of differentiable manifold on the orbit set N_v , for which the mappings $\bar{\chi}$ defined by Eq. (14) are local charts on the “space” manifold N_v . I.e., if $N = 3$, *also the global space N_v is browsed (locally) by precisely the*

triplet $\mathbf{x} \equiv (x^j)$ made with the spatial projection of the spacetime coordinates $\mathbf{X} \equiv (x^\mu) \equiv \chi(X)$ of a v -adapted chart χ .

(c) In Section 4 we establish the link with the previously defined space manifold M_F , associated with a given local reference frame F — defined as an equivalence class of charts for the relation (5). We show in Theorem 7 that, when the charts belonging to F are v -adapted and with the mapping $l \mapsto \mathbf{x}$ being injective, then M_F is naturally identified with an open subset of N_v .

The definitions of a local reference frame F and the corresponding “space” manifold M_F do not need any metrical structure on the “spacetime” manifold V [8]. Just the same can be said for the definition of the global “space” manifold N_v , beyond the very fact that V should have a metrizable topology.

2 Local existence of adapted charts

2.1 Definitions

Let V be an $(N + 1)$ -dimensional real differentiable manifold, with $N \geq 1$.² We consider a given global, smooth vector field v on V . A continuously differentiable (\mathcal{C}^1) mapping C from an open interval I of \mathbb{R} into V defines an integral curve of v iff $\frac{dC}{ds} = v(C(s))$ for $s \in I$. For any $X \in V$, let C_X be the solution of

$$\frac{dC}{ds} = v(C(s)), \quad C(0) = X \quad (9)$$

for which the open interval I is *maximal*, and denote this maximal interval by I_X [12]. That is, I_X is an open interval defined as the union of all open intervals I , each containing 0, in which a solution of (9) is defined. The solution C_X is defined on I_X and is unique [12]. Let $s \in I_X$ and set $Y = C_X(s)$. These definitions imply easily [12] that

$$I_Y = I_X - s \quad \text{and} \quad \forall t \in I_Y, \quad C_Y(t) = C_X(s + t). \quad (10)$$

For any $X \in V$, call the *range* $l_X \equiv C_X(I_X) \subset V$ the “maximal integral curve at X ”. From (10), “ l_X does not depend on the point $X \in l_X$ ”: if $Y \in l_X$,

² We understand “differentiable manifold” as a topological space endowed with an atlas of compatible charts, hence with the corresponding equivalence class of compatible atlases — with the restriction that that space should be metrizable and separable [11].

then $l_Y = l_X$. We define the set of the maximal integral curves (or orbits) of v :

$$N_v \equiv \{l_X; X \in V\}. \quad (11)$$

Once endowed with further structure, N_v will be the global space manifold associated with the global vector field v (when the latter is non-vanishing and obeys another assumption). Note that, if the set $U \subset V$ is not empty, then the following subset of N_v :

$$D_U \equiv \{l \in N_v; l \cap U \neq \emptyset\} \quad (12)$$

is not empty. Indeed, for any $X \in U$, the world line $l \equiv l_X$ belongs to D_U . Let $P_S : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N$, $(x^\mu) \mapsto (x^j)$ ($\mu = 0, \dots, N$; $j = 1, \dots, N$) be the “spatial” projection.

Definition 1. *A mapping $\chi : U \rightarrow \mathbb{R}^{N+1}$ with $U \subset V$ is said “ v -adapted” iff for any $l \in D_U$, there exists $\mathbf{x} \in \mathbb{R}^N$ such that*

$$\forall Y \in l \cap U, \quad P_S(\chi(Y)) = \mathbf{x}. \quad (13)$$

If Eq. (13) is verified by some world line $l \in N_v$, then necessarily $l \in D_U$, and \mathbf{x} is obviously unique. Thus, for any v -adapted mapping χ , the mapping

$$\bar{\chi} : D_U \rightarrow \mathbb{R}^N, \quad l \mapsto \mathbf{x} \text{ such that (13) is verified} \quad (14)$$

is well defined. In Section 3, we will endow the set N_v with first a topology and then a structure of differentiable manifold, for which the charts (of N_v) will be mappings $\bar{\chi}$, where χ is a v -adapted *chart* of V . Since any chart is in particular a one-to-one mapping, we need to restrict ourselves to v -adapted charts χ such that the associated mapping $\bar{\chi}$ is injective. Thus, we define that a v -adapted chart χ is “*nice*” iff $\bar{\chi}$ is injective on D_U , with $U \subset V$ the (open) domain of χ .

2.2 Straightening-out vs v -adapted charts

In the remainder of this section, we investigate whether there exist nice v -adapted charts in the neighborhood of any point $X_0 \in V$. If the vector field v does not vanish, a well-known theorem applies at any point $X_0 \in V$:

Straightening-out theorem (e.g. [13]). *Let v be a vector field of class C^∞ defined on V . Suppose that at $X_0 \in V$ we have $v(X_0) \neq 0$. There is a “straightening-out chart” χ defined on an open neighborhood U of X_0 , i.e. χ is such that:*

- (i) $\chi(U) = I \times \Omega$, $I =] - a, +a[$, $a \neq 0$, Ω open set in \mathbb{R}^N .
- (ii) For any $\mathbf{x} \in \Omega$, $\chi^{-1}(I \times \{\mathbf{x}\})$ is an integral curve of v .
- (iii) In U , we have $v = \partial_0$, where (∂_μ) is the natural basis associated with the chart χ .

However, the direct link with the notion of a v -adapted chart works in the wrong direction:

Proposition 0. *Let (χ, U) be a v -adapted chart. (i) We have $v|_U = f\partial_0$ with $f : U \rightarrow \mathbb{R}$ a smooth function. (ii) Given any point $X \in U$ such that $v(X) \neq 0$, one may obtain a straightening-out chart (χ', U') , with $U' \subset U$ being an open neighborhood of X , by changing merely the y^0 coordinate.*

Proof. (i) To say that (χ, U) is v -adapted means, according to Definition 1, that for any given $X \in U$, we have for any $Y \in l_X \cap U$: $P_S(\chi(Y)) = P_S(\chi(X))$. In the coordinates $\chi(X) = (x^\mu) \equiv \mathbf{X}$, $\chi(Y) = (y^\mu) \equiv \mathbf{Y}$ ($\mu = 0, \dots, N$), the latter rewrites as

$$y^j = x^j \quad (j = 1, \dots, N). \quad (15)$$

On the other hand, remembering the definition of l_X as a (maximal) integral curve of v , Eq. (9) and below, let J be the connected component of 0 in $I'_{XU} \equiv \{s \in I_X; C_X(s) \in U\}$: J is an open interval containing 0 and we have $l' \equiv C_X(J) \subset l_X \cap U$. Let us denote this as $l' = \{Y(s); s \in J\}$, with

$$\frac{dy^\mu}{ds} = v^\mu(\mathbf{Y}(s)) \quad (\mu = 0, \dots, N), \quad \mathbf{Y}(0) = \mathbf{X}, \quad (16)$$

$v^\mu = v^\mu(\mathbf{Y})$ being the components of v in the chart χ . It follows from (15) and (16) that for $s \in J$ we have $v^j(\mathbf{Y}(s)) = 0$ ($j = 1, \dots, N$), i.e. $v(\mathbf{Y}(s)) = v^0(\mathbf{Y}(s))\partial_0(Y(s))$, thus in particular $v(X) = v^0(\mathbf{X})\partial_0(X) \equiv f(X)\partial_0(X)$. Since this is true at any point $X \in U$, our statement (i) is proved.

(ii) If we leave the coordinates y^j ($j = 1, \dots, N$) unchanged: $\chi'(Y) \equiv \mathbf{Y}' = (g(\mathbf{Y}), (y^j))$ where $\mathbf{Y} = (y^0, (y^j)) = \chi(Y)$ ($Y \in U' \subset U$), then the components of v in the new chart (χ', U') are $v'^j = v^j = 0$ ($j = 1, \dots, N$) and $v'^0(\mathbf{Y}') = \frac{\partial g}{\partial y^0} v^0(\mathbf{Y}) = \frac{\partial g}{\partial y^0} f(Y)$. The latter must be equal to 1 for a straightening-out chart. Since $v|_U = f\partial_0$ and $v(X) \neq 0$, we have $f(Y) \neq 0$ when Y is in some neighborhood $U'' \subset U$ of X . Hence, we get $v'^0 = 1$ in U' if we take $U' = \chi^{-1}(B)$ with $B =]x^0 - r, x^0 + r[\times \dots \times]x^N - r, x^N + r[\subset \chi(U'')$ and define $y'^0 \equiv g(\mathbf{Y}') = \int_{x^0}^{y^0} du/f(u, (y^j))$ for $\mathbf{Y}' \in B$. Property (i) in the straightening-out theorem is then got by a mere shift $y'^0 \mapsto y'^0 + \delta$, and its Property (ii) is a straightforward consequence of Properties (i) and (iii). \square

Conversely, if (χ, U) is a straightening-out chart, i.e. it fulfills conditions (i) to (iii) of the theorem above, then let $\mathbf{x} \in \Omega$ and set $l' \equiv \chi^{-1}(I \times \{\mathbf{x}\})$. From (ii), l' is an integral curve of v and, since $l' \subset U$ by construction, we have (13):

$$\forall Y \in l' \cap U, \quad P_S(\chi(Y)) = \mathbf{x}. \quad (17)$$

Hence, at first sight, it might seem that χ is a v -adapted chart. However, let $X = \chi^{-1}(s, \mathbf{x}) \in l'$ and let $l \equiv l_X \in N_v$ be the *maximal* integral curve of v passing at X . We have $l' \subset (l \cap U)$ since l' is an integral curve of v that is included in U and that passes at X . But nothing guarantees that $l' \supset (l \cap U)$: the intersection $l \cap U$ may contain other arcs, say $l'_1 \equiv \chi^{-1}(I \times \{\mathbf{x}_1\})$ with $\mathbf{x}_1 \in \Omega$ and $\mathbf{x}_1 \neq \mathbf{x}$. In such a case, the straightening-out chart χ is not v -adapted, since for $Y \in l \cap U$, $P_S(\chi(Y))$ may take different values $\mathbf{x}, \mathbf{x}_1, \dots$

As we already noted, Property (ii) in the straightening-out theorem follows easily from Properties (i) and (iii). More generally, if a chart (χ, U) satisfies Property (iii), i.e. $v = \partial_0$ in U , and if $\chi(U)$ contains a set $I \times \{\mathbf{x}\}$, with $\mathbf{x} \in \mathbb{R}^N$ and I an open interval, then Property (ii) applies for this $\mathbf{x} \in \mathbb{R}^N$. Later on, we will need that the boundary of the open set U be a smooth hypersurface, hence we must consider charts for which (iii) is true, but not (i).

Proposition 1. *Let (χ, U) be a chart such that $v = \partial_0$ in U . Assume there is an open subset $\Omega \subset \mathbb{R}^N$ such that*

$$\chi(U) = \bigcup_{\mathbf{x} \in \Omega} I_{\mathbf{x}} \times \{\mathbf{x}\}, \quad (18)$$

where $I_{\mathbf{x}}$ ($\mathbf{x} \in \Omega$) are open intervals.

(i) In order that the chart (χ, U) be v -adapted, it is necessary and sufficient that

$$\forall X \in U, \chi(l_X \cap U) \text{ has the form } I_{\mathbf{x}} \times \{\mathbf{x}\} \text{ for some } \mathbf{x} \in \Omega. \quad (19)$$

(ii) Moreover, if that is the case, then the v -adapted chart (χ, U) is nice.

Proof. (i) If Condition (19) is satisfied, let $l \in D_U$. Thus, $l \cap U \neq \emptyset$, so let $X \in l \cap U$, hence $l = l_X$. Let $\mathbf{x} \in \Omega$ be given by (19) for precisely the maximal integral curve $l = l_X$. For any $Y \in l \cap U$, we have thus $\chi(Y) = (s, \mathbf{x})$ for some $s \in I_{\mathbf{x}}$. Hence, we have (13). Therefore, according to Definition 1, χ is v -adapted. Conversely, assume that (χ, U) is v -adapted. Let $X \in U$ and set $l \equiv l_X$. Further, let \mathbf{x} be given by (13). That is, any point $\mathbf{Y} \in \chi(l \cap U)$ has the form (s, \mathbf{x}) for some $s \in \mathbb{R}$. Since moreover $\chi(U)$ has the form (18), we have also $\mathbf{Y} = (t, \mathbf{x}')$ for some $\mathbf{x}' \in \Omega$ and some $t \in I_{\mathbf{x}'}$, so $\mathbf{x} = \mathbf{x}'$ and $s = t \in I_{\mathbf{x}}$. Hence $\chi(l \cap U) \subset I_{\mathbf{x}} \times \{\mathbf{x}\}$. But also $I_{\mathbf{x}} \times \{\mathbf{x}\} \subset \chi(l \cap U)$. Indeed, setting $l' \equiv \chi^{-1}(I_{\mathbf{x}} \times \{\mathbf{x}\})$, we have $l' \subset (l \cap U)$ because, as noted before the statement of this Proposition 1, l' is an integral curve of v that is included in U and that passes at X .

(ii) Assuming that the chart (χ, U) obeys Condition (19) [and hence, by (i), is a v -adapted chart], let us show that the mapping $\bar{\chi}$ defined by (14) is injective. Thus, let $l, l' \in N_v$, assume that both intersect U , and let \mathbf{x} and \mathbf{x}' be the images of l and l' by $\bar{\chi}$. This means, according to the definition (14), that we have (13), and similarly

$$\forall Y \in l' \cap U, \quad P_S(\chi(Y)) = \mathbf{x}'. \quad (20)$$

From Condition (19), we get thus:

$$\chi(l \cap U) = I_{\mathbf{x}} \times \{\mathbf{x}\} \quad \text{and} \quad \chi(l' \cap U) = I_{\mathbf{x}'} \times \{\mathbf{x}'\}. \quad (21)$$

Therefore, if $\mathbf{x} = \mathbf{x}'$, it is clear that $l = l'$. The proof is complete. \square

The condition that the open set $A \equiv \chi(U) \subset \mathbb{R}^{N+1}$ have the form (18) is fulfilled, in particular, if A is convex. So it is fulfilled if one restricts the chart χ to $\chi^{-1}(A)$ with A a convex open subset of $\chi(U_0)$. Unfortunately, what we have in a rather general situation is the following result, which does not ensure the applicability of Proposition 1:

Theorem 0 [14]. *Suppose that some maximal integral curve l of the C^∞ vector field v is closed in V and is not reduced to a point. Since v then does not vanish on l , let $\chi : U \rightarrow I \times \Omega$ be a straightening-out chart in an open neighborhood U of some point of l . Then we have $\chi(l \cap U) = I \times E$, where E is a closed countable subset of Ω , and any point $X \in l \cap U$ is “transversally isolated”, i.e. $\mathbf{x} \equiv P_S(\chi(X))$ is isolated in E .*

(Note that the set E is closed in Ω , which is an open set in \mathbb{R}^N . So E is not necessarily closed in \mathbb{R}^N .) Assume that some straightening-out chart (χ, U) is such that, for any $X \in U$, the maximal integral curve l_X is closed in V . From Point (i) in Proposition 1, we get that this is a v -adapted chart iff, in addition, for any point $X \in U$, the countable closed subset E_X of Ω , whose existence is ensured by Theorem 0 for the curve l_X , is actually reduced to a point. We can further characterize this desired situation:

Proposition 2. *Let $\chi : U_0 \rightarrow I \times \Omega_0$ be a straightening-out chart for the C^∞ vector field v . Let $U \subset U_0$ be an open set such that $\chi(U)$ has the form (18). Assume that, for any $X \in U$, the maximal integral curve l_X is closed in V .*

- (i) *For any given $X \in U$, set $\chi(X) = (s, \mathbf{x})$. The connected component λ'' of (s, \mathbf{x}) in $\lambda \equiv \chi(l_X \cap U)$ is equal to $\lambda' \equiv I_{\mathbf{x}} \times \{\mathbf{x}\}$.*
- (ii) *In order that (χ, U) be a v -adapted chart, it is necessary and sufficient that, for any $X \in U$, the intersection $l_X \cap U$ be a connected set.*
- (iii) *Let W be an open subset of U such that, for any $X \in W$, $l_X \cap U$ be a connected set. Then the restriction (χ, W) is a nice v -adapted chart.*

Proof. (i) Since $I_{\mathbf{x}}$ is an interval, the set $\lambda' \equiv I_{\mathbf{x}} \times \{\mathbf{x}\}$ is connected. To show that $\lambda'' = \lambda'$ is always true, we note first that, $\chi(U)$ having the form (18), $\chi(X) = (s, \mathbf{x})$ is such that $s \in I_{\mathbf{x}}$. Hence $(s, \mathbf{x}) \in \lambda'$, and since λ' is connected, we have $\lambda' \subset \lambda''$. To prove that in fact $\lambda' = \lambda''$, we will show that λ' is open and closed in λ'' . We know from Theorem 0 that $\lambda_0 \equiv \chi(l_X \cap U_0)$ has the form $\lambda_0 = I \times E$, where E is a subset of Ω_0 having only isolated points. Thus, \mathbf{x} being isolated in E , let $r > 0$ be such that $B \cap E = \{\mathbf{x}\}$, where $B \equiv B(\mathbf{x}, r)$ is the open ball of radius r in \mathbb{R}^N , centered at \mathbf{x} . Hence, we have

$$(I_{\mathbf{x}} \times B) \cap \lambda'' \subset (I_{\mathbf{x}} \times B) \cap \lambda_0 = (I_{\mathbf{x}} \times B) \cap (I \times E) = (I_{\mathbf{x}} \cap I) \times (B \cap E) = I_{\mathbf{x}} \times \{\mathbf{x}\} \equiv \lambda'. \quad (22)$$

We have also $\lambda' \subset (\mathbb{I}_{\mathbf{x}} \times \mathbb{B}) \cap \lambda''$, because $\lambda' \subset \mathbb{I}_{\mathbf{x}} \times \mathbb{B}$ and $\lambda' \subset \lambda''$. So $\lambda' = (\mathbb{I}_{\mathbf{x}} \times \mathbb{B}) \cap \lambda''$ is an open subset of λ'' . On the other hand, if a sequence of points of λ' tends towards a limit in λ'' , say $(s_n, \mathbf{x}) \rightarrow (s', \mathbf{y}) \in \lambda''$, then $\mathbf{y} = \mathbf{x}$ and, since $\lambda'' \subset \chi(\mathbb{U})$ which is given by (18), we have $(s', \mathbf{x}) \in (\mathbb{R} \times \{\mathbf{x}\}) \cap \chi(\mathbb{U}) = \mathbb{I}_{\mathbf{x}} \times \{\mathbf{x}\} \equiv \lambda'$, hence the limit (s', \mathbf{x}) is in λ' , so that λ' is a closed subset of λ'' . Being non-empty and an open-and-closed subset of the connected set λ'' , λ' is equal to λ'' .

(ii) Since $\chi : \mathbb{U}_0 \rightarrow \mathbb{I} \times \Omega_0$ is a bicontinuous mapping, it is of course equivalent to say that $l_X \cap \mathbb{U}$ or $\lambda_X \equiv \chi(l_X \cap \mathbb{U})$ is connected. Therefore, (ii) follows immediately from (i) and from Statement (i) in Proposition 1.

(iii) Let X be any point in \mathbb{W} and set $\chi(X) = (s, \mathbf{x})$. Since $l_X \cap \mathbb{U}$ is connected, we have $\chi(l_X \cap \mathbb{U}) = \mathbb{I}_{\mathbf{x}} \times \{\mathbf{x}\}$ from (i). Thus, $P_S(\chi(Y)) = \mathbf{x}$ is true for any $Y \in l_X \cap \mathbb{U}$, hence a fortiori for any $Y \in l_X \cap \mathbb{W}$. Hence the chart (χ, \mathbb{W}) is v -adapted. In a similar way, it is easy to adapt the proof of Statement (ii) in Proposition 1 to conclude that the v -adapted chart (χ, \mathbb{W}) is nice. \square

Proposition 3. *Let $\chi : \mathbb{U}_0 \rightarrow \mathbb{I} \times \Omega_0$ be a straightening-out chart for the \mathcal{C}^∞ vector field v and assume that, for some $X \in \mathbb{U}_0$, the maximal integral curve l_X is closed. Let \mathbb{E} be the closed countable subset of Ω_0 given by Theorem 0 for the maximal curve $l \equiv l_X$, thus $\chi(l_X \cap \mathbb{U}_0) = \mathbb{I} \times \mathbb{E}$. Set $\mathbf{x} \equiv P_S(\chi(X))$ and, as ensured by Theorem 0, let $\Omega \subset \Omega_0$ be any open neighborhood of \mathbf{x} such that $\mathbb{E} \cap \Omega = \{\mathbf{x}\}$. Let $\mathbb{U} \subset \mathbb{U}_0$ be any open set such that $\chi(\mathbb{U})$ has the form (18) with this set Ω . Then $l_X \cap \mathbb{U}$ is connected, $\chi(l_X \cap \mathbb{U}) = \mathbb{I}_{\mathbf{x}} \times \{\mathbf{x}\}$.*

Proof. Since $\chi : \mathbb{U}_0 \rightarrow \mathbb{I} \times \Omega_0$ is a bijection, we have

$$\begin{aligned}
\chi(l_X \cap \mathbb{U}) &= \chi((l_X \cap \mathbb{U}_0) \cap \mathbb{U}) = \chi(l_X \cap \mathbb{U}_0) \cap \chi(\mathbb{U}) & (23) \\
&= (\mathbb{I} \times \mathbb{E}) \cap \left(\bigcup_{\mathbf{x}' \in \Omega} \mathbb{I}_{\mathbf{x}'} \times \{\mathbf{x}'\} \right) = \bigcup_{\mathbf{x}' \in \Omega} (\mathbb{I} \cap \mathbb{I}_{\mathbf{x}'}) \times (\mathbb{E} \cap \{\mathbf{x}'\}) \\
&= \mathbb{I}_{\mathbf{x}} \times \{\mathbf{x}\}. & \square
\end{aligned}$$

2.3 Intersections of straight lines with inverse images under the flow

Recall that the flow of the global vector field v on V is the mapping $F : \mathcal{D} \rightarrow V$, $(s, X) \mapsto F(s, X) \equiv C_X(s)$, where \mathcal{D} is the domain of the flow F :

$$\mathcal{D} \equiv \bigcup_{X \in V} I_X \times \{X\} \subset \mathbb{R} \times V. \quad (24)$$

If v is \mathcal{C}^q ($q \geq 1$, possibly $q = \infty$), \mathcal{D} is an open set in $\mathbb{R} \times V$, moreover F is \mathcal{C}^q on \mathcal{D} [12, 13].

Proposition 4. *Let U be an open subset of V . (i) For any $X \in V$, we have*

$$l_X \cap U = F_X(I'_{XU}) = F(I'_{XU} \times \{X\}), \quad (25)$$

where $F_X \equiv F(\cdot, X) = C_X$ is defined on $I_X \subset \mathbb{R}$, and where

$$I'_{XU} \equiv F_X^{-1}(U) = \{s \in I_X; F(s, X) \in U\}. \quad (26)$$

(ii) Further, we have for any $X \in V$:

$$I'_{XU} \times \{X\} = (\mathbb{R} \times \{X\}) \cap \mathcal{D}_U = (I_X \times \{X\}) \cap \mathcal{D}_U, \quad (27)$$

where

$$\mathcal{D}_U \equiv F^{-1}(U) = \bigcup_{X \in V} I'_{XU} \times \{X\}. \quad (28)$$

(iii) Assume that v is \mathcal{C}^∞ and that, for some $X \in V$, l_X is closed in V and not reduced to a point. Then (a) l_X is a submanifold of V and the mapping $F_X : I_X \rightarrow l_X$ is a local diffeomorphism at any point $s \in I_X$. (b) Assume moreover that F_X is non-periodic. Then it is a (global) diffeomorphism of I_X onto l_X . The connected components of $l_X \cap U$ are the images by F_X of the connected components of $I'_{XU} \equiv F_X^{-1}(U) = F_X^{-1}(l_X \cap U)$, which are open intervals of \mathbb{R} . In particular, in order that $l_X \cap U$ be connected, it is necessary and sufficient that I'_{XU} be an open interval.

Proof. Points (i) and (ii) follow immediately from the definitions. Let us prove Point (iii). (a) Since the maximal integral curve l_X is closed in V , this is a submanifold of V [14]. Since l_X is not reduced to a point, the vector field v does not vanish on l_X . [If $v(Y) = 0$ for some $Y \in l_X$, then we have

$I_Y = \mathbb{R}$ and $C_Y(s) = Y \forall s \in \mathbb{R}$ from the uniqueness of the maximal integral curve, hence $X = Y$ from the translation invariance (10).] Therefore, $\frac{dF_X}{ds} = v(F_X(s)) \neq 0$ implies that F_X is a local diffeomorphism, at any point $s \in I_X$, between the one-dimensional manifolds I_X and l_X . (b) Since l_X is closed in V , not reduced to a point, and since F_X is non-periodic, it follows that F_X is injective [14]. In view of (a) and since F_X is surjective by the definition of l_X , it is a diffeomorphism of I_X onto l_X . Hence, the connected components l_j of $l_X \cap U$ are the images by F_X of the connected components of the open set $I'_{XU} \equiv F_X^{-1}(U) = F_X^{-1}(l_X \cap U) \subset \mathbb{R}$, which are open intervals and make a finite or countable set (I_j) {Ref. [15], §(3.19.6)}. \square

An argument of transversality. Assume that v does not vanish and that each maximal integral curve is non-periodic and is closed in V . Given an arbitrary point $X \in V$, let $\chi : U_0 \rightarrow I \times \Omega_0$ be a straightening-out chart in the neighborhood of X , and let E be the closed countable subset of Ω_0 , having only isolated points, such that $\chi(l_X \cap U_0) = I \times E$. As Proposition 3 states: by restricting χ to an open subset $U \subset U_0$ for which $\chi(U)$ has the form (18) with $\Omega \subset \Omega_0$ any open neighborhood of $\mathbf{x} \equiv P_S(\chi(X))$ such that $E \cap \Omega = \{\mathbf{x}\}$, we ensure that $l_X \cap U$ is connected. From Point (iii) of Proposition 2, it follows that we will obtain a nice v -adapted chart by restricting χ to an open neighborhood $W \subset U$ of X — if it exists — such that, for any $Y \in W$, $l_Y \cap U$ be connected. To this purpose, we observe that, when the boundary of U is a regular hypersurface, then the same should be true for the boundary of the open set $\mathcal{D}_U \equiv F^{-1}(U) \subset \mathbb{R} \times V$: that boundary $\Sigma_U \equiv \text{Fr}(\mathcal{D}_U)$ should “normally” be a hypersurface in $\mathbb{R} \times V$. I.e., Σ_U should be a submanifold of codimension 1 of the $(N + 2)$ -dimensional manifold $\mathbb{R} \times V$. Then, “generically”, a straight line $\mathbb{R} \times \{X\} \subset \mathbb{R} \times V$ that intersects Σ_U is nowhere tangent to it, i.e. it is transverse to it at each intersection point. Thus, when Y is sufficiently close to X , the intersection points should be slightly displaced but should remain in the same number, hence the structure of $I'_{YU} \times \{Y\} = (\mathbb{R} \times \{Y\}) \cap \mathcal{D}_U$ should be the same as for X . In particular, if I'_{XU} is an interval, i.e. (according to Proposition 4) if $l_X \cap U$ is connected, then also I'_{YU} should be an interval, i.e., also $l_Y \cap U$ should be connected.

To make this line of reasoning precise, we first introduce, for a general hypersurface Σ in $\mathbb{R} \times V$, the set S_Σ of the points X in V for which the straight

line $\mathbb{R} \times \{X\}$ is tangent to Σ at one point at least. (Our aim is to use this when Σ is the boundary hypersurface $\Sigma_U \equiv \text{Fr}(\mathcal{D}_U)$ introduced above.) Note that the tangent space to $\mathbb{R} \times V$ at some point (s, X) is $T_{(s,X)}(\mathbb{R} \times V) \simeq T_s\mathbb{R} \times T_XV \simeq \mathbb{R} \times T_XV$. The tangent space to $\mathbb{R} \times \{X\}$ at (s, X) is $\mathbb{R}\xi$, where $\xi \equiv (1, 0_X)$, 0_X being the zero element of T_XV . Thus we define

$$S_\Sigma \equiv \{X \in V; \exists s \in \mathbb{R} : (s, X) \in \Sigma \text{ and } (1, 0_X) \in T_{(s,X)}\Sigma\}. \quad (29)$$

However, we note that $\mathcal{D}_U \equiv F^{-1}(U)$ is in general not a bounded domain of $\mathbb{R} \times V$, even if the open set $U \subset V$ is bounded.³ Therefore, even if $X \in V$ is such that the straight line $\mathbb{R} \times \{X\}$ is not tangent to the boundary hypersurface Σ_U , i.e. $X \notin S_{\Sigma_U}$, it may happen that $\mathbb{R} \times \{X\}$ is “tangent to Σ_U at infinity”, in which case any line $\mathbb{R} \times \{Y\}$, however close Y can be to X , may have “new” intersections with the hypersurface Σ_U . For a general hypersurface Σ in $\mathbb{R} \times V$, in fact for *any subset* Σ of $\mathbb{R} \times V$, we thus introduce:

$$S_{\Sigma_\infty} \equiv \{X \in V; \liminf_{r \rightarrow \infty} \inf_{|s| \geq r} d(X, \Sigma_s) = 0\}, \quad (30)$$

where, for any subset \mathcal{B} of $\mathbb{R} \times V$, we define \mathcal{B}_s to be its slice in V at $s \in \mathbb{R}$:

$$\mathcal{B}_s \equiv \{X \in V; (s, X) \in \mathcal{B}\} \subset V, \quad (31)$$

and where, given any distance d that generates the (metrizable) topology of V , one defines for any subset A of V and any point $X \in V$: $d(X, A) \equiv \inf_{Z \in A} d(X, Z)$.

2.4 Relevant theorems

Theorem 1. *Let Σ be a hypersurface of $\mathbb{R} \times V$ that is closed in $\mathbb{R} \times V$. Let $K = [\alpha, \beta]$ be a compact interval of \mathbb{R} ($\alpha < \beta$). Suppose that, for some $X_0 \in V$, the intersection $(K \times \{X_0\}) \cap \Sigma$ be a singleton (s_0, X_0) with $\alpha < s_0 < \beta$ and that this intersection be transverse, i.e. $X_0 \notin S_\Sigma$. Then there is a neighborhood W of X_0 such that, for any $X \in W$, $(K \times \{X\}) \cap \Sigma$ is a singleton $\Phi(X)$ and this intersection is transverse. Moreover, the function*

³ For instance, consider a constant vector field v on $V \equiv \mathbb{R}^n$. Then we have simply $F(s, X) = X + sv$. Hence, for any $X_0 \in V$, its inverse image is the unbounded straight line $F^{-1}(X_0) = \{(s, X) \in \mathbb{R} \times V; s \in \mathbb{R}, X = X_0 - sv\}$. From this, one may deduce more general examples by applying a diffeomorphism.

$\Phi : W \rightarrow \Sigma$ is smooth.

Proof. Since $(s_0, X_0) \in \Sigma$, and since Σ is a submanifold of dimension $N + 1$ of the $(N + 2)$ -dimensional differentiable manifold $\mathbb{R} \times V$, there is a chart (Ψ, \mathcal{U}) on $\mathbb{R} \times V$, with $(s_0, X_0) \in \mathcal{U}$, for which the function

$$g : \mathcal{U} \rightarrow \mathbb{R}, (s, X) \mapsto g(s, X) \equiv \Psi^{N+2}(s, X) \quad (32)$$

[the last component of $\Psi(s, X)$] is such that

$$(s, X) \in \mathcal{U} \cap \Sigma \iff g(s, X) = 0 \quad (33)$$

{e.g. [11], §(16.8.3)}. Then, if $(s, X) \in \mathcal{U} \cap \Sigma$, a vector $\eta = (a, u) \in T_{(s, X)}(\mathbb{R} \times V) \simeq \mathbb{R} \times T_X V$ is in the tangent space $T_{(s, X)}\Sigma$, iff $dg_{(s, X)}(\eta) = 0$. Since the intersection $(K \times \{X_0\}) \cap \Sigma = \{(s_0, X_0)\}$ is transverse, we have

$$dg_{(s_0, X_0)}(\xi_0) \neq 0, \quad \xi_0 \equiv (1, 0_{T_{X_0}V}). \quad (34)$$

Since K is a neighborhood of s_0 : replacing \mathcal{U} with a smaller neighborhood of (s_0, X_0) if necessary, we may assume that, if $(s, X) \in \mathcal{U}$, then $s \in K$. Consider a chart (χ, U) of V in a neighborhood U of X_0 , thus $\chi(X) = \mathbf{X} = (x^\mu) \in \mathbb{R}^{N+1}$ for $X \in U$. The mapping $\Xi : (s, X) \mapsto (s, \chi(X))$ is a local chart of $\mathbb{R} \times V$ defined in the neighborhood $\mathbb{R} \times U$ of (s_0, X_0) , while Ψ is also a local chart of $\mathbb{R} \times V$, defined in the neighborhood \mathcal{U} of (s_0, X_0) . Let

$$f(s, \mathbf{X}) \equiv g(\Xi^{-1}(s, \mathbf{X})) = g(s, \chi^{-1}(\mathbf{X})) \quad (35)$$

be the local expression of g in the chart Ξ . This function f is defined and \mathcal{C}^∞ on the open subset $\mathcal{O} \equiv \Xi((\mathbb{R} \times U) \cap \mathcal{U})$ of \mathbb{R}^{N+2} . Note that \mathcal{O} contains (s_0, \mathbf{X}_0) , where $\mathbf{X}_0 \equiv \chi(X_0)$. With this local expression, we have for any vector $\eta = (a, u) \in T_{(s, X)}(\mathbb{R} \times V)$:

$$dg_{(s, X)}(\eta) = \frac{\partial f}{\partial s}(s, \mathbf{X}) a + \frac{\partial f}{\partial x^\mu}(s, \mathbf{X}) u^\mu, \quad (36)$$

where $\mathbf{X} \equiv \chi(X)$, and where $(a, (u^\mu))$ ($\mu = 0, \dots, N$) are the components of η in the product chart Ξ . From (33), we have

$$f(s_0, \mathbf{X}_0) = 0. \quad (37)$$

From (34) and (36) we have

$$dg_{(s_0, X_0)}(\xi_0) = \frac{\partial f}{\partial s}(s_0, \mathbf{X}_0) \neq 0. \quad (38)$$

We can thus apply the implicit function theorem: there is an open neighborhood A of \mathbf{X}_0 and a unique smooth function $\varphi : A \rightarrow \mathbb{R}$, such that $\varphi(\mathbf{X}_0) = s_0$ and that, for any $\mathbf{X} \in A$:

$$(\varphi(\mathbf{X}), \mathbf{X}) \in \mathcal{O}, \quad f(\varphi(\mathbf{X}), \mathbf{X}) = 0, \quad \frac{\partial f}{\partial s}(\varphi(\mathbf{X}), \mathbf{X}) \neq 0. \quad (39)$$

We define a smooth mapping Φ by setting for any $X \in W' \equiv \chi^{-1}(A) \subset V$:

$$\Phi(X) \equiv (\varphi(\chi(X)), X). \quad (40)$$

Thus, for any $X \in W'$, with $\mathbf{X} \equiv \chi(X) \in A$, we have $\Phi(X) = (\varphi(\mathbf{X}), \chi^{-1}(\mathbf{X})) = \Xi^{-1}(\varphi(\mathbf{X}), \mathbf{X}) \in \Xi^{-1}(\mathcal{O}) \subset \mathcal{U}$. In particular, $\Phi(X_0) = (s_0, X_0)$. For any $X \in W'$, we have from (35), (39) and (40): $g(\Phi(X)) = 0$, thus $\Phi(X) \in \Sigma \cap \mathcal{U}$. As with (34) and (38), (39) means that the intersection $\Phi(X) \in (K \times \{X\}) \cap \Sigma$ is transverse. The definition (40) entails also that $\text{Pr}_2 \circ \Phi = \text{Id}_{W'}$, where $\text{Pr}_2 : \mathbb{R} \times V \rightarrow V$, $(s, X) \mapsto X$. Thus, the rank of $\text{Pr}_2 \circ \Phi$ is $\dim V = N + 1$, and since the rank of $\text{Pr}_2 \circ \Phi$ is not larger than that of Φ , this latter is also $N + 1 = \dim \Sigma$, i.e. Φ is a submersion. (All of this is true at any point $X \in W'$.) It follows that $W'' \equiv \Phi(W')$ is an open set in the manifold Σ {[11], §(16.7.5)}, hence is a neighborhood of $(s_0, X_0) = \Phi(X_0)$ in Σ . We have

$$\forall (s, X) \in W'', \quad (s, X) = \Phi(X). \quad (41)$$

We claim that there is some neighborhood $W \subset W'$ of X_0 , such that

$$\forall X \in W, \quad [s \in K \text{ and } (s, X) \in \Sigma] \implies [(s, X) = \Phi(X)]. \quad (42)$$

Note that, if $W \subset W'$, the reverse implication is also true for any $X \in W$, by the construction of the mapping Φ . Thus, if (42) is true, W is as stated by Theorem 1. We will reason *ab absurdo*. If it does not exist such a neighborhood W , then, for any integer $n > 0$, one can find $s_n \in K$ and $X_n \in W' \cap B(X_0, 1/n)$ such that $(s_n, X_n) \in \Sigma$ and $(s_n, X_n) \neq \Phi(X_n)$. Thus $X_n \rightarrow X_0$ and, by extraction in the compact set K , we may assume that s_n has a limit $s \in K$, so that $(s_n, X_n) \rightarrow (s, X_0)$. If it happened that $s = s_0$, then, since W'' is a neighborhood of (s_0, X_0) and $(s_n, X_n) \rightarrow (s, X_0)$, we would have $(s_n, X_n) \in W''$ for large enough n , hence $(s_n, X_n) = \Phi(X_n)$ from (41), which is a contradiction. Thus $s \neq s_0$. But since Σ is closed, we have $(s, X_0) \in \Sigma$, which contradicts the assumption that $(K \times \{X_0\}) \cap \Sigma = \{(s_0, X_0)\}$. This completes the proof of Theorem 1. \square

Theorem 2. *Let \mathcal{U} be an open domain in $\mathbb{R} \times V$, whose boundary Σ be a hypersurface of $\mathbb{R} \times V$. Assume that, for some $X \in V$, all intersections of the straight line $\mathcal{L} \equiv \mathbb{R} \times \{X\}$ with Σ are transverse. Then the boundary of $\mathcal{L} \cap \mathcal{U}$ in \mathcal{L} is $\text{Fr}_{(\mathcal{L})}(\mathcal{L} \cap \mathcal{U}) = \mathcal{L} \cap \Sigma$.*

Proof. The boundary of a subset is of course relative to which containing set is considered. Here, the boundary Σ of the open set $\mathcal{U} \subset \mathbb{R} \times V$ is relative to the whole manifold $\mathcal{V} \equiv \mathbb{R} \times V$, i.e. $\Sigma \equiv \overline{\mathcal{U}} \cap \overline{\mathcal{C}\mathcal{U}}$, where the upper bar $\overline{}$ denotes the adherence in \mathcal{V} and \mathcal{C} the complementary set in \mathcal{V} . Thus

$$\Sigma = \overline{\mathcal{U}} \setminus \mathcal{U}, \quad (43)$$

since \mathcal{U} is open. The boundary of some subset $\mathcal{B} \subset \mathcal{L}$ in \mathcal{L} (or relative to \mathcal{L}) is $\text{Fr}_{(\mathcal{L})}(\mathcal{B}) \equiv \overline{\mathcal{B}^\mathcal{L}} \cap \overline{\mathcal{C}_\mathcal{L}\mathcal{B}^\mathcal{L}}$, where $\overline{\mathcal{B}^\mathcal{L}}$ denotes the adherence of \mathcal{B} in \mathcal{L} , and where $\mathcal{C}_\mathcal{L}\mathcal{B} \equiv \mathcal{L} \setminus \mathcal{B}$ is the complementary set of \mathcal{B} in \mathcal{L} . However, here $\mathcal{L} \equiv \mathbb{R} \times \{X\}$ is closed in $\mathcal{V} = \mathbb{R} \times V$, hence we have $\overline{\mathcal{B}^\mathcal{L}} = \overline{\mathcal{B}}$, the adherence in the whole set \mathcal{V} . Thus

$$\text{Fr}_{(\mathcal{L})}(\mathcal{L} \cap \mathcal{U}) \equiv \overline{\mathcal{L} \cap \mathcal{U}}^\mathcal{L} \cap \overline{\mathcal{C}_\mathcal{L}(\mathcal{L} \cap \mathcal{U})}^\mathcal{L} = \overline{\mathcal{L} \cap \mathcal{U}} \cap \overline{\mathcal{C}_\mathcal{L}\mathcal{U}}. \quad (44)$$

Again because \mathcal{L} is closed in \mathcal{V} , we have $\overline{\mathcal{L} \cap \mathcal{U}} \subset \mathcal{L} \cap \overline{\mathcal{U}}$. We will show that we have exactly

$$\overline{\mathcal{L} \cap \mathcal{U}} = \mathcal{L} \cap \overline{\mathcal{U}}. \quad (45)$$

We shall in fact show that

$$\mathcal{L} \cap \Sigma \subset \overline{\mathcal{L} \cap \mathcal{U}}. \quad (46)$$

Since (43) implies that $(\mathcal{L} \cap \overline{\mathcal{U}}) \setminus \overline{\mathcal{L} \cap \mathcal{U}} \subset (\mathcal{L} \cap \overline{\mathcal{U}}) \setminus (\mathcal{L} \cap \mathcal{U}) = \mathcal{L} \cap \Sigma$, this will prove that $\mathcal{L} \cap \overline{\mathcal{U}} \subset \overline{\mathcal{L} \cap \mathcal{U}}$, whence (45).

To prove (46), consider an arbitrary point $p_0 = (s_0, X) \in \mathcal{L} \cap \Sigma$. As in the proof of Theorem 1, let (Ψ, \mathcal{W}) be a chart on \mathcal{V} , with $p_0 \in \mathcal{W}$, such that $p \in \mathcal{W} \cap \Sigma \Leftrightarrow g(p) = 0$, where $g = \Psi^n$, with $n = \dim(\mathcal{V}) = N + 2$. Since we assume that the intersection $p_0 \in \mathcal{L} \cap \Sigma$ is transverse, we have again (34). Hence, there is an interval $J =]s_0 - r, s_0 + r[$ in which $s = s_0$ is the only zero of the smooth function $\varphi(s) \equiv g(s, X)$. (That is, p_0 is the only intersection of $J \times \{X\}$ with Σ .) Thus, we may assume that, say, $g(s, X) > 0$ for $s_0 < s < s_0 + r$, so that

$$g(s, X) < 0 \text{ for } s_0 - r < s < s_0. \quad (47)$$

Replacing $\Psi(p)$ by $\Psi(p) - \Psi(p_0)$, we may assume that $\Psi(p_0) = \mathbf{0}_{\mathbb{R}^n}$. There is some open ball $W = B(\mathbf{0}, r) \subset \Psi(\mathcal{W})$. Replacing \mathcal{W} by $\Psi^{-1}(W)$, we have that $\mathcal{W}_+ \equiv \{p \in \mathcal{W}; x^n \equiv \Psi^n(p) > 0\}$ is just $\mathcal{W}_+ = \Psi^{-1}(W_+)$, where $W_+ \equiv \{\mathbf{P} \in W; x^n > 0\}$, hence \mathcal{W}_+ is non-empty and connected as is W_+ . The same is true for $\mathcal{W}_- \equiv \{p \in \mathcal{W}; \Psi^n(p) < 0\}$. Since $p_0 \in \Sigma \subset \overline{\mathcal{U}}$, and since \mathcal{W} is a neighborhood of p_0 , we have $\mathcal{U} \cap \mathcal{W} \neq \emptyset$, so let $p'_0 \in \mathcal{U} \cap \mathcal{W}$. Because \mathcal{W} is the disjointed union $\mathcal{W} = \mathcal{W}_+ \cup \mathcal{W}_- \cup (\mathcal{W} \cap \Sigma)$, and because $p'_0 \in \mathcal{U}$ cannot belong to $\Sigma = \overline{\mathcal{U}} \setminus \mathcal{U}$, we have either $p'_0 \in \mathcal{W}_+$ or $p'_0 \in \mathcal{W}_-$. Let us assume that, for instance, $p'_0 \in \mathcal{W}_-$, so that $\mathcal{U} \cap \mathcal{W}_- \neq \emptyset$. It follows that $\mathcal{W}_- \subset \mathcal{U} \cap \mathcal{W}$, for otherwise the connected set \mathcal{W}_- would intersect both \mathcal{U} and $\mathcal{C}\mathcal{U}$, hence would intersect the boundary Σ — which is impossible, since we have $x^n = 0$ in Σ , not $x^n < 0$. Therefore, we get from (47) that $]s_0 - r, s_0[\times \{X\} \subset \mathcal{U} \cap \mathcal{W}$, so that $p_0 \equiv (s_0, X) \in \overline{\mathcal{L} \cap \mathcal{U}}$. This proves (46).

Combining (44) and (45) gives us:

$$\text{Fr}_{(\mathcal{L})}(\mathcal{L} \cap \mathcal{U}) = \mathcal{L} \cap \overline{\mathcal{U}} \cap \overline{\mathcal{C}\mathcal{U}}. \quad (48)$$

But $\mathcal{C}\mathcal{L}\mathcal{U} = \mathcal{L} \cap \mathcal{C}\mathcal{U}$ is closed in \mathcal{V} , for \mathcal{L} is closed and \mathcal{U} is open. Hence $\mathcal{L} \cap \overline{\mathcal{C}\mathcal{U}} = \mathcal{L} \cap \mathcal{C}\mathcal{U} = \mathcal{L} \cap \mathcal{C}\mathcal{U} = \mathcal{L} \cap \overline{\mathcal{C}\mathcal{U}}$. Therefore, (48) rewrites as

$$\text{Fr}_{(\mathcal{L})}(\mathcal{L} \cap \mathcal{U}) = \mathcal{L} \cap \overline{\mathcal{U}} \cap \overline{\mathcal{C}\mathcal{U}} \equiv \mathcal{L} \cap \Sigma, \quad (49)$$

which proves Theorem 2. \square

Remark 2.1. With small and straightforward modifications, the foregoing proof shows the result in the much more general case that $\mathbb{R} \times V$ (with V a differentiable manifold) is replaced by a general differentiable manifold \mathcal{V} and \mathcal{L} is the range $\mathcal{L} = C(J)$, assumed closed, of a smooth curve $C : J \rightarrow \mathcal{V}$ (with J an interval of \mathbb{R}), such that all intersections $p \in \mathcal{L} \cap \mathcal{V}$ are transverse. It is easy to see that the latter assumption is necessary.

Remark 2.2. In the course of the proof, the following intuitively obvious result was proved: Suppose that the line $\mathcal{L} = C(J)$ intersects transversely at point $p = C(s_0)$ the boundary Σ , assumed to be a regular hypersurface, of some open domain $\mathcal{U} \subset \mathcal{V}$. Then, among the two parts of \mathcal{L} : $s < s_0$ and $s > s_0$, at least one is such that, when s is close enough to s_0 , we have $C(s) \in \mathcal{U}$.

Theorem 3. *Let U be an open subset of V . (i) Assume that $\overline{F^{-1}(U)} \subset \mathcal{D}$, with \mathcal{D} the domain of the flow F . (This is true, in particular, if the flow is complete, i.e. $\mathcal{D} = \mathcal{V}$.) Then we have:*

$$\text{Fr}(F^{-1}(U)) = F^{-1}(\text{Fr}(U)). \quad (50)$$

(ii) *If $\text{Fr}(U)$ is a hypersurface of V , then $F^{-1}(\text{Fr}(U))$ is a hypersurface of $\mathcal{V} \equiv \mathbb{R} \times V$.*

Proof of Point (i). First, consider the more general context that \mathcal{V} and V are merely metric spaces and $F : \mathcal{D} \rightarrow V$ is merely a continuous map, with \mathcal{D} an open subset of \mathcal{V} . Then, if $\overline{F^{-1}(U)} \subset \mathcal{D}$, we have

$$\text{Fr}(F^{-1}(U)) \subset F^{-1}(\text{Fr}(U)). \quad (51)$$

Indeed, since U is open in V , we have $\text{Fr}(U) = \overline{U}^V \setminus U$, as with Eq. (43). Since F is continuous on \mathcal{D} , $F^{-1}(U)$ is open in \mathcal{D} ; hence, \mathcal{D} being an open subset of \mathcal{V} , $F^{-1}(U)$ is open in \mathcal{V} . Therefore, we have similarly $\text{Fr}(F^{-1}(U)) = \overline{F^{-1}(U)} \setminus F^{-1}(U)$. Again because F is continuous on \mathcal{D} , we have $\overline{F^{-1}(U)}^{\mathcal{D}} \subset F^{-1}(\overline{U}^V)$. But $\overline{F^{-1}(U)}^{\mathcal{D}} = \overline{F^{-1}(U)}$ since $\overline{F^{-1}(U)} \subset \mathcal{D}$. Thus $\text{Fr}(F^{-1}(U)) \subset F^{-1}(\overline{U}^V) \setminus F^{-1}(U)$, whence (51).

To prove the reverse inclusion, we consider any $p = (s, X) \in F^{-1}(\text{Fr}(U))$, thus $Y \equiv F(s, X) \in \text{Fr}(U)$, and we will show that $p \in \text{Fr}(F^{-1}(U))$. There exists an open neighborhood \mathcal{U} of p , having the form $\mathcal{U} = J \times W$, with J an open interval containing s and 0 , and with W an open neighborhood of X in V , such that $\mathcal{U} \subset \mathcal{D}$ {Ref. [12], §(18.2.5)}. For all $t \in J$, $F_t \equiv F(t, \cdot)$ is a homeomorphism of W onto $W_t \equiv F_t(W)$, moreover F_{-t} is defined over W_t and is the inverse homeomorphism of F_t [these two points result from (10)] {Ref. [12], §(18.2.8)}. (We note for the proof of Point (ii) that F_t , as well as F_{-t} , is of class \mathcal{C}^q if the vector field v is itself \mathcal{C}^q [12].) Now we consider any neighborhood \mathcal{U}' of p and we show that it intersects both $F^{-1}(U)$ and $\mathcal{C}F^{-1}(U)$. We may assume that $\mathcal{U}' \subset \mathcal{U}$ and has the form $\mathcal{U}' = J' \times W'$, with $J' \subset J$ an open interval containing s , and with $W' \subset W$ an open neighborhood of X in V . Thus $F_s(W')$ is an open neighborhood of $Y = F_s(X)$. Since $Y \in \text{Fr}(U)$, both $F_s(W') \cap U$ and $F_s(W') \cap \mathcal{C}U$ are non-empty, so let $Y' \in F_s(W') \cap U$ and $Y'' \in F_s(W') \cap \mathcal{C}U$. Therefore, $X' \equiv F_{-s}(Y') \in W'$. Thus, $F(s, X') = Y' \in U$, and also $(s, X') \in J' \times W'$, so that $(s, X') \in (J' \times W') \cap F^{-1}(U)$. In just the

same way, with $X'' \equiv F_{-s}(Y'')$, we get that $(s, X'') \in (J' \times W') \cap F^{-1}(\mathbb{C}U)$. Since $F^{-1}(\mathbb{C}U) = \mathbb{C}_{\mathcal{D}}F^{-1}(U) \subset \mathbb{C}F^{-1}(U) \equiv \mathcal{V} \setminus F^{-1}(U)$, we have shown that any neighborhood \mathcal{U}' of p intersects both $F^{-1}(U)$ and $\mathbb{C}F^{-1}(U)$, thus $p \in \text{Fr}(F^{-1}(U))$. Therefore, we have indeed $F^{-1}(\text{Fr}(U)) \subset \text{Fr}(F^{-1}(U))$. Together with (51), this proves (50).

Proof of Point (ii). Let $p = (s, X) \in F^{-1}(\text{Fr}(U)) \subset \mathcal{V}$. Define J (with $s \in J$), $W \subset V$ (with $X \in W$), $\mathcal{U} = J \times W$, and F_t ($t \in J$) just as at the beginning of the foregoing paragraph. Since $S \equiv \text{Fr}(U)$ is assumed to be a hypersurface of V , and since by hypothesis $Y \equiv F(s, X) \in S$, let $\chi : Y' \mapsto \mathbf{Y} \equiv (y^1, \dots, y^{n-1})$ be a chart of V (with $n - 1 = \dim(\mathcal{V}) - 1 = \dim(V)$), defined in the neighborhood of Y , and such that

$$Y' \in S \cap \text{Dom}(\chi) \Leftrightarrow y^{n-1} \equiv \chi^{n-1}(Y') = 0. \quad (52)$$

We may assume that $\text{Dom}(\chi)$, the domain of χ , is just $W_s \equiv F_s(W)$. Then the mapping

$$\Xi : \mathcal{U} = J \times W \rightarrow \mathbb{R}^n, \quad (t, X') \mapsto (t, \chi(F_s(X'))) = (t, \mathbf{Y}) \quad (53)$$

is a chart of \mathcal{V} in the neighborhood of p . Consider the mapping

$$\Psi : \mathcal{U}' \equiv F^{-1}(W_s) \rightarrow \mathbb{R}^n, \quad (t, X') \mapsto (t, \chi(F(t, X'))). \quad (54)$$

(Note that \mathcal{U}' is an open neighborhood of p .) We have (cf. Eq. (10)):

$$F(t, X') = F(s + u, X') = F(u, F(s, X')) = F(u, F_s(X')), \quad u \equiv t - s. \quad (55)$$

Hence, the local expression of Ψ in the chart Ξ is:

$$\mathbf{G}(t, \mathbf{Y}) \equiv \Psi(\Xi^{-1}(t, \mathbf{Y})) = (t, \mathbf{Z}(t, \mathbf{Y})) \quad (56)$$

with

$$\mathbf{Z}(t, \mathbf{Y}) = (z^1, \dots, z^{n-1}) \equiv \chi(F(u, \chi^{-1}(\mathbf{Y}))). \quad (57)$$

Let $\mathbf{v} = (v^1, \dots, v^{n-1}) = \mathbf{v}(\mathbf{Y})$ be the local expression of v in the chart χ . Thus \mathbf{Z} is the value at $u \equiv t - s$ of the solution of $\frac{d\mathbf{Z}'}{dw} = \mathbf{v}(\mathbf{Z}'(w))$, $\mathbf{Z}'(w = 0) = \mathbf{Y}$. Hence we have, uniformly w.r.t. (t, \mathbf{Y}) in some neighborhood of $(s, \mathbf{Y}_0) \equiv \Xi(p)$:

$$\mathbf{Z}(t, \mathbf{Y}) = \mathbf{Y} + (t - s)\mathbf{v}(\mathbf{Y}) + O(u^2). \quad (58)$$

The Jacobian matrix of $\mathbf{G}(t, \mathbf{Y}) = (t, \mathbf{Z}(t, \mathbf{Y}))$ at point (s, \mathbf{Y}_0) is therefore:

$$J = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ v^1(\mathbf{Y}_0) & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ v^{n-1}(\mathbf{Y}_0) & 0 & 0 & \dots & 1 \end{pmatrix}, \quad (59)$$

a triangular matrix with 1 on the diagonal, so $\det J = 1$. It follows that Ψ also is a chart of \mathcal{V} in some neighborhood $\mathcal{U}'' \subset \mathcal{U} \cap \mathcal{U}'$ of p . When $(t, X') \in \mathcal{U}''$, we have from (52) and (54):

$$(t, X') \in F^{-1}(S) \Leftrightarrow z^{n-1} \equiv \Psi^n(t, X') \equiv \chi^{n-1}(F(t, X')) = 0. \quad (60)$$

Hence, $F^{-1}(S)$ is a hypersurface of \mathcal{V} . The proof of Theorem 3 is complete. \square

Remark 3.1. From (54) and (56), we have also $\mathbf{Z}(t, \mathbf{Y}) = \chi(F(\Xi^{-1}(t, \mathbf{Y})))$, thus $\mathbf{Z}(t, \mathbf{Y})$ is the local expression of F in the charts Ξ on \mathcal{V} and χ on V . Equation (59) shows also that the Jacobian matrix of \mathbf{Z} at $(s, \mathbf{Y}_0) \equiv \Xi(p)$ has rank $n - 1 = \dim V$. [Relation (52), hence the fact that $p \in F^{-1}(S)$, are not used to get this; hence this is true at any point $p \in \mathcal{D} = \text{Dom}(F)$.] Thus, F is a submersion. Hence, it is transversal to any submanifold of V . Point (ii) of Theorem 3 follows also from this [16].

Proposition 5. *Let Σ be a subset of $\mathbb{R} \times V$. If $X \in V \setminus S_{\Sigma_\infty}$ [cf. (30)], there is a neighborhood W of X and a real $R > 0$ such that*

$$(Y \in W \text{ and } |s| \geq R) \Rightarrow (s, Y) \notin \Sigma. \quad (61)$$

Proof. Clearly, $f(X) \equiv \lim_{r \rightarrow \infty} \inf_{|s| \geq r} d(X, (\Sigma)_s)$ is well defined for any $X \in V$ and verifies $0 \leq f(X) \leq +\infty$. Hence, if $X \notin S_{\Sigma_\infty}$ it means that $f(X) > 0$, or equivalently that there exists $R > 0$ and $\delta > 0$ such that

$$|s| \geq R \Rightarrow d(X, \Sigma_s) \geq \delta, \quad (62)$$

hence $d(X, Z) \geq \delta$ is true for any $Z \in \Sigma_s$ if $|s| \geq R$. We deduce from this that, if $Y \in V$ verifies $d(X, Y) < \delta/2$, and if $|s| \geq R$, we have $d(Y, Z) > \delta/2$ for any $Z \in \Sigma_s$, hence $d(Y, \Sigma_s) \geq \delta/2$. (Thus, $V \setminus S_{\Sigma_\infty}$ is open in V , in other words S_{Σ_∞} is closed.) In particular, if $d(X, Y) < \delta/2$ and $|s| \geq R$, then $Y \notin \Sigma_s$, i.e. $(s, Y) \notin \Sigma$. \square

Theorem 4. *Let U be an open subset of V and set $\mathcal{D}_U \equiv F^{-1}(U)$ and $\Sigma_U \equiv \text{Fr}(\mathcal{D}_U)$. Assume that Σ_U is a hypersurface of $\mathcal{V} \equiv \mathbb{R} \times V$ which is closed in \mathcal{V} , that for some $X \in U$, the open set I'_{XU} is a bounded interval, and that $X \notin (S_{\Sigma_U} \cup S_{\Sigma_U \infty})$.*

(i) *There is a neighborhood W of X , $W \subset U$, such that for $Y \in W$, also I'_{YU} is a bounded interval. (ii) *If v does not vanish and all maximal integral curves are closed in V and non-periodic, then $l_Y \cap U$ is connected for any $Y \in W$. (iii) *If moreover $\chi(U)$ has the form (18) and there is a straightening-out chart (χ, U_0) with $U_0 \supset U$, then the restriction of χ to W is a nice v -adapted chart.***

Proof. (i) Since $X \notin S_{\Sigma_U \infty}$, by Proposition 5 there is a neighborhood W_0 of X and a real $R > 0$ such that

$$(Y \in W_0 \text{ and } |s| \geq R) \Rightarrow (s, Y) \notin \Sigma_U. \quad (63)$$

Since I'_{XU} is assumed to be a bounded interval (and $I'_{XU} \neq \emptyset$ since $0 \in I'_{XU}$), let $I'_{XU} =]a, b[$, with $a, b \in \mathbb{R}$, $a < b$. In (63) we may assume that $-R < a < b < R$. By Proposition 4, we have

$$I'_{XU} \times \{X\} =]a, b[\times \{X\} = (\mathbb{R} \times \{X\}) \cap \mathcal{D}_U. \quad (64)$$

Since by assumption $X \notin S_{\Sigma_U}$, we get by Theorem 2 (setting $\mathcal{L} \equiv \mathbb{R} \times \{X\}$):

$$\{(a, X), (b, X)\} = \text{Fr}_{(\mathcal{L})}(\mathcal{L} \cap \mathcal{D}_U) = \mathcal{L} \cap \Sigma_U. \quad (65)$$

Thus, considering the compact intervals $K_1 \equiv [-R, \frac{a+b}{2}]$ and $K_2 \equiv [\frac{a+b}{2}, R]$, we have

$$(K_1 \times \{X\}) \cap \Sigma_U = \{(a, X)\}, \quad (K_2 \times \{X\}) \cap \Sigma_U = \{(b, X)\}. \quad (66)$$

Therefore, by Theorem 1, there are neighborhoods W_1 and W_2 of X , and smooth functions $\Phi_j : W_j \rightarrow \Sigma_U$ ($j = 1, 2$), such that:

$$Y \in W_j \Rightarrow (K_j \times \{Y\}) \cap \Sigma_U = \{\Phi_j(Y)\} \quad (j = 1, 2), \quad (67)$$

with transverse intersection. We have $\Phi_j(Y) = (\varphi_j(Y), Y)$, with $\varphi_j : W_j \rightarrow \mathbb{R}$ a smooth function. From (66) and (67), we get: $\varphi_1(X) = a$ and $\varphi_2(X) = b$. Thus $\varphi_1(X) = a \neq b = \varphi_2(X)$; hence, by considering a small enough neighborhood W of X , with $W \subset W_0 \cap W_1 \cap W_2$, we get $\varphi_1(Y) \neq \varphi_2(Y)$ for $Y \in W$. With (63) and (67), this implies that

$$Y \in W \Rightarrow (\mathbb{R} \times \{Y\}) \cap \Sigma_U = \{\Phi_1(Y), \Phi_2(Y)\}. \quad (68)$$

Another application of Theorem 2 proves that

$$\text{Fr}_{(\mathbb{R} \times \{Y\})}((\mathbb{R} \times \{Y\}) \cap \mathcal{D}_U) = (\mathbb{R} \times \{Y\}) \cap \Sigma_U. \quad (69)$$

Together with (68), this implies that I'_{YU} , the open subset of \mathbb{R} such that $I'_{YU} \times \{Y\} = (\mathbb{R} \times \{Y\}) \cap \mathcal{D}_U$, has boundary $\{\varphi_1(Y), \varphi_2(Y)\}$. Therefore, $I'_{YU} =]\varphi_1(Y), \varphi_2(Y)[$.

(ii) This follows from Point (iii) in Proposition 4.

(iii) This follows from Point (iii) in Proposition 2. □

2.5 Adapted charts and “normal” vector fields

With Theorem 4, we formalized our transversality argument in Subsect. 2.3 to investigate the problem of the existence, in the neighborhood of any point $X \in V$, of a nice v -adapted chart. Assuming that v does not vanish and that all maximal integral curves are closed in V and non-periodic, let us check if Theorem 4 applies. Due to Proposition 4, $I'_{XU} = F_X^{-1}(l_X \cap U)$ is an interval iff $l_X \cap U$ is connected, and I'_{XU} is bounded if U is relatively compact. As shown by Proposition 3, the assumption that $l_X \cap U$ is connected may be fulfilled by starting from a straightening-out chart $\chi : U_0 \rightarrow I \times \Omega_0$ in the neighborhood of the arbitrary point $X \in V$ and by restricting χ to an open subset $U \subset U_0$ such that $\chi(U)$ has the form (18) with $\Omega \subset \Omega_0$ a small enough open neighborhood of $\mathbf{x} \equiv P_S(\chi(X))$.

As shown by Theorem 3, the assumption that Σ_U is a hypersurface of \mathcal{V} that is closed in \mathcal{V} is fulfilled, in particular, if the boundary of the open set $U \subset V$ is itself a hypersurface of V that is closed in V , and if moreover $\overline{F^{-1}(U)} \subset \mathcal{D}$. The latter inclusion is true, in particular, if $\mathcal{D} = \mathcal{V}$, i.e., if the vector field v is complete (in other words, if every maximal integral curve of v is defined over the whole real line). Actually, this does not restrict in any way the set of the maximum integral curves, $N_v \equiv \{l_X; X \in V\}$ (the “congruence of world lines”, in the physical context with $N = 3$). Indeed, there always exists a smooth function $\lambda : V \rightarrow \mathbb{R}_+$, such that the vector field λv is complete, moreover the mappings C_X corresponding to the maximal integral curves of λv are mere reparameterizations of those of v , so that the curves l_X themselves are unchanged [17]. Thus, the assumption that Σ_U is a hypersurface of \mathcal{V} that is closed in \mathcal{V} is not a restrictive one.

The assumption “ $X \notin (S_{\Sigma_U} \cup S_{\Sigma_U \infty})$ ” means that the straight line $\mathbb{R} \times \{X\}$ is not tangent to the hypersurface Σ_U , and is not “tangent to it at infinity”. For a given hypersurface Σ_U , the points thus excluded form a kind of apparent contour (of that hypersurface Σ_U), having “normally” measure zero in V , hence this is true for a “generic” point X . However, here the hypersurface $\Sigma_U = F^{-1}(\text{Fr}(U))$ of $\mathbb{R} \times V$ depends on the selected neighborhood U of the given point $X \in V$. There is much freedom in the choice of this neighborhood, since it is merely required to have a regular boundary and have the form (18) with Ω a small enough open neighborhood of $\mathbf{x} \equiv P_S(\chi(X))$. If it turns out that $X \in (S_{\Sigma_U} \cup S_{\Sigma_U \infty})$ for some U satisfying these requirements, then a slightly deformed neighborhood U' does also satisfy them, but the boundary $\Sigma_{U'}$ is also deformed w.r.t. Σ_U . Hence, it seems plausible that, due to this freedom, there always exists U satisfying these requirements and such that $X \notin (S_{\Sigma_U} \cup S_{\Sigma_U \infty})$ — unless v has some “pathology” that we were not able to describe in a more explicit way.

Thus, for any point $X \in V$, the assumptions of Theorem 4 should be fulfilled in a suitable neighborhood U of X if the vector field v does not vanish, has all maximal integral curves closed in V and non-periodic, and does not suffer from the “pathology” alluded to. Therefore, we set the following definition, the word “normal” being justified by the foregoing discussion.

Definition 2. *A non-vanishing C^∞ vector field v is called “normal” iff all maximal integral curves are closed in V and, moreover, any point $X \in V$ has nested open neighborhoods $W \subset U \subset U_0$ such that: (i) There is a straightening-out chart (χ, U_0) . (ii) $\chi(U)$ has the form (18). (iii) For any maximal integral curve l of v intersecting W , the line $l \cap U$ is connected.*

The following result shows that this concept is relevant. Note that we do not need to assume that the maximal integral curves are non-periodic.

Theorem 5. *Let v be a non-vanishing global vector field, such that all maximal integral curves are closed in V . (i) In order that, for any point $X \in V$, there exist a nice v -adapted chart whose domain be an open neighborhood of X , it is necessary and sufficient that v be normal. (ii) Also, in order that v be normal, it is necessary and sufficient that any point $X \in V$ have an open neighborhood W on which there is a straightening-out chart (χ, W) , and such*

that, for any maximal integral curve l of v , the line $l \cap W$ is connected.

Proof. (i) The sufficiency is an immediate consequence of Point (iii) in Proposition 2. Conversely, if for any $X \in V$ there is a nice v -adapted chart (χ_1, U_1) with $X \in U_1$, then by Point (ii) of Proposition 0 we get a chart (χ, W) , with an open set $W \subset U_1$ and $X \in W$, which (a) differs from χ_1 merely by the time coordinate y^0 , and (b) is a straightening-out chart. From (a), (χ, W) is also a v -adapted chart. Therefore, by Point (ii) of Proposition 2: for any $Y \in W$, the intersection $l_Y \cap W$ is a connected set. This implies that for any maximal integral curve l of v , the line $l \cap W$ is a connected (possibly empty) set. By this and (b) above, and since $X \in V$ is arbitrary, the vector field v is normal. (This is the case $W = U = U_0$ in the definition.)

(ii) For the reason just invoked, the condition is sufficient in order that v be normal. Conversely, if v is normal, consider any $X \in V$. We know that there is a nice v -adapted chart (χ_1, U_1) with $X \in U_1$, and the proof of the necessity at Point (i) shows that from it we deduce a straightening-out chart (χ, W) , with $X \in W$, and such that for any maximal integral curve l of v , the line $l \cap W$ is a connected set. \square

Examples. (i) Take $V = \mathbb{R}^{N+1}$ and consider any constant vector field $v(\mathbf{X}) = \mathbf{v} = \text{Constant} \neq \mathbf{0}$. The maximal integral curve at $\mathbf{X} \in V$ is $l_{\mathbf{X}} = \{\mathbf{Y} = \mathbf{X} + t\mathbf{v}; t \in \mathbb{R}\}$. To define a straightening-out chart explicitly, take $\mathbf{u}_1, \dots, \mathbf{u}_N$ such that the vectors $\mathbf{u}_0 \equiv \mathbf{v}, \mathbf{u}_1, \dots, \mathbf{u}_N$ form a basis of \mathbb{R}^{N+1} , and define an invertible linear transformation L of \mathbb{R}^{N+1} by $L(x^\mu \mathbf{u}_\mu) = x^\mu \mathbf{e}_\mu$, where (\mathbf{e}_μ) ($\mu = 0, \dots, N$) is the canonical basis of \mathbb{R}^{N+1} . Now consider any open set of the form $U = L^{-1}(I \times \Omega)$ with $I =] - a, +a[$ and Ω an open subset of \mathbb{R}^N : we have explicitly $U = \{s\mathbf{v} + x^j \mathbf{u}_j; s \in I, \mathbf{x} \equiv (x^j) \in \Omega\}$. The restriction χ of L to U defines a straightening-out chart, because $L(\mathbf{v}) = \mathbf{e}_0$ means that $v = \partial_0$ in that chart. Moreover, given $\mathbf{X} = s\mathbf{v} + x^j \mathbf{u}_j \in U$ (thus $s \in I, \mathbf{x} \equiv (x^j) \in \Omega$), a point $\mathbf{Y} = \mathbf{X} + t\mathbf{v}$ of $l_{\mathbf{X}}$ belongs to U , iff $s + t \in I$, so $l_{\mathbf{X}} \cap U$ is connected. Hence, a constant vector field is normal. And indeed, we have for $\mathbf{Y} \in l_{\mathbf{X}} \cap U$: $\chi(\mathbf{Y}) = L(\mathbf{X} + t\mathbf{v}) = \chi(\mathbf{X}) + t\mathbf{e}_0$, hence χ is v -adapted. Moreover, let $l \in D_U$, where D_U is defined in Eq. (12). Thus $l = l_{\mathbf{X}}$, where $\mathbf{X} = s\mathbf{v} + x^j \mathbf{u}_j \in U$, i.e. $s \in I, \mathbf{x} \equiv (x^j) \in \Omega$. Then we have from the

definition (14): $\bar{\chi}(l) = \mathbf{x} = L(\mathbf{x}_0)$ with $\mathbf{x}_0 \equiv x^j \mathbf{u}_j$. Hence, the mapping $\bar{\chi} : D_U \rightarrow \mathbb{R}^N$, $l = l_{\mathbf{x}} = l_{s\mathbf{v} + \mathbf{x}_0} \mapsto \mathbf{x} = L(\mathbf{x}_0)$ is injective, i.e., the v -adapted chart χ is nice.

(ii) If $\phi : V \rightarrow V'$ is a diffeomorphism, set $v' \equiv \phi^*v : v'(X') \equiv D\phi_{\phi^{-1}(X')}(v(\phi^{-1}(X')))$, $X' \in V'$. The maximal integral curves of v' , as well as the associate flow F' , are just the images of their counterparts for v : $F'(s, X') \equiv C'_{X'}(s) = \phi(C_{\phi^{-1}(X')}(s)) = \phi(F(s, \phi^{-1}(X')))$. Moreover, if (χ, U_0) is a straightening-out chart in the neighborhood of $X \in V$, then so is $(\chi \circ \phi^{-1}, \phi(U_0))$ in the neighborhood of $X' \equiv \phi(X) \in V'$. Therefore, if v is normal, so is v' .

(iii) If we have a normal vector field v on some differentiable manifold V , having non-periodic orbits, and if U is any open subset of V , let us show that the restriction $v' \equiv v|_U$ is a normal vector field on the differentiable manifold U . Given any $X \in U$, it is easy to check from the definitions that the maximal open interval I'_X defining the orbit (maximal integral curve) l'_X of v' at X is the connected component of 0 in I'_{XU} , the latter being defined in Eq. (26). (This result is true for any vector field.) It follows by Point (iii) in Proposition 4 that l'_X is the connected component of X in $l_X \cap U$, where l_X is the orbit of v in V at X . Therefore, since the orbits of v are closed subsets of V , the orbits of v' are closed subsets of U . If $X \in U$, there exists by Point (ii) of Theorem 5 a straightening-out chart (χ, W) of (V, v) , with $X \in W$ and $\chi(W) = I \times \Omega$, such that for any $Y \in W$, $l_Y \cap W$ is connected. Let $B =]x^0 - r, x^0 + r[\times \dots \times]x^N - r, x^N + r[$ be a ball centered at $\chi(X) = (x^\mu)$ and such that $B \subset \chi(U)$, and set $W' \equiv \chi^{-1}(B)$. Then the restriction $\chi' \equiv \chi|_{W'}$ is a straightening-out chart for v' (up to a shift in y^0). For $Y \in W'$, set $\mathbf{y} \equiv P_S(\chi(Y))$. Since $l_Y \cap W$ is connected, we have $l_Y \cap W = \chi^{-1}(I \times \{\mathbf{y}\})$ by Point (i) of Proposition 2. Hence $l_Y \cap W' = \chi^{-1}(]x^0 - r, x^0 + r[\times \{\mathbf{y}\})$, thus a connected set $\subset l_Y \cap U$. Since l'_Y is the connected component of Y in $l_Y \cap U$, we have therefore $l'_Y \cap W' = l_Y \cap W' = \chi^{-1}(]x^0 - r, x^0 + r[\times \{\mathbf{y}\})$. The conclusion follows by Point (ii) of Theorem 5. \square

(iv) By combining the three former examples, we get that, if a manifold V is diffeomorphic to an *open subset* Γ of \mathbb{R}^{N+1} and $\phi : \Gamma \rightarrow V$ is any diffeomorphism, then for any constant vector field $\mathbf{v} \neq \mathbf{0}$ on Γ , its pushforward vector field by ϕ , $v = \phi^*\mathbf{v}$, is a normal vector field on V . In the application to physics (for which $N = 3$, as far as we know), this describes already a wide

variety of spacetimes and vector fields, together with the associated reference fluids. Those are deformable in a very general way with respect to each other, by changing ϕ , i.e. by transforming the integral curves by any diffeomorphism of V . Of course, we expect that much more general normal vector fields do exist, due to the discussion at the beginning of this subsection.

3 The set of orbits of v as a differentiable manifold

The set N_v of the maximal integral curves of v has been defined in Eq. (11). In this section, we will show that, when v is a normal vector field on the differentiable manifold V , the set N_v can be endowed with a canonical structure of differentiable manifold.

Proposition 6. *Let v be a normal vector field on V . Define the set \mathcal{F}_v made of all nice v -adapted charts on V . For any chart $\chi \in \mathcal{F}_v$, with domain $U \subset V$, let D_U be defined by (12), and, for any subset $\mathcal{O} \subset N_v$, define $\bar{\chi}(\mathcal{O}) \equiv \bar{\chi}(\mathcal{O} \cap D_U)$, where $\bar{\chi}$ is defined in Eq. (14) on $\text{Dom}(\bar{\chi}) \equiv D_U$. Let \mathcal{T}' be the set of the subsets $\mathcal{O} \subset N_v$ such that*

$$\forall \chi \in \mathcal{F}_v, \quad \bar{\chi}(\mathcal{O}) \text{ is an open set in } \mathbb{R}^N. \quad (70)$$

The set \mathcal{T}' is a topology on N_v .

Proof. This is an adaptation of the proof of Proposition C in Ref. [8], replacing M by N_v , F by \mathcal{F}_v , $\tilde{\chi}$ by $\bar{\chi}$, and \mathbb{R}^3 by \mathbb{R}^N . In particular, the proof that the whole set N_v (instead of M) belongs to \mathcal{T}' is exactly identical. Also, by definition of a nice v -adapted chart, the mapping $\bar{\chi} : D_U \rightarrow \mathbb{R}^N$ is injective. Therefore, if $\mathcal{O}_1 \in \mathcal{T}'$ and $\mathcal{O}_2 \in \mathcal{T}'$, we have

$$\begin{aligned} \bar{\chi}(\mathcal{O}_1 \cap \mathcal{O}_2) &\equiv \bar{\chi}((\mathcal{O}_1 \cap \mathcal{O}_2) \cap D_U) = \bar{\chi}((\mathcal{O}_1 \cap D_U) \cap (\mathcal{O}_2 \cap D_U)) \\ &= \bar{\chi}(\mathcal{O}_1 \cap D_U) \cap \bar{\chi}(\mathcal{O}_2 \cap D_U) \equiv \bar{\chi}(\mathcal{O}_1) \cap \bar{\chi}(\mathcal{O}_2), \end{aligned} \quad (71)$$

which is thus an open set of \mathbb{R}^N , for any $\chi \in \mathcal{F}_v$, so that $\mathcal{O}_1 \cap \mathcal{O}_2 \in \mathcal{T}'$. It is also trivial to check that the union of any family of subsets $\mathcal{O}_i \in \mathcal{T}'$ is still an element of \mathcal{T}' . \square

To prove that the mappings $\bar{\chi}$ are continuous for this topology, and to prove the compatibility of any two mappings $\bar{\chi}, \bar{\chi}'$ on N_v , associated with two nice v -adapted charts $\chi, \chi' \in \mathcal{F}_v$, the following difficulty arises: χ and χ' have in general different domains U and U' . We may have $U \cap U' = \emptyset$, although there is some $l \in N_v$ with

$$l \cap U \neq \emptyset, \quad l \cap U' \neq \emptyset. \quad (72)$$

I.e., it may happen that the domains of the charts χ and χ' do not overlap, and that the domains of the mappings $\bar{\chi}$ and $\bar{\chi}'$ do. To overcome this difficulty, we use the flow F of the vector field v to associate smoothly with any point Y in some neighborhood $W \subset U$ of a point X , a point $g(Y) \in U'$:

Lemma. *Let v be a \mathcal{C}^∞ vector field on V . Let $\chi, \chi' \in \mathcal{F}_v$, with domains U and U' , be such that $D_U \cap D_{U'} \neq \emptyset$. Let $l \in D_U \cap D_{U'}$ and $X \in l \cap U$ and set $\chi(X) = (t, \mathbf{x})$, so that $\bar{\chi}(l) = \mathbf{x}$. There is an open neighborhood Ω of \mathbf{x} in \mathbb{R}^N and a \mathcal{C}^∞ mapping g defined on an open neighborhood $W \subset U$ of X , such that for any $\mathbf{y} \in \Omega$ we have $(t, \mathbf{y}) \in \chi(W)$ (so that $\mathbf{y} \in \bar{\chi}(D_U)$), $\bar{\chi}^{-1}(\mathbf{y}) \in D_{U'}$, and*

$$\forall \mathbf{y} \in \Omega, \quad \bar{\chi}' \circ \bar{\chi}^{-1}(\mathbf{y}) = P_S(\chi'(g(\chi^{-1}(t, \mathbf{y}))).) \quad (73)$$

Proof. Since $l \in D_{U'}$, there is some point $X' \in l \cap U'$, and since $X \in l$ there is some $s \in I_X$ such that $X' = F(s, X)$. Thus, the domain \mathcal{D} of F being open in $\mathbb{R} \times V$ and F being continuous, there is an interval I centered at s and an open neighborhood $W \subset U$ of X in V , such that $I \times W \subset \mathcal{D}$ and $F(I \times W) \subset U'$. For $Y \in W$, set $g(Y) = F(s, Y)$. This defines a \mathcal{C}^∞ -mapping $g : W \rightarrow U'$. Because $\chi(W)$ is open in \mathbb{R}^{N+1} and $(t, \mathbf{x}) \in \chi(W)$, there is an interval J centered at t and an open neighborhood Ω of \mathbf{x} in \mathbb{R}^N , with $J \times \Omega \subset \chi(W)$. Hence, if $\mathbf{y} \in \Omega$, then indeed $(t, \mathbf{y}) \in \chi(W)$, thus $Y \equiv \chi^{-1}(t, \mathbf{y}) \in W \subset U$, which implies that $l_Y \in D_U$ and that

$$\mathbf{y} = P_S(\chi(Y)) = \bar{\chi}(l_Y), \quad (74)$$

so $\mathbf{y} \in \bar{\chi}(D_U)$. Moreover, we have $Y' \equiv g(Y) = F(s, Y) \in l_Y \cap U'$, so $l_Y = \bar{\chi}^{-1}(\mathbf{y}) \in D_{U'}$ and

$$\mathbf{y}' \equiv P_S(\chi'(Y')) = \bar{\chi}'(l_Y), \quad (75)$$

whence follows (73). \square

Theorem 6. *Let v be a normal vector field on V . Let \mathcal{A} be the set of all mappings $\bar{\chi}$, where $\chi \in \mathcal{F}_v$. This set \mathcal{A} is an atlas on the topological space (N_v, \mathcal{T}') .*

Proof. (i) Consider any $\chi' \in \mathcal{F}_v$, with domain U' . Let us prove that $\bar{\chi}'$, which is defined on $D_{U'}$, is continuous for the topology induced on $D_{U'}$ by the topology \mathcal{T}' on N_v . Thus, A being any open set in \mathbb{R}^N , we must show that the set $\mathcal{O}_1 \equiv \bar{\chi}'^{-1}(A)$ has the form $\mathcal{O}_1 = \mathcal{O} \cap D_{U'}$, where \mathcal{O} is such that we have (70). We shall actually show that $\mathcal{O}_1 \in \mathcal{T}'$, i.e., that we have (70) with $\mathcal{O} \equiv \mathcal{O}_1 \subset D_{U'}$. To prove this, we may assume that $\mathcal{O}_1 \neq \emptyset$. Moreover, let $\chi \in \mathcal{F}_v$, with domain U . We may also assume that $\bar{\chi}(\mathcal{O}_1) \neq \emptyset$, i.e. $\mathcal{O}_1 \cap D_U \neq \emptyset$, so let $\mathbf{x} \in \bar{\chi}(\mathcal{O}_1)$. We have to find a neighborhood B of \mathbf{x} in \mathbb{R}^N , such that $B \subset \bar{\chi}(\mathcal{O}_1)$, i.e., $B \subset \bar{\chi}(\bar{\chi}'^{-1}(A))$. Since $\mathbf{x} \in \bar{\chi}(\mathcal{O}_1) \equiv \bar{\chi}(\mathcal{O}_1 \cap D_U)$, we have that $l \equiv \bar{\chi}^{-1}(\mathbf{x}) \in (\bar{\chi}'^{-1}(A)) \cap D_U \subset D_{U'} \cap D_U$. In particular, there is some $X \in l \cap U$, with $\chi(X) = (t, \mathbf{x})$ for some $t \in \mathbb{R}$. Hence, we may apply the Lemma and get the corresponding open neighborhood Ω of \mathbf{x} . Thus (73) shows that the mapping $\mathbf{f} \equiv \bar{\chi}' \circ \bar{\chi}^{-1}$ is well defined and continuous over Ω . Hence, since $\mathbf{x} \in \Omega$ and $\mathbf{x}' \equiv \mathbf{f}(\mathbf{x}) = \bar{\chi}'(l) \in A$, which is open in \mathbb{R}^N , there is a neighborhood $B \subset \Omega$ of \mathbf{x} in \mathbb{R}^N , such that $\mathbf{f}(B) \subset A$. Set $\mathbf{f}' \equiv \bar{\chi} \circ \bar{\chi}'^{-1} = \mathbf{f}^{-1}$. For $\mathbf{y} \in B$, we have thus $\mathbf{y}' \equiv \mathbf{f}(\mathbf{y}) \in A$, that is, $\mathbf{y} = \mathbf{f}'(\mathbf{y}') \in \mathbf{f}'(A) = (\bar{\chi} \circ \bar{\chi}'^{-1})(A)$. Thus, for any open set A in \mathbb{R}^N , the set $\mathcal{O}_1 \equiv \bar{\chi}'^{-1}(A)$ belongs to \mathcal{T}' , as announced. This proves that $\bar{\chi}'$ is continuous for the topology induced on $D_{U'}$ by the topology \mathcal{T}' on N_v . Moreover, taking $A = \mathbb{R}^N$, we get that $D_{U'} = \bar{\chi}'^{-1}(\mathbb{R}^N)$ is open in N_v , $D_{U'} \in \mathcal{T}'$.

(ii) Given any $\chi \in \mathcal{F}_v$, with domain U , let us show that the mapping $\bar{\chi}^{-1} : \bar{\chi}(D_U) \rightarrow D_U \subset N_v$, is continuous. Since D_U is open as seen at the end of (i), we have to show that, for any $\mathcal{O} \in \mathcal{T}'$ such that $\mathcal{O} \subset D_U$, the set $\Omega \equiv (\bar{\chi}^{-1})^{-1}(\mathcal{O})$ is open in \mathbb{R}^N . But since $\mathcal{O} \subset D_U = \text{Dom}(\bar{\chi})$, and since $\bar{\chi}$ is injective, we have $\Omega = \bar{\chi}(\mathcal{O})$. The fact that this is open in \mathbb{R}^N follows from the very definition of the topology \mathcal{T}' in Eq. (70). With (i), this means that, for any $\chi \in \mathcal{F}_v$, the mapping $\bar{\chi} : D_U \rightarrow \bar{\chi}(D_U) \subset \mathbb{R}^N$ is bicontinuous, thus is indeed a *chart* on the topological space (N_v, \mathcal{T}') .

(iii) Let us show that the domains of definition of the charts $\bar{\chi}$, for $\chi \in \mathcal{F}_v$, cover the whole set N_v . Given $l \in N_v$, let $X \in l$. Since v is a normal vector field on V , we know from Theorem 5 that there is a nice v -adapted chart χ whose domain U is a neighborhood of X . Thus $X \in l \cap U$, so $l \in D_U$, Q.E.D.

(iv) Finally, given any two nice v -adapted charts $\chi, \chi' \in \mathcal{F}_v$, having domains U and U' respectively, let us show the compatibility of the two charts $\bar{\chi}, \bar{\chi}'$ on N_v , with domains D_U and $D_{U'}$. The relevant case is when $D_U \cap D_{U'} \neq \emptyset$, so that $\text{Dom}(\bar{\chi}' \circ \bar{\chi}^{-1}) = \bar{\chi}(D_U \cap D_{U'}) \neq \emptyset$. Thus we may apply the Lemma. Its Eq. (73) shows that, given any $\mathbf{x} \in \bar{\chi}(D_U \cap D_{U'})$, it has a neighborhood Ω in which the function $\bar{\chi}' \circ \bar{\chi}^{-1}$ is given as a composition of \mathcal{C}^∞ functions, hence $\bar{\chi}' \circ \bar{\chi}^{-1}$ is \mathcal{C}^∞ on its domain. \square

Thus, we have endowed the set N_v with first the topology \mathcal{T}' defined by (70), and then with a canonical atlas \mathcal{A} of compatible charts, which are simply the mappings $\bar{\chi}$, where $\chi \in \mathcal{F}_v$ is any nice v -adapted chart. To call this a differentiable manifold in the rather usual sense of Note 2 needs that the topological space (N_v, \mathcal{T}') be metrizable and separable — hence, in particular, that it be Hausdorff. We do not have very general results on the latter point.

Proposition 7. *Let v be a normal vector field on V . (i) Any two points $l \neq l'$ in the orbit space N_v are topologically distinguishable, i.e., there exists an open set $\mathcal{O} \in \mathcal{T}'$ such that $l \in \mathcal{O}$ and $l' \notin \mathcal{O}$.*

(ii) *Suppose $l \neq l'$ but l and l' both belong to the domain D_U of a chart $\bar{\chi}$, where $\chi \in \mathcal{F}_v$ with $\text{Dom } \chi = U$. Then l and l' are separated by neighborhoods, i.e., there are two open sets $\mathcal{O}, \mathcal{O}' \in \mathcal{T}'$, such that $l \in \mathcal{O}$, $l' \in \mathcal{O}'$, and $\mathcal{O} \cap \mathcal{O}' = \emptyset$.*

(iii) *Suppose there is a chart $\chi \in \mathcal{F}_v$, such that any maximal integral curve $l \in N_v$ intersects its domain U , so that $D_U = N_v$. Then the topological space (N_v, \mathcal{T}') is metrizable and separable. [Hence, in particular, it has the Hausdorff property, i.e., any two distinct elements $l, l' \in N_v$ are separated by neighborhoods — as follows also from Point (ii).]*

Proof. (i) Let $X \in l$. There is some open neighborhood U of X , such that $U \cap l' = \emptyset$: if that were not true, then, taking any distance on V that defines its topology, and considering $U_n \equiv B(X, 1/n)$, we would get a sequence (X_n) with $X_n \in l'$ and $X_n \rightarrow X$, hence $X \in l'$ since we defined that a normal vector field has all its maximal integral curves closed; but this implies that $l = l'$, which is a contradiction. By Theorem 5, let $\chi \in \mathcal{F}_v$ whose domain U_2 is an open neighborhood of X , and set $U_1 \equiv U \cap U_2$. The restriction χ_1 of χ to $U_1 \subset U$ is still a nice v -adapted chart: it is obviously v -adapted, we have

$D_{U_1} \subset D_U$, and the mapping $\bar{\chi}_1 : D_{U_1} \rightarrow \mathbb{R}^N$ is clearly the restriction of $\bar{\chi}$ to D_{U_1} , hence it also is injective. Therefore, $\mathcal{O} \equiv D_{U_1}$ is an open set, $\mathcal{O} \in \mathcal{T}'$. Since $X \in l \cap U_1$, we have $l \in \mathcal{O}$; and since $l' \cap U_1 \subset l' \cap U = \emptyset$, we have $l' \notin \mathcal{O}$.

(ii) Since $l \in D_U$ and $l' \in D_U$ with $l \neq l'$, and since $\bar{\chi}$ is defined on D_U and injective, we have

$$\mathbf{x} \equiv \bar{\chi}(l) \neq \mathbf{x}' \equiv \bar{\chi}(l'). \quad (76)$$

Let Ω and Ω' be open neighborhoods in \mathbb{R}^N of \mathbf{x} and \mathbf{x}' respectively, such that $\Omega \cap \Omega' = \emptyset$. Set $\mathcal{O} \equiv \bar{\chi}^{-1}(\Omega)$ and $\mathcal{O}' \equiv \bar{\chi}^{-1}(\Omega')$. These are open sets such that $l \in \mathcal{O}$ and $l' \in \mathcal{O}'$, and we have $\mathcal{O} \cap \mathcal{O}' = \bar{\chi}^{-1}(\Omega \cap \Omega') = \emptyset$.

(iii) This follows from the fact that $\bar{\chi}$ is a homeomorphism of its domain D_U onto its range $\bar{\chi}(D_U) \subset \mathbb{R}^N$. \square

Example. The assumption made for Point (iii) is fulfilled, in particular, in the following case, which occurs frequently in relativistic theories of gravitation. Assume a chart (χ, U) is defined on the whole of the manifold: $U = V$, which means that χ is a diffeomorphism of V onto the open subset $\Gamma \equiv \chi(V)$ of \mathbb{R}^{N+1} . Then, the tangent vector field v to the world lines $l_{\mathbf{a}}$ given by (3) for $N = 3$, with constant component vector $\mathbf{v}_0 = (1, 0, \dots, 0)$ in the chart χ , is the pushforward vector field of the constant vector field $\mathbf{v}(\mathbf{X}) = \mathbf{v}_0$ for $\mathbf{X} \in \Gamma$ by the diffeomorphism χ^{-1} . Hence, by No. (iv) in the Examples above, v is a normal vector field on V . Due to No. (ii) in these examples, the orbits of v are the images of the orbits of \mathbf{v} by χ^{-1} , hence [by No. (iii)] are the connected components of the lines $l_{\mathbf{a}}$ [$\mathbf{a} = (a^j) \in P_S(\Gamma)$]. Hence, the chart χ is v -adapted, for $P_S(\chi(X)) = \mathbf{a}$ if $X \in l_{\mathbf{a}}$. If actually all lines $l_{\mathbf{a}}$ defined in (3) are *connected* (which happens iff the domain of the time coordinate x^0 is an interval for any such line), then χ is nice ($\mathbf{a} = \mathbf{a}' \Rightarrow l_{\mathbf{a}} = l_{\mathbf{a}'}$). Thus, in that case, $\chi \in \mathcal{F}_v$. Since the domain of χ is $U = V$, we have then $D_U = N_v$, so N_v is metrizable and separable. This case includes of course standard situations, e.g. an inertial frame (e.g. with Cartesian coordinates) in Minkowski spacetime; a uniformly rotating frame (e.g. with “rotating Cartesian coordinates” [10]) in Minkowski spacetime [even though in that case the lines (3) are spacelike when $\rho \equiv \sqrt{x^2 + y^2} > c/\omega$]; harmonic coordinates in an asymptotically flat spacetime [18]; etc. It also includes known singular solutions of general relativity such as the singular Schwarzschild-Kruskal-Szekeres spacetime: the Kruskal-Szekeres coordinates (T, ξ, θ, ϕ) [19, 20] cover the whole

of the “maximally extended” Schwarzschild manifold. Since the domain of the coordinates T, ξ is: $\xi \in \mathbb{R}, T^2 - \xi^2 < 1$, i.e. $T \in] - \sqrt{1 + \xi^2}, +\sqrt{1 + \xi^2}[$, each line $l_{\mathbf{a}}$ [with $\mathbf{a} \equiv (\xi, \theta, \phi)$] is connected. Thus this global chart on the Schwarzschild spacetime does define a global space manifold. Moreover, the tangent vector field v to these lines (3) is time-like.

Proposition 8. *Assume that v is a normal vector field on V . (i) There is a countable cover of V by open sets U_n such that, for any integer n , there is a nice v -adapted chart, $\chi_n \in \mathcal{F}_v$, having domain U_n . (ii) Then, setting $D_n \equiv D_{U_n}$, the sequence (D_n) is a countable cover of N_v by metrizable open subsets. Hence the topological space (N_v, \mathcal{T}') is separable.*

Proof. (i) By Theorem 5, for any $X \in V$ there is a nice v -adapted chart $\chi_X \in \mathcal{F}_v$, such that its domain U_X is an open neighborhood of X . But, since V is metrizable and separable, there exists a countable basis $(V_n)_{n \in \mathbb{N}}$ for the open sets of V . Hence, for any $X \in V$, there is some integer $\tilde{n}(X)$ such that

$$X \in V_{\tilde{n}(X)} \subset U_X. \quad (77)$$

This defines a mapping $\tilde{n} : V \rightarrow \mathbb{N}$ and we have

$$V = \bigcup_{n \in \tilde{n}(V)} V_n. \quad (78)$$

We may define a mapping $\tilde{n}(V) \rightarrow V$, $n \mapsto X_n$, by choosing X_n , for any $n \in \tilde{n}(V)$, as being one of the points $X \in V$ such that $n = \tilde{n}(X)$. From (77), it follows then that, for any $n \in \tilde{n}(V)$, we have

$$V_n = V_{\tilde{n}(X_n)} \subset U_{X_n}. \quad (79)$$

For $n \in \tilde{n}(V)$, define $U_n \equiv U_{X_n}$ and $\chi_n \equiv \chi_{X_n} \in \mathcal{F}_v$. From (78) and (79), it results that the countable family $(U_n)_{n \in \tilde{n}(V)}$ is as in Statement (i).

(ii) Note first that, since $\chi_n \in \mathcal{F}_v$, with domain U_n , it follows from Theorem 6 that D_n , the domain of the associated chart $\bar{\chi}_n$ on N_v , is open in N_v . If $l \in N_v$, let $X \in l$ and, since (U_n) is a cover of V , let n be such that $X \in U_n$. We have thus $l \cap U_n \neq \emptyset$, i.e. $l \in D_n$. So (D_n) is a countable open cover of N_v . Since $\bar{\chi}_n$ is a homeomorphism of D_n onto $\bar{\chi}_n(D_n) \subset \mathbb{R}^N$, it follows that D_n is a metrizable and separable space. Hence it is second-countable, i.e., there exists a countable basis $(\mathcal{O}_{nm})_{m \in \mathbb{N}}$ for the open subsets

of D_n . Since any open subset \mathcal{O} of N_v is the countable union of the open subsets $\mathcal{O}_n \equiv \mathcal{O} \cap D_n$ of D_n , we have that $(\mathcal{O}_{nm})_{n,m \in \mathbb{N}}$ is a countable basis for the topology \mathcal{T}' of N_v . Thus also N_v is second-countable, hence it is separable. \square

4 The local manifold as an open subset of the global one

Let v be a normal vector field on V . In addition, as in Subsect. 1.2, let F be a (local) reference frame, thus an equivalence class of charts for the relation (5), in which U is a given open subset of V .⁴ In Subsect. 1.2, the local space manifold M_F associated with F was defined as the set of the world lines (6). On the other hand, the orbit set N_v defined in Subsect. 2.1: the set of the maximal integral curves of v , was endowed in Sect. 3 with a topology \mathcal{T}' and an atlas \mathcal{A} , which (assuming that \mathcal{T}' is metrizable and separable) makes it a differentiable manifold. Thus, we have also a global space manifold: N_v . When the charts $\chi \in F$, all having domain U , are nice v -adapted charts, i.e. belong to \mathcal{F}_v , we have the following tight relation between M_F and N_v :

Theorem 7. *Assume that $F \subset \mathcal{F}_v$. For any $l \in M_F$, there is a unique maximal integral curve $l' \in N_v$ such that, for any $X \in l$, we have $l' = l_X$. It holds $l = l' \cap U$. The mapping $I : l \mapsto l'$ is a diffeomorphism of M_F onto the open subset D_U of N_v .*

Proof. Let $l \in M_F$. By the definition of M_F near Eq. (6), there is some chart $\chi \in F$ and some $\mathbf{x} \in P_S(\chi(U)) \subset \mathbb{R}^N$, such that

$$l = \{ X \in U; P_S(\chi(X)) = \mathbf{x} \}. \quad (80)$$

Let $X_1 \in l$ and $X_2 \in l$. Denote the maximal integral curves of v at X_1 and X_2 as $l'_1 \equiv l_{X_1}$, $l'_2 \equiv l_{X_2}$. Since χ is v -adapted, there exist $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^N$, such that $\forall X \in l'_1 \cap U$, $P_S(\chi(X)) = \mathbf{x}_1$; and $\forall X \in l'_2 \cap U$, $P_S(\chi(X)) = \mathbf{x}_2$. In particular, since $X_j \in l'_j \cap U$, we have $P_S(\chi(X_j)) = \mathbf{x}_j$ ($j = 1, 2$). But since

⁴ As in Sects. 2 and 3, the dimension of V is $N + 1$, where N is any integer ≥ 1 . All results summarized in Subsect. 1.2 hold true if one substitutes any integer $N \geq 1$ for the integer 3, and $N + 1$ for 4 [8].

$X_j \in l$, we have also $P_S(\chi(X_j)) = \mathbf{x}$ by (80), so $\mathbf{x} = \mathbf{x}_1 = \mathbf{x}_2$. Because the v -adapted chart χ is nice, it follows that $l'_1 = l'_2$. Therefore, we define a mapping $I : M_F \rightarrow N_v$ by associating with any $l \in M_F$, the unique maximal integral curve $l' \in N_v$, such that for any $X \in l$, we have $l' = l_X$. Note that actually $l' \in D_U$. Owing to the definitions (7) and (14), we have $\mathbf{x} = \tilde{\chi}(l) = \bar{\chi}(l')$, and since here l is any element of M_F , this shows that $\tilde{\chi}(M_F) \subset \bar{\chi}(D_U)$. Thus

$$I(l) = \bar{\chi}^{-1}(\tilde{\chi}(l)), \quad (81)$$

so that we have simply $I = \bar{\chi}^{-1} \circ \tilde{\chi}$, for whatever chart $\chi \in F$. Let us show that $l = l' \cap U$. By definition, for any $X \in l$, we have $l' = l_X$, hence $X \in l'$, and since $l \subset U$ by the definition (80), we have $l \subset l' \cap U$. Conversely, consider any $\chi \in F$; this is by assumption a v -adapted chart and, as we showed before (81), we have $\bar{\chi}(l') = \tilde{\chi}(l) \equiv \mathbf{x}$. Therefore, by the definition (13), we have for any $X \in l' \cap U$: $P_S(\chi(X)) = \mathbf{x}$. Then (80) implies that $X \in l$, so $l' \cap U \subset l$.

As we showed, the mapping I is defined on the whole of M_F and ranges into D_U , which is an open subset of N_v . Let us show that in fact $I(M_F) = D_U$. Let $l' \in D_U$, so there exists $X \in l' \cap U$. Let $\chi \in F$ and set $\mathbf{x} \equiv P_S(\chi(X))$ and $l \equiv \{Y \in U; P_S(\chi(Y)) = \mathbf{x}\}$. Clearly $l \in M_F$ and $X \in l$. By the definition of I , we have that, for any $Y \in l$, $l_Y = I(l)$. In particular, $l_X = I(l)$. But since $X \in l'$, we have $l_X = l'$, hence $l' = I(l)$: thus indeed $D_U \subset I(M_F)$. Note that again here, from the definitions (7) and (14) we have $\mathbf{x} = \tilde{\chi}(l) = \bar{\chi}(l')$, and since now l' is any element of D_U , this shows that $\bar{\chi}(D_U) \subset \tilde{\chi}(M_F)$. Since the reverse inclusion has been proved before (81), we have $\bar{\chi}(D_U) = \tilde{\chi}(M_F)$.

As shown in Ref. [8], $\tilde{\chi}$ is a global chart on the differentiable manifold M_F , for any $\chi \in F$. As shown in Theorem 6, $\bar{\chi}$ is a chart with domain D_U on the differentiable manifold N_v , also for any $\chi \in F$. Moreover, as we just saw, we have $\bar{\chi}(D_U) = \tilde{\chi}(M_F)$. It follows that the one-to-one mapping $I = \bar{\chi}^{-1} \circ \tilde{\chi}$, from $\text{Dom}(\tilde{\chi}) = M_F$ onto $I(M_F) = D_U = \text{Dom}(\bar{\chi})$, is a diffeomorphism. Therefore, I is an immersion of M_F into N_v . Actually, recall that D_U is more specifically an *open subset* of N_v . \square

Thus, if $F \subset \mathcal{F}_v$, we may identify the local space M_F with the open subset $I(M_F) = D_U$ of the global space N_v . Since we proved that $l = I(l) \cap U$, we can say that M_F is made of the intersections with the local domain U of the

maximal integral curves of v . Given that each world line $l \in M_F$ is invariant under any exchange of the chart $\chi \in F$ for another chart $\chi' \in F$, to say that $F \subset \mathcal{F}_v$ is equivalent to say that *one* chart $\chi \in F$ is a nice v -adapted chart.

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