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Defining the space(s) in a general spacetime

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Abstract

A global vector field v on a “spacetime” differentiable manifold V , of dimension $N + 1$, defines a congruence of world lines: the maximal integral curves of v , or orbits. The associated global space N_v is the set of these orbits. A “ v -adapted” chart on V is one for which the \mathbb{R}^N vector $\mathbf{x} \equiv (x^j)$ ($j = 1, \dots, N$) of the “spatial” coordinates remains constant on any orbit l . We consider non-vanishing vector fields v that have non-periodic orbits, each of which is a closed set. We show by transversality arguments that, among those vector fields, those for which there exists in the neighborhood of any point $X \in V$ a chart that is v -adapted and “nice”, i.e., such that the mapping $\bar{\chi} : l \mapsto \mathbf{x}$ is injective, have a generic character. For any “normal” vector field, the mappings $\bar{\chi}$ build an atlas of charts, thus providing N_v with a canonical structure of differentiable manifold. Previously, a local space manifold M_F had been associated with any “reference frame” F , defined as an equivalence class of charts. We show that, if F is made of nice v -adapted charts, M_F is naturally identified with an open subset of the global space manifold N_v .

Keywords: Physical space; global vector field; orbit space; adapted chart; differentiable manifold.

MSC codes: 37C20 (Generic properties, structural stability);
57R99 (Differential topology);
58Z05 (Global analysis, analysis on manifolds: Applications to physics).

1 Introduction

1.1 Physical motivation

The theory of relativity says that space and time merge into “a kind of union of the two” (in Minkowski’s words): the spacetime. However, the notion of a *physical space* should be useful also in relativistic physics. In our opinion it is even needed, for the following two reasons. (i) In the experimental/observational literature, one of course needs to define the spatial position of the experimental apparatus and/or of the observed system, and this is true also for the part of that literature for which relativistic effects have to be considered. (ii) In quantum mechanics, the “state” of a quantum-mechanical particle is a function ψ of the position x belonging to some 3-D “physical space” M , $\psi : M \rightarrow \mathbb{C}$. Note that defining such a space as a spacelike 3-D *submanifold of the spacetime manifold* (e.g. [1]) can work to define an initial condition for a field in spacetime, but does not allow one to define a spatial position in the way that is needed in the two foregoing examples: in those, one needs to identify spatial points that exist at least for some open interval of time — e.g. to state that some objects maintain a fixed spatial position in some reference frame, or to define the stationary states of a quantum particle. In practice, the spatial position is taken to be the triplet of the spatial coordinates. However, *a priori*, such a triplet does not have a precise geometric meaning in a theory starting from a spacetime structure. Only a notion of “spatial tensors” has been defined for a general spacetime of relativistic gravity. This definition was based on the concept of “reference fluid” [2, 3, 4], also named “reference body” [5] — i.e., a three-dimensional congruence of world lines, whose the tangent vector is assumed to be a time-like vector field. The latter can be normed to become a four-velocity field v , i.e. $\mathbf{g}(v, v) = 1$ where \mathbf{g} is the spacetime metric. The data of the four-velocity field v allows one to define the spatial projection operator Π_X (depending on the point X in the spacetime manifold V) [2, 3, 4, 5, 6, 7]. A “spatial vector” is then defined as a *spacetime vector* which is equal to its spatial projection. A full algebra of “spatial tensors” can be defined in the same way, and also, once a relevant connection has been defined, a spatial tensor analysis [2, 3, 4, 5, 6].

However, it is possible in a general spacetime manifold V to define a relevant physical space as a *3-D differentiable manifold*, at least *locally* in V .

To see this, consider a coordinate system or chart:

$$\chi : U \rightarrow \mathbb{R}^4, \quad X \mapsto \chi(X) = \mathbf{X} \equiv (x^\mu) \quad (\mu = 0, \dots, 3), \quad (1)$$

where U is an open subset of V : the domain of the chart. Then one may define a set of world lines, each of which, l , has constant spatial coordinates a^j in the chart χ :¹

$$l = \{X \in U; \chi(X) = (x^\mu) \text{ is such that } x^j = a^j \text{ for } j = 1, 2, 3\}. \quad (3)$$

Let us suppose for a moment that the chart χ is in fact a Cartesian coordinate system on the Minkowski spacetime. Then that chart defines an inertial reference frame. In that case, it is clear that, for any event X , with $\chi(X) = (x^\mu)$, the triplet $\mathbf{x} \equiv (x^j)$ ($j = 1, 2, 3$) defines the spatial position associated in that chart with the event X . Note that the data of \mathbf{x} is equivalent to specifying a unique world line in the “congruence (3)”. [By this we mean the set of the world lines (3), when $\mathbf{a} \equiv (a^j)$ takes any value in \mathbb{R}^3 such that the corresponding world line (3) is not empty.] That world line is thus uniquely determined by the event X and may be noted $l(X)$. Events X' that have different values of the time coordinate x^0 , but that have the same values of the spatial coordinates x^j ($j = 1, 2, 3$), can be said to occur at the same spatial position in the inertial frame as does X . Thus, the whole of $l(X)$ is needed. However, each world line in the congruence (3) stays invariant if we change the coordinate system by a purely spatial coordinate change:

$$x'^0 = x^0, \quad x'^k = \phi^k((x^j)). \quad (4)$$

It is clear that this transformation leaves us in the same inertial reference frame. With the new chart χ' , the new triplet $\mathbf{x}' \equiv (x'^k) \equiv \phi(\mathbf{x})$ corresponds to the same spatial position in the inertial frame as does \mathbf{x} with the first chart χ . And indeed, that world line of the congruence which is defined in the chart χ' by the data of \mathbf{x}' is just the same as that world line of the congruence which is defined in the chart χ by the data of \mathbf{x} . The spatial position

¹ Note that, if we assume that V is endowed with a Lorentzian metric \mathbf{g} whose component g_{00} in the chart χ verifies $g_{00} > 0$ in U , then each among the world lines l is time-like, because in the chart χ the tangent vector to l has components $\propto (1, 0, 0, 0)$, which may be normed to

$$v^0 \equiv \frac{1}{\sqrt{g_{00}}}, \quad v^j = 0. \quad (2)$$

of the event X in the inertial frame is therefore most precisely defined by *the world line $l(X)$ of the congruence which passes at X .*

Now note that very little in the foregoing paragraph actually depends on whether or not the chart χ is a Cartesian coordinate system on the Minkowski spacetime: only the qualification of the reference frame as being inertial depends on that. It is just that we are accustomed to consider a spatial position in an inertial frame in a flat spacetime, and special relativity makes it natural to accept that it is actually the world line $l(X)$ which best represents that spatial position. Hence, consider a general spacetime, and define a congruence of world lines from the data of a coordinate system as in Eq. (3). In the domain of the chart, we may then define the spatial position of an event X as the unique world line $l(X)$ of the congruence (3) which passes at X — i.e., $l(X)$ is the unique world line of the congruence (3), such that $X \in l(X)$. Thus, *the data of a coordinate system on the spacetime defines a three-dimensional space M , of which the points (the elements of M) are the world lines of the congruence (3) associated with that coordinate system.*

1.2 Local reference frame and local space manifold

The foregoing approach can be used to define precise notions of a reference frame and its unique associated space manifold [8]. First, the invariance of the congruence (3) under the purely spatial coordinate changes (4) allows one to define a *reference frame* as being an equivalence class of charts related by a change (4). More exactly, the following is an equivalence relation between charts which are defined on a given open subspace U of the spacetime manifold V :

$$\chi \mathcal{R}_U \chi' \iff [\forall \mathbf{X} \in \chi(U), \quad \phi^0(\mathbf{X}) = x^0 \text{ and } \frac{\partial \phi^k}{\partial x^0}(\mathbf{X}) = 0 \ (k = 1, \dots, 3)], \quad (5)$$

where $f \equiv \chi' \circ \chi^{-1} \equiv (\phi^\mu)$ is the transition map, which is defined on $\chi(U)$. Thus a reference frame F is a set of charts defined on the same open domain U and exchanging by a purely spatial coordinate change (4). Then the *space manifold* M or M_F associated with the reference frame F is defined as the set of the world lines (3). In detail: let $P_S : \mathbb{R}^4 \rightarrow \mathbb{R}^3$, $\mathbf{X} \equiv (x^\mu) \mapsto \mathbf{x} \equiv (x^j)$, be the spatial projection. A world line l is an element of M_F iff there is a chart $\chi \in F$ and a triplet $\mathbf{x} \in P_S(\chi(U))$, such that l is the set of *all* points X in

the domain U , whose spatial coordinates are \mathbf{x} :

$$l \equiv \{ X \in U; P_S(\chi(X)) = \mathbf{x} \}. \quad (6)$$

(Thus, l is not necessarily a connected set.) It results easily from (4) that (6) holds true then in any chart $\chi' \in F$, of course with the transformed spatial projection triplet $\mathbf{x}' = \phi(\mathbf{x}) \equiv (\phi^j(\mathbf{x}))$ [8]. For a chart $\chi \in F$, one defines the “associated chart” as the mapping which associates, with a world line $l \in M$, the constant triplet of the spatial coordinates of the points $X \in l$:

$$\tilde{\chi} : M \rightarrow \mathbb{R}^3, \quad l \mapsto \mathbf{x} \text{ such that } \forall X \in l, P_S(\chi(X)) = \mathbf{x}. \quad (7)$$

One shows then that the set \mathcal{T} of the subsets $\Omega \subset M$ such that,

$$\forall \chi \in F, \quad \tilde{\chi}(\Omega) \text{ is an open set in } \mathbb{R}^3 \quad (8)$$

is a topology on M . Finally one shows that the set of the associated charts: $\tilde{F} \equiv \{\tilde{\chi}; \chi \in F\}$, is an atlas on the topological space (M, \mathcal{T}) , hence defines a structure of *differentiable manifold* on M [8]. Thus the space M is browsed by precisely the triplet $\mathbf{x} \equiv (x^j)$ made with the spatial projection of the spacetime coordinates $\mathbf{X} \equiv (x^\mu) \equiv \chi(X)$, see Eq. (7).

Using these results, one may define the space of quantum-mechanical states, for a given reference frame F in a given spacetime (V, \mathbf{g}) , as being the set \mathcal{H} of the square-integrable functions defined on the corresponding space manifold M [9]. One may also define the full algebra of spatial tensors: the pointwise algebra is defined simply as the tensor algebra of the tangent vector space TM_x to the space manifold M at some arbitrary point $x \in M$ [10].

1.3 Goal and summary

Thus, by defining a reference frame as a set F of charts that all have the same open domain U and that exchange by a purely spatial coordinate change (4), one can then define the associated space manifold M_F as the set of the world lines (6) [8]. These definitions are relevant to physical applications [9, 10]. However, they apply to a *parametrizable* domain U of the spacetime manifold V , i.e., to an open set U , such that at least one regular chart can be defined over the whole of U . Since the manifold V itself as a whole is in general not

parametrizable, no global reference frame exists in general, and in the same way the space manifolds do not look “maximal”. *The aim of the present work is to define global notions and to establish their link with the formerly defined local notions.* In that work, as in the former one [8], we do not need to consider a true “spacetime” in the precise sense, but merely a differentiable manifold V having dimension $N + 1$, thus N is the dimension of the “space” manifold to be defined. Our starting concept is simply that of a global vector field v on V . We define the corresponding “global space” as the set N_v of the maximal integral curves (or “orbits”) of v . To reach our goal, we take the following steps:

(a) Section 2 studies when a given vector field v on a differentiable manifold V is such that there locally exists charts of V which are *adapted* [2] to the congruence associated with v ; i.e., charts in which the “spatial” position $\mathbf{x} \equiv (x^j)$ ($j = 1, \dots, N$) is constant on any orbit l of v ; see Definition 1. We need also that the mapping $l \mapsto \mathbf{x}$ be injective. The desired situation is defined by Proposition 2. According to a transversality argument, this situation should be attainable, in general, if v does not vanish and each of its orbits is non-periodic and is closed in V . In Subsect. 2.4, two theorems of transversality and another theorem pertaining to differential topology allow us to formalize that argument in Theorem 4. This justifies us in introducing a notion of “normal” vector field by Definition 2, which ensures the local existence of v -adapted charts through Theorem 5.

(b) For any v -adapted chart χ , one may define the mapping $\bar{\chi}$ which associates with any orbit l the constant spatial position \mathbf{x} . We show in Section 3 that, using the set \mathcal{A} of the injective mappings $\bar{\chi}$, one can endow the global orbit set N_v with a topology \mathcal{T}' for which this set is an atlas of charts. See Theorem 6. Thus, when \mathcal{T}' is metrizable and separable, we do have a canonical structure of differentiable manifold on the orbit set N_v , for which the mappings $\bar{\chi}$ defined by Eq. (14) are local charts on the “space” manifold N_v . I.e., if $N = 3$, also the global space N_v is browsed (locally) by precisely the triplet $\mathbf{x} \equiv (x^j)$ made with the spatial projection of the spacetime coordinates $\mathbf{X} \equiv (x^\mu) \equiv \chi(X)$ — provided the spacetime chart χ is v -adapted.

(c) In Section 4 we establish the link with the previously defined space manifold M_F , associated with a given local reference frame F — defined as an equivalence class of charts for the relation (5). We show in Theorem 7 that,

when the charts belonging to F are v -adapted and with the mapping $l \mapsto \mathbf{x}$ being injective, then M_F is naturally identified with an open subset of N_v .

The definitions of a local reference frame F and the corresponding “space” manifold M_F do not need any metrical structure on the “spacetime” manifold V [8]. Just the same can be said for the definition of the global “space” manifold N_v , beyond the very fact that V should have a metrizable topology.

2 Local existence of adapted charts

2.1 Definitions

Let V be an $(N + 1)$ -dimensional real differentiable manifold, with $N \geq 1$.² We consider a given global, smooth vector field v on V . We shall not actually need that $N = 3$, nor that V be endowed with a Lorentzian metric \mathbf{g} for which v be a time-like vector field. However, we will often need that v be *non-vanishing*. Note that a time-like vector field on a Lorentzian manifold (V, \mathbf{g}) is non-vanishing; and that, if a Lorentzian manifold (V, \mathbf{g}) is time-oriented, which indeed is usually required for a spacetime, then by definition there exists at least one global time-like vector field on V . A continuously differentiable (\mathcal{C}^1) mapping C from an open interval I of \mathbb{R} into V defines an integral curve of v iff $\frac{dC}{ds} = v(C(s))$ for $s \in I$. For any $X \in V$, let C_X be the solution of

$$\frac{dC}{ds} = v(C(s)), \quad C(0) = X \quad (9)$$

for which the open interval I is *maximal*, and denote this maximal interval by I_X [12]. That is, I_X is an open interval defined as the union of all open intervals I , each containing 0, in which a solution of (9) is defined. The solution C_X is defined on I_X and is unique [12]. Let $s \in I_X$ and set $Y = C_X(s)$. These definitions imply easily [12] that

$$I_Y = I_X - s \quad \text{and} \quad \forall t \in I_Y, \quad C_Y(t) = C_X(s + t). \quad (10)$$

For any $X \in V$, call the *range* $l_X \equiv C_X(I_X) \subset V$ the “maximal integral curve at X ”. From (10), “ l_X does not depend on the point $X \in l_X$ ”: if $Y \in l_X$,

² We understand “differentiable manifold” as a topological space endowed with an atlas of compatible charts, hence with the corresponding equivalence class of compatible atlases — with the restriction that that space should be metrizable and separable [11].

then $l_Y = l_X$. We define the set of the maximal integral curves (or orbits) of v :

$$N_v \equiv \{l_X; X \in V\}. \quad (11)$$

Once endowed with further structure, N_v will be the global space manifold associated with the global vector field v (when the latter is non-vanishing and obeys another assumption). Note that, if the set $U \subset V$ is not empty, then the following subset of N_v :

$$D_U \equiv \{l \in N_v; l \cap U \neq \emptyset\} \quad (12)$$

is not empty. Indeed, for any $X \in U$, the world line $l \equiv l_X$ belongs to D_U . Let $P_S : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N$, $(x^\mu) \mapsto (x^j)$ ($\mu = 0, \dots, N$; $j = 1, \dots, N$) be the “spatial” projection.

Definition 1. *A mapping $\chi : U \rightarrow \mathbb{R}^{N+1}$ with $U \subset V$ is said “ v -adapted” iff for any $l \in D_U$, there exists $\mathbf{x} \in \mathbb{R}^N$ such that*

$$\forall Y \in l \cap U, \quad P_S(\chi(Y)) = \mathbf{x}. \quad (13)$$

If Eq. (13) is verified by some world line $l \in N_v$, then necessarily $l \in D_U$, and \mathbf{x} is obviously unique. Thus, for any v -adapted mapping χ , the mapping

$$\bar{\chi} : D_U \rightarrow \mathbb{R}^N, \quad l \mapsto \mathbf{x} \text{ such that (13) is verified} \quad (14)$$

is well defined. In Section 3, we will endow the set N_v with first a topology and then a structure of differentiable manifold, for which the charts (of N_v) will be mappings $\bar{\chi}$, where χ is a v -adapted *chart* of V . Since any chart is in particular a one-to-one mapping, we need to restrict ourselves to v -adapted charts χ such that the associated mapping $\bar{\chi}$ is injective. Thus, we define that a v -adapted chart χ is “*nice*” iff $\bar{\chi}$ is injective on D_U , with $U \subset V$ the (open) domain of χ .

2.2 Straightening-out vs v -adapted charts

In the remainder of this section, we investigate whether there exist nice v -adapted charts in the neighborhood of any point $X_0 \in V$. If the vector field v does not vanish, a well-known theorem applies at any point $X_0 \in V$:

Straightening-out theorem (e.g. [13]). *Let v be a vector field of class C^∞ defined on V . Suppose that at $X_0 \in V$ we have $v(X_0) \neq 0$. There is a “straightening-out chart” χ defined on an open neighborhood U of X_0 , i.e. χ is such that:*

- (i) $\chi(U) = I \times \Omega$, $I =] - a, +a[$, $a \neq 0$, Ω open set in \mathbb{R}^N .
- (ii) For any $\mathbf{x} \in \Omega$, $\chi^{-1}(I \times \{\mathbf{x}\})$ is an integral curve of v .
- (iii) In U , we have $v = \partial_0$, where (∂_μ) is the natural basis associated with the chart χ .

However, the direct link with the notion of a v -adapted chart is tenuous. In one direction, we have:

Proposition 0. *If (χ, U) is a v -adapted chart, then $v|_U = f\partial_0$ with $f : U \rightarrow \mathbb{R}$ a smooth function.*

Proof. To say that (χ, U) is v -adapted means, according to Definition 1, that for any given $X \in U$, we have for any $Y \in l_X \cap U$: $P_S(\chi(Y)) = P_S(\chi(X))$. In the coordinates $\chi(X) = (x^\mu) \equiv \mathbf{X}$, $\chi(Y) = (y^\mu) \equiv \mathbf{Y}$ ($\mu = 0, \dots, N$), the latter rewrites as

$$y^j = x^j \quad (j = 1, \dots, N). \quad (15)$$

On the other hand, remembering the definition of l_X as a (maximal) integral curve of v , Eq. (9) and below, let J be the connected component of 0 in $I'_{XU} \equiv \{s \in I_X; C_X(s) \in U\}$: J is an open interval containing 0 and we have $l' \equiv C_X(J) \subset l_X \cap U$. Let us denote this as $l' = \{Y(s); s \in J\}$, with

$$\frac{dy^\mu}{ds} = v^\mu(\mathbf{Y}(s)) \quad (\mu = 0, \dots, N), \quad \mathbf{Y}(0) = \mathbf{X}, \quad (16)$$

$v^\mu = v^\mu(\mathbf{Y})$ being the components of v in the chart χ . It follows from (15) and (16) that for $s \in J$ we have $v^j(\mathbf{Y}(s)) = 0$ ($j = 1, \dots, N$), i.e. $v(Y(s)) = v^0(\mathbf{Y}(s))\partial_0(Y(s))$, thus in particular $v(X) = v^0(\mathbf{X})\partial_0(X) \equiv f(X)\partial_0(X)$. Since this is true at any point $X \in U$, our statement is proved. \square

Note that the function f is as smooth as is v , but apart from this it is a priori quite general, so a straightening-out chart cannot be directly deduced

from the data of a v -adapted chart. Conversely, if (χ, U) is a straightening-out chart, i.e. it fulfills conditions (i) to (iii) of the theorem above, then let $\mathbf{x} \in \Omega$ and set $l' \equiv \chi^{-1}(I \times \{\mathbf{x}\})$. From (ii), l' is an integral curve of v and, since $l' \subset U$ by construction, we have (13):

$$\forall Y \in l' \cap U, \quad P_S(\chi(Y)) = \mathbf{x}. \quad (17)$$

Hence, at first sight, it might seem that χ is a v -adapted chart. However, let $X = \chi^{-1}(s, \mathbf{x}) \in l'$ and let $l \equiv l_X \in \mathcal{N}_v$ be the *maximal* integral curve of v passing at X . We have $l' \subset (l \cap U)$ since l' is an integral curve of v that is included in U and that passes at X . But nothing guarantees that $l' \supset (l \cap U)$: the intersection $l \cap U$ may contain other arcs, say $l'_1 \equiv \chi^{-1}(I \times \{\mathbf{x}_1\})$ with $\mathbf{x}_1 \in \Omega$ and $\mathbf{x}_1 \neq \mathbf{x}$. In such a case, the straightening-out chart χ is not v -adapted, since for $Y \in l \cap U$, $P_S(\chi(Y))$ may take different values $\mathbf{x}, \mathbf{x}_1, \dots$

Property (ii) in the straightening-out theorem is a straightforward consequence of Properties (i) and (iii). More generally, if a chart (χ, U) satisfies Property (iii), i.e. $v = \partial_0$ in U , and if $\chi(U)$ contains a set $I \times \{\mathbf{x}\}$, with $\mathbf{x} \in \mathbb{R}^N$ and I an open interval, then Property (ii) applies for this $\mathbf{x} \in \mathbb{R}^N$. Later on, we will need that the boundary of the open set U be a smooth hypersurface, hence we must consider charts for which (iii) is true, but not (i).

Proposition 1. *Let (χ, U) be a chart such that $v = \partial_0$ in U . Assume there is an open subset $\Omega \subset \mathbb{R}^N$ such that*

$$\chi(U) = \bigcup_{\mathbf{x} \in \Omega} I_{\mathbf{x}} \times \{\mathbf{x}\}, \quad (18)$$

where $I_{\mathbf{x}}$ ($\mathbf{x} \in \Omega$) are open intervals.

(i) *In order that the chart (χ, U) be v -adapted, it is necessary and sufficient that*

$$\forall X \in U, \quad \chi(l_X \cap U) \text{ has the form } I_{\mathbf{x}} \times \{\mathbf{x}\} \text{ for some } \mathbf{x} \in \Omega. \quad (19)$$

(ii) *Moreover, if that is the case, then the v -adapted chart (χ, U) is nice.*

Proof. (i) If Condition (19) is satisfied, let $l \in \mathcal{D}_U$. Thus, $l \cap U \neq \emptyset$, so let $X \in l \cap U$, hence $l = l_X$. Let $\mathbf{x} \in \Omega$ be given by (19) for precisely the maximal integral curve $l = l_X$. For any $Y \in l \cap U$, we have thus $\chi(Y) = (s, \mathbf{x})$ for

some $s \in I_{\mathbf{x}}$. Hence, we have (13). Therefore, according to Definition 1, χ is v -adapted. Conversely, assume that (χ, U) is v -adapted. Let $X \in U$ and set $l \equiv l_X$. Further, let \mathbf{x} be given by (13). That is, any point $\mathbf{Y} \in \chi(l \cap U)$ has the form (s, \mathbf{x}) for some $s \in \mathbb{R}$. Since moreover $\chi(U)$ has the form (18), we have also $\mathbf{Y} = (t, \mathbf{x}')$ for some $\mathbf{x}' \in \Omega$ and some $t \in I_{\mathbf{x}'}$, so $\mathbf{x} = \mathbf{x}'$ and $s = t \in I_{\mathbf{x}}$. Hence $\chi(l \cap U) \subset I_{\mathbf{x}} \times \{\mathbf{x}\}$. But also $I_{\mathbf{x}} \times \{\mathbf{x}\} \subset \chi(l \cap U)$. Indeed, setting $l' \equiv \chi^{-1}(I_{\mathbf{x}} \times \{\mathbf{x}\})$, we have $l' \subset (l \cap U)$ because, as noted before the statement of this Proposition 1, l' is an integral curve of v that is included in U and that passes at X .

(ii) Assuming that the chart (χ, U) obeys Condition (19) [and hence, by (i), is a v -adapted chart], let us show that the mapping $\bar{\chi}$ defined by (14) is injective. Thus, let $l, l' \in N_v$, assume that both intersect U , and let \mathbf{x} and \mathbf{x}' be the images of l and l' by $\bar{\chi}$. This means, according to the definition (14), that we have (13), and similarly

$$\forall Y \in l' \cap U, \quad P_S(\chi(Y)) = \mathbf{x}'. \quad (20)$$

From Condition (19), we get thus:

$$\chi(l \cap U) = I_{\mathbf{x}} \times \{\mathbf{x}\} \quad \text{and} \quad \chi(l' \cap U) = I_{\mathbf{x}'} \times \{\mathbf{x}'\}. \quad (21)$$

Therefore, if $\mathbf{x} = \mathbf{x}'$, it is clear that $l = l'$. The proof is complete. \square

The condition that the open set $A \equiv \chi(U) \subset \mathbb{R}^{N+1}$ have the form (18) is fulfilled, in particular, if A is convex. So it is fulfilled if one restricts the chart χ to $\chi^{-1}(A)$ with A a convex open subset of $\chi(U_0)$. Unfortunately, what we have in a rather general situation is the following result, which does not ensure the applicability of Proposition 1:

Theorem 0 [14]. *Suppose that some maximal integral curve l of the C^∞ vector field v is closed in V and is not reduced to a point. Since v then does not vanish on l , let $\chi : U \rightarrow I \times \Omega$ be a straightening-out chart in an open neighborhood U of some point of l . Then we have $\chi(l \cap U) = I \times E$, where E is a closed countable subset of Ω , and any point $X \in l \cap U$ is “transversally isolated”, i.e. $\mathbf{x} \equiv P_S(\chi(X))$ is isolated in E .*

(Note that the set E is closed in Ω , which is an open set in \mathbb{R}^N . So E is not necessarily closed in \mathbb{R}^N .) Assume that some straightening-out chart (χ, U)

is such that, for any $X \in U$, the maximal integral curve l_X is closed in V . From Point (i) in Proposition 1, we get that this is a v -adapted chart iff, in addition, for any point $X \in U$, the countable closed subset E_X of Ω , whose existence is ensured by Theorem 0 for the curve l_X , is actually reduced to a point. We can further characterize this desired situation:

Proposition 2. *Let $\chi : U_0 \rightarrow I \times \Omega_0$ be a straightening-out chart for the C^∞ vector field v . Let $U \subset U_0$ be an open set such that $\chi(U)$ has the form (18). Assume that, for any $X \in U$, the maximal integral curve l_X is closed in V .*

- (i) *For any given $X \in U$, set $\chi(X) = (s, \mathbf{x})$. The connected component λ'' of (s, \mathbf{x}) in $\lambda \equiv \chi(l_X \cap U)$ is equal to $\lambda' \equiv I_{\mathbf{x}} \times \{\mathbf{x}\}$.*
- (ii) *In order that (χ, U) be a v -adapted chart, it is necessary and sufficient that, for any $X \in U$, the intersection $l_X \cap U$ be a connected set.*
- (iii) *Let W be an open subset of U such that, for any $X \in W$, $l_X \cap U$ be a connected set. Then the restriction (χ, W) is a nice v -adapted chart.*

Proof. (i) Since $I_{\mathbf{x}}$ is an interval, the set $\lambda' \equiv I_{\mathbf{x}} \times \{\mathbf{x}\}$ is connected. To show that $\lambda'' = \lambda'$ is always true, we note first that, $\chi(U)$ having the form (18), $\chi(X) = (s, \mathbf{x})$ is such that $s \in I_{\mathbf{x}}$. Hence $(s, \mathbf{x}) \in \lambda'$, and since λ' is connected, we have $\lambda' \subset \lambda''$. To prove that in fact $\lambda' = \lambda''$, we will show that λ' is open and closed in λ'' . We know from Theorem 0 that $\lambda_0 \equiv \chi(l_X \cap U_0)$ has the form $\lambda_0 = I \times E$, where E is a subset of Ω_0 having only isolated points. Thus, \mathbf{x} being isolated in E , let $r > 0$ be such that $B \cap E = \{\mathbf{x}\}$, where $B \equiv B(\mathbf{x}, r)$ is the open ball of radius r in \mathbb{R}^N , centered at \mathbf{x} . Hence, we have

$$(I_{\mathbf{x}} \times B) \cap \lambda'' \subset (I_{\mathbf{x}} \times B) \cap \lambda_0 = (I_{\mathbf{x}} \times B) \cap (I \times E) = (I_{\mathbf{x}} \cap I) \times (B \cap E) = I_{\mathbf{x}} \times \{\mathbf{x}\} \equiv \lambda'. \quad (22)$$

We have also $\lambda' \subset (I_{\mathbf{x}} \times B) \cap \lambda''$, because $\lambda' \subset I_{\mathbf{x}} \times B$ and $\lambda' \subset \lambda''$. So $\lambda' = (I_{\mathbf{x}} \times B) \cap \lambda''$ is an open subset of λ'' . On the other hand, if a sequence of points of λ' tends towards a limit in λ'' , say $(s_n, \mathbf{x}) \rightarrow (s', \mathbf{y}) \in \lambda''$, then $\mathbf{y} = \mathbf{x}$ and, since $\lambda'' \subset \chi(U)$ which is given by (18), we have $(s', \mathbf{x}) \in (\mathbb{R} \times \{\mathbf{x}\}) \cap \chi(U) = I_{\mathbf{x}} \times \{\mathbf{x}\} \equiv \lambda'$, hence the limit (s', \mathbf{x}) is in λ' , so that λ' is a closed subset of λ'' . Being non-empty and an open-and-closed subset of the connected set λ'' , λ' is equal to λ'' .

- (ii) Since $\chi : U_0 \rightarrow I \times \Omega_0$ is a bicontinuous mapping, it is of course equivalent

to say that $l_X \cap U$ or $\lambda_X \equiv \chi(l_X \cap U)$ is connected. Therefore, (ii) follows immediately from (i) and from Statement (i) in Proposition 1.

(iii) Let X be any point in W and set $\chi(X) = (s, \mathbf{x})$. Since $l_X \cap U$ is connected, we have $\chi(l_X \cap U) = I_{\mathbf{x}} \times \{\mathbf{x}\}$ from (i). Thus, $P_S(\chi(Y)) = \mathbf{x}$ is true for any $Y \in l_X \cap U$, hence a fortiori for any $Y \in l_X \cap W$. Hence the chart (χ, W) is v -adapted. In a similar way, it is easy to adapt the proof of Statement (ii) in Proposition 1 to conclude that the v -adapted chart (χ, W) is nice. \square

Proposition 3. *Let $\chi : U_0 \rightarrow I \times \Omega_0$ be a straightening-out chart for the C^∞ vector field v and assume that, for some $X \in U_0$, the maximal integral curve l_X is closed. Let E be the closed countable subset of Ω_0 given by Theorem 0 for the maximal curve $l \equiv l_X$, thus $\chi(l_X \cap U_0) = I \times E$. Set $\mathbf{x} \equiv P_S(\chi(X))$ and, as ensured by Theorem 0, let $\Omega \subset \Omega_0$ be any open neighborhood of \mathbf{x} such that $E \cap \Omega = \{\mathbf{x}\}$. Let $U \subset U_0$ be any open set such that $\chi(U)$ has the form (18) with this set Ω . Then $l_X \cap U$ is connected, $\chi(l_X \cap U) = I_{\mathbf{x}} \times \{\mathbf{x}\}$.*

Proof. Since $\chi : U_0 \rightarrow I \times \Omega_0$ is a bijection, we have

$$\begin{aligned} \chi(l_X \cap U) &= \chi((l_X \cap U_0) \cap U) = \chi(l_X \cap U_0) \cap \chi(U) & (23) \\ &= (I \times E) \cap \left(\bigcup_{\mathbf{x}' \in \Omega} I_{\mathbf{x}'} \times \{\mathbf{x}'\} \right) = \bigcup_{\mathbf{x}' \in \Omega} (I \cap I_{\mathbf{x}'}) \times (E \cap \{\mathbf{x}'\}) \\ &= I_{\mathbf{x}} \times \{\mathbf{x}\}. & \square \end{aligned}$$

2.3 Intersections of straight lines with inverse images under the flow

Recall that the flow of the global vector field v on V is the mapping $F : \mathcal{D} \rightarrow V$, $(s, X) \mapsto F(s, X) \equiv C_X(s)$, where \mathcal{D} is the domain of the flow F :

$$\mathcal{D} \equiv \bigcup_{X \in V} I_X \times \{X\} \subset \mathbb{R} \times V. \quad (24)$$

If v is C^q ($q \geq 1$, possibly $q = \infty$), \mathcal{D} is an open set in $\mathbb{R} \times V$, moreover F is C^q on \mathcal{D} [12, 13].

Proposition 4. *Let U be an open subset of V . (i) For any $X \in V$, we have*

$$l_X \cap U = F_X(I'_{XU}) = F(I'_{XU} \times \{X\}), \quad (25)$$

where $F_X \equiv F(\cdot, X) = C_X$ is defined on $I_X \subset \mathbb{R}$, and where

$$I'_{XU} \equiv F_X^{-1}(U) = \{s \in I_X; F(s, X) \in U\}. \quad (26)$$

(ii) *Further, we have for any $X \in V$:*

$$I'_{XU} \times \{X\} = (\mathbb{R} \times \{X\}) \cap \mathcal{D}_U = (I_X \times \{X\}) \cap \mathcal{D}_U, \quad (27)$$

where

$$\mathcal{D}_U \equiv F^{-1}(U) = \bigcup_{X \in V} I'_{XU} \times \{X\}. \quad (28)$$

(iii) *Assume that v is C^∞ and that, for some $X \in V$, l_X is closed in V and not reduced to a point. Then (a) l_X is a submanifold of V and the mapping $F_X : I_X \rightarrow l_X$ is a local diffeomorphism at any point $s \in I_X$. (b) Assume moreover that F_X is non-periodic. Then it is a (global) diffeomorphism of I_X onto l_X . The connected components of $l_X \cap U$ are the images by F_X of the connected components of $I'_{XU} \equiv F_X^{-1}(U) = F_X^{-1}(l_X \cap U)$, which are open intervals of \mathbb{R} . In particular, in order that $l_X \cap U$ be connected, it is necessary and sufficient that I'_{XU} be an open interval.*

Proof. Points (i) and (ii) follow immediately from the definitions. Let us prove Point (iii). (a) Since the maximal integral curve l_X is closed in V , this is a submanifold of V [14]. Since l_X is not reduced to a point, the vector field v does not vanish on l_X . [If $v(Y) = 0$ for some $Y \in l_X$, then we have $I_Y = \mathbb{R}$ and $C_Y(s) = Y \forall s \in \mathbb{R}$ from the uniqueness of the maximal integral curve, hence $X = Y$ from the translation invariance (10).] Therefore, $\frac{dF_X}{ds} = v(F_X(s)) \neq 0$ implies that F_X is a local diffeomorphism, at any point $s \in I_X$, between the one-dimensional manifolds I_X and l_X . (b) Since l_X is closed in V , not reduced to a point, and since F_X is non-periodic, it follows that F_X is injective [14]. In view of (a) and since F_X is surjective by the definition of l_X , it is a diffeomorphism of I_X onto l_X . Hence, the connected components l_j of $l_X \cap U$ are the images by F_X of the connected components of the open set $I'_{XU} \equiv F_X^{-1}(U) = F_X^{-1}(l_X \cap U) \subset \mathbb{R}$, which are open intervals and make a finite or countable set $(I_j) \{\text{Ref. [15], §(3.19.6)}\}$. \square

An argument of transversality. Assume that v does not vanish and that each maximal integral curve is non-periodic and is closed in V . Given an arbitrary point $X \in V$, let $\chi : U_0 \rightarrow I \times \Omega_0$ be a straightening-out chart in the neighborhood of X , and let E be the closed countable subset of Ω_0 , having only isolated points, such that $\chi(l_X \cap U_0) = I \times E$. As Proposition 3 states: by restricting χ to an open subset $U \subset U_0$ for which $\chi(U)$ has the form (18) with $\Omega \subset \Omega_0$ any open neighborhood of $\mathbf{x} \equiv P_S(\chi(X))$ such that $E \cap \Omega = \{\mathbf{x}\}$, we ensure that $l_X \cap U$ is connected. From Point (iii) of Proposition 2, it follows that we will obtain a nice v -adapted chart by restricting χ to an open neighborhood $W \subset U$ of X — if it exists — such that, for any $Y \in W$, $l_Y \cap U$ be connected. To this purpose, we observe that, when the boundary of U is a regular hypersurface, then the same should be true for the boundary of the open set $\mathcal{D}_U \equiv F^{-1}(U) \subset \mathbb{R} \times V$: that boundary $\Sigma_U \equiv \text{Fr}(\mathcal{D}_U)$ should “normally” be a hypersurface in $\mathbb{R} \times V$. I.e., Σ_U should be a submanifold of codimension 1 of the $(N + 2)$ -dimensional manifold $\mathbb{R} \times V$. Then, “generically”, a straight line $\mathbb{R} \times \{X\} \subset \mathbb{R} \times V$ that intersects Σ_U is nowhere tangent to it, i.e. it is transverse to it at each intersection point. Thus, when Y is sufficiently close to X , the intersection points should be slightly displaced but should remain in the same number, hence the structure of $I'_{YU} \times \{Y\} = (\mathbb{R} \times \{Y\}) \cap \mathcal{D}_U$ should be the same as for X . In particular, if I'_{XU} is an interval, i.e. (according to Proposition 4) if $l_X \cap U$ is connected, then also I'_{YU} should be an interval, i.e., also $l_Y \cap U$ should be connected.

To make this line of reasoning precise, we first introduce, for a general hypersurface Σ in $\mathbb{R} \times V$, the set S_Σ of the points X in V for which the straight line $\mathbb{R} \times \{X\}$ is tangent to Σ at one point at least. (Our aim is to use this when Σ is the boundary hypersurface $\Sigma_U \equiv \text{Fr}(\mathcal{D}_U)$ introduced above.) Note that the tangent space to $\mathbb{R} \times V$ at some point (s, X) is $T_{(s,X)}(\mathbb{R} \times V) \simeq T_s \mathbb{R} \times T_X V \simeq \mathbb{R} \times T_X V$. The tangent space to $\mathbb{R} \times \{X\}$ at (s, X) is $\mathbb{R}\xi$, where $\xi \equiv (1, 0_X)$, 0_X being the zero element of $T_X V$. Thus we define

$$S_\Sigma \equiv \{X \in V; \exists s \in \mathbb{R} : (s, X) \in \Sigma \text{ and } (1, 0_X) \in T_{(s,X)}\Sigma\}. \quad (29)$$

However, we note that $\mathcal{D}_U \equiv F^{-1}(U)$ is in general not a bounded domain of $\mathbb{R} \times V$, even if the open set $U \subset V$ is bounded. ³ Therefore, even if

³ For instance, consider a constant vector field v on $V \equiv \mathbb{R}^n$. Then we have simply $F(s, X) = X + sv$. Hence, for any $X_0 \in V$, its inverse image is the unbounded straight

$X \in V$ is such that the straight line $\mathbb{R} \times \{X\}$ is not tangent to the boundary hypersurface Σ_U , i.e. $X \notin S_{\Sigma_U}$, it may happen that $\mathbb{R} \times \{X\}$ is “tangent to Σ_U at infinity”, in which case any line $\mathbb{R} \times \{Y\}$, however close Y can be to X , may have “new” intersections with the hypersurface Σ_U . For a general hypersurface Σ in $\mathbb{R} \times V$, in fact for *any subset* Σ of $\mathbb{R} \times V$, we thus introduce:

$$S_{\Sigma_\infty} \equiv \{X \in V; \liminf_{r \rightarrow \infty} \inf_{|s| \geq r} d(X, \Sigma_s) = 0\}, \quad (30)$$

where, for any subset \mathcal{B} of $\mathbb{R} \times V$, we define \mathcal{B}_s to be its slice in V at $s \in \mathbb{R}$:

$$\mathcal{B}_s \equiv \{X \in V; (s, X) \in \mathcal{B}\} \subset V, \quad (31)$$

and where, given any distance d that generates the (metrizable) topology of V , one defines for any subset A of V and any point $X \in V$: $d(X, A) \equiv \inf_{Z \in A} d(X, Z)$.

2.4 Relevant theorems

Theorem 1. *Let Σ be a hypersurface of $\mathbb{R} \times V$ that is closed in $\mathbb{R} \times V$. Let $K = [\alpha, \beta]$ be a compact interval of \mathbb{R} ($\alpha < \beta$). Suppose that, for some $X_0 \in V$, the intersection $(K \times \{X_0\}) \cap \Sigma$ be a singleton (s_0, X_0) with $\alpha < s_0 < \beta$ and that this intersection be transverse, i.e. $X_0 \notin S_\Sigma$. Then there is a neighborhood W of X_0 such that, for any $X \in W$, $(K \times \{X\}) \cap \Sigma$ is a singleton $\Phi(X)$ and this intersection is transverse. Moreover, the function $\Phi : W \rightarrow \Sigma$ is smooth.*

Proof. Since $(s_0, X_0) \in \Sigma$, and since Σ is a submanifold of dimension $N + 1$ of the $(N + 2)$ -dimensional differentiable manifold $\mathbb{R} \times V$, there is a chart (Ψ, \mathcal{U}) on $\mathbb{R} \times V$, with $(s_0, X_0) \in \mathcal{U}$, for which the function

$$g : \mathcal{U} \rightarrow \mathbb{R}, \quad (s, X) \mapsto g(s, X) \equiv \Psi^{N+2}(s, X) \quad (32)$$

[the last component of $\Psi(s, X)$] is such that

$$(s, X) \in \mathcal{U} \cap \Sigma \iff g(s, X) = 0 \quad (33)$$

line $F^{-1}(X_0) = \{(s, X) \in \mathbb{R} \times V; s \in \mathbb{R}, X = X_0 - sv\}$. From this, one may deduce more general examples by applying a diffeomorphism.

{e.g. [11], §(16.8.3)}. Then, if $(s, X) \in \mathcal{U} \cap \Sigma$, a vector $\eta = (a, u) \in T_{(s,X)}(\mathbb{R} \times V) \simeq \mathbb{R} \times T_X V$ is in the tangent space $T_{(s,X)}\Sigma$, iff $dg_{(s,X)}(\eta) = 0$. Since the intersection $(K \times \{X_0\}) \cap \Sigma = \{(s_0, X_0)\}$ is transverse, we have

$$dg_{(s_0, X_0)}(\xi_0) \neq 0, \quad \xi_0 \equiv (1, 0_{T_{X_0}V}). \quad (34)$$

Since K is a neighborhood of s_0 : replacing \mathcal{U} with a smaller neighborhood of (s_0, X_0) if necessary, we may assume that, if $(s, X) \in \mathcal{U}$, then $s \in K$. Consider a chart (χ, U) of V in a neighborhood U of X_0 , thus $\chi(X) = \mathbf{X} = (x^\mu) \in \mathbb{R}^{N+1}$ for $X \in U$. The mapping $\Xi : (s, X) \mapsto (s, \chi(X))$ is a local chart of $\mathbb{R} \times V$ defined in the neighborhood $\mathbb{R} \times U$ of (s_0, X_0) , while Ψ is also a local chart of $\mathbb{R} \times V$, defined in the neighborhood \mathcal{U} of (s_0, X_0) . Let

$$f(s, \mathbf{X}) \equiv g(\Xi^{-1}(s, \mathbf{X})) = g(s, \chi^{-1}(\mathbf{X})) \quad (35)$$

be the local expression of g in the chart Ξ . This function f is defined and \mathcal{C}^∞ on the open subset $\mathcal{O} \equiv \Xi((\mathbb{R} \times U) \cap \mathcal{U})$ of \mathbb{R}^{N+2} . Note that \mathcal{O} contains (s_0, \mathbf{X}_0) , where $\mathbf{X}_0 \equiv \chi(X_0)$. With this local expression, we have for any vector $\eta = (a, u) \in T_{(s,X)}(\mathbb{R} \times V)$:

$$dg_{(s,X)}(\eta) = \frac{\partial f}{\partial s}(s, \mathbf{X}) a + \frac{\partial f}{\partial x^\mu}(s, \mathbf{X}) u^\mu, \quad (36)$$

where $\mathbf{X} \equiv \chi(X)$, and where $(a, (u^\mu))$ ($\mu = 0, \dots, N$) are the components of η in the product chart Ξ . From (33), we have

$$f(s_0, \mathbf{X}_0) = 0. \quad (37)$$

From (34) and (36) we have

$$dg_{(s_0, X_0)}(\xi_0) = \frac{\partial f}{\partial s}(s_0, \mathbf{X}_0) \neq 0. \quad (38)$$

We can thus apply the implicit function theorem: there is an open neighborhood A of \mathbf{X}_0 and a unique smooth function $\varphi : A \rightarrow \mathbb{R}$, such that $\varphi(\mathbf{X}_0) = s_0$ and that, for any $\mathbf{X} \in A$:

$$(\varphi(\mathbf{X}), \mathbf{X}) \in \mathcal{O}, \quad f(\varphi(\mathbf{X}), \mathbf{X}) = 0, \quad \frac{\partial f}{\partial s}(\varphi(\mathbf{X}), \mathbf{X}) \neq 0. \quad (39)$$

We define a smooth mapping Φ by setting for any $X \in W' \equiv \chi^{-1}(A) \subset V$:

$$\Phi(X) \equiv (\varphi(\chi(X)), X). \quad (40)$$

Thus, for any $X \in W'$, with $\mathbf{X} \equiv \chi(X) \in A$, we have $\Phi(X) = (\varphi(\mathbf{X}), \chi^{-1}(\mathbf{X})) = \Xi^{-1}(\varphi(\mathbf{X}), \mathbf{X}) \in \Xi^{-1}(\mathcal{O}) \subset \mathcal{U}$. In particular, $\Phi(X_0) = (s_0, X_0)$. For any $X \in W'$, we have from (35), (39) and (40): $g(\Phi(X)) = 0$, thus $\Phi(X) \in \Sigma \cap \mathcal{U}$. As with (34) and (38), (39) means that the intersection $\Phi(X) \in (K \times \{X\}) \cap \Sigma$ is transverse. The definition (40) entails also that $\text{Pr}_2 \circ \Phi = \text{Id}_{W'}$, where $\text{Pr}_2 : \mathbb{R} \times V \rightarrow V$, $(s, X) \mapsto X$. Thus, the rank of $\text{Pr}_2 \circ \Phi$ is $\dim V = N + 1$, and since the rank of $\text{Pr}_2 \circ \Phi$ is not larger than that of Φ , this latter is also $N + 1 = \dim \Sigma$, i.e. Φ is a submersion. (All of this is true at any point $X \in W'$.) It follows that $W'' \equiv \Phi(W')$ is an open set in the manifold Σ {[11], §(16.7.5)}, hence is a neighborhood of $(s_0, X_0) = \Phi(X_0)$ in Σ . We have

$$\forall (s, X) \in W'', (s, X) = \Phi(X). \quad (41)$$

We claim that there is some neighborhood $W \subset W'$ of X_0 , such that

$$\forall X \in W, [s \in K \text{ and } (s, X) \in \Sigma] \implies [(s, X) = \Phi(X)]. \quad (42)$$

Note that, if $W \subset W'$, the reverse implication is also true for any $X \in W$, by the construction of the mapping Φ . Thus, if (42) is true, W is as stated by Theorem 1. We will reason *ab absurdo*. If it does not exist such a neighborhood W , then, for any integer $n > 0$, one can find $s_n \in K$ and $X_n \in W' \cap B(X_0, 1/n)$ such that $(s_n, X_n) \in \Sigma$ and $(s_n, X_n) \neq \Phi(X_n)$. Thus $X_n \rightarrow X_0$ and, by extraction in the compact set K , we may assume that s_n has a limit $s \in K$, so that $(s_n, X_n) \rightarrow (s, X_0)$. If it happened that $s = s_0$, then, since W'' is a neighborhood of (s_0, X_0) and $(s_n, X_n) \rightarrow (s, X_0)$, we would have $(s_n, X_n) \in W''$ for large enough n , hence $(s_n, X_n) = \Phi(X_n)$ from (41), which is a contradiction. Thus $s \neq s_0$. But since Σ is closed, we have $(s, X_0) \in \Sigma$, which contradicts the assumption that $(K \times \{X_0\}) \cap \Sigma = \{(s_0, X_0)\}$. This completes the proof of Theorem 1. \square

Theorem 2. *Let \mathcal{U} be an open domain in $\mathbb{R} \times V$, whose boundary Σ be a hypersurface of $\mathbb{R} \times V$. Assume that, for some $X \in V$, all intersections of the straight line $\mathcal{L} \equiv \mathbb{R} \times \{X\}$ with Σ are transverse. Then the boundary of $\mathcal{L} \cap \mathcal{U}$ in \mathcal{L} is $\text{Fr}_{(\mathcal{L})}(\mathcal{L} \cap \mathcal{U}) = \mathcal{L} \cap \Sigma$.*

Proof. The boundary of a subset is of course relative to which containing set is considered. Here, the boundary Σ of the open set $\mathcal{U} \subset \mathbb{R} \times V$ is relative to the whole manifold $\mathcal{V} \equiv \mathbb{R} \times V$, i.e. $\Sigma \equiv \overline{\mathcal{U}} \cap \overline{\mathcal{U}}$, where the upper bar -

denotes the adherence in \mathcal{V} and \mathcal{C} the complementary set in \mathcal{V} . Thus

$$\Sigma = \overline{\mathcal{U}} \setminus \mathcal{U}, \quad (43)$$

since \mathcal{U} is open. The boundary of some subset $\mathcal{B} \subset \mathcal{L}$ in \mathcal{L} (or relative to \mathcal{L}) is $\text{Fr}_{(\mathcal{L})}(\mathcal{B}) \equiv \overline{\mathcal{B}}^{\mathcal{L}} \cap \overline{\mathcal{C}_{\mathcal{L}}\mathcal{B}}^{\mathcal{L}}$, where $\overline{\mathcal{B}}^{\mathcal{L}}$ denotes the adherence of \mathcal{B} in \mathcal{L} , and where $\mathcal{C}_{\mathcal{L}}\mathcal{B} \equiv \mathcal{L} \setminus \mathcal{B}$ is the complementary set of \mathcal{B} in \mathcal{L} . However, here $\mathcal{L} \equiv \mathbb{R} \times \{X\}$ is closed in $\mathcal{V} = \mathbb{R} \times \mathbb{V}$, hence we have $\overline{\mathcal{B}}^{\mathcal{L}} = \overline{\mathcal{B}}$, the adherence in the whole set \mathcal{V} . Thus

$$\text{Fr}_{(\mathcal{L})}(\mathcal{L} \cap \mathcal{U}) \equiv \overline{\mathcal{L} \cap \mathcal{U}}^{\mathcal{L}} \cap \overline{\mathcal{C}_{\mathcal{L}}(\mathcal{L} \cap \mathcal{U})}^{\mathcal{L}} = \overline{\mathcal{L} \cap \mathcal{U}} \cap \overline{\mathcal{C}_{\mathcal{L}}\mathcal{U}}. \quad (44)$$

Again because \mathcal{L} is closed in \mathcal{V} , we have $\overline{\mathcal{L} \cap \mathcal{U}} \subset \mathcal{L} \cap \overline{\mathcal{U}}$. We will show that we have exactly

$$\overline{\mathcal{L} \cap \mathcal{U}} = \mathcal{L} \cap \overline{\mathcal{U}}. \quad (45)$$

We shall in fact show that

$$\mathcal{L} \cap \Sigma \subset \overline{\mathcal{L} \cap \mathcal{U}}. \quad (46)$$

Since (43) implies that $(\mathcal{L} \cap \overline{\mathcal{U}}) \setminus \overline{\mathcal{L} \cap \mathcal{U}} \subset (\mathcal{L} \cap \overline{\mathcal{U}}) \setminus (\mathcal{L} \cap \mathcal{U}) = \mathcal{L} \cap \Sigma$, this will prove that $\mathcal{L} \cap \overline{\mathcal{U}} \subset \overline{\mathcal{L} \cap \mathcal{U}}$, whence (45).

To prove (46), consider an arbitrary point $p_0 = (s_0, X) \in \mathcal{L} \cap \Sigma$. As in the proof of Theorem 1, let (Ψ, \mathcal{W}) be a chart on \mathcal{V} , with $p_0 \in \mathcal{W}$, such that $p \in \mathcal{W} \cap \Sigma \Leftrightarrow g(p) = 0$, where $g = \Psi^n$, with $n = \dim(\mathcal{V}) = N + 2$. Since we assume that the intersection $p_0 \in \mathcal{L} \cap \Sigma$ is transverse, we have again (34). Hence, there is an interval $J =]s_0 - r, s_0 + r[$ in which $s = s_0$ is the only zero of the smooth function $\varphi(s) \equiv g(s, X)$. (That is, p_0 is the only intersection of $J \times \{X\}$ with Σ .) Thus, we may assume that, say, $g(s, X) > 0$ for $s_0 < s < s_0 + r$, so that

$$g(s, X) < 0 \text{ for } s_0 - r < s < s_0. \quad (47)$$

Replacing $\Psi(p)$ by $\Psi(p) - \Psi(p_0)$, we may assume that $\Psi(p_0) = \mathbf{0}_{\mathbb{R}^n}$. There is some open ball $W = B(\mathbf{0}, r) \subset \Psi(\mathcal{W})$. Replacing \mathcal{W} by $\Psi^{-1}(W)$, we have that $\mathcal{W}_+ \equiv \{p \in \mathcal{W}; x^n \equiv \Psi^n(p) > 0\}$ is just $\mathcal{W}_+ = \Psi^{-1}(W_+)$, where $W_+ \equiv \{\mathbf{P} \in W; x^n > 0\}$, hence \mathcal{W}_+ is non-empty and connected as is W_+ . The same is true for $\mathcal{W}_- \equiv \{p \in \mathcal{W}; \Psi^n(p) < 0\}$. Since $p_0 \in \Sigma \subset \overline{\mathcal{U}}$, and

since \mathcal{W} is a neighborhood of p_0 , we have $\mathcal{U} \cap \mathcal{W} \neq \emptyset$, so let $p'_0 \in \mathcal{U} \cap \mathcal{W}$. Because \mathcal{W} is the disjoint union $\mathcal{W} = \mathcal{W}_+ \cup \mathcal{W}_- \cup (\mathcal{W} \cap \Sigma)$, and because $p'_0 \in \mathcal{U}$ cannot belong to $\Sigma = \overline{\mathcal{U}} \setminus \mathcal{U}$, we have either $p'_0 \in \mathcal{W}_+$ or $p'_0 \in \mathcal{W}_-$. Let us assume that, for instance, $p'_0 \in \mathcal{W}_-$, so that $\mathcal{U} \cap \mathcal{W}_- \neq \emptyset$. It follows that $\mathcal{W}_- \subset \mathcal{U} \cap \mathcal{W}$, for otherwise the connected set \mathcal{W}_- would intersect both \mathcal{U} and $\mathcal{C}\mathcal{U}$, hence would intersect the boundary Σ — which is impossible, since we have $x^n = 0$ in Σ , not $x^n < 0$. Therefore, we get from (47) that $]s_0 - r, s_0[\times \{X\} \subset \mathcal{U} \cap \mathcal{W}$, so that $p_0 \equiv (s_0, X) \in \overline{\mathcal{L} \cap \mathcal{U}}$. This proves (46).

Combining (44) and (45) gives us:

$$\text{Fr}_{(\mathcal{L})}(\mathcal{L} \cap \mathcal{U}) = \mathcal{L} \cap \overline{\mathcal{U}} \cap \overline{\mathcal{C}\mathcal{U}}. \quad (48)$$

But $\mathcal{C}\mathcal{L}\mathcal{U} = \mathcal{L} \cap \mathcal{C}\mathcal{U}$ is closed in \mathcal{V} , for \mathcal{L} is closed and \mathcal{U} is open. Hence $\mathcal{L} \cap \overline{\mathcal{C}\mathcal{U}} = \mathcal{L} \cap \mathcal{C}\mathcal{U} = \mathcal{L} \cap \mathcal{C}\mathcal{U} = \mathcal{L} \cap \overline{\mathcal{C}\mathcal{U}}$. Therefore, (48) rewrites as

$$\text{Fr}_{(\mathcal{L})}(\mathcal{L} \cap \mathcal{U}) = \mathcal{L} \cap \overline{\mathcal{U}} \cap \overline{\mathcal{C}\mathcal{U}} \equiv \mathcal{L} \cap \Sigma, \quad (49)$$

which proves Theorem 2. \square

Remark 2.1. With small and straightforward modifications, the foregoing proof shows the result in the much more general case that $\mathbb{R} \times V$ (with V a differentiable manifold) is replaced by a general differentiable manifold \mathcal{V} and \mathcal{L} is the range $\mathcal{L} = C(J)$, assumed closed, of a smooth curve $C : J \rightarrow \mathcal{V}$ (with J an interval of \mathbb{R}), such that all intersections $p \in \mathcal{L} \cap \mathcal{V}$ are transverse. It is easy to see that the latter assumption is necessary.

Remark 2.2. In the course of the proof, the following intuitively obvious result was proved: Suppose that the line $\mathcal{L} = C(J)$ intersects transversely at point $p = C(s_0)$ the boundary Σ , assumed to be a regular hypersurface, of some open domain $\mathcal{U} \subset \mathcal{V}$. Then, among the two parts of \mathcal{L} : $s < s_0$ and $s > s_0$, at least one is such that, when s is close enough to s_0 , we have $C(s) \in \mathcal{U}$. Note that it may be the case that, when s is close enough to s_0 , $C(s)$ belongs to \mathcal{U} for both $s < s_0$ and $s > s_0$: e.g. if $\mathcal{U} \cup \Sigma = \mathcal{V}$.

Theorem 3. *Let U be an open subset of V . (i) Assume that $\overline{F^{-1}(U)} \subset \mathcal{D}$, with \mathcal{D} the domain of the flow F . (This is true, in particular, if the flow is complete, i.e. $\mathcal{D} = \mathcal{V}$.) Then we have:*

$$\text{Fr}(F^{-1}(U)) = F^{-1}(\text{Fr}(U)). \quad (50)$$

(ii) If $\text{Fr}(U)$ is a hypersurface of V , then $F^{-1}(\text{Fr}(U))$ is a hypersurface of $\mathcal{V} \equiv \mathbb{R} \times V$.

Proof of Point (i). First, consider the more general context that \mathcal{V} and V are merely metric spaces and $F : \mathcal{D} \rightarrow V$ is merely a continuous map, with \mathcal{D} an open subset of \mathcal{V} . Then, if $\overline{F^{-1}(U)} \subset \mathcal{D}$, we have

$$\text{Fr}(F^{-1}(U)) \subset F^{-1}(\text{Fr}(U)). \quad (51)$$

Indeed, since U is open in V , we have $\text{Fr}(U) = \overline{U}^V \setminus U$, as with Eq. (43). Since F is continuous on \mathcal{D} , $F^{-1}(U)$ is open in \mathcal{D} ; hence, \mathcal{D} being an open subset of \mathcal{V} , $F^{-1}(U)$ is open in \mathcal{V} . Therefore, we have similarly $\text{Fr}(F^{-1}(U)) = \overline{F^{-1}(U)} \setminus F^{-1}(U)$. Again because F is continuous on \mathcal{D} , we have $\overline{F^{-1}(U)}^{\mathcal{D}} \subset F^{-1}(\overline{U}^V)$. But $\overline{F^{-1}(U)}^{\mathcal{D}} = \overline{F^{-1}(U)}$ since $\overline{F^{-1}(U)} \subset \mathcal{D}$. Thus $\text{Fr}(F^{-1}(U)) \subset F^{-1}(\overline{U}^V) \setminus F^{-1}(U)$, whence (51).

To prove the reverse inclusion, we consider any $p = (s, X) \in F^{-1}(\text{Fr}(U))$, thus $Y \equiv F(s, X) \in \text{Fr}(U)$, and we will show that $p \in \text{Fr}(F^{-1}(U))$. There exists an open neighborhood \mathcal{U} of p , having the form $\mathcal{U} = J \times W$, with J an open interval containing s and 0 , and with W an open neighborhood of X in V , such that $\mathcal{U} \subset \mathcal{D}$ {Ref. [12], §(18.2.5)}. For all $t \in J$, $F_t \equiv F(t, \cdot)$ is a homeomorphism of W onto $W_t \equiv F_t(W)$, moreover F_{-t} is defined over W_t and is the inverse homeomorphism of F_t [these two points result from (10)] {Ref. [12], §(18.2.8)}. (We note for the proof of Point (ii) that F_t , as well as F_{-t} , is of class \mathcal{C}^q if the vector field v is itself \mathcal{C}^q [12].) Now we consider any neighborhood \mathcal{U}' of p and we show that it intersects both $F^{-1}(U)$ and $\mathcal{C}F^{-1}(U)$. We may assume that $\mathcal{U}' \subset \mathcal{U}$ and has the form $\mathcal{U}' = J' \times W'$, with $J' \subset J$ an open interval containing s , and with $W' \subset W$ an open neighborhood of X in V . Thus $F_s(W')$ is an open neighborhood of $Y = F_s(X)$. Since $Y \in \text{Fr}(U)$, both $F_s(W') \cap U$ and $F_s(W') \cap \mathcal{C}U$ are non-empty, so let $Y' \in F_s(W') \cap U$ and $Y'' \in F_s(W') \cap \mathcal{C}U$. Therefore, $X' \equiv F_{-s}(Y') \in W'$. Thus, $F(s, X') = Y' \in U$, and also $(s, X') \in J' \times W'$, so that $(s, X') \in (J' \times W') \cap F^{-1}(U)$. In just the same way, with $X'' \equiv F_{-s}(Y'')$, we get that $(s, X'') \in (J' \times W') \cap F^{-1}(\mathcal{C}U)$. Since $F^{-1}(\mathcal{C}U) = \mathcal{C}_{\mathcal{D}}F^{-1}(U) \subset \mathcal{C}F^{-1}(U) \equiv \mathcal{V} \setminus F^{-1}(U)$, we have shown that any neighborhood \mathcal{U}' of p intersects both $F^{-1}(U)$ and $\mathcal{C}F^{-1}(U)$, thus $p \in \text{Fr}(F^{-1}(U))$. Therefore, we have indeed $F^{-1}(\text{Fr}(U)) \subset \text{Fr}(F^{-1}(U))$. Together with (51), this proves (50).

Proof of Point (ii). Let $p = (s, X) \in F^{-1}(\text{Fr}(\mathcal{U})) \subset \mathcal{V}$. Define J (with $s \in J$), $W \subset V$ (with $X \in W$), $\mathcal{U} = J \times W$, and F_t ($t \in J$) just as at the beginning of the foregoing paragraph. Since $S \equiv \text{Fr}(\mathcal{U})$ is assumed to be a hypersurface of V , we have thus $Y \equiv F(s, X) \in S$, so let $\chi : Y' \mapsto \mathbf{Y} \equiv (y^1, \dots, y^{n-1})$ be a chart of V (with $n-1 = \dim(\mathcal{V}) - 1 = \dim(V)$), defined in the neighborhood of Y , and such that

$$Y' \in S \cap \text{Dom}(\chi) \Leftrightarrow y^{n-1} \equiv \chi^{n-1}(Y') = 0. \quad (52)$$

We may assume that $\text{Dom}(\chi)$, the domain of χ , is just $W_s \equiv F_s(W)$. Then the mapping

$$\Xi : \mathcal{U} = J \times W \rightarrow \mathbb{R}^n, \quad (t, X') \mapsto (t, \chi(F_s(X'))) = (t, \mathbf{Y}) \quad (53)$$

is a chart of \mathcal{V} in the neighborhood of p . Consider the mapping

$$\Psi : \mathcal{U}' \equiv F^{-1}(W_s) \rightarrow \mathbb{R}^n, \quad (t, X') \mapsto (t, \chi(F(t, X'))). \quad (54)$$

(Note that \mathcal{U}' is an open neighborhood of p .) We have (cf. Eq. (10)):

$$F(t, X') = F(s + u, X') = F(u, F(s, X')) = F(u, F_s(X')), \quad u \equiv t - s. \quad (55)$$

Hence, the local expression of Ψ in the chart Ξ is:

$$\mathbf{G}(t, \mathbf{Y}) \equiv \Psi(\Xi^{-1}(t, \mathbf{Y})) = (t, \mathbf{Z}(t, \mathbf{Y})) \quad (56)$$

with

$$\mathbf{Z}(t, \mathbf{Y}) = (z^1, \dots, z^{n-1}) \equiv \chi(F(u, \chi^{-1}(\mathbf{Y}))). \quad (57)$$

Let $\mathbf{v} = (v^1, \dots, v^{n-1}) = \mathbf{v}(\mathbf{Y})$ be the local expression of v in the chart χ . Thus \mathbf{Z} is the value at $u \equiv t - s$ of the solution of $\frac{d\mathbf{Z}'}{dw} = \mathbf{v}(\mathbf{Z}'(w))$, $\mathbf{Z}'(w = 0) = \mathbf{Y}$. Hence we have, uniformly w.r.t. (t, \mathbf{Y}) in some neighborhood of $(s, \mathbf{Y}_0) \equiv \Xi(p)$:

$$\mathbf{Z}(t, \mathbf{Y}) = \mathbf{Y} + (t - s)\mathbf{v}(\mathbf{Y}) + O(u^2). \quad (58)$$

The Jacobian matrix of $\mathbf{G}(t, \mathbf{Y}) = (t, \mathbf{Z}(t, \mathbf{Y}))$ at point (s, \mathbf{Y}_0) is therefore:

$$J = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ v^1(\mathbf{Y}_0) & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ v^{n-1}(\mathbf{Y}_0) & 0 & 0 & \dots & 1 \end{pmatrix}, \quad (59)$$

a triangular matrix with 1 on the diagonal, so $\det J = 1$. It follows that Ψ also is a chart of \mathcal{V} in some neighborhood $\mathcal{U}'' \subset \mathcal{U} \cap \mathcal{U}'$ of p . When $(t, X') \in \mathcal{U}''$, we have from (52) and (54):

$$(t, X') \in F^{-1}(S) \Leftrightarrow z^{n-1} \equiv \Psi^n(t, X') \equiv \chi^{n-1}(F(t, X')) = 0. \quad (60)$$

Hence, $F^{-1}(S)$ is a hypersurface of \mathcal{V} . The proof of Theorem 3 is complete. \square

Proposition 5. *Let Σ be a subset of $\mathbb{R} \times V$. If $X \in V \setminus S_{\Sigma\infty}$ [cf. (30)], there is a neighborhood W of X and a real $R > 0$ such that*

$$(Y \in W \text{ and } |s| \geq R) \Rightarrow (s, Y) \notin \Sigma. \quad (61)$$

Proof. Clearly, $f(X) \equiv \lim_{r \rightarrow \infty} \inf_{|s| \geq r} d(X, (\Sigma)_s)$ is well defined for any $X \in V$ and verifies $0 \leq f(X) \leq +\infty$. Hence, if $X \notin S_{\Sigma\infty}$ it means that $f(X) > 0$, or equivalently that there exists $R > 0$ and $\delta > 0$ such that

$$|s| \geq R \Rightarrow d(X, \Sigma_s) \geq \delta, \quad (62)$$

hence $d(X, Z) \geq \delta$ is true for any $Z \in \Sigma_s$ if $|s| \geq R$. We deduce from this that, if $Y \in V$ verifies $d(X, Y) < \delta/2$, and if $|s| \geq R$, we have $d(Y, Z) > \delta/2$ for any $Z \in \Sigma_s$, hence $d(Y, \Sigma_s) \geq \delta/2$. (Thus, $V \setminus S_{\Sigma\infty}$ is open in V , in other words $S_{\Sigma\infty}$ is closed.) In particular, if $d(X, Y) < \delta/2$ and $|s| \geq R$, then $Y \notin \Sigma_s$, i.e. $(s, Y) \notin \Sigma$. \square

Theorem 4. *Let U be an open subset of V and set $\mathcal{D}_U \equiv F^{-1}(U)$ and $\Sigma_U \equiv \text{Fr}(\mathcal{D}_U)$. Assume that Σ_U is a hypersurface of $\mathcal{V} \equiv \mathbb{R} \times V$ which is closed in \mathcal{V} , that for some $X \in U$, the open set I'_{XU} is a bounded interval, and that $X \notin (S_{\Sigma_U} \cup S_{\Sigma_U\infty})$.*

(i) *There is a neighborhood W of X , $W \subset U$, such that for $Y \in W$, also I'_{YU} is a bounded interval. (ii) If v does not vanish and all maximal integral curves are closed in V and non-periodic, then $l_Y \cap U$ is connected for any $Y \in W$. (iii) If moreover $\chi(U)$ has the form (18) and there is a straightening-out chart (χ, U_0) with $U_0 \supset U$, then the restriction of χ to W is a nice v -adapted chart.*

Proof. (i) Since $X \notin S_{\Sigma_U\infty}$, by Proposition 5 there is a neighborhood W_0 of X and a real $R > 0$ such that

$$(Y \in W_0 \text{ and } |s| \geq R) \Rightarrow (s, Y) \notin \Sigma_U. \quad (63)$$

Since I'_{XU} is assumed to be a bounded interval (and $I'_{XU} \neq \emptyset$ since $0 \in I'_{XU}$), let $I'_{XU} =]a, b[$, with $a, b \in \mathbb{R}$, $a < b$. In (63) we may assume that $-R < a < b < R$. By Proposition 4, we have

$$I'_{XU} \times \{X\} =]a, b[\times \{X\} = (\mathbb{R} \times \{X\}) \cap \mathcal{D}_U. \quad (64)$$

Since by assumption $X \notin S_{\Sigma_U}$, we get by Theorem 2 (setting $\mathcal{L} \equiv \mathbb{R} \times \{X\}$):

$$\{(a, X), (b, X)\} = \text{Fr}_{(\mathcal{L})}(\mathcal{L} \cap \mathcal{D}_U) = \mathcal{L} \cap \Sigma_U. \quad (65)$$

Thus, considering the compact intervals $K_1 \equiv [-R, \frac{a+b}{2}]$ and $K_2 \equiv [\frac{a+b}{2}, R]$, we have

$$(K_1 \times \{X\}) \cap \Sigma_U = \{(a, X)\}, \quad (K_2 \times \{X\}) \cap \Sigma_U = \{(b, X)\}. \quad (66)$$

Therefore, by Theorem 1, there are neighborhoods W_1 and W_2 of X , and smooth functions $\Phi_j : W_j \rightarrow \Sigma_U$ ($j = 1, 2$), such that:

$$Y \in W_j \Rightarrow (K_j \times \{Y\}) \cap \Sigma_U = \{\Phi_j(Y)\} \quad (j = 1, 2), \quad (67)$$

with transverse intersection. We have $\Phi_j(Y) = (\varphi_j(Y), Y)$, with $\varphi_j : W_j \rightarrow \mathbb{R}$ a smooth function. From (66) and (67), we get: $\varphi_1(X) = a$ and $\varphi_2(X) = b$. Thus $\varphi_1(X) = a \neq b = \varphi_2(X)$; hence, by considering a small enough neighborhood W of X , with $W \subset W_0 \cap W_1 \cap W_2$, we get $\varphi_1(Y) \neq \varphi_2(Y)$ for $Y \in W$. With (63) and (67), this implies that

$$Y \in W \Rightarrow (\mathbb{R} \times \{Y\}) \cap \Sigma_U = \{\Phi_1(Y), \Phi_2(Y)\}. \quad (68)$$

Another application of Theorem 2 proves that

$$\text{Fr}_{(\mathbb{R} \times \{Y\})}((\mathbb{R} \times \{Y\}) \cap \mathcal{D}_U) = (\mathbb{R} \times \{Y\}) \cap \Sigma_U. \quad (69)$$

Together with (68), this implies that I'_{YU} , the open subset of \mathbb{R} such that $I'_{YU} \times \{Y\} = (\mathbb{R} \times \{Y\}) \cap \mathcal{D}_U$, has boundary $\{\varphi_1(Y), \varphi_2(Y)\}$. Therefore, $I'_{YU} =]\varphi_1(Y), \varphi_2(Y)[$.

(ii) This follows from Point (iii) in Proposition 4.

(iii) This follows from Point (iii) in Proposition 2. \square

2.5 Adapted charts and “normal” vector fields

With Theorem 4, we formalized our transversality argument in Subsect. 2.3 to investigate the problem of the existence, in the neighborhood of any point $X \in V$, of a nice v -adapted chart. Assuming that v does not vanish and that all maximal integral curves are closed in V and non-periodic, let us check if Theorem 4 applies. Due to Proposition 4, $I'_{XU} = F_X^{-1}(l_X \cap U)$ is an interval iff $l_X \cap U$ is connected, and I'_{XU} is bounded if U is relatively compact. As shown by Proposition 3, the assumption that $l_X \cap U$ is connected may be fulfilled by starting from a straightening-out chart $\chi : U_0 \rightarrow I \times \Omega_0$ in the neighborhood of the arbitrary point $X \in V$ and by restricting χ to an open subset $U \subset U_0$ such that $\chi(U)$ has the form (18) with $\Omega \subset \Omega_0$ a small enough open neighborhood of $\mathbf{x} \equiv P_S(\chi(X))$.

As shown by Theorem 3, the assumption that Σ_U is a hypersurface of \mathcal{V} that is closed in \mathcal{V} is fulfilled, in particular, if the boundary of the open set $U \subset V$ is itself a hypersurface of V that is closed in V , and if moreover $\overline{F^{-1}(U)} \subset \mathcal{D}$. The latter inclusion is true, in particular, if $\mathcal{D} = \mathcal{V}$, i.e., if the vector field v is complete (in other words, if every maximal integral curve of v is defined over the whole real line). Actually, this does not restrict in any way the set of the maximum integral curves, $N_v \equiv \{l_X; X \in V\}$ (the “congruence of world lines”, in the physical context with $N = 3$). Indeed, there always exists a smooth function $\lambda : V \rightarrow \mathbb{R}_+$, such that the vector field λv is complete, moreover the mappings C_X corresponding to the maximal integral curves of λv are mere reparameterizations of those of v , so that the curves l_X themselves are unchanged [16]. Thus, the assumption that Σ_U is a hypersurface of \mathcal{V} that is closed in \mathcal{V} is not a restrictive one.

The assumption “ $X \notin (S_{\Sigma_U} \cup S_{\Sigma_U \infty})$ ” means that the straight line $\mathbb{R} \times \{X\}$ is not tangent to the hypersurface Σ_U , and is not “tangent to it at infinity”. For a given hypersurface, the points thus excluded form a kind of apparent contour, hence this is true for a generic point X . However, here the hypersurface $\Sigma_U = F^{-1}(\text{Fr}(U))$ of $\mathbb{R} \times V$ depends on the selected neighborhood U of the point $X \in V$. Nevertheless, there is much freedom in the “choice” of this neighborhood U , since it is merely assigned to have a regular boundary and have the form (18) with Ω a small enough open neighborhood of $\mathbf{x} \equiv P_S(\chi(X))$.

Thus, for any point $X \in V$, the assumptions of Theorem 4 should be fulfilled in a suitable neighborhood U of X if the vector field v does not vanish and has all maximal integral curves closed in V and non-periodic — unless v has some “pathology” that we were not able to describe in a more explicit way. Therefore, we set the following definition, the word “normal” being justified by the foregoing discussion.

Definition 2. *A non-vanishing C^∞ vector field v is called “normal” iff all maximal integral curves are closed in V and, moreover, any point $X \in V$ has nested open neighborhoods $W \subset U \subset U_0$ such that: (i) There is a straightening-out chart (χ, U_0) . (ii) $\chi(U)$ has the form (18). (iii) For any maximal integral curve l of v intersecting W , the line $l \cap U$ is connected.*

In particular, a vector field v is normal in that sense if all maximal integral curves are closed in V and, moreover, any point $X \in V$ has an open neighborhood U on which there is a straightening-out chart (χ, U) , and such that, for any maximal integral curve l of v , the line $l \cap U$ is connected. (This is the case $W = U = U_0$ in the definition.) The following is an immediate consequence of Point (iii) in Proposition 2. Note that we do not need to assume that the maximal integral curves are non-periodic.

Theorem 5. *Let v be a normal vector field defined on the manifold V . For any point $X \in V$, there is a nice v -adapted chart whose domain is an open neighborhood of X .*

Examples. (i) Take $V = \mathbb{R}^{N+1}$ and consider any constant vector field $v(\mathbf{X}) = \mathbf{v} = \text{Constant} \neq \mathbf{0}$. The maximal integral curve at $\mathbf{X} \in V$ is $l_{\mathbf{X}} = \{\mathbf{Y} = \mathbf{X} + t\mathbf{v}; t \in \mathbb{R}\}$. To define a straightening-out chart explicitly, take $\mathbf{u}_1, \dots, \mathbf{u}_N$ such that the vectors $\mathbf{u}_0 \equiv \mathbf{v}, \mathbf{u}_1, \dots, \mathbf{u}_N$ form a basis of \mathbb{R}^{N+1} , and define an invertible linear transformation L of \mathbb{R}^{N+1} by $L(x^\mu \mathbf{u}_\mu) = x^\mu \mathbf{e}_\mu$, where (\mathbf{e}_μ) ($\mu = 0, \dots, N$) is the canonical basis of \mathbb{R}^{N+1} . Now consider any open set of the form $U = L^{-1}(I \times \Omega)$ with $I =] - a, +a[$ and Ω an open subset of \mathbb{R}^N : we have explicitly $U = \{s\mathbf{v} + x^j \mathbf{u}_j; s \in I, \mathbf{x} \equiv (x^j) \in \Omega\}$. The restriction χ of L to U defines a straightening-out chart, because $L(\mathbf{v}) = \mathbf{e}_0$ means that $v = \partial_0$ in that chart. Moreover, given $\mathbf{X} = s\mathbf{v} + x^j \mathbf{u}_j \in U$ (thus $s \in I, \mathbf{x} \equiv (x^j) \in \Omega$), a point $\mathbf{Y} = \mathbf{X} + t\mathbf{v}$ of $l_{\mathbf{X}}$ belongs to U , iff $s + t \in I$, so $l_{\mathbf{X}} \cap U$ is connected. Hence, a constant vector field is normal. And indeed, we

have for $\mathbf{Y} \in l_{\mathbf{x}} \cap U$: $\chi(\mathbf{Y}) = L(\mathbf{X} + t\mathbf{v}) = \chi(\mathbf{X}) + t\mathbf{e}_0$, hence χ is v -adapted. Moreover, let $l \in D_U$, where D_U is defined in Eq. (12). Thus $l = l_{\mathbf{x}}$, where $\mathbf{X} = s\mathbf{v} + x^j\mathbf{u}_j \in U$, i.e. $s \in I$, $\mathbf{x} \equiv (x^j) \in \Omega$. Then we have from the definition (14): $\bar{\chi}(l) = \mathbf{x} = L(\mathbf{x}_0)$ with $\mathbf{x}_0 \equiv x^j\mathbf{u}_j$. Hence, the mapping $\bar{\chi} : D_U \rightarrow \mathbb{R}^N$, $l = l_{\mathbf{x}} = l_{s\mathbf{v} + \mathbf{x}_0} \mapsto \mathbf{x} = L(\mathbf{x}_0)$ is injective, i.e., the v -adapted chart χ is nice.

(ii) If $\phi : V \rightarrow V'$ is a diffeomorphism, set $v' \equiv \phi^*v : v'(X') \equiv D\phi_{\phi^{-1}(X')}(v(\phi^{-1}(X')))$, $X' \in V'$. The maximal integral curves of v' , as well as the associate flow F' , are just the images of their counterparts for v : $F'(s, X') \equiv C'_{X'}(s) = \phi(C_{\phi^{-1}(X')}(s)) = \phi(F(s, \phi^{-1}(X')))$. Moreover, if (χ, U_0) is a straightening-out chart in the neighborhood of $X \in V$, then so is $(\chi \circ \phi^{-1}, \phi(U_0))$ in the neighborhood of $X' \equiv \phi(X) \in V'$. Therefore, if v is normal, so is v' .

(iii) These two examples are rather trivial from the mathematical point of view. By combining the two, we get that, if a manifold V is diffeomorphic to \mathbb{R}^{N+1} and $\phi : \mathbb{R}^{N+1} \rightarrow V$ is any diffeomorphism, then for any constant vector field $\mathbf{v} \neq \mathbf{0}$ on \mathbb{R}^{N+1} , its pushforward vector field by ϕ , $v = \phi^*\mathbf{v}$, is a normal vector field on V . In the application to physics (for which $N = 3$, as far as we know), this describes already a wide variety of vector fields, together with the associated reference fluids. Those are deformable in a very general way with respect to each other, by changing ϕ , i.e. by transforming the integral curves by any diffeomorphism of V . Of course, we expect that much more general normal vector fields do exist, due to the discussion at the beginning of this subsection.

3 The set of orbits of v as a differentiable manifold

The set N_v of the maximal integral curves of v , defined in (11), can be seen as the quotient space of the set V by the equivalence relation “ $X \sim Y$ iff $l_X = l_Y$ ”. In this section, we will show that, when v is a normal vector field on the differentiable manifold V , the set N_v can be endowed with a canonical structure of differentiable manifold.

Proposition 6. *Let v be a normal vector field on V . Define the set \mathcal{F}_v made of all nice v -adapted charts on V . For any chart $\chi \in \mathcal{F}_v$, with domain $U \subset V$, let D_U be defined by (12), and, for any subset $\mathcal{O} \subset N_v$, define $\bar{\chi}(\mathcal{O}) \equiv \bar{\chi}(\mathcal{O} \cap D_U)$, where $\bar{\chi}$ is defined in Eq. (14) on $\text{Dom}(\bar{\chi}) \equiv D_U$. Let \mathcal{T}' be the set of the subsets $\mathcal{O} \subset N_v$ such that*

$$\forall \chi \in \mathcal{F}_v, \quad \bar{\chi}(\mathcal{O}) \text{ is an open set in } \mathbb{R}^N. \quad (70)$$

The set \mathcal{T}' is a topology on N_v .

Proof. This is an adaptation of the proof of Proposition C in Ref. [8], replacing M by N_v , F by \mathcal{F}_v , $\tilde{\chi}$ by $\bar{\chi}$, and \mathbb{R}^3 by \mathbb{R}^N . In particular, the proof that the whole set N_v (instead of M) belongs to \mathcal{T}' is exactly identical. Also, by definition of a nice v -adapted chart, the mapping $\bar{\chi} : D_U \rightarrow \mathbb{R}^N$ is injective. Therefore, if $\mathcal{O}_1 \in \mathcal{T}'$ and $\mathcal{O}_2 \in \mathcal{T}'$, we have

$$\begin{aligned} \bar{\chi}(\mathcal{O}_1 \cap \mathcal{O}_2) &\equiv \bar{\chi}((\mathcal{O}_1 \cap \mathcal{O}_2) \cap D_U) = \bar{\chi}((\mathcal{O}_1 \cap D_U) \cap (\mathcal{O}_2 \cap D_U)) \\ &= \bar{\chi}(\mathcal{O}_1 \cap D_U) \cap \bar{\chi}(\mathcal{O}_2 \cap D_U) \equiv \bar{\chi}(\mathcal{O}_1) \cap \bar{\chi}(\mathcal{O}_2), \end{aligned} \quad (71)$$

which is thus an open set of \mathbb{R}^N , for any $\chi \in \mathcal{F}_v$, so that $\mathcal{O}_1 \cap \mathcal{O}_2 \in \mathcal{T}'$. It is also trivial to check that the union of any family of subsets $\mathcal{O}_i \in \mathcal{T}'$ is still an element of \mathcal{T}' . \square

To prove that the mappings $\bar{\chi}$ are continuous for this topology, and to prove the compatibility of any two mappings $\bar{\chi}, \bar{\chi}'$ on N_v , associated with two nice v -adapted charts $\chi, \chi' \in \mathcal{F}_v$, the following difficulty arises: χ and χ' have in general different domains U and U' . We may have $U \cap U' = \emptyset$, although there is some $l \in N_v$ with

$$l \cap U \neq \emptyset, \quad l \cap U' \neq \emptyset. \quad (72)$$

I.e., it may happen that the domains of the charts χ and χ' do not overlap, and that the domains of the mappings $\bar{\chi}$ and $\bar{\chi}'$ do. To overcome this difficulty, we use the flow F of the vector field v to associate smoothly with any point Y in some neighborhood $W \subset U$ of a point X , a point $g(Y) \in U'$:

Lemma. *Let v be a C^∞ vector field on V . Let $\chi, \chi' \in \mathcal{F}_v$, with domains U and U' , be such that $D_U \cap D_{U'} \neq \emptyset$. Let $l \in D_U \cap D_{U'}$ and $X \in l \cap U$ and set $\chi(X) = (t, \mathbf{x})$, so that $\bar{\chi}(l) = \mathbf{x}$. There is an open neighborhood Ω of \mathbf{x} in \mathbb{R}^N and a C^∞ mapping g defined on an open neighborhood $W \subset U$*

of X , such that for any $\mathbf{y} \in \Omega$ we have $(t, \mathbf{y}) \in \chi(W)$ (so that $\mathbf{y} \in \bar{\chi}(D_U)$), $\bar{\chi}^{-1}(\mathbf{y}) \in D_{U'}$, and

$$\forall \mathbf{y} \in \Omega, \quad \bar{\chi}' \circ \bar{\chi}^{-1}(\mathbf{y}) = P_S(\chi'(g(\chi^{-1}(t, \mathbf{y}))))). \quad (73)$$

Proof. Since $l \in D_{U'}$, there is some point $X' \in l \cap U'$, and since $X \in l$ there is some $s \in I_X$ such that $X' = F(s, X)$. Thus, the domain \mathcal{D} of F being open in $\mathbb{R} \times V$ and F being continuous, there is an interval I centered at s and an open neighborhood $W \subset U$ of X in V , such that $I \times W \subset \mathcal{D}$ and $F(I \times W) \subset U'$. For $Y \in W$, set $g(Y) = F(s, Y)$. This defines a \mathcal{C}^∞ -mapping $g : W \rightarrow U'$. Because $\chi(W)$ is open in \mathbb{R}^{N+1} and $(t, \mathbf{x}) \in \chi(W)$, there is an interval J centered at t and an open neighborhood Ω of \mathbf{x} in \mathbb{R}^N , with $J \times \Omega \subset \chi(W)$. Hence, if $\mathbf{y} \in \Omega$, then indeed $(t, \mathbf{y}) \in \chi(W)$, thus $Y \equiv \chi^{-1}(t, \mathbf{y}) \in W \subset U$, which implies that $l_Y \in D_U$ and that

$$\mathbf{y} = P_S(\chi(Y)) = \bar{\chi}(l_Y), \quad (74)$$

so $\mathbf{y} \in \bar{\chi}(D_U)$. Moreover, we have $Y' \equiv g(Y) = F(s, Y) \in l_Y \cap U'$, so $l_Y = \bar{\chi}^{-1}(\mathbf{y}) \in D_{U'}$ and

$$\mathbf{y}' \equiv P_S(\chi'(Y')) = \bar{\chi}'(l_Y), \quad (75)$$

whence follows (73). \square

Theorem 6. *Let v be a normal vector field on V . Let \mathcal{A} be the set of all mappings $\bar{\chi}$, where $\chi \in \mathcal{F}_v$. This set \mathcal{A} is an atlas on the topological space (N_v, \mathcal{T}') .*

Proof. (i) Consider any $\chi' \in \mathcal{F}_v$, with domain U' . Let us prove that $\bar{\chi}'$, which is defined on $D_{U'}$, is continuous for the topology induced on $D_{U'}$ by the topology \mathcal{T}' on N_v . Thus, A being any open set in \mathbb{R}^N , we must show that the set $\mathcal{O}_1 \equiv \bar{\chi}'^{-1}(A)$ has the form $\mathcal{O}_1 = \mathcal{O} \cap D_{U'}$, where \mathcal{O} is such that we have (70). We shall actually show that $\mathcal{O}_1 \in \mathcal{T}'$, i.e., that we have (70) with $\mathcal{O} \equiv \mathcal{O}_1 \subset D_{U'}$. To prove this, we may assume that $\mathcal{O}_1 \neq \emptyset$. Moreover, let $\chi \in \mathcal{F}_v$, with domain U . We may also assume that $\bar{\chi}(\mathcal{O}_1) \neq \emptyset$, i.e. $\mathcal{O}_1 \cap D_U \neq \emptyset$, so let $\mathbf{x} \in \bar{\chi}(\mathcal{O}_1)$. We have to find a neighborhood B of \mathbf{x} in \mathbb{R}^N , such that $B \subset \bar{\chi}(\mathcal{O}_1)$, i.e., $B \subset \bar{\chi}(\bar{\chi}'^{-1}(A))$. Since $\mathbf{x} \in \bar{\chi}(\mathcal{O}_1) \equiv \bar{\chi}(\mathcal{O}_1 \cap D_U)$, we have that $l \equiv \bar{\chi}^{-1}(\mathbf{x}) \in (\bar{\chi}'^{-1}(A)) \cap D_U \subset D_{U'} \cap D_U$. In particular, there

is some $X \in l \cap U$, with $\chi(X) = (t, \mathbf{x})$ for some $t \in \mathbb{R}$. Hence, we may apply the Lemma and get the corresponding open neighborhood Ω of \mathbf{x} . Thus (73) shows that the mapping $\mathbf{f} \equiv \bar{\chi}' \circ \bar{\chi}^{-1}$ is well defined and continuous over Ω . Hence, since $\mathbf{x} \in \Omega$ and $\mathbf{x}' \equiv \mathbf{f}(\mathbf{x}) = \bar{\chi}'(l) \in A$, which is open in \mathbb{R}^N , there is a neighborhood $B \subset \Omega$ of \mathbf{x} in \mathbb{R}^N , such that $\mathbf{f}(B) \subset A$. Set $\mathbf{f}' \equiv \bar{\chi} \circ \bar{\chi}'^{-1} = \mathbf{f}^{-1}$. For $\mathbf{y} \in B$, we have thus $\mathbf{y}' \equiv \mathbf{f}(\mathbf{y}) \in A$, that is, $\mathbf{y} = \mathbf{f}'(\mathbf{y}') \in \mathbf{f}'(A) = (\bar{\chi} \circ \bar{\chi}'^{-1})(A)$. Thus, for any open set A in \mathbb{R}^N , the set $\mathcal{O}_1 \equiv \bar{\chi}'^{-1}(A)$ belongs to \mathcal{T}' , as announced. This proves that $\bar{\chi}'$ is continuous for the topology induced on $D_{U'}$ by the topology \mathcal{T}' on N_v . Moreover, taking $A = \mathbb{R}^N$, we get that $D_{U'} = \bar{\chi}'^{-1}(\mathbb{R}^N)$ is open in N_v , $D_{U'} \in \mathcal{T}'$.

(ii) Given any $\chi \in \mathcal{F}_v$, with domain U , let us show that the mapping $\bar{\chi}^{-1} : \bar{\chi}(D_U) \rightarrow D_U \subset N_v$, is continuous. Since D_U is open as seen at the end of (i), we have to show that, for any $\mathcal{O} \in \mathcal{T}'$ such that $\mathcal{O} \subset D_U$, the set $\Omega \equiv (\bar{\chi}^{-1})^{-1}(\mathcal{O})$ is open in \mathbb{R}^N . But since $\mathcal{O} \subset D_U = \text{Dom}(\bar{\chi})$, and since $\bar{\chi}$ is injective, we have $\Omega = \bar{\chi}(\mathcal{O})$. The fact that this is open in \mathbb{R}^N follows from the very definition of the topology \mathcal{T}' in Eq. (70). With (i), this means that, for any $\chi \in \mathcal{F}_v$, the mapping $\bar{\chi} : D_U \rightarrow \bar{\chi}(D_U) \subset \mathbb{R}^N$ is bicontinuous, thus is indeed a *chart* on the topological space (N_v, \mathcal{T}') .

(iii) Let us show that the domains of definition of the charts $\bar{\chi}$, for $\chi \in \mathcal{F}_v$, cover the whole set N_v . Given $l \in N_v$, let $X \in l$. Since v is a normal vector field on V , we know from Theorem 5 that there is a nice v -adapted chart χ whose domain U is a neighborhood of X . Thus $X \in l \cap U$, so $l \in D_U$, Q.E.D.

(iv) Finally, given any two nice v -adapted charts $\chi, \chi' \in \mathcal{F}_v$, having domains U and U' respectively, let us show the compatibility of the two charts $\bar{\chi}, \bar{\chi}'$ on N_v , with domains D_U and $D_{U'}$. The relevant case is when $D_U \cap D_{U'} \neq \emptyset$, so that $\text{Dom}(\bar{\chi}' \circ \bar{\chi}^{-1}) = \bar{\chi}(D_U \cap D_{U'}) \neq \emptyset$. Thus we may apply the Lemma. Its Eq. (73) shows that, given any $\mathbf{x} \in \bar{\chi}(D_U \cap D_{U'})$, it has a neighborhood Ω in which the function $\bar{\chi}' \circ \bar{\chi}^{-1}$ is given as a composition of \mathcal{C}^∞ functions, hence $\bar{\chi}' \circ \bar{\chi}^{-1}$ is \mathcal{C}^∞ on its domain. \square

Thus, we have endowed the set N_v with first the topology \mathcal{T}' defined by (70), and then with a canonical atlas \mathcal{A} of compatible charts, which are simply the mappings $\bar{\chi}$, where $\chi \in \mathcal{F}_v$ is any nice v -adapted chart. To call this a differentiable manifold in the rather usual sense of Note 2 needs that the

topological space (N_v, \mathcal{T}') be metrizable and separable — hence, in particular, that it be Hausdorff. We do not have very general results on this point.

Proposition 7. *Let v be a normal vector field on V . (i) Any two points $l \neq l'$ in the orbit space N_v are topologically distinguishable, i.e., there exists an open set $\mathcal{O} \in \mathcal{T}'$ such that $l \in \mathcal{O}$ and $l' \notin \mathcal{O}$.*

(ii) Suppose $l \neq l'$ but l and l' both belong to the domain D_U of a chart $\bar{\chi}$, where $\chi \in \mathcal{F}_v$ with $\text{Dom } \chi = U$. Then l and l' are separated by neighborhoods, i.e., there are two open sets $\mathcal{O}, \mathcal{O}' \in \mathcal{T}'$, such that $l \in \mathcal{O}$, $l' \in \mathcal{O}'$, and $\mathcal{O} \cap \mathcal{O}' = \emptyset$.

(iii) Suppose there is a chart $\chi \in \mathcal{F}_v$, such that any maximal integral curve $l \in N_v$ intersects its domain U , so that $D_U = N_v$. Then the topological space (N_v, \mathcal{T}') is metrizable and separable.

Proof. (i) Let $X \in l$. There is some open neighborhood U of X , such that $U \cap l' = \emptyset$: if that were not true, then, taking any distance on V that defines its topology, and considering $U_n \equiv B(X, 1/n)$, we would get a sequence (X_n) with $X_n \in l'$ and $X_n \rightarrow X$, hence $X \in l'$ since we defined that a normal vector field has all its maximal integral curves closed; but this implies that $l = l'$, which is a contradiction. By Theorem 5, let $\chi \in \mathcal{F}_v$ whose domain U_2 is an open neighborhood of X , and set $U_1 \equiv U \cap U_2$. The restriction χ_1 of χ to $U_1 \subset U$ is still a nice v -adapted chart: it is obviously v -adapted, we have $D_{U_1} \subset D_U$, and the mapping $\bar{\chi}_1 : D_{U_1} \rightarrow \mathbb{R}^N$ is clearly the restriction of $\bar{\chi}$ to D_{U_1} , hence it also is injective. Therefore, $\mathcal{O} \equiv D_{U_1}$ is an open set, $\mathcal{O} \in \mathcal{T}'$. Since $X \in l \cap U_1$, we have $l \in \mathcal{O}$; and since $l' \cap U_1 \subset l' \cap U = \emptyset$, we have $l' \notin \mathcal{O}$.

(ii) Since $l \in D_U$ and $l' \in D_U$ with $l \neq l'$, and since $\bar{\chi}$ is defined on D_U and injective, we have

$$\mathbf{x} \equiv \bar{\chi}(l) \neq \mathbf{x}' \equiv \bar{\chi}(l'). \quad (76)$$

Let Ω and Ω' be open neighborhoods in \mathbb{R}^N of \mathbf{x} and \mathbf{x}' respectively, such that $\Omega \cap \Omega' = \emptyset$. Set $\mathcal{O} \equiv \bar{\chi}^{-1}(\Omega)$ and $\mathcal{O}' \equiv \bar{\chi}^{-1}(\Omega')$. These are open sets such that $l \in \mathcal{O}$ and $l' \in \mathcal{O}'$, and we have $\mathcal{O} \cap \mathcal{O}' = \bar{\chi}^{-1}(\Omega \cap \Omega') = \emptyset$.

(iii) This follows from the fact that $\bar{\chi}$ is a homeomorphism of its domain D_U onto its range $\bar{\chi}(D_U) \subset \mathbb{R}^N$. \square

Proposition 8. *Let v be a normal vector field on V . Assume there is a countable cover of V by open sets U_n such that, for any integer n , there is a nice v -adapted chart, $\chi_n \in \mathcal{F}_v$, having domain U_n . Then, setting $D_n \equiv D_{U_n}$, the sequence (D_n) is a countable cover of N_v by metrizable open subsets, and the topological space (N_v, \mathcal{T}') is separable.*

Proof. Note first that, since $\chi_n \in \mathcal{F}_v$, with domain U_n , it follows from Theorem 6 that D_n , the domain of the associated chart $\bar{\chi}_n$ on N_v , is open in N_v . If $l \in N_v$, let $X \in l$ and, (U_n) being a cover of V , let n be such that $X \in U_n$. We have thus $l \cap U_n \neq \emptyset$, i.e. $l \in D_n$. So (D_n) is a countable open cover of N_v . Since $\bar{\chi}_n$ is a homeomorphism of D_n onto $\bar{\chi}_n(D_n) \subset \mathbb{R}^N$, it follows that D_n is a metrizable and separable space. Hence it is second-countable, i.e., there exists a countable basis $(\mathcal{O}_{nm})_{m \in \mathbb{N}}$ for the open subsets of D_n . Since any open subset \mathcal{O} of N_v is the countable union of the open subsets $\mathcal{O}_n \equiv \mathcal{O} \cap D_n$ of D_n , we have that $(\mathcal{O}_{nm})_{n,m \in \mathbb{N}}$ is a countable basis for the topology \mathcal{T}' of N_v . Thus also N_v is second-countable, hence it is separable. \square

Example. It follows from Point (iii) in Proposition 7 that Example (iii) considered after Theorem 5 provides us with a physically important case of normal vector field v for which the orbit set N_v is metrizable and separable — and hence, by Theorem 6, is a differentiable manifold. Namely, if a manifold V is diffeomorphic to \mathbb{R}^{N+1} and $\phi : \mathbb{R}^{N+1} \rightarrow V$ is any diffeomorphism, then for any constant vector field $\mathbf{v} \neq \mathbf{0}$ on \mathbb{R}^{N+1} , the orbit set N_v of the pushforward vector field by ϕ , $v = \phi^*\mathbf{v}$, is a differentiable manifold. Indeed, the unique chart $L \circ \phi^{-1}$ covers V [where L is defined in Example (i)] and is a nice v -adapted chart.

4 The local manifold as an open subset of the global one

Let v be a normal vector field on V . In addition, as in Subsect. 1.2, let F be a (local) reference frame, thus an equivalence class of charts for the relation (5), in which U is a given open subset of V .⁴ In Subsect. 1.2, the local space

⁴ As in Sects. 2 and 3, the dimension of V is $N + 1$, where N is any integer ≥ 1 . All results summarized in Subsect. 1.2 hold true if one substitutes any integer $N \geq 1$ for the

manifold M_F associated with F was defined as the set of the world lines (6). On the other hand, the orbit set N_v defined in Subsect. 2.1: the set of the maximal integral curves of v , was endowed in Sect. 3 with a topology \mathcal{T}' and an atlas \mathcal{A} , which (assuming that \mathcal{T}' is metrizable and separable) makes it a differentiable manifold. Thus, we have also a global space manifold: N_v . When the charts $\chi \in F$, all having domain U , are nice v -adapted charts, i.e. belong to \mathcal{F}_v , we have the following tight relation between M_F and N_v :

Theorem 7. *Assume that $F \subset \mathcal{F}_v$. For any $l \in M_F$, there is a unique maximal integral curve $l' \in N_v$ such that, for any $X \in l$, we have $l' = l_X$. It holds $l = l' \cap U$. The mapping $I : l \mapsto l'$ is a diffeomorphism of M_F onto the open subset D_U of N_v .*

Proof. Let $l \in M_F$. By the definition of M_F near Eq. (6), there is some chart $\chi \in F$ and some $\mathbf{x} \in P_S(\chi(U)) \subset \mathbb{R}^N$, such that

$$l = \{ X \in U; P_S(\chi(X)) = \mathbf{x} \}. \quad (77)$$

Let $X_1 \in l$ and $X_2 \in l$. Denote the maximal integral curves of v at X_1 and X_2 as $l'_1 \equiv l_{X_1}$, $l'_2 \equiv l_{X_2}$. Since χ is v -adapted, there exist $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^N$, such that $\forall X \in l'_1 \cap U$, $P_S(\chi(X)) = \mathbf{x}_1$; and $\forall X \in l'_2 \cap U$, $P_S(\chi(X)) = \mathbf{x}_2$. In particular, since $X_j \in l'_j \cap U$, we have $P_S(\chi(X_j)) = \mathbf{x}_j$ ($j = 1, 2$). But since $X_j \in l$, we have also $P_S(\chi(X_j)) = \mathbf{x}$ by (77), so $\mathbf{x} = \mathbf{x}_1 = \mathbf{x}_2$. Because the v -adapted chart χ is nice, it follows that $l'_1 = l'_2$. Therefore, we define a mapping $I : M_F \rightarrow N_v$ by associating with any $l \in M_F$, the unique maximal integral curve $l' \in N_v$, such that for any $X \in l$, we have $l' = l_X$. Note that actually $l' \in D_U$. Owing to the definitions (7) and (14), we have $\mathbf{x} = \tilde{\chi}(l) = \bar{\chi}(l')$, and since here l is any element of M_F , this shows that $\tilde{\chi}(M_F) \subset \bar{\chi}(D_U)$. Thus

$$I(l) = \bar{\chi}^{-1}(\tilde{\chi}(l)), \quad (78)$$

so that we have simply $I = \bar{\chi}^{-1} \circ \tilde{\chi}$, for whatever chart $\chi \in F$. Let us show that $l = l' \cap U$. By definition, for any $X \in l$, we have $l' = l_X$, hence $X \in l'$, and since $l \subset U$ by the definition (77), we have $l \subset l' \cap U$. Conversely, consider any $\chi \in F$; this is by assumption a v -adapted chart and, as we showed before (78), we have $\bar{\chi}(l') = \tilde{\chi}(l) \equiv \mathbf{x}$. Therefore, by the definition (13), we have for any $X \in l' \cap U$: $P_S(\chi(X)) = \mathbf{x}$. Then (77) implies that $X \in l$, so

integer 3, and $N + 1$ for 4 [8].

$l' \cap U \subset l$.

As we showed, the mapping I is defined on the whole of M_F and ranges into D_U , which is an open subset of N_v . Let us show that in fact $I(M_F) = D_U$. Let $l' \in D_U$, so there exists $X \in l' \cap U$. Let $\chi \in F$ and set $\mathbf{x} \equiv P_S(\chi(X))$ and $l \equiv \{Y \in U; P_S(\chi(Y)) = \mathbf{x}\}$. Clearly $l \in M_F$ and $X \in l$. By the definition of I , we have that, for any $Y \in l$, $l_Y = I(l)$. In particular, $l_X = I(l)$. But since $X \in l'$, we have $l_X = l'$, hence $l' = I(l)$: thus indeed $D_U \subset I(M_F)$. Note that again here, from the definitions (7) and (14) we have $\mathbf{x} = \tilde{\chi}(l) = \bar{\chi}(l')$, and since now l' is any element of D_U , this shows that $\bar{\chi}(D_U) \subset \tilde{\chi}(M_F)$. Since the reverse inclusion has been proved before (78), we have $\bar{\chi}(D_U) = \tilde{\chi}(M_F)$.

As shown in Ref. [8], $\tilde{\chi}$ is a global chart on the differentiable manifold M_F , for any $\chi \in F$. As shown in Theorem 6, $\bar{\chi}$ is a chart with domain D_U on the differentiable manifold N_v , also for any $\chi \in F$. Moreover, as we just saw, we have $\bar{\chi}(D_U) = \tilde{\chi}(M_F)$. It follows that the one-to-one mapping $I = \bar{\chi}^{-1} \circ \tilde{\chi}$, from $\text{Dom}(\tilde{\chi}) = M_F$ onto $I(M_F) = D_U = \text{Dom}(\bar{\chi})$, is a diffeomorphism. Therefore, I is an immersion of M_F into N_v . Actually, recall that D_U is more specifically an *open subset* of N_v . \square

Thus, if $F \subset \mathcal{F}_v$, we may identify the local space M_F with the open subset $I(M_F) = D_U$ of the global space N_v . Since we proved that $l = I(l) \cap U$, we can say that M_F is made of the intersections with the local domain U of the maximal integral curves of v . Given that each world line $l \in M_F$ is invariant under any exchange of the chart $\chi \in F$ for another chart $\chi' \in F$, to say that $F \subset \mathcal{F}_v$ is equivalent to say that *one* chart $\chi \in F$ is a nice v -adapted chart.

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