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Duality and integrability of a supermatrix model with an external source

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 We study the Hermitian supermatrix model involving an external source. We derive the determinantal formula for the supermatrix partition function, and also for the expectation value of the characteristic polynomial ratio, which yields the duality between the characteristic polynomial and the external source with an arbitrary matrix potential function. We also show that the supermatrix integral satisfies the 1- and 2D Toda lattice equations as well as the ordinary matrix model.

Subject Index A10

1. Introduction

The matrix model has played a crucial role in quantum field theory (QFT) for the last few decades, as a 0D QFT model, or rather a toy model for the infinite-dimensional path integral. Since it is just defined by the finite-dimensional integral of a matrix itself, one can obtain an exact solution in various situations, which provides a significant insight for the understanding of QFT. This methodology is also referred to as random matrix theory (RMT), and is now applied to an extremely wide range of research fields [1].

In QFT, in order to compute correlation functions, it is convenient to introduce the generating function by adding the extra source term

$$\mathcal{Z}[J] = \int \mathcal{D}\phi e^{-\frac{1}{\hbar}S[\phi] + \int d^D x J(x)\phi(x)}. \tag{1}$$

The correlation function can be obtained by taking the functional derivative with the source field

$$\langle \phi(x_1) \cdots \phi(x_k) \rangle = \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_k)} \log \mathcal{Z}[J] \Big|_{J=0}. \tag{2}$$

This generating function is defined in the sense of a path integral, and thus it is quite difficult to compute in general. On the other hand, the matrix model with an external source, which is just given by a finite-dimensional integral, plays a similar role to the generating function in QFT,

$$\mathcal{Z}_N(A) = \int dX e^{-\frac{1}{\hbar}\text{Tr}W(X) + \text{Tr}XA}, \tag{3}$$

where the integral is taken over a size- N matrix. We can compute the correlation function by taking the derivative with respect to the source matrix A , as well as the QFT generating function.

A remarkable feature of the matrix model with an external source is the duality with a characteristic polynomial, which was found in Ref. [2] especially for the Gaussian matrix model. The claim of the duality is as follows: The M -point correlation function of characteristic polynomials in the size- N matrix model with an external source is equivalent to the N -point function with a size- M matrix, by exchanging the arguments of the characteristic polynomial and the external source. This duality is now extended to the generic β ensemble [3], the chiral ensemble [4], and beyond the Gaussian model [5], and has various interesting interpretations in terms of conformal field theory, string/M-theory, knot theory, algebraic geometry, and so on.

Another interesting property of the external source matrix model is the connection with the integrable system. The matrix integral with an external source and also the characteristic polynomial average satisfy the integrable equation, e.g., the Toda lattice equation, and they can be interpreted as the corresponding τ -function; see, e.g., Ref. [6]. Such integrability of the matrix model is quite useful to determine the spectral density and also the correlation function.

In this paper we study a supersymmetric version of the matrix model involving an external source based on a Hermitian supermatrix. The supermatrix method is now well known both in high-energy and condensed-matter physics. For example, one can avoid technical difficulty in dealing with the normalization factor of the correlation function by applying the supermatrix method, in a similar way to the replica method. This property helps us to take the disorder average in a random system [7]. In Sect. 2 we will compute the $U(N|M)$ symmetric supermatrix partition function involving the external source. As in the case of the ordinary Hermitian matrix model, we can utilize the determinantal structure in order to perform the integral. We will obtain a determinantal formula for the partition function, which can be expressed as a size- N determinant of the $U(1|1)$ partition function as a determinantal kernel especially for $N = M$.

As in the case of the ordinary matrix model, there have been some proposals for the duality of the supermatrix model. For example, in particular for the Gaussian supermatrix model, one can show an explicit duality between the external source and the characteristic polynomial [8]. In Sect. 3 we will exhibit that this duality generally holds for the supermatrix with an arbitrary potential function, and is just interpreted as a Fourier transformation. In addition, it will be pointed out that the relation between the characteristic polynomial ratio with the ordinary matrix model [9] and supermatrix models [10–12] is naturally explained as some specialization of the characteristic polynomial expectation value in the supermatrix.

In Sect. 4 we will also show the integrable equations for the supermatrix integral with the external source and the characteristic polynomial, based on the determinantal formula derived in this article. Since we have more parameters for the partition function than the ordinary matrix model, we can obtain several integrable equations corresponding to variables parametrizing the external source and the characteristic polynomial.

2. Supermatrix model with external source

Let us introduce a Hermitian supermatrix model with a potential function $W(x)$, involving the external source matrix C , as a natural generalization of the Hermitian matrix model (3),

$$\mathcal{Z}_{N,M}(\{a_i\}_{i=1}^N, \{b_j\}_{j=1}^M) = \int dZ e^{-\frac{1}{\hbar} \text{Str} W(Z) + \text{Str} ZC}, \quad (4)$$

where \hbar is the coupling constant. The Hermitian supermatrices Z and C are given by

$$Z = \begin{pmatrix} X & \xi \\ \xi^\dagger & Y \end{pmatrix}, \quad C = \begin{pmatrix} A & \eta \\ \eta^\dagger & B \end{pmatrix}, \quad (5)$$

where X, A are $N \times N$, and Y, B are $M \times M$ (bosonic) Hermitian matrices. ξ and η are $N \times M$ fermionic matrices, whose elements are given by Grassmannian variables. We can assume that C is a diagonal matrix without loss of generality, and in this paper we also assume $N \geq M$.

The matrix measure of (4) is invariant under the supergroup transformation, $dZ = d(UZU^{-1})$ with $U \in U(N|M)$, and is normalized by the corresponding volume factor $\text{Vol. } U(N|M)$. As well as the ordinary Hermitian matrix integral, one can decompose this measure into diagonal and angular parts,

$$dZ = \Delta_{N,M}(x; y)^2 d^N x d^N y dU, \quad (6)$$

and the corresponding Jacobian $\Delta_{N,M}(x; y)$ is now given by the generalized Cauchy determinant [13],

$$\begin{aligned} \Delta_{N,M}(x; y) &\equiv \Delta_N(x) \Delta_M(y) \prod_{i=1}^N \prod_{j=1}^M (x_i - y_j)^{-1} \\ &= \det \begin{pmatrix} x_i^{k-1} \\ (x_i - y_j)^{-1} \end{pmatrix} \quad \text{with} \quad \begin{cases} i = 1, \dots, N \\ j = 1, \dots, M \\ k = 1, \dots, N - M \end{cases}, \end{aligned} \quad (7)$$

where $\Delta_N(x)$ is the Vandermonde determinant,

$$\Delta_N(x) = \det_{1 \leq i, j \leq N} x_i^{j-1} = \prod_{i < j} (x_i - x_j). \quad (8)$$

The fact that the Jacobian factor can be written as a determinant will play a crucial role in the following discussion.

To perform the angular part integral of (4), we now apply the supergroup version of the Harish-Chandra–Itzykson–Zuber (HCIZ) formula, which is written in terms of the Cauchy determinant [14–16],

$$\begin{aligned} \mathcal{I}_{N,M}(Z, C) &\equiv \int_{U(N|M)} dU e^{\text{Str} ZUCU^{-1}} \\ &= \frac{1}{\Delta_{N,M}(x; y) \Delta_{N,M}(a; b)} \det_{1 \leq i, j \leq N} e^{x_i a_j} \det_{1 \leq i, j \leq M} e^{-y_i b_j}. \end{aligned} \quad (9)$$

This is a quite natural extension of the original formula for the $U(N)$ group, which is given by replacing the Cauchy determinant with the Vandermonde determinant [17,18]:

$$\begin{aligned} \mathcal{I}_N(X, A) &\equiv \int_{U(N)} dU e^{\text{Tr} XUAU^{-1}} \\ &= \frac{1}{\Delta_N(x) \Delta_N(a)} \det_{1 \leq i, j \leq N} e^{x_i a_j}. \end{aligned} \quad (10)$$

Applying the formula (9), we obtain the expression only in terms of eigenvalues:

$$\mathcal{Z}_{N,M} = \frac{1}{\Delta_{N,M}(a; b)} \int \prod_{i=1}^N dx_i e^{-\frac{1}{\hbar} W(x_i) + x_i a_i} \prod_{j=1}^M dy_j e^{\frac{1}{\hbar} W(y_j) - y_j b_j} \Delta_{N,M}(x; y). \quad (11)$$

Let us comment on the convergence of this integral. Naively speaking, it is not a convergent integral in general. We can regard this as a formal integral, but one possible way to avoid this divergence is to modify the matrix potential by inserting a constant supermatrix $I_{N,M} = \text{diag}(\mathbb{1}_N, i\mathbb{1}_M) \in U(N|M)$, i.e., $\text{Str}Z^2 = \text{Tr}X^2 - \text{Tr}Y^2 \rightarrow \text{Str}I_{N,M}Z^2 = \text{Tr}X^2 + \text{Tr}Y^2$ [3]. However, this treatment does not work for the generic potential function $W(x)$. Another possibility is to consider the coupling constant as an imaginary number, $\hbar \in i\mathbb{R}$. In this case, although the integrand becomes an oscillating function, one can provide an interpretation of it as the Fresnel integral. For example, this interpretation works well for the Chern–Simons matrix model [19] and its supermatrix generalization, which is called the ABJM matrix model [20–22], although the corresponding coupling constant is pure imaginary, e.g., $\hbar = 2\pi i/(k + N)$, or $\hbar = 2\pi i/k$.

Then, due to the Cauchy determinantal formula (7), it can be written as a size- N determinant, consisting of $N \times M$ and $N \times (N - M)$ blocks,

$$\mathcal{Z}_{N,M}(\{a_i\}_{i=1}^N, \{b_j\}_{j=1}^M) = \frac{1}{\Delta_{N,M}(a; b)} \det \begin{pmatrix} Q_{k-1}(a_i) \\ R(a_i; b_j) \end{pmatrix}, \tag{12}$$

where the indices run as $i = 1, \dots, N, j = 1, \dots, M, k = 1, \dots, N - M$, and we have introduced the following functions:

$$P_{i-1}(x) = x^{i-1}, \quad \tilde{R}(x; y) = \frac{1}{x - y}, \tag{13}$$

$$Q_{i-1}(a) = \int dx P_{i-1}(x) e^{-\frac{1}{\hbar}W(x)+xa}, \tag{14}$$

$$R(a; b) = \int dx dy \tilde{R}(x; y) e^{-\frac{1}{\hbar}W(x)+\frac{1}{\hbar}W(y)+xa-yb}. \tag{15}$$

Note that the function (14) obeys

$$Q_i(a) = \frac{d}{da} Q_{i-1}(a), \tag{16}$$

and $Q_{i=0}(a)$ is seen as a generalized Airy function:

$$Q_{i=0}(a) = \int dx e^{-\frac{1}{\hbar}W(x)+xa}. \tag{17}$$

As we will see later, these functions are just seen as a (double) Fourier or rather Laplace transform of $P_{i-1}(x)e^{-\frac{1}{\hbar}W(x)}$ and $\tilde{R}(x; y)e^{-\frac{1}{\hbar}W(x)+\frac{1}{\hbar}W(y)}$, respectively.¹

Let us comment on some specialization of the formula (12). If we take the limit $M = 0$, the determinantal expression (12) is reduced to the well known formula for the ordinary Hermitian matrix

¹ If we modify the source term in the supermatrix integral

$$\mathcal{Z}_{N,M}(\{a_i\}_{i=1}^N, \{b_j\}_{j=1}^M) = \int dZ e^{-\frac{1}{\hbar}\text{Str}W(Z)+i\text{Str}ZC}, \tag{18}$$

they just become Fourier transforms:

$$Q_{i-1}(a) = \int dx P_{i-1}(x) e^{-\frac{1}{\hbar}W(x)+ixa}, \quad R(a; b) = \int dx dy \tilde{R}(x; y) e^{-\frac{1}{\hbar}W(x)+\frac{1}{\hbar}W(y)+ixa-iyb}. \tag{19}$$

model with the external source

$$\begin{aligned} \mathcal{Z}_N \left(\{a_i\}_{i=1}^N \right) &= \int dX e^{-\frac{1}{\hbar} \text{Tr}W(X) + \text{Tr}XA} \\ &= \frac{1}{\Delta_N(a)} \det_{1 \leq i, j \leq N} Q_{j-1}(a_i); \end{aligned} \tag{20}$$

see, e.g., Ref. [6] for details. Then, in the case with $M = N$, the partition function becomes

$$\mathcal{Z}_{N,N} \left(\{a_i\}_{i=1}^N, \{b_j\}_{j=1}^N \right) = \frac{1}{\Delta_{N,N}(a; b)} \det_{1 \leq i, j \leq N} R(a_i; b_j). \tag{21}$$

This shows that the formula for $U(N|N)$ theory is reduced to a size- N determinant of the “dual” kernel function $R(a; b)$, which is given by the $U(1|1)$ partition function

$$R(a; b) = \frac{\mathcal{Z}_{1,1}(a, b)}{a - b}. \tag{22}$$

This factorization property is called Giambelli compatibility [23], and follows the Fay identity [24].

3. Characteristic polynomial average and duality

In this section we consider expectation values of the characteristic polynomial ratio with the supermatrix model in the presence of an external source:

$$\begin{aligned} \Psi_{N,M; p,q} \left(\{a_i\}_{i=1}^N, \{b_j\}_{j=1}^M; \{\lambda_\alpha\}_{\alpha=1}^p, \{\mu_\beta\}_{\beta=1}^q \right) \\ = \int dZ e^{-\frac{1}{\hbar} \text{Str}W(Z) + \text{Str}ZC} \prod_{\alpha=1}^p \text{Sdet}(\lambda_\alpha - Z) \prod_{\beta=1}^q \text{Sdet}(\mu_\beta - Z)^{-1}, \end{aligned} \tag{23}$$

and then show the duality between the $(p|q)$ -point function with $U(N|M)$ theory and $(N|M)$ -point function with $U(p|q)$ theory. The characteristic polynomial average (23) includes several useful situations:

- $M = q = 0$: Characteristic polynomial with the Hermitian matrix model

$$\Psi_{N; p} \left(\{a_i\}_{i=1}^N; \{\lambda_\alpha\}_{\alpha=1}^p \right) = \int dX e^{-\frac{1}{\hbar} \text{Tr}W(X) + \text{Tr}XA} \prod_{\alpha=1}^p \det(\lambda_\alpha - X). \tag{24}$$

- $M = p = 0$: Characteristic polynomial inverse with the Hermitian matrix model

$$\Psi_{N; q} \left(\{a_i\}_{i=1}^N; \{\mu_\beta\}_{\beta=1}^q \right) = \int dX e^{-\frac{1}{\hbar} \text{Tr}W(X) + \text{Tr}XA} \prod_{\beta=1}^q \det(\mu_\beta - X)^{-1}. \tag{25}$$

- $M = 0$: Characteristic polynomial ratio with the Hermitian matrix model

$$\begin{aligned} \Psi_{N; p,q} \left(\{a_i\}_{i=1}^N; \{\lambda_\alpha\}_{\alpha=1}^p, \{\mu_\beta\}_{\beta=1}^q \right) \\ = \int dX e^{-\frac{1}{\hbar} \text{Tr}W(X) + \text{Tr}XA} \prod_{\alpha=1}^p \det(\lambda_\alpha - X) \prod_{\beta=1}^q \det(\mu_\beta - X)^{-1}. \end{aligned} \tag{26}$$

Therefore, expression (23) provides a *master formula* for the characteristic polynomial average in various matrix models. For example, the duality in the case with $M = 0$ (26) claims that the characteristic polynomial ratio with the ordinary Hermitian matrix model is dual to another supermatrix model, as shown in Ref. [11].

As discussed in Sect. 2, the angular part of the integral is performed using the HCIZ formula (9), and then we obtain the expression only in terms of eigenvalues:

$$\begin{aligned} \Psi_{N,M;p,q} &= \frac{1}{\Delta_{N,M}(a; b)} \int \prod_{i=1}^N dx_i e^{-\frac{1}{\hbar}W(x_i)+x_i a_i} \prod_{j=1}^M dy_j e^{\frac{1}{\hbar}W(y_j)-y_j b_j} \Delta_{N,M}(x; y) \\ &\times \prod_{\alpha=1}^p \left(\frac{\prod_{i=1}^N (\lambda_\alpha - x_i)}{\prod_{j=1}^M (\lambda_\alpha - y_j)} \right) \prod_{\beta=1}^q \left(\frac{\prod_{j=1}^M (\mu_\beta - y_j)}{\prod_{i=1}^N (\mu_\beta - x_i)} \right). \end{aligned} \tag{27}$$

It is now convenient to apply the following identity to this expression:

$$\begin{aligned} \Delta_{N+p,M+q}(x, \lambda; y, \mu) &= \Delta_{N,M}(x; y) \Delta_{p,q}(\lambda; \mu) \prod_{\alpha=1}^p \left(\frac{\prod_{i=1}^N (\lambda_\alpha - x_i)}{\prod_{j=1}^M (\lambda_\alpha - y_j)} \right) \prod_{\beta=1}^q \left(\frac{\prod_{j=1}^M (\mu_\beta - y_j)}{\prod_{i=1}^N (\mu_\beta - x_i)} \right). \end{aligned} \tag{28}$$

If we assume $N + p > M + q$, the LHS of this equation is a large Cauchy determinant of the size $N + p$, with several block structures,

$$\Delta_{N+p,M+q}(a, \lambda; b, \mu) = \det \begin{pmatrix} x_i^{k-1} & \lambda_\alpha^{k-1} \\ (x_i - y_j)^{-1} & (\lambda_\alpha - y_j)^{-1} \\ (x_i - \mu_\beta)^{-1} & (\lambda_\alpha - \mu_\beta)^{-1} \end{pmatrix} \tag{29}$$

$$= (-1)^{Np+Mq} \det \begin{pmatrix} \lambda_\alpha^{k-1} & x_i^{k-1} \\ (\lambda_\alpha - \mu_\beta)^{-1} & (x_i - \mu_\beta)^{-1} \\ (\lambda_\alpha - y_j)^{-1} & (x_i - y_j)^{-1} \end{pmatrix}, \tag{30}$$

where $i = 1, \dots, N, j = 1, \dots, M, \alpha = 1, \dots, p, \beta = 1, \dots, q$, and $k = 1, \dots, N + p - M - q$. From this expression, we obtain the determinantal formula for the characteristic polynomial ratio expectation value

$$\begin{aligned} \Psi_{N,M;p,q}(\{a_i\}_{i=1}^N, \{b_j\}_{j=1}^M; \{\lambda_\alpha\}_{\alpha=1}^p, \{\mu_\beta\}_{\beta=1}^q) &= \frac{1}{\Delta_{N,M}(a; b) \Delta_{p,q}(\lambda; \mu)} \int \prod_{i=1}^N dx_i e^{-\frac{1}{\hbar}W(x_i)+x_i a_i} \prod_{j=1}^M dy_j e^{\frac{1}{\hbar}W(y_j)-y_j b_j} \Delta_{N+p,M+q}(a, \lambda; b, \mu) \\ &= \frac{1}{\Delta_{N,M}(a; b) \Delta_{p,q}(\lambda; \mu)} \det \begin{pmatrix} Q_{k-1}(a_i) & P_{k-1}(\lambda_\alpha) \\ R(a_i; b_j) & S_R(\lambda_\alpha; b_j) \\ S_L(a_i; \mu_\beta) & \tilde{R}(\lambda_\alpha; \mu_\beta) \end{pmatrix}, \end{aligned} \tag{31}$$

where we have introduced the following auxiliary functions in addition to (13), (14), and (15):

$$S_L(a; \mu) = \int dx \frac{1}{x - \mu} e^{-\frac{1}{\hbar}W(x)+xa}, \quad S_R(\lambda; b) = \int dy \frac{1}{\lambda - y} e^{\frac{1}{\hbar}W(y)-yb}. \tag{32}$$

The formula (31) shows a duality between the external source and the characteristic polynomials in the supermatrix model with an arbitrary potential function $W(x)$, since it can

also be written as follows:

$$\begin{aligned} & \Psi_{N,M;p,q} \left(\{a_i\}_{i=1}^N, \{b_j\}_{j=1}^M; \{\lambda_\alpha\}_{\alpha=1}^p, \{\mu_\beta\}_{\beta=1}^q \right) \\ &= \frac{(-1)^{Np+Mq}}{\Delta_{N,M}(a; b) \Delta_{p,q}(\lambda; \mu)} \det \begin{pmatrix} P_{k-1}(\lambda_\alpha) & Q_{k-1}(a_i) \\ \tilde{R}(\lambda_\alpha; \mu_\beta) & S_L(a_i; \mu_\beta) \\ S_R(\lambda_\alpha; b_j) & R(a_i; b_j) \end{pmatrix}. \end{aligned} \tag{33}$$

This duality was originally shown in the Gaussian model with the harmonic potential $W(x) = \frac{1}{2}x^2$ [8]. Under this duality, we see the correspondence between the auxiliary functions:

$$P_k(\lambda) \longleftrightarrow Q_k(a), \quad S_L(a; \mu) \longleftrightarrow S_R(\lambda; b), \quad R(a; b) \longleftrightarrow \tilde{R}(\lambda; \mu), \tag{34}$$

and then the $(p|q)$ -point function for the $U(N|M)$ supermatrix model with the potential $W(x)$ is converted into the $(N|M)$ -point function for the $U(p|q)$ supermatrix model with another potential $\tilde{W}(x)$, which is obtained through Fourier transformation:

$$\begin{aligned} & \Psi_{N,M;p,q} \left(\{a_i\}_{i=1}^N, \{b_j\}_{j=1}^M; \{\lambda_\alpha\}_{\alpha=1}^p, \{\mu_\beta\}_{\beta=1}^q \right) \\ & \stackrel{\text{F.T.}}{=} \Psi_{p,q;N,M} \left(\{\lambda_\alpha\}_{\alpha=1}^p, \{\mu_\beta\}_{\beta=1}^q; \{a_i\}_{i=1}^N, \{b_j\}_{j=1}^M \right). \end{aligned} \tag{35}$$

In this sense, it is easier to see this duality, especially for the Gaussian matrix model, because the Gaussian function is self-dual with respect to Fourier transformation. We remark that this kind of duality can also be found in the Chern–Simons matrix model and the ABJM matrix model, which are obtained by replacing the Vandermonde determinant with the exponentiated one:

$$\Delta(x) = \prod_{i < j}^N (x_i - x_j) \longrightarrow \prod_{i < j}^N \left(2 \sinh \frac{x_i - x_j}{2} \right). \tag{36}$$

In this case, insertion of an external source implies the expectation value of the Wilson loop operator, which is characterized by the Schur function [25].

To see this duality more explicitly, we now introduce another correlation function by multiplying the weight functions:

$$\begin{aligned} & \Phi_{N,M;p,q} \left(\{a_i\}_{i=1}^N, \{b_j\}_{j=1}^M; \{\lambda_\alpha\}_{\alpha=1}^p, \{\mu_\beta\}_{\beta=1}^q \right) \\ &= \Psi_{N,M;p,q} \left(\{a_i\}_{i=1}^N, \{b_j\}_{j=1}^M; \{\lambda_\alpha\}_{\alpha=1}^p, \{\mu_\beta\}_{\beta=1}^q \right) \prod_{\alpha=1}^p e^{-\frac{1}{\hbar} W(\lambda_\alpha)} \prod_{\beta=1}^q e^{\frac{1}{\hbar} W(\mu_\beta)}. \end{aligned} \tag{37}$$

This can also be written in the determinantal form (31), but by replacing the auxiliary functions as follows:

$$P_k(\lambda) \rightarrow \lambda^k e^{\mp \frac{1}{\hbar} W(\lambda)}, \quad Q_k(a) \rightarrow \int dx x^k e^{\mp \frac{1}{\hbar} W(x) \pm xa}, \tag{38}$$

$$R(a; b) \rightarrow \int dx dy \frac{1}{x-y} e^{-\frac{1}{\hbar} W(x) + \frac{1}{\hbar} W(y) + xa - yb}, \quad \tilde{R}(\lambda; \mu) \rightarrow \frac{1}{\lambda - \mu} e^{-\frac{1}{\hbar} W(\lambda) + \frac{1}{\hbar} W(\mu)}, \tag{39}$$

$$S_L(a; \mu) \rightarrow \int dx \frac{1}{x - \mu} e^{-\frac{1}{\hbar} W(x) + \frac{1}{\hbar} W(\mu) + xa}, \quad S_R(\lambda; b) \rightarrow \int dy \frac{1}{\lambda - y} e^{-\frac{1}{\hbar} W(\lambda) + \frac{1}{\hbar} W(y) - yb}. \tag{40}$$

The sign factor in (38) depends on whether $M + p > N + q$ or $M + p < N + q$. In this case, it is easier to see the relation between auxiliary functions, which is summarized in Fig. 1.

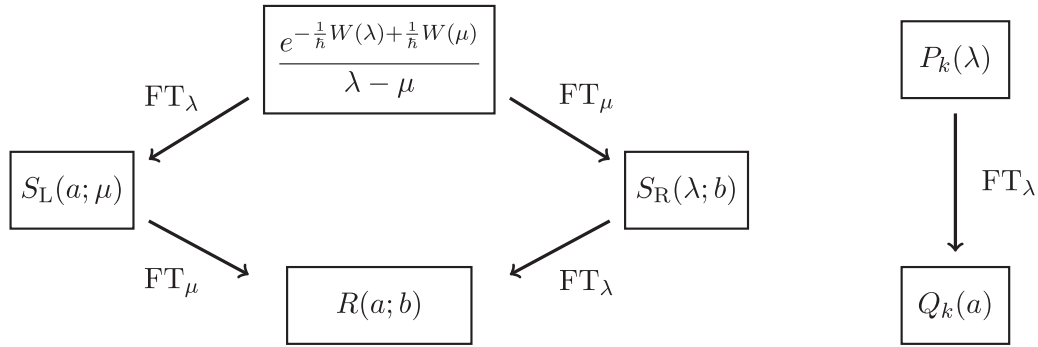


Fig. 1. *Fourier transform web:* the relation between the auxiliary functions. FT_x means Fourier transformation with respect to x variable. The duality (35) is seen as a double Fourier transformation with λ and μ variables.

4. Integrable equations

In the previous sections, we have shown that the partition functions of the Hermitian supermatrix models have a determinantal formula, and such a determinant structure plays an important role in the relation to the integrable systems [6]: it behaves as a τ -function for the corresponding integrable equation. In this section we study the connection between the supermatrix partition function and the integrable system.

4.1. Supermatrix model with external source

We now show that the supermatrix integral involving the external source (12) can be interpreted as the τ -function for the Toda lattice equations. In this paper we especially derive the integrable equation, obtained by taking all the external source parameters to the same value.

We first split the external source parameters into the center of mass and relative parts:

$$a_i = \delta a_i + a, \quad b_j = \delta b_j + b, \quad \begin{cases} i = 1, \dots, N \\ j = 1, \dots, M \end{cases} . \tag{41}$$

With this separation, the determinant (7) is given by

$$\Delta_{N,M}(a; b) = \Delta_N(\delta a) \Delta_M(\delta b) \prod_{i=1}^N \prod_{j=1}^M (\delta a_i - \delta b_j + a - b)^{-1}. \tag{42}$$

Let us then consider the Taylor expansion around the center of mass for the functions (14) and (15):

$$Q_{k-1}(a_i) = \sum_{l=1}^{\infty} \frac{(\delta a_i)^{l-1}}{(l-1)!} Q_{k+l-2}(a), \tag{43}$$

$$R(a_i; b_j) = \sum_{l,m=1}^{\infty} \frac{(\delta a_i)^{l-1}}{(l-1)!} \frac{(\delta b_j)^{m-1}}{(m-1)!} R^{(l-1,m-1)}(a; b), \tag{44}$$

with

$$R^{(i,j)}(a; b) = \frac{\partial^i}{\partial a^i} \frac{\partial^j}{\partial b^j} R(a; b). \tag{45}$$

By taking the limit $\delta a_i = \delta b_j = 0$ for $\forall i, j$, the determinant in (12) is written

$$\det \begin{pmatrix} Q_{k-1}(a_i) \\ R(a_i; b_j) \end{pmatrix} = \det_{1 \leq i, j \leq N} \frac{(\delta a_i)^{j-1}}{(j-1)!} \cdot \det_{1 \leq i, j \leq M} \frac{(\delta b_i)^{j-1}}{(j-1)!} \cdot \det \begin{pmatrix} Q_{i+k-2}(a) \\ R^{(i-1, j-1)}(a; b) \end{pmatrix}, \tag{46}$$

where the range of indices is given by $i = 1, \dots, N$, $j = 1, \dots, M$, and $k = 1, \dots, N - M$ in the determinants of the LHS and the third one in the RHS. Therefore, after some cancellation, we obtain the following formula:

$$\tilde{Z}_{N,M}(\{a_i = a\}_{i=1}^N, \{b_j = b\}_{j=1}^M) = c_{N,M} \tilde{Z}_{N,M}(a, b), \tag{47}$$

$$\tilde{Z}_{N,M}(a, b) = \det \left(R^{(i-1, j-1)}(a; b) \right), \tag{48}$$

with the constant

$$c_{N,M} = (a - b)^{NM} G(N + 1)^{-1} G(M + 1)^{-1}, \tag{49}$$

where $G(n) = \prod_{i=0}^{n-2} i! = \prod_{i=1}^{n-1} \Gamma(i)$ is the Barnes G-function. We remark that this factor corresponds to the volume element for the $U(N)$ and $U(M)$ groups. This shows that the supermatrix integral (4) is finally written as a kind of Wronskian in the equal parameter limit, $\delta a_i = \delta b_j = 0$.

In order to derive the integrable equation for the supermatrix integral, we now apply the Jacobi identity for a determinant

$$D \cdot D \begin{pmatrix} i & j \\ k & l \end{pmatrix} = D \begin{pmatrix} i \\ k \end{pmatrix} \cdot D \begin{pmatrix} j \\ l \end{pmatrix} - D \begin{pmatrix} i \\ l \end{pmatrix} \cdot D \begin{pmatrix} j \\ k \end{pmatrix}, \tag{50}$$

where D is a size- n determinant, and the size- $n - 1$ minor determinant $D \begin{pmatrix} i \\ j \end{pmatrix}$ is obtained by removing the i th row and the j th column from the matrix. Similarly, $D \begin{pmatrix} i & j \\ k & l \end{pmatrix}$ is the size- $n - 2$ determinant obtained by getting rid of the i, j th rows and the k, l th columns. Setting $i = k = N$ and $j = l = N - 1$, we have

$$\tilde{Z}_{N,M} \cdot \tilde{Z}_{N-2, M-2} = \tilde{Z}_{N-1, M-1} \cdot \partial_a \partial_b \tilde{Z}_{N-1, M-1} - \partial_a \tilde{Z}_{N-1, M-1} \cdot \partial_b \tilde{Z}_{N-1, M-1}, \tag{51}$$

and by putting $i = k = N - M$ and $j = l = N - M - 1$, we obtain

$$\tilde{Z}_{N,M} \cdot \tilde{Z}_{N-2, M} = \tilde{Z}_{N-1, M} \cdot \partial_a^2 \tilde{Z}_{N-1, M} - \left(\partial_a \tilde{Z}_{N-1, M} \right)^2. \tag{52}$$

These yield the Toda lattice equations [26]:

- 2D Toda equation:

$$\frac{\tilde{Z}_{N+1, M+1} \cdot \tilde{Z}_{N-1, M-1}}{\left(\tilde{Z}_{N, M} \right)^2} = \frac{\partial^2}{\partial a \partial b} \log \tilde{Z}_{N, M}. \tag{53}$$

- 1D Toda equation:

$$\frac{\tilde{Z}_{N+1, M} \cdot \tilde{Z}_{N-1, M}}{\left(\tilde{Z}_{N, M} \right)^2} = \frac{\partial^2}{\partial a^2} \log \tilde{Z}_{N, M}. \tag{54}$$

This means that the partition function is the τ -function for the Toda lattice system both in one and two dimensions simultaneously. This property can also be found in the ordinary Hermitian matrix model [5].

4.2. Characteristic polynomial with external source

We then show that the characteristic polynomial average with an external source (31) similarly obeys the integrable equations. In addition to the external source, we split the parameters for characteristic

polynomials:

$$\lambda_\alpha = \delta\lambda_\alpha + \lambda, \quad \mu_\beta = \delta\mu_\beta + \mu, \quad \begin{cases} \alpha = 1, \dots, p \\ \beta = 1, \dots, q \end{cases}. \quad (55)$$

As in the previous case, by considering the Taylor expansion around the center of mass, we obtain a Wronskian-type determinantal formula in the equal parameter limit $\delta a_i = \delta b_j = \delta\lambda_\alpha = \delta\mu_\beta = 0$ for $\forall i, j, \alpha, \beta$,

$$\Psi_{N,M;p,q}(a, b; \lambda, \mu) = c_{N,M;p,q} \tilde{\Psi}_{N,M;p,q}(a, b; \lambda, \mu), \quad (56)$$

$$\tilde{\Psi}_{N,M;p,q}(a, b; \lambda, \mu) = \det \begin{pmatrix} Q_{i+k-2}(a) & P_{k-1}^{(\alpha-1)}(\lambda) \\ R^{(i-1,j-1)}(a; b) & S_L^{(\alpha-1,j-1)}(\lambda; b) \\ S_R^{(\beta-1,i-1)}(a; \mu) & \tilde{R}^{(\alpha-1,\beta-1)}(\lambda; \mu) \end{pmatrix}, \quad (57)$$

with the constant

$$c_{N,M;p,q} = \frac{(a-b)^{NM}(\lambda-\mu)^{pq}}{G(N+1)G(M+1)G(p+1)G(q+1)}. \quad (58)$$

Here $P_k^{(\alpha)}(\lambda) = \frac{d^\alpha}{d\lambda^\alpha} P_k(\lambda)$, $S_L^{(\alpha,j)}(\lambda; b) = \frac{\partial^\alpha}{\partial \lambda^\alpha} \frac{\partial^j}{\partial b^j} S_L(\lambda; b)$, and so on. In this case, since the determinantal formula (57) consists of six blocks, we correspondingly obtain six equations by applying the identity (50):

$$\begin{aligned} &\tilde{\Psi}_{N,M;p,q} \cdot \tilde{\Psi}_{N,M;p-2,q-2} \\ &= \tilde{\Psi}_{N,M;p-1,q-1} \cdot \partial_\lambda \partial_\mu \tilde{\Psi}_{N,M;p-1,q-1} - \partial_\lambda \tilde{\Psi}_{N,M;p-1,q-1} \cdot \partial_\mu \tilde{\Psi}_{N,M;p-1,q-1}, \end{aligned} \quad (59)$$

$$\begin{aligned} &\tilde{\Psi}_{N,M;p,q} \cdot \tilde{\Psi}_{N-2,M;p,q-2} \\ &= \tilde{\Psi}_{N-1,M;p,q-1} \cdot \partial_a \partial_\mu \tilde{\Psi}_{N-1,M;p,q-1} - \partial_a \tilde{\Psi}_{N-1,M;p,q-1} \cdot \partial_\mu \tilde{\Psi}_{N-1,M;p,q-1}, \end{aligned} \quad (60)$$

$$\begin{aligned} &\tilde{\Psi}_{N,M;p,q} \cdot \tilde{\Psi}_{N,M-2;p-2,q} \\ &= \tilde{\Psi}_{N,M-1;p-1,q} \cdot \partial_\lambda \partial_b \tilde{\Psi}_{N,M-1;p-1,q} - \partial_\lambda \tilde{\Psi}_{N,M-1;p-1,q} \cdot \partial_b \tilde{\Psi}_{N,M-1;p-1,q}, \end{aligned} \quad (61)$$

$$\begin{aligned} &\tilde{\Psi}_{N,M;p,q} \cdot \tilde{\Psi}_{N-2,M-2;p,q} \\ &= \tilde{\Psi}_{N-1,M-1;p,q} \cdot \partial_a \partial_b \tilde{\Psi}_{N-1,M-1;p,q} - \partial_a \tilde{\Psi}_{N-1,M-1;p,q} \cdot \partial_b \tilde{\Psi}_{N-1,M-1;p,q}, \end{aligned} \quad (62)$$

$$\begin{aligned} &\tilde{\Psi}_{N,M;p,q} \cdot \tilde{\Psi}_{N,M;p-2,q} \\ &= (p-1) \left(\tilde{\Psi}_{N,M;p-1,q} \cdot \partial_a \partial_\lambda \tilde{\Psi}_{N,M;p-1,q} - \partial_\lambda \tilde{\Psi}_{N,M;p-1,q} \cdot \lambda \partial_a \tilde{\Psi}_{N,M;p-1,q} \right), \end{aligned} \quad (63)$$

$$\begin{aligned} &\tilde{\Psi}_{N,M;p,q} \cdot \tilde{\Psi}_{N-2,M;p,q} \\ &= p \left(\tilde{\Psi}_{N-1,M;p,q} \cdot \lambda \partial_a^2 \tilde{\Psi}_{N,M;p-1,q} - \partial_a \tilde{\Psi}_{N,M;p-1,q} \cdot \lambda \partial_a \tilde{\Psi}_{N,M;p-1,q} \right). \end{aligned} \quad (64)$$

We have used the following identity to derive some of these equations:

$$\det \begin{pmatrix} (x^N)^{(0)} & \dots & (x^N)^{(M-1)} \\ \vdots & & \vdots \\ (x^{N+M-2})^{(0)} & \dots & (x^{N+M-2})^{(M-1)} \\ (x^{N+M})^{(0)} & \dots & (x^{N+M})^{(M-1)} \end{pmatrix} = M x \prod_{1 \leq i, j \leq M} (x^{N+i-1})^{(j-1)}, \quad (65)$$

where we denote $(x^j)^{(k)} = \frac{d^k}{dx^k} x^j$. From the first equation (59), e.g., we obtain the 2D Toda lattice equation

$$\frac{\tilde{\Psi}_{N,M;p+1,q-1} \cdot \tilde{\Psi}_{N,M;p-1,q-1}}{\tilde{\Psi}_{N,M;p,q}} = \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \mu} \log \tilde{\Psi}_{N,M;p,q}. \quad (66)$$

We can similarly obtain the 2D integrable equations from the others, except for (64), which leads to the 1D Toda lattice equation. Therefore, in this sense, the characteristic polynomial average with the supermatrix model plays the role of the τ -function for the Toda lattice system.

Let us comment on higher-order integrable equations, which the matrix partition function would satisfy under generic parametrization of the source. In order to derive such equations, it is convenient to introduce the Miwa coordinate to the time variables,

$$t_n(A) = \frac{1}{n} \text{Tr} A^{-n}. \quad (67)$$

In this sense, since there are two and four kinds of source parameters in (48) and (57), we can correspondingly apply two and four series of the time variables, respectively. Although these variables are decoupled in the equal parameter limit, and we obtain the well known Toda lattice equations, they might interact with each other when we consider more generic parametrization. If so, it would be interesting to study such a situation providing more involved integrable equations.

5. Discussion

In this paper we have studied the supermatrix model involving the arbitrary potential function $W(x)$ with the external source term. We have derived the determinantal formula for the partition function and the characteristic polynomial expectation value with this supermatrix model. Based on this formula, we have exhibited the duality between the external source and the characteristic polynomial, and then pointed out that this duality is just interpreted as a Fourier transformation. We have also shown that the partition function and the characteristic polynomial average satisfy the Toda lattice equation both in one and two dimensions, especially in the equal parameter limit. This implies that we can obtain the corresponding τ -function as the supermatrix integral with an external source as well as the ordinary Hermitian matrix model.

As pointed out in Sect. 3, we can obtain various situations from the supermatrix model by taking the corresponding limit. From this point of view, it is interesting to study not only simple reductions of the characteristic polynomial average (23), e.g., $M = 0$, $q = 0$, but also the analytic continuation to negative numbers. Actually, the $U(N|M)$ supermatrix integral is, at least perturbatively, equivalent to the non-supersymmetric matrix model through the analytic continuation $M \rightarrow -M$, but with the two-cut solution, which breaks the originally symmetry, $U(N+M) \rightarrow U(N) \times U(M)$. This kind of equivalence has recently been gathering a great deal of attention in the area of string/M-theory [22]. For example, it would be interesting to see how the duality relation is affected by such an analytic continuation. It is expected that we can obtain more various kinds of correlation functions, and all the reductions could preserve the determinantal structure [27,28], through the analytic continuation.

From the random matrix theoretical point of view, it would be interesting to extend the present result to other situations, e.g., the $OSp(N|M)$ supermatrix model, chiral ensemble, complex matrix model, two-matrix model, and so on. To investigate the matrix model with an external source, the HCIZ formula plays an important role in integrating out the angular part of the matrix. Thus it seems better to start by deriving the corresponding formula, e.g., by considering the differential equation for the matrix integral.

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