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# COUPLING SYMMETRIES WITH POISSON STRUCTURES

CAMILLE LAURENT-GENGOUX AND EVA MIRANDA

**ABSTRACT.** We study local normal forms for completely integrable systems on Poisson manifolds in the presence of additional symmetries. The symmetries that we consider are encoded in actions of compact Lie groups. The existence of Weinstein's splitting theorem for the integrable system is also studied giving some examples in which such a splitting does not exist, i.e. when the integrable system is not, locally, a product of an integrable system on the symplectic leaf and an integrable system on a transversal. The problem of splitting for integrable systems with additional symmetries is also considered.

## 1. INTRODUCTION

Integrable Hamiltonian systems have been widely studied in the context of symplectic manifolds. The existence of action-angle coordinates (semilocal and global) under some additional conditions have become a main goal in this area. For an extensive study of the existence of action-angle coordinates in symplectic manifolds, we refer to Arnold [1] and Duistermaat [9]. Several extensions of the concept of complete integrability have been done in the symplectic context. We refer to Dazord and Delzant [7] and Nekhoroshev [22] for more details. However, the most natural framework for several dynamical systems is the framework of Poisson manifolds. For instance, the Gelfand-Cetlin system, whose underlying Poisson structure is the dual of a Lie algebra [11], arises from a non-symplectic Poisson structure.

In [13], we proved an action-angle theorem for completely integrable systems within the Poisson context. Indeed, we proved an action-angle theorem for non-commutative integrable systems, non-commutative integrable systems being systems for which there are more constants of motion than required in order to prove Liouville integrability. An important point is that a non-commutative integrable systems may be regular even at singular points of the Poisson structure.

The construction of these *action-angle* coordinates goes through the construction of a natural Hamiltonian  $\mathbb{T}^n$  action tangent to the fibers of the

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moment map. So, as it happens in the symplectic case, we can naturally let a compact Abelian group act in the integrable system. In [13], however, we did not address what happens when there are additional symmetries encoded in actions of compact Lie groups. As a first step in that direction, we prove a local equivariant Carathéodory-Jacobi-Lie theorem<sup>1</sup> for the Poisson structure. The existence of an equivariant analogue of the action-angle theorem for non-commutative integrable systems will be considered in a future work.

Another interesting problem in the Poisson context is that of splitting of completely integrable systems. Weinstein's celebrated splitting theorem [23] asserts that locally a Poisson manifold is a direct product of a symplectic manifold and a Poisson manifold of rank-0 (transversal Poisson structures). One can attempt to apply a similar strategy taking into account additional symmetries. Nguyen Tien Zung and the second author of this paper considered this problem in [19] where an equivariant splitting theorem was proved under some mild assumptions. Still the problem of studying the possibility of splitting, not only the Poisson structure, but an integrable system and a Poisson structure simultaneously was not considered before. We also consider the problem of splitting the integrable system and the Poisson structure taking into account additional symmetries (equivariant splitting theorem for completely integrable systems). We do not claim to have settled the question in the present note, but we claim to have at least clarified some elementary and less elementary facts that shall be the basis of a more general study.

### **Organization of this paper:**

In Section 2, we obtain an equivariant Carathéodory-Jacobi-Lie theorem for Poisson structures which can be seen as a variant of Weinstein's splitting theorem [23] in which normal forms for a set of functions are also taken into account.

In Section 3, we consider the problem of splitting completely integrable systems. In particular, we give some counterexamples that make the question more precise. In [12], we will present the obstruction of integrability in terms of foliated Poisson cohomology.

In Section 4, we combine results and techniques in Section 2 and Section 3 to study equivariant normal forms for regular completely integrable systems which are split.

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<sup>1</sup>We could also call this theorem equivariant Carathéodory-Weinstein theorem as its equivalent for symplectic forms is often called the Darboux-Carathéodory theorem. But we prefer to stick to the old denomination Carathéodory-Jacobi-Lie since Lie already worked partial aspects of this result in (see Satz 3 on p. 198 in [16]).

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## 2. EQUIVARIANT CARATHÉODORY-JACOBI-LIE THEOREM FOR POISSON MANIFOLDS

We recall the definitions of a (Liouville) completely integrable system and of an involutive family on a Poisson manifold. We refer to [14] and [13] for a more detailed introduction to that matter.

**Definition 2.1.** Let  $(M, \Pi = \{.,.\})$  be a Poisson manifold of rank  $2r$  and of dimension  $n$ . An  $s$ -tuple of functions  $\mathbf{F} = (f_1, \dots, f_s)$  on  $M$  is said to define a (Liouville) *integrable system* on  $(M, \Pi)$  if

- (1) the functions  $f_1, \dots, f_s$  are independent,
- (2) for all  $i, j = 1, \dots, s$ , the functions  $f_i$  and  $f_j$  are in involution, i.e.  $\{f_i, f_j\} = 0$ ,
- (3)  $r + s = n$ .

When only items (1) and (2) hold true, we shall speak of a *involutive family*.

The map  $\mathbf{F} : M \rightarrow \mathbf{R}^s$  is called the *momentum map* of  $(M, \Pi, \mathbf{F})$ . For the moment, we leave aside Liouville completely integrable systems, and start by giving equivariant normal forms when only the first and second items in the previous definition are satisfied, but, a priori, not the third one. The following theorem is the classical Carathéodory-Jacobi-Lie theorem, but stated within the context of Poisson manifolds, as stated in [13].

**Theorem 2.2.** Let  $(M, \Pi)$  be a Poisson manifold of dimension  $n = 2r + s$ . Let  $p_1, \dots, p_r$  an involutive family, defined on a neighborhood of some given point  $m \in M$ , which vanish at  $m$  and whose Hamiltonian vector fields are linearly independent at  $m$ . There exist a neighborhood  $U$  on  $m$  and functions  $q_1, \dots, q_r, z_1, \dots, z_s$  on  $U$ , such that

- (1) the  $n$  functions  $(p_1, q_1, \dots, p_r, q_r, z_1, \dots, z_s)$  form a system of local coordinates, centered at  $m$ ;
- (2) the Poisson structure  $\Pi$  is given in these coordinates by

$$\Pi = \sum_{i=1}^r \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i,j=1}^s g_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \quad (2.1)$$

where each function  $g_{ij}(z)$  is a smooth function and is independent from  $p_1, \dots, p_r, q_1, \dots, q_r$ .

The rank of  $\Pi$  at  $m$  is  $2r$  if and only if all the matrix  $(g_{ij}(z))_{i,j=1}^s$  vanishes for  $z = 0$ .

In this section we prove an equivariant version of Theorem 2.2. Our proof follows roughly the lines of [13].

**Theorem 2.3.** *Let  $(M, \Pi)$  be a Poisson manifold of dimension  $n = 2r + s$ , acted upon by a compact Lie group  $G$ , whose action preserves the Poisson structure. Let  $p_1, \dots, p_r$  be an involutive family of  $G$ -invariant functions, defined on a neighborhood of a fixed point  $m$  of the  $G$ -action, whose Hamiltonian vector fields are linearly independent at  $m$ . There exist a neighborhood  $U$  of  $m$  and functions  $q_1, \dots, q_r, z_1, \dots, z_s$  on  $U$ , such that*

- (1) *the  $n$  functions  $(p_1, q_1, \dots, p_r, q_r, z_1, \dots, z_s)$  form a system of local coordinates, centered at  $m$ ;*
- (2) *the Poisson structure  $\Pi$  is given in these coordinates by*

$$\Pi = \sum_{i=1}^r \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i,j=1}^s g_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \quad (2.2)$$

*where each function  $g_{ij}(z)$  is a smooth function and is independent from  $p_1, \dots, p_r, q_1, \dots, q_r$ .*

- (3) *There exists a group homomorphism from  $G$  to the group of  $s \times s$  invertible matrices, denoted by  $g \rightarrow A(g)$ , so that the  $G$ -action on  $U$  is given, in the previous coordinates, for all  $g \in G$ , by*

$$\rho(g, (p_1, q_1, \dots, p_r, q_r, z_1, \dots, z_s)) = (p_1, q_1, \dots, p_r, q_r, \sum_{i=1}^s m_{i,1}(g)z_i, \dots, \sum_{i=1}^s m_{i,s}(g)z_i).$$

*The rank of  $\Pi$  at  $m$  is  $2r$  if and only if all the matrix  $(g_{ij}(z))_{i,j=1}^s$  vanishes for  $z = 0$ .*

*Proof.* We show the theorem by induction on  $r$ . For  $r = 0$ , the theorem reduces to the classical theorem of Bochner, that states that there exists a local system of coordinates  $z_1, \dots, z_n$ , centered at  $m \in M$ , on which the action of  $G$  is linear. Any such a system of coordinates satisfies the three items of theorem 2.3.

Now assume that the result holds true for every point in every Poisson manifold and every  $r - 1$  tuple of functions as above, with  $r \geq 1$ , and let us prove that this still holds for arbitrary such  $r$ -tuples. For that purpose, we consider an arbitrary point  $m$  in a Poisson manifold  $(M, \Pi)$ , assumed to be a fixed point for the action of  $G$ , and we assume that we are given  $G$ -invariant and Poisson commuting functions  $p_1, \dots, p_r$ , defined on a neighborhood of  $m$ , which vanish at  $m$ , and whose Hamiltonian vector fields are linearly independent at  $m$ . On a neighbourhood of  $m$ , the distribution  $\mathcal{D} := \langle \mathcal{X}_{p_1}, \dots, \mathcal{X}_{p_r} \rangle$  has constant rank  $r$  and is an involutive distribution because  $[\mathcal{X}_{p_i}, \mathcal{X}_{p_j}] = -\mathcal{X}_{\{p_i, p_j\}} = 0$ . By Frobenius Theorem, there exist local coordinates  $g_1, \dots, g_n$ , centered at  $m$ , where  $n := \dim M$ , such that  $\mathcal{X}_{p_i} = \frac{\partial}{\partial g_i}$  for  $i = 1, \dots, r$ , on a neighbourhood  $V$  of  $m$ . Setting  $\tilde{q}_r := g_r$  we

have,

$$\mathcal{X}_{\tilde{q}_r}[p_i] = -\mathcal{X}_{p_i}[\tilde{q}_r] = \{\tilde{q}_r, p_i\} = -\delta_{i,r}, \quad i = 1, \dots, r. \quad (2.3)$$

Without any loss of generality, one can assume that the neighbourhood  $V$  is  $G$ -invariant. Let  $q_r = \int_{g \in G} g^* \tilde{q}_r d\mu$  where  $\mu$  is the normalized Haar measure on  $G$  and  $g^* f$  is, for any function  $f$  defined on a  $g$ -invariant subset of  $M$ , a shorthand for the function  $m \rightarrow f(\rho(g, m))$ . The function  $q_r$  is  $G$ -invariant by construction. Since  $G$  acts by preserving the Poisson structure, and since  $p_1, \dots, p_r$  are  $G$ -invariant functions, we have, for all  $g \in G$ ,

$$\{g^* \tilde{q}_r, p_i\} = \{g^* \tilde{q}_r, g^* p_i\} = g^* \{\tilde{q}_r, p_i\} = -g^* \delta_{i,r} = -\delta_{i,r} \quad i = 1, \dots, r.$$

Integrating this last inequality we obtain  $\{q_r, p_i\} = -\delta_{i,r}$ , and

$$\mathcal{X}_{q_r}[p_i] = -\mathcal{X}_{p_i}[q_r] = \{q_r, p_i\} = -\delta_{i,r}, \quad i = 1, \dots, r. \quad (2.4)$$

In particular, the  $r+1$  covectors  $d_m p_1, \dots, d_m p_r$  and  $d_m q_r$  of  $T_m^* M$  are linearly independent.

The distribution  $\mathcal{D}' = \langle \mathcal{X}_{p_r}, \mathcal{X}_{q_r} \rangle$  has rank 2 at  $m$ , hence in a neighborhood of  $m$ . It is an integrable distribution because  $[\mathcal{X}_{q_r}, \mathcal{X}_{p_r}] = -\mathcal{X}_{\{q_r, p_r\}} = 0$ . Applied to  $\mathcal{D}'$ , Frobenius Theorem shows that there exist local coordinates  $v_1, \dots, v_n$ , centered at  $m$ , such that

$$\mathcal{X}_{p_r} = \frac{\partial}{\partial v_{n-1}} \quad \text{and} \quad \mathcal{X}_{q_r} = \frac{\partial}{\partial v_n}. \quad (2.5)$$

It is clear that the covectors  $d_m v_1, \dots, d_m v_{n-2}$  vanish on  $\mathcal{X}_{p_r}(m)$  and on  $\mathcal{X}_{q_r}(m)$ , so that  $(d_m v_1, \dots, d_m v_{n-2}, d_m p_r, d_m q_r)$  is a basis of  $T_m^* M$ . Therefore, the  $n$  functions  $(v_1, \dots, v_{n-2}, p_r, q_r)$  form a system of local coordinates, centered at  $m$ . It follows from (2.4) and (2.5) that the Poisson structure takes in terms of these coordinates the following form:

$$\Pi = \frac{\partial}{\partial q_r} \wedge \frac{\partial}{\partial p_r} + \sum_{i,j=1}^{n-2} h_{ij}(p_r, q_r, v_1, \dots, v_{n-2}) \frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial v_j}.$$

The Jacobi identity, applied to the triples  $(p_r, v_i, v_j)$  and  $(q_r, v_i, v_j)$ , implies that the functions  $h_{ij}$  do not depend on the variables  $p_r, q_r$ , so that

$$\Pi = \frac{\partial}{\partial q_r} \wedge \frac{\partial}{\partial p_r} + \sum_{i,j=1}^{n-2} h_{ij}(v_1, \dots, v_{n-2}) \frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial v_j}, \quad (2.6)$$

which means that  $\Pi$  is, in a  $G$ -invariant neighborhood of  $m$ , the product of a symplectic structure (on a neighborhood  $V_S$  of the origin in  $\mathbf{R}^2$ ) and a Poisson structure (on a neighborhood  $V_P$  of the origin in  $\mathbf{R}^{n-2}$ ).

In order to apply the induction hypothesis, we need to show that in case  $r-1 > 0$ ,

- (1) the functions  $p_1, \dots, p_{r-1}$  depend only on the coordinates  $v_1, \dots, v_{n-2}$ , i.e., are independent of  $p_r$  and  $q_r$ ,

$$\frac{\partial p_i}{\partial p_r} = 0 = \frac{\partial p_i}{\partial q_r} \quad i = 1, \dots, r-1. \quad (2.7)$$

- (2) the action of  $G$  depends only on the coordinates  $v_1, \dots, v_{n-2}$ , i.e.  $g^*v_j$  is, for all  $j = 1, \dots, s$ , a function that depends only on  $v_1, \dots, v_{n-2}$ .

Both equalities in (2.7) follow from the fact that  $p_i$  Poisson commutes with  $p_r$  and  $q_r$ , for  $i = 1, \dots, r-1$ , combined with (2.6):

$$0 = \{p_i, p_r\} = \frac{\partial p_i}{\partial q_r}, \quad 0 = \{p_i, q_r\} = -\frac{\partial p_i}{\partial p_r}.$$

This proves the first point above.

For the second point, we proceed as follows: for any  $g \in G$ ,  $\rho(g, \cdot)$  is defined in coordinates by

$$(p_r, q_r, v_1, \dots, v_{n-2}) \rightarrow (p_r, q_r, \alpha_1(p_r, q_r, v_1, \dots, v_{n-2}), \dots, \alpha_{n-2}(p_r, q_r, v_1, \dots, v_{n-2}))$$

where we have exploited the fact that  $p_r, q_r$  are  $G$ -invariant, and where, by definition,  $\alpha_i = g^*v_i$  for  $i = 1, \dots, n-2$ . Now, the invariance of the Poisson bracket amounts to:

$$\frac{\partial \alpha_i}{\partial q_r} = \{p_r, \alpha_i\} = \{p_r, g^*v_i\} = g^*\{p_r, v_i\} = \{p_r, v_i\} = 0$$

for  $i = 1, \dots, n-2$ . Applying the same procedure to the functions  $p_r$  yields  $\frac{\partial \alpha_i}{\partial p_r} = 0$ .

We may now apply the induction hypothesis on the second term in (2.6) and to the  $r-1$ -tuple of commuting functions  $p_1, \dots, p_{r-1}$  to build coordinates that will satisfy the three items of the theorem. By recursion, the theorem is valid for any integer  $r$ .  $\square$

**Corollary 2.4.** *Assume that the conditions of theorem 2.3 are satisfied, and that, moreover, the rank of  $\Pi$  at  $m$  is  $2r$  and that the linearized part of the transversal Poisson structure of  $\Pi$  at  $m$  is a semisimple compact Lie algebra  $\mathfrak{k}$ . Then the coordinates  $(p_1, q_1, \dots, p_r, q_r, z_1, \dots, z_s)$  can be chosen such that, in addition to the properties of theorem 2.3, we have:*

$$\Pi = \sum_{i=1}^r \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} + \frac{1}{2} \sum_{i,j,k} c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

where  $c_{ij}^k$  are structural constants of  $\mathfrak{k}$ .

*Proof.* The corollary simply follows from a famous result by Conn [4], stating that any smooth Poisson structure, which vanishes at a point and whose linear part at that point is of semisimple compact type, is locally smoothly linearizable. But this result has to be adapted, since, a priori, this change of coordinates might make the action non-linear. Fortunately, we can now use a result due to Ginzburg [10] stating that if a Poisson structure  $\Pi$  vanishes at a point  $p$  and is smoothly linearizable near  $p$ , if there is an action of a

compact Lie group  $G$  which fixes  $p$  and preserves  $\Pi$ , then  $\Pi$  and this action of  $G$  can be simultaneously linearized.<sup>2</sup>  $\square$

### 3. SPLIT COMPLETELY INTEGRABLE SYSTEMS

In this section, we will assume that our *completely integrable system* defines a regular foliation of dimension  $r$ . By an *isomorphism of completely integrable system*, we mean a diffeomorphism which preserves both the Poisson structure and the foliation defining the integrable system.

Given two completely integrable systems, the product of both Poisson manifolds, endowed with the product of both foliations, is again a completely integrable system, defining henceforth the *direct product of completely integrable system*.

**Convention.** We denote by  $\mathcal{F}$  the sheaf of functions constant on the leaves of the foliation, which means that, for a given open subset  $U \subset M$ ,  $\mathcal{F}_U$  stands for sections over  $U$ . Also  $T_m\mathcal{F}$  stands for the tangent space of the leaf through  $m \in M$ .

*Remark 3.1.* There are various alternative definitions of completely integrable systems : we insist on the requirement the the foliation is regular - which can be obtained in any case by considering only the open subset of regular points.

*Remark 3.2.* Note that not every Poisson structure can admit a completely integrable system defined as before. An obvious condition is, for instance, that at each point  $m \in M$  where  $\pi_m = 0$ , the natural Lie algebra structure that the cotangent space  $T_m^*M$  is endowed with needs to admit an Abelian Lie sub-algebra of dimension  $n - r$ , namely  $T_m\mathcal{F}^\perp$ .

Let  $\mathcal{S}$  be a symplectic leaf of a Poisson structure  $(M, \pi)$ . We make no assumption of regularity on  $\mathcal{S}$ , which means that the dimension  $2r'$  of  $\mathcal{S}$  may be strictly smaller than  $2r$  - in fact the singular case is the one we are really interested in. Our basic assumption is the following.

**Assumption.** From now, we assume that the restriction of the integrable system to  $\mathcal{S}$  is an integrable system on a neighbourhood (in  $\mathcal{S}$ ) of some fixed point  $s \in \mathcal{S}$ , that is to say, we assume that one of the following equivalent conditions is satisfied:

- (1) there exists, in a neighbourhood  $U$  of every  $s \in \mathcal{S}$  functions  $f_1, \dots, f_{r'} \in \mathcal{F}_U$  whose restrictions to  $\mathcal{S}$  are independent

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<sup>2</sup>One could envisage to implement here the ideas of Crainic and Fernandes in [5] and [6] that use the idea of stability to give a different proof of this equivariant Conn's linearization theorem. The authors did not try this approach.



(2) the following condition holds

$$\dim(T_s \mathcal{F} \cap T_s \mathcal{S}) = r'$$

for all  $s \in \mathcal{S}$  (recall that  $r' = \frac{\dim(\mathcal{S})}{2}$  by definition).

We call  $\mathcal{F}$ -regular such a symplectic leaf.

We leave it to the reader to check the equivalence of these conditions.

Notice that every regular leaf is  $\mathcal{F}$ -regular. But it is *not* automatically true when the leaf is singular. Here is a counter-example.

**Counter-example.** On  $M = \mathbb{R}^4$ , define a Poisson structure by:

$$\pi = xy \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p}.$$

Then  $\mathcal{T} = \{x = 0, y = 0\}$  is a singular leaf of dimension 2. So that  $n = 4, r = 2$  and  $r' = 1$ .

Let  $\mathcal{F}$  be the foliation defined by the pair of functions  $\mathbf{F} = (xe^p, ye^{-q})$ . These functions are independent at all point of  $M$ , hence this foliation is regular. One checks easily the relation

$$\{xe^p, ye^{-q}\} = 0,$$

which guaranties that  $\mathbf{F}$  is an integrable system. Now,  $\dim T_s(\mathcal{F}) \cap T_s \mathcal{S} = 2 \neq r'$ .

Let us recall that Weinstein's Splitting Theorem ([23]) states that, for every transverse submanifold  $\mathcal{T}$  of  $\mathcal{S}$  (i.e. a submanifold of  $M$  crossing transversally  $\mathcal{S}$  at a point  $s \in \mathcal{S}$ )

- (1)  $\mathcal{T}$  is a Poisson-Dirac submanifold (in a neighborhood of  $s$  in  $\mathcal{T}$  at least) - and therefore inherits a natural Poisson structure  $\pi_{\mathcal{T}}$ .
- (2) the Poisson manifold  $(M, \pi)$  is (in a neighborhood of  $s$  in  $M$ ) the direct product of the symplectic structure of  $\mathcal{S}$  (restricted to a neighborhood of  $s$  in  $\mathcal{S}$ ) with the induced Poisson structure on  $\mathcal{T}$  (restricted to a neighborhood of  $s$  in  $\mathcal{T}$ ).

We wish to see whether or not a similar result can be established for a Poisson structure endowed with a completely integrable system. More precisely, given a  $\mathcal{F}$ -regular symplectic leaf  $\mathcal{S}$  on a Poisson manifold  $(M, \pi)$  endowed with a completely integrable system  $\mathcal{F}$ , and a transversal  $\mathcal{T}$  through  $s \in \mathcal{S}$ . We want to address the following problems.

- (1) Does the transversal  $\mathcal{T}$  inherits an integrable system (with respect to its induced Poisson-Dirac structure  $\pi_{\mathcal{T}}$ ) ?
- (2) When this is the case, is the integrable system  $(M, \pi, \mathcal{F})$  the direct product of the restricted integrable system on  $\mathcal{S}$  and the induced integrable system on  $\mathcal{T}$ ?

The answer to both question is negative in general, but necessary and sufficient conditions can be given to give a positive answer. Let us start with a definition:

**Definition 3.3.** We say that a transversal  $\mathcal{T}$  through  $s \in \mathcal{S}$  is  $\mathcal{F}$ -compatible when  $\mathcal{F}$  restricts (at least in a neighborhood of  $s$  in  $\mathcal{T}$ ) to a regular foliation  $\mathcal{F}_{\mathcal{T}}$  of rank  $r - r'$  on  $\mathcal{T}$ .

Let us relate regularity of a symplectic leaf and existence of  $\mathcal{F}$ -compatible transversal,

**Lemma 3.4.** *Let  $(M, \pi)$  be a Poisson manifold of,  $\mathcal{F}$  a completely integrable system, and  $\mathcal{S}$  a symplectic leaf of dimension  $2r'$*

- (1) *The symplectic leaf  $\mathcal{S}$  is  $\mathcal{F}$ -regular at  $s \in \mathcal{S}$  if and only if there exists a  $\mathcal{F}$ -compatible transversal through  $s$ .*
- (2) *In this case, there exists, in a neighborhood of  $s$ , a foliation for which all leaves are  $\mathcal{F}$ -compatible transversals.*

*Proof.* 1) If there exists a  $\mathcal{F}$ -compatible transversal through  $s$ , then the intersection of  $T_s \mathcal{S}$  with  $T_s \mathcal{F}$  has to have dimension  $r'$ , i.e. the symplectic leaf  $\mathcal{S}$  is  $\mathcal{F}$ -regular at  $s \in \mathcal{S}$ .

Conversely, let us assume that  $s \in \mathcal{S}$ . Let  $f_1, \dots, f_{r'}$  be functions in  $\mathcal{F}_U$  ( $U$  a neighborhood of  $s$  in  $M$ ) whose restriction to  $\mathcal{S}$  form an integrable system on it.

By shrinking the neighbourhood  $U$  if necessary, we can extend  $f_1, \dots, f_{r'}$  to a family  $f_1, \dots, f_{n-r}$  of independent functions defining the foliation  $\mathcal{F}_U$ . Shrinking again  $U$  if necessary, we can construct, according to the Carathéodory-Jacobi-Lie theorem 2.2, functions  $g_1, \dots, g_{r'}$  such that  $\{f_i, g_j\}(s) = \delta_i^j$  for all  $i, j = 1, \dots, r'$  for every  $s \in \mathcal{S} \cap U$ . We can assume without any loss of generality that all the previously constructed functions vanish at  $s$ .

The functions  $f_1, \dots, f_{r'}, g_1, \dots, g_{r'}$  define local coordinates on  $\mathcal{S}$ , so that the following manifold is transversal to  $\mathcal{S}$

$$\mathcal{T} = \{f_1 = \dots = f_{r'} = g_1 = \dots = g_{r'} = 0\} \cap U.$$

Shrinking  $U$  again, one can assume that the functions  $f_1, \dots, f_{n-r}, g_1, \dots, g_{r'}$  are independent, so that the intersection

$$T_x \mathcal{F} \cap T_x \mathcal{T}$$

is the dual of the space

$$d_x f_{r'+1}, \dots, d_x f_{n-r}$$

and has therefore dimension  $r - r'$ . This proves the first claim.

2) The second claim is obtained by defining a foliation by:

$$\mathcal{T}_{AB} = \{f_1 = a_1, \dots, f_{r'} = a_{r'}, g_1 = b_1, \dots, g_{r'} = b_{r'}\} \cap U,$$

where  $A = (a_1, \dots, a_{r'})$  and  $B = (b_1, \dots, b_{r'})$  for all  $A, B$  in a neighborhood of  $0 \in \mathbf{R}^n$ .  $\square$

The following proposition replies to the first question above

**Proposition 3.5.** *Let  $(M, \pi)$  be a Poisson manifold of dimension  $n$  and rank  $2r$ ,  $\mathcal{F}$  a completely integrable system, and  $\mathcal{S}$  a  $\mathcal{F}$ -regular symplectic leaf of dimension  $2r'$ . Then, for every  $\mathcal{F}$ -compatible transversal  $\mathcal{T}$  through  $s \in \mathcal{S}$ , the induced regular foliation  $\mathcal{F}_{\mathcal{T}}$  is an integrable system with respect to the induced Poisson structure  $\pi_{\mathcal{T}}$ .*

*Proof.* The Poisson-Dirac structure on  $\mathcal{T}$  has rank  $2(r - r')$ , and the foliation  $\mathcal{F}_{\mathcal{T}}$  has rank  $r - r'$ . The condition on dimensions is therefore satisfied, and we are left with the task of proving that every two functions in  $\mathcal{F}_{\mathcal{T}}$  Poisson-commute.

This point is not trivial, for commuting functions of the ambient space do not need to commute any more, when restricted to a Poisson-Dirac submanifold. Let  $m \in \mathcal{T}$ , and  $f, f' \in \mathcal{F}$  be defined in a neighborhood of  $m$ . A priori  $\mathcal{X}_f(m)$  is not tangent to  $\mathcal{T}$ . To overcome this issue, let us consider functions  $f_1, \dots, f_{r'} \in \mathcal{F}$ , whose restrictions to  $\mathcal{S}$  are independent. We can moreover assume that the restriction to  $\mathcal{T}$  of these functions are equal to zero. The vectors  $\mathcal{X}_{f_1}(m), \dots, \mathcal{X}_{f_{r'}}(m)$ , for  $m$  close enough from  $s$ , do generate a subspace in  $T_m \mathcal{F}$  which has no intersection with  $T_m \mathcal{T}$ . Hence, there exists a linear combination of the form  $\sum_{i=1}^{r'} \lambda_i \mathcal{X}_{f_i}(m)$  such that

$$\mathcal{X}_f(m) - \sum_{i=1}^{r'} \lambda_i \mathcal{X}_{f_i}(m) \in T_m \mathcal{T}$$

Now the restriction to  $\mathcal{T}$  of  $f$  and  $f - \sum_{i=1}^{r'} \lambda_i f_i$  are equal, so that, by definition of the Poisson-Dirac bracket on  $\mathcal{T}$ , we have:

$$\{f, f'\}_{\mathcal{T}}(m) = \{f - \sum_i \lambda_i f_i, f'\}(m) = 0.$$

This completes the proof.  $\square$

**Definition 3.6.** We say that the integrable system  $\mathcal{F}$  is split at a point  $s$  in a  $\mathcal{F}$ -regular symplectic leaf  $\mathcal{S}$  if there exists a neighborhood  $U$  of  $s \in M$  which is isomorphic, as a completely integrable system, to the direct product of,

- (1) the induced completely integrable system on  $\mathcal{S} \cap U$ ,
- (2) and a completely integrable system on an open ball of dimension  $B^{n-2r'}$ , called transverse completely integrable system.

Said differently,  $\mathcal{F}$  is split at a point  $s$  in a  $\mathcal{F}$ -regular symplectic leaf  $\mathcal{S}$  if and only if there exists local coordinates  $p_1, \dots, p_{r'}, q_1, \dots, q_{r'}, z_1, \dots, z_s$  defined in a neighborhood  $U$  of  $s$  in  $M$ , where  $p_1, \dots, p_{r'}, z_1, \dots, z_{n-2r'}$  generate  $\mathcal{F}_U$ , and in which  $\pi$  reads:

$$\pi = \sum_{i=1}^{r'} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{0 \leq i < j \leq n-2r'} \Phi_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}.$$

**Proposition 3.7.** *Let  $(M, \pi)$  be a Poisson manifold of dimension  $n$  and rank  $2r$ ,  $\mathcal{F}$  a completely integrable system, and  $\mathcal{S}$  a  $\mathcal{F}$ -regular symplectic leaf of dimension  $2r'$ . then if the integrable system is split at  $\mathcal{S}$ , then all the compatible transversals have induced completely integrable systems isomorphic to the transverse integrable system.*

*Proof.* Let  $p_1, \dots, p_{r'}, q_1, \dots, q_{r'}, z_1, \dots, z_s$  be split coordinates, where  $p_1, \dots, p_{r'}, z_1, \dots, z_{n-r-r'}$  generate  $\mathcal{F}$ . The transverse manifold  $\mathcal{T}$  is given by equations of the form

$$p_i = \phi_i(z_1, \dots, z_s), q_i = \psi_i(z_1, \dots, z_s) \text{ for } i = 1, \dots, r'.$$

Requiring the transversal to be  $\mathcal{F}$ -compatible amounts to requiring that the functions  $\phi_1, \dots, \phi_{r'}$  depend only on  $z_1, \dots, z_{n-r-r'}$ . Said differently, the functions  $z_1, \dots, z_{n-r-r'}$  define the induced integrable system on  $T$  and on  $\mathcal{T}$  as well.

The induced Poisson structure on  $\mathcal{T}$  is precisely the Poisson structure obtained from  $\pi_T$  with the help of the gauge  $B = \sum_i d\phi_i \wedge d\psi_i$  (which is a 2-form) (see for instance lemma 2.2 in [23]) Recall that this implies that for every 1-form  $\theta$  with  $d\theta = B$  (form that we can assume to vanish at  $z = 0$ ), the flow  $(\phi_t)$  of  $\pi^\# \theta$ , evaluated at the time  $t = 1$ , is a Poisson diffeomorphism between  $\pi_T$  and  $\pi_{\mathcal{T}}$ . Now observe that  $B$  vanishes when restricted to  $\mathcal{F}_{\mathcal{T}}$ , i.e. the foliation defined by  $z_1, \dots, z_{n-r-r'}$ , so that  $\theta$  can be assumed to satisfy the same property by the foliated version of the Poincaré Lemma. This, in turn, amounts to the fact that  $\pi^\# \theta$  is a vector field tangent to the foliation  $\mathcal{F}_{\mathcal{T}}$ . The flow  $(\Phi)_t$  computed with the help of that particular  $\theta$  preserves the foliation, hence it is a isomorphism of integrable system.  $\square$

Notice that this lemma implies that the notion of transverse completely integrable system is well-defined for split completely integrable systems.

**Counter-example.** A completely integrable system is not, in general, split in a neighborhood of a point in a  $\mathcal{F}$ -regular symplectic leaf. Consider the following counter-example. On  $\mathbb{R}^4$ , define a Poisson structure by

$$\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + p \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}$$

and a completely integrable system with the help of the functions  $x, p + xq$ .

The  $\mathcal{F}$ -compatible transversal  $x = 0, y = 0$  admits for induced foliation defined by the function  $p$ , while the transversal  $x = 1, y = 0$  admits the induced foliation defined by the function  $p + q$ . For the first one, the singular locus of the induced Poisson structure is a leaf. This is not the case for the second one. So that there exists two transversals which admit non-isomorphic induced completely integrable system. By proposition 3.7, this integrable system is not trivial.

In [12], we give a necessary and sufficient condition for local triviality of regular completely integrable systems on Poisson manifolds. This characterization is done in terms of *foliated Poisson cohomology*. Also, the conditions of a system to be split and rigid are related in [17].

#### 4. EQUIVARIANT SPLIT INTEGRABLE SYSTEMS

To conclude this article, we present a last statement where the completely integrable systems under consideration is split and a part of it is equivariant with respect to the action of a compact Lie group  $G$ .

**Theorem 4.1.** *Let  $(M, \Pi)$  be a Poisson manifold of dimension  $n = 2r + s$ , acted upon by a compact Lie group  $G$ , whose action preserves the Poisson structure. Let  $p_1, \dots, p_{n-r}$  be an integrable system which is globally invariant, i.e. such that the  $G$ -action preserves the foliation. Given a symplectic leaf all whose points are fixed by the  $G$ -action and for which the integrable system is split, there exist, in a neighborhood  $U$  of  $m$ , functions  $q_1, \dots, q_r, z_1, \dots, z_s$ , such that*

- (1) *the  $n$  functions  $(p_1, q_1, \dots, p_r, q_r, z_1, \dots, z_s)$  form a system of local coordinates, centered at  $m$ ;*
- (2) *the Poisson structure  $\Pi$  is given in these coordinates by*

$$\Pi = \sum_{i=1}^r \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i,j=1}^s g_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \quad (4.1)$$

*where each function  $g_{ij}(z)$  is a smooth function and is independent from  $p_1, \dots, p_r, q_1, \dots, q_r$ .*

- (3) *There exists a group homomorphism from  $G$  to the group of  $s \times s$  invertible matrices, denoted by  $g \rightarrow A(g)$ , so that the  $G$ -action on  $U$  is given, in the previous coordinates, for all  $g \in G$ , by*

$$\rho(g, (p_1, q_1, \dots, p_r, q_r, z_1, \dots, z_s)) = (p_1, q_1, \dots, p_r, q_r, \sum_{i=1}^s m_{i,1}(g)z_i, \dots, \sum_{i=1}^s m_{i,s}(g)z_i).$$

- (4) *the family  $(p_1, \dots, p_r, z_1, \dots, z_{n-2r})$  generates the integrable system.*

*Proof.* The proof consists in repeating one by one the steps of the proof of theorem 2.3 with the additional condition that all the systems of coordinates considered there must be split, i.e. we want that the coordinates  $(v_1, \dots, v_n)$  that appear in the proof of theorem 2.3 be such that the integrable system is given by  $v_1$  and by  $r - 1$  of the functions  $v_2, \dots, v_n$ . This can be done using a foliated version of the Frobenius Theorem that states that, since  $\mathcal{X}_{p_r}$  and  $\mathcal{X}_{q_r}$  generate a foliation of dimension 2 whose intersection with the foliation defined by the integrable system is a foliation of rank 1, there exist local coordinates  $v_1, \dots, v_n$ , centered at  $m$ , such that

$$\mathcal{X}_{p_r} = \frac{\partial}{\partial v_{n-1}} \quad \text{and} \quad \mathcal{X}_{q_r} = \frac{\partial}{\partial v_n}.$$

and such that the integrable system is generated by  $v_1$  and  $r - 1$  of the functions  $v_2, \dots, v_n$ . The rest of the proof follows then exactly the same lines.  $\square$

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