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**ON THE EXACT CONTROLLABILITY AND OBSERVABILITY OF  
NEUTRAL TYPE SYSTEMS**

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**Abstract**

Neutral type systems considered in infinite-dimensional Hilbert space are analyzed for exact controllability characterization. The approach is based on the problem of moments using a Riesz basis of eigenvectors. The duality with observability is investigated. A criterion of exact observability is deduced.

**AMS Subject Classification:** 93B05, 93B23.

**Keywords:** Exact controllability, moment problem, exact observability, neutral type systems, Riesz basis.

## 1 Introduction

In the present paper, we consider the problems of controllability and observability for a large class of neutral type systems given by the following equation

$$\frac{d}{dt}[z(t) - Kz_t] = Lz_t + Bu(t), \quad t \geq 0, \quad z_0(\cdot) = f_0(\cdot), \quad (1.1)$$

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where  $z_t : [-1, 0] \rightarrow \mathbb{C}^n$  is the history of  $z$  defined by  $z_t(s) = z(t + s)$ . The difference and delay operators  $K$  and  $L$ , respectively, are defined by

$$Kf = A_{-1}f(-1) \quad \text{and} \quad Lf = \int_{-1}^0 [A_2(\theta)f'(\theta) + A_3(\theta)f(\theta)] d\theta$$

for  $f \in H^1([-1, 0], \mathbb{C}^n)$ , where  $A_{-1}$  is a constant  $n \times n$  matrix,  $A_2, A_3$  are  $n \times n$  matrices whose elements belong to  $L_2(-1, 0)$ ,  $B$  is a constant  $n \times r$  matrix, and the control  $u$  is an  $L_2$ -function.

Among many approaches to controllability problems for delay systems, a very fruitful one is to consider a delay system as an abstract operator system in a functional space. We consider the operator model of the neutral-type system (1.1) introduced by Burns, Herdman, and Stech [3] in product spaces (see also [5, 25]).

The state space is  $M_2(-1, 0; \mathbb{C}^n) = \mathbb{C}^n \times L_2(-1, 0; \mathbb{C}^n)$ , briefly  $M_2$ , and (1.1) can be reformulated as

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \quad x(t) = \begin{pmatrix} v(t) \\ z_t(\cdot) \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & L \\ 0 & \frac{d}{d\theta} \end{pmatrix}, \quad (1.2)$$

with  $\mathcal{D}(\mathcal{A}) = \{(v, z(\cdot)) \in M_2 : z \in H^1([-1, 0]; \mathbb{C}^n), v = z(0) - A_{-1}z(-1)\}$ . If  $L = 0$ , i.e. if  $A_2(\theta) = A_2(\theta) \equiv 0$ , the operator  $\mathcal{A}$  is noted by  $\mathcal{A}$ .

The problem of controllability is to find all the states  $x_T$  that can be reached from a fixed initial state (say 0) at a finite time  $T$  by the choice of the controls  $u(\cdot) \in L_2(0, T; U)$ , where  $U$  is the Hilbert space of controls. Let us denote by  $R_T \subset M_2$  the reachable subspace of the system (1.2):

$$R_T = \text{Im} \mathcal{R}_T = \left\{ \mathcal{R}_T u(\cdot) = \int_0^T e^{\mathcal{A}t} \mathcal{B}u(t) dt : u(t) \in L_2(0, T; \mathbb{R}^r) \right\},$$

where  $\mathcal{R}_T : L_2 \mapsto M_2$  is a linear bounded operator. In contrast to finite-dimensional systems, Kalman controllability concept ( $R_T = X$ ), does not hold generally for infinite-dimensional systems. Moreover, for neutral type systems, it can be shown that  $R_T \subset \mathcal{D}(\mathcal{A})$  for all  $T > 0$  (see [5]). Thus it may be possible to pose the problem of reaching the set  $\mathcal{D}(\mathcal{A})$ , which leads to the notion of exact controllability in this sense.

For neutral systems of the form (1.1), the semigroup is not explicitly known, in contrast to the situation of several discrete delays (see [10, 6, 2, 19]). To study controllability, we use the moment problem approach. Namely, the steering conditions of controllable states are interpreted as a vector moment problem with respect to a special Riesz basis. We analyze the resolvability of the non-Fourier trigonometric moment problem obtained using methods developed by [1].

In the general case (an arbitrary matrix  $A_{-1}$ ) the procedure for the choice of a Riesz basis is quite sophisticated. Moreover, due to the form of these bases, the moment problem becomes more complicated, which makes further manipulations with it technically difficult. However, for a class of systems (matrices  $A_{-1}$ ), there is a Riesz basis of the state space of eigenvectors (but this is not the common situation, see [13, 14, 17]) and the expression of the moment problem is simplified (see [16]).

The last remark gave us the idea that, by means of a change of control, it is possible to pass to an equivalent controllability problem for a system with matrix  $A_{-1}$  of a simple

structure. For such systems there is a Riesz basis of eigenvectors and the form of the corresponding moment problem is much simpler, which makes the proofs of the main results more illustrative. Here, we give the proof of Theorem 2.2 for the system (1.1) with  $A_{-1}$  of a special form and show that this fact leads to the proof for systems with an arbitrary matrix  $A_{-1}$ . Moreover, we found that the proof of the main theorem in the case of multivariable control might also be simplified. Here, we give the proof of the fact that the system (1.1) is uncontrollable when  $A_{-1}$  is singular and the pair  $(A_{-1}, B)$  is uncontrollable (there is no clear proof of this fact in [12]).

Our purpose is also to investigate the problem of observability of delay systems of neutral type by duality with controllability. Usually the results on controllability are easily reformulated in terms of observability. However in the case of delay systems, and specially for neutral type systems, the situation is not so simple.

The question of observability consists of measuring some output  $y = \mathcal{C}x(t)$ , with a linear bounded or  $\mathcal{A}$ -bounded operator (in fact *admissible* in the sense that will be detailed later). The system is said to be observable if one can determine the initial state  $x_0$  by measuring the output over a finite interval of time  $[0, T]$ . It is said exactly to be observable if this operation of determining the initial state is continuous. In the case of finite dimensional state spaces, we have the following duality equivalence: *The system  $(\mathcal{C}, \mathcal{A}, \mathcal{B})$  is controllable (observable) if and only if the system  $(\mathcal{B}^*, \mathcal{A}^*, \mathcal{C}^*)$  is observable (controllable)*. This is the Kalman principle of duality (see for example [23]).

The notions of approximate and spectral controllability and observability for linear neutral type systems were widely discussed in the book by Salamon [21]. The principle of duality for such notions and systems is not so simple.

Here, we consider the relationship between exact controllability and exact observability. The main part is devoted to the criterion of exact controllability, for which give a scheme of proof. Although we refer to technical results obtained in the paper by two authors [12], the main principle here is simpler. In the part concerning observability, we describe the dual neutral type system corresponding to the adjoint state space operator. There is something slightly different in the duality relationship, but the proof is not trivial.

The paper is organized as follows. In Section 3, for simplified systems, we expand the steering condition using a spectral Riesz basis and consider the resolvability of a moment problem. In Section 4 we give short proofs of our main results. In Section 5, we prove the main result for arbitrary systems of the form (1.1). In Section 6, we consider the problem of exact observability.

## 2 Preliminaries and formulation of main results

Let us now define the notions of exact controllability and observability and give the main results.

**Definition 2.1.** The system (1.2) is exactly null-controllable by controls from  $L_2$  at the time  $T$  if  $R_T = \mathcal{D}(\mathcal{A})$ .

To the best of our knowledge, the very important results concerning the characterization of exact controllability were obtained in [2, 9, 7, 24] for systems with discrete delays. Our

result concerns a large class of systems including distributed delays. In [12] the following criterion of controllability is obtained by the coauthors of the present paper.

**Theorem 2.2.** *The system (1.1) is exactly null-controllable if and only if the following conditions are verified.*

- (i) *There is no  $\lambda \in \mathbb{C}$  and  $v \in \mathbb{C}^n, v \neq 0$ , such that  $\Delta_{\mathcal{A}}^*(\lambda)v = 0$  and  $B^*v = 0$ , where*

$$\Delta_{\mathcal{A}}^*(\lambda) = \lambda I - \lambda e^{-\lambda} A_{-1}^* - \lambda \int_{-1}^0 e^{\lambda s} A_2^*(s) ds - \int_{-1}^0 e^{\lambda s} A_3^*(s) ds,$$

*or equivalently  $\text{rank}(\Delta_{\mathcal{A}}(\lambda) B) = n$  for all  $\lambda \in \mathbb{C}$ .*

- (ii) *There is no  $\mu \in \sigma(A_{-1})$  and  $v \in \mathbb{C}^n, v \neq 0$ , such that  $A_{-1}^* v = \bar{\mu} v$  and  $B^* v = 0$ , equivalently  $\text{rank}(\lambda I - A_{-1} B) = n$  for all  $\lambda \in \mathbb{C}$ .*

*If conditions (i) and (ii) hold, then the system is controllable at the time  $T > n_1$  and not controllable at the time  $T \leq n_1$ , where  $n_1$  is the controllability index of the pair  $(A_{-1}, B)$ .*

*If the delay is  $h$  instead of 1, then the exact time of controllability is  $n_1 h$ .*

One of the important contributions of Theorem 2.2 consists in giving the precise time of controllability, which may be very important in the minimal time problem and other related problems of optimal control.

We consider the finite-dimensional observation

$$y(t) = \mathcal{C}x(t) \tag{2.1}$$

where  $\mathcal{C}$  is a linear operator and  $y(t) \in \mathbb{R}^p$  is a finite dimensional output. There are several ways to design the output operator  $\mathcal{C}$  [20, 21, 8]. One of our goals in this paper is to investigate how to design a minimal output operator like

$$\mathcal{C}x(t) = Cz(t) \quad \text{or} \quad \mathcal{C}x(t) = Cz(t-1), \tag{2.2}$$

where  $C$  is a  $p \times n$  matrix. More general outputs, for example with several and/or distributed delays are not considered in this paper. We want to use the results on exact controllability in order to analyze, by duality, the exact observability property in the infinite dimensional setting like, for example, in [22]. The operator  $\mathcal{C}$  defined in (2.2) is linear but not bounded in  $M_2$ . However, in both cases it is admissible in the following sense:

$$\int_0^T \|\mathcal{C}e^{\mathcal{A}t}x_0\|_{\mathbb{R}^n}^2 dt \leq \kappa^2 \|x_0\|_{M_2}^2, \quad \forall x_0 \in \mathcal{D}(\mathcal{A}),$$

because it is bounded on  $\mathcal{D}(\mathcal{A})$ . We recall that if  $x_0 \in \mathcal{D}(\mathcal{A})$  then  $e^{\mathcal{A}t}x_0 \in \mathcal{D}(\mathcal{A})$ ,  $t \geq 0$  (see for example [11]). In fact,  $\mathcal{C}$  is admissible in the resolvent norm:  $\|x_0\|_{-1} = \|(\lambda I - \mathcal{A})^{-1}x_0\| = \|\mathcal{R}(\lambda, \mathcal{A})x_0\|$ ,  $\lambda \in \rho(\mathcal{A})$ . This is because  $\mathcal{C}$  is a closed operator and takes values in a finite dimensional space (see [22, Definition 4.3.1] and comments on this definition).

**Definition 2.3.** Let  $\mathcal{H}$  be the output operator  $\mathcal{H} : M_2 \rightarrow L_2(0, T; \mathbb{R}^p)$  defined by  $x_0 \mapsto \mathcal{H} x_0 = \mathcal{C} e^{\mathcal{A}t} x_0$ . The system (1.1) is said to be observable (or approximately observable) if  $\text{Ker } \mathcal{H} = \{0\}$  and exactly observable if

$$\int_0^T \|\mathcal{C} e^{\mathcal{A}t} x_0\|_{\mathbb{R}^p}^2 dt \geq \delta^2 \|x_0\|^2, \quad (2.3)$$

for some constant  $\delta$ .

The main result about exact observability is the following precise characterization [18]. It is a non-trivial consequence of Theorem 2.2.

**Theorem 2.4.** Let  $\Delta_{\mathcal{A}}^*(\lambda) = \lambda I - \lambda e^{-\lambda} A_{-1}^* - \lambda \int_{-1}^0 e^{\lambda s} A_2^*(s) ds - \int_{-1}^0 e^{\lambda s} A_3^*(s) ds$ , then:

1. The system (1.1) with the output  $y = Cz(t-1)$  is exactly observable over  $[0, T]$  iff

i) For all  $\lambda \in \mathbb{C}$ ,  $\text{rank} \begin{pmatrix} \Delta_{\mathcal{A}}^*(\lambda) & C^* \end{pmatrix} = n$ ,

ii) For all  $\lambda \in \mathbb{C}$ ,  $\text{rank} \begin{pmatrix} \lambda I - A_{-1}^* & C^* \end{pmatrix} = n$ ,

iii)  $T > n_1(A_{-1}^*, C^*)$ , where  $n_1$  is the first index of controllability for the pair  $(A_{-1}^*, C^*)$ .

2. If  $\det A_{-1} \neq 0$ , then assertion 1 is verified for the output  $y(t) = Cz(t)$ .

### 3 Exact controllability and the problem of moments: the simple case

Below, we assume that the spectrum of matrix  $A_{-1}$  satisfies the following constraints

$$\sigma(A_{-1}) = \{\mu_1, \dots, \mu_n\} \subset \mathbb{R}, \quad \mu_i \neq \mu_j, i \neq j, \quad \mu_i \notin \{0, 1\}, i = 1, \dots, n. \quad (3.1)$$

Due to our assumption, the matrix  $A_{-1}$  is non-singular.

#### 3.1 The Riesz basis of eigenvectors

The eigenvalues of the operator  $\widetilde{\mathcal{A}}$  (the state operator corresponding to the case  $A_2 = A_3 = 0$ ) are the roots of the equation  $\det \Delta_{\widetilde{\mathcal{A}}}(\lambda) = \det(\lambda I - \lambda e^{-\lambda} A_{-1}) = 0$ , and, taking into account (3.1), we obtain

$$\sigma(\widetilde{\mathcal{A}}) = \{\widetilde{\lambda}_m^k = \ln |\mu_m| + 2k\pi i, m = 1, \dots, n; k \in \mathbb{Z}\} \cup \{0\},$$

where  $\{\mu_1, \dots, \mu_n\} = \sigma(A_{-1})$ . Due to the specific form of  $A_{-1}$  the operator  $\widetilde{\mathcal{A}}$  possesses eigenvectors only (no root-vectors). These eigenvectors can be expressed using the eigenvectors of the matrix  $A_{-1}$  and the eigenvalues  $\widetilde{\lambda}_m^k$ .

Let us pass to the operator  $\mathcal{A}$ . Its spectrum allows the following characterization

$$\sigma(\mathcal{A}) = \{\ln |\mu_m| + 2k\pi i + O(1/k), m = 1, \dots, n; k \in \mathbb{Z}\}.$$

Due to [15, Theorem 4], an integer  $N \in \mathbb{N}$  exists such that the total multiplicity of the eigenvalues of  $\mathcal{A}$ , contained in the circles  $L_m^k(r^k)$ , equals 1 for all  $m = 1, \dots, n$  and  $k : |k| > N$ ,

where  $L_m^k(r^k) = L_m^k$  are circles centered at  $\tilde{\lambda}_m^k$  and their radii  $r^k$  satisfy the relation  $\sum_{k \in \mathbb{Z}} (r^k)^2 < \infty$ . We denote these eigenvalues of the operator  $\mathcal{A}$  as  $\lambda_m^k, m = 1, \dots, n, |k| > N$ .

Moreover, the space  $M_2$  possesses a Riesz basis of  $\mathcal{A}$ -invariant finite-dimensional subspaces  $\{V\} = \{V_m^k, |k| > N, m = 1, \dots, n\} \cup \{\widehat{V}_N\}$ , where

$$V_m^k = P_m^k M_2, \quad P_m^k = \frac{1}{2\pi i} \int_{L_m^k} R(\lambda, \mathcal{A}) d\lambda,$$

and  $\widehat{V}_N$  is the subspace spanned over all the generalized eigenvectors of  $\mathcal{A}$  whose eigenvalues are located outside the circles  $L_m^k, |k| > N, m = 1, \dots, n$ , with  $\dim \widehat{V}_N = 2(N+1)n$ . Since the multiplicity of  $\lambda_m^k$  equals 1, then

$$V_m^k = \text{Lin}\{\varphi_{m,k}\}, \quad \mathcal{A} \varphi_{m,k} = \lambda_m^k \varphi_{m,k}, \quad m = 1, \dots, n; |k| > N.$$

We note also that  $\varphi_{m,k} = \left( (I - e^{\lambda_m^k} A_{-1}) x_{m,k}, e^{\lambda_m^k \theta} x_{m,k} \right)^T$ , where  $x_{m,k} \in \text{Ker } \Delta_{\mathcal{A}}(\lambda_m^k)$ .

The sequences of eigenvectors of  $\mathcal{A}$  and  $\widetilde{\mathcal{A}}$  are quadratically close (after normalization if needed):  $\sum_{|k| > N} \sum_{m=1}^n \|\varphi_{m,k} - \widetilde{\varphi}_{m,k}\|^2 < \infty$ .

Let  $\widehat{\lambda}_j, j = 1, \dots, 2(N+1)n$  be the (non-distinct) eigenvalues of  $\mathcal{A}|_{\widehat{V}_N}$ , and  $\{\widehat{\varphi}_j\}$  the corresponding Jordan basis of generalized eigenvectors of the operator  $\mathcal{A}|_{\widehat{V}_N}$ . The family  $\{\varphi\} = \{\varphi_{m,k}\} \cup \{\widehat{\varphi}_j\}$  forms a Riesz basis of the state space  $M_2$ . The family  $\{\psi\} = \{\psi_{m,k}\} \cup \{\widehat{\psi}_j\}$ , biorthogonal to the family  $\{\varphi\} = \{\varphi_{m,k}\} \cup \{\widehat{\varphi}_j\}$ , is a Riesz basis of eigenvectors and generalized eigenvectors of the adjoint operator  $\mathcal{A}^*$  for the space  $M_2$ .

### 3.2 Expansion of the steering condition in the Riesz basis

Let us expand the steering condition  $x_T = \begin{pmatrix} v_T \\ z_T(\cdot) \end{pmatrix} = \int_0^T e^{\mathcal{A}t} \mathcal{B}u(t) dt$  with respect to the basis  $\{\varphi\}$  and to the biorthogonal basis  $\{\psi\}$  constructed above. A state  $x_T \in M_2$  is reachable at time  $T$  if and only if

$$\sum_{\varphi \in \{\varphi\}} \langle x_T, \psi \rangle \varphi = \sum_{\varphi \in \{\varphi\}} \int_0^T \langle e^{\mathcal{A}t} \mathcal{B}u(t), \psi \rangle dt \cdot \varphi, \quad u(\cdot) \in L_2(0, T; \mathbb{C}^r).$$

Then the steering condition can be substituted by the following system of equalities:

$$\langle x_T, \psi \rangle = \int_0^T \langle e^{\mathcal{A}t} \mathcal{B}u(t), \psi \rangle dt, \quad \psi \in \{\psi\}, u(\cdot) \in L_2(0, T; \mathbb{C}^r). \quad (3.2)$$

Let  $\{b_1, \dots, b_r\}$  be an arbitrary basis in  $\text{Im} B$ , the image of the matrix  $B$ , and  $\mathbf{b}_d = (b_d, 0)^T \in M_2, d = 1, \dots, r$  (more details about the choice of this basis will be given below). Then, the right-hand side of (3.2) takes the form

$$\int_0^T \langle e^{\mathcal{A}t} \mathcal{B}u(t), \psi \rangle dt = \sum_{d=1}^r \int_0^T \langle e^{\mathcal{A}t} \mathbf{b}_d, \psi \rangle u_d(t) dt. \quad (3.3)$$

Let us transform the term  $\langle e^{\mathcal{A}t} \mathbf{b}_d, \psi \rangle$  for  $\psi = \psi_{m,k}$ ,  $m = 1, \dots, n$ ,  $|k| > N$  as follows:

$$\langle e^{\mathcal{A}t} \mathbf{b}_d, \psi_{m,k} \rangle_{M_2} = \langle \mathbf{b}_d, e^{\mathcal{A}^*t} \psi_{m,k} \rangle_{M_2} = e^{\lambda_m^k t} \langle \mathbf{b}_d, \psi_{m,k} \rangle_{M_2} = e^{\lambda_m^k t} \langle b_d, y_{m,k} \rangle_{M_2}, \quad (3.4)$$

where  $y_{m,k} \in \text{Ker } \Delta_{\mathcal{A}}^* (\overline{\lambda_m^k})$ .

Let us introduce the notation:  $q_{m,k}^d = k \langle \mathbf{b}_d, \psi_{m,k} \rangle_{M_2}$ .

Due to (3.3) and (3.4), the infinite part of the system (3.2) corresponding to  $\{\psi_{m,k}, |k| > N, m = 1, \dots, n\}$ , reads as

$$k \langle x_T, \psi_{m,k} \rangle = \sum_{d=1}^r \int_0^T e^{\lambda_m^k t} q_{m,k}^d u_d(t) dt. \quad (3.5)$$

Next we observe that if  $\psi = \widehat{\psi}_j$ ,  $j = 1, \dots, 2(N+1)n$ , then

$$\langle e^{\mathcal{A}t} \mathbf{b}_d, \psi \rangle = \langle \mathbf{b}_d, e^{\mathcal{A}^*t} \psi \rangle = \widetilde{q}_j^d(t) e^{\widetilde{\lambda}_j t},$$

where  $\widetilde{q}_j^d(t)$  are polynomials in  $t$ . Therefore, the finite part of the system (3.2) corresponding to  $\psi \in \{\widehat{\psi}_j\}$  reads as

$$\langle x_T, \widehat{\psi}_j \rangle = \sum_{d=1}^r \int_0^T e^{\widetilde{\lambda}_j t} \widetilde{q}_j^d(t) u_d(t) dt. \quad (3.6)$$

Thus, we conclude that the state  $x_T \in M_2$  is reachable from 0 at the time  $T > 0$  if and only if the equalities (3.5) and (3.6) hold for some controls  $u_d(\cdot) \in L_2(0, T)$ ,  $d = 1, \dots, r$ . These equalities pose a kind of *moment problem*.

We complete this section with two estimates that play a significant role in our further considerations. First, it is important to notice that the constants  $q_{m,k}^d$  are uniformly bounded:  $|q_{m,k}^d| \leq C$ ,  $C > 0$ . More precisely, one can prove that for some constant  $\delta_1$ , we have

$$\left| \langle \mathbf{b}_d, \psi_{m,k} \rangle \right| \leq \frac{\delta_1}{|k|}, \quad m = 1, \dots, n; |k| > N; d = 1, \dots, r.$$

The second estimate is formulated as a lemma.

**Lemma 3.1** ([12]). *There is a sequence  $\{\alpha_k\}$ ,  $\sum_{|k|>N} \alpha_k^2 < \infty$ , such that for all  $m = 1, \dots, n$  and  $d = 1, \dots, r$  the following estimates hold:*

$$\left| e^{\lambda_m^k t} \langle \mathbf{b}_d, \psi_{m,k} \rangle_{M_2} - e^{\widetilde{\lambda}_m^k t} \langle \mathbf{b}_d, \widetilde{\psi}_{m,k} \rangle_{M_2} \right| \leq \frac{\alpha_k}{|k|}, \quad |k| > N, \quad t \in [0, T]. \quad (3.7)$$

This means that the projections of each vector  $\mathbf{b}_d$  on the eigenvectors of  $\mathcal{A}$  and  $\widetilde{\mathcal{A}}$  are *dynamically* and *quadratically* close.

### 3.3 The problem of moments and the Riesz basis property

Let us recall the general properties of the problem of moments that will be applied to the analysis of the problem (3.5)–(3.6) given in the previous section.



Consider a collection of functions  $\{g_k(t), t \in [0, \infty[ ]_{k \in \mathbb{N}}$  assuming that for any  $k \in \mathbb{N}$ ,  $T > 0$ :  $g_k(\cdot) \in L_2(0, T)$ , and consider the following problem of moments:

$$s_k = \int_0^T g_k(t)u(t) dt, \quad k \in \mathbb{N}. \quad (3.8)$$

The following well-known fact is a consequence of the Bari theorem (see [4, Chapter 6] and [26, Chapter 4]).

**Proposition 3.2.** *The following statements are equivalent:*

- (i) *For the scalars  $s_k$ ,  $k \in \mathbb{N}$ , the problem (3.8) has a solution  $u(\cdot) \in L_2(0, T)$  if and only if  $\{s_k\} \in \ell_2$ , i.e.,  $\sum_{k \in \mathbb{N}} s_k^2 < \infty$ ;*
- (ii) *the family  $\{g_k(t)\}_{k \in \mathbb{N}}$ ,  $t \in [0, T]$  forms a Riesz basis in the closure of its linear span*

$$\mathcal{L}(0, T) \stackrel{\text{def}}{=} \text{ClLin}\{g_k(t), k \in \mathbb{N}\} \subset L_2(0, T).$$

The following propositions are important steps in our consideration.

**Proposition 3.3** ([12]). *Let us suppose that for some  $T_1 > 0$ , the functions  $\{g_k(t)\}_{k \in \mathbb{N}}$ ,  $t \in [0, T_1]$ , form a Riesz basis in  $\mathcal{L}(0, T_1) \subset L_2(0, T_1)$  and  $\text{codim} \mathcal{L}(0, T_1) < \infty$ . Then for any  $0 < T < T_1$ , there is an infinite-dimensional subspace  $\ell^T \subset \ell_2$  such that the problem of moments (3.8) is unsolvable for  $\{s_k\} \in \ell^T$  if  $\{s_k\} \neq \{0\}$ .*

**Proposition 3.4** ([12]). *Let us consider the moment problem*

$$s_k = \sum_{d=1}^r \int_0^T g_k^d(t)u_d(t) dt, \quad k \in \mathbb{N}. \quad (3.9)$$

*If  $\sum_{k \in \mathbb{N}} \int_0^T |g_k^d(t)|^2 dt < \infty$ ,  $d = 1, \dots, r$ , then the set  $S_T$  of sequences  $\{s_k\}$  for which problem (3.9) is solvable is a non-trivial submanifold of  $\ell_2$ , i.e.,  $S_T \neq \ell_2$ .*

In the following, our analysis will be based on the theory of families of exponentials developed by S. Avdonin and S. Ivanov in [1]. We are particularly interested in the basis properties of such families.

Let  $\delta_1, \dots, \delta_n$  be different, modulus  $2\pi i$ , complex numbers, let  $N$  be a natural integer and let the set  $\{\varepsilon_{m,k}, |k| > N, m = 1, \dots, n\} \subset \mathbb{C}^n$  be such that  $\sum_{m,k} |\varepsilon_{m,k}|^2 < \infty$ . Let us denote by  $\widetilde{\mathcal{E}}_N$  the family

$$\widetilde{\mathcal{E}}_N = \left\{ e^{(\delta_m + 2\pi i k + \varepsilon_{m,k})t}, \quad |k| > N, m = 1, \dots, n \right\}.$$

Next, let  $\varepsilon_1, \dots, \varepsilon_r$  be another collection of different complex numbers such that  $\varepsilon_j \neq \delta_m + 2\pi i k + \varepsilon_{m,k}$ ,  $j = 1, \dots, r$ ,  $m = 1, \dots, n$ ,  $|k| > N$ , and let  $m'_1, \dots, m'_r$  be positive integers. Let us denote by  $\mathcal{E}_0$  the collection

$$\mathcal{E}_0 = \left\{ e^{\varepsilon_j t}, t e^{\varepsilon_j t}, \dots, t^{m'_j - 1} e^{\varepsilon_j t} \right\}_{j=1, \dots, r}.$$

The following theorem, which is based on the results of [1], is the main tool of our further analysis.

**Theorem 3.5.** (i) If  $\sum_{j=1}^r m'_j = (2N+1)n$ , then the family  $\mathcal{E} = \widetilde{\mathcal{E}}_N \cup \mathcal{E}_0$  constitutes a Riesz basis in  $L_2(0, n)$ .

(ii) If  $T > n$ , then independently of the number of elements in  $\mathcal{E}_0$ , the family  $\mathcal{E}$  forms a Riesz basis of the closure of its linear span in the space  $L_2(0, T)$ .

Now we apply Theorem 3.5 to the collection of functions appearing in (3.5). Let us fix  $d \in \{1, \dots, r\}$  and choose an arbitrary subset  $L \subset \{1, \dots, n\}$ .

**Theorem 3.6** ([12]). For any choice of  $d, L$ , for any  $T \geq n' = |L|$  the collection of functions

$$\Phi_1 = \left\{ e^{\lambda_m^k t} q_{m,k}^d, \quad |k| > N; m \in L \right\}$$

constitutes a Riesz basis of  $\text{C} \text{Lin} \Phi_1$  in  $L_2(0, T)$ .

If  $T = n'$ , the subspace  $\text{C} \text{Lin} \Phi_1$  is of finite codimension  $(2N+1)n'$  in  $L_2(0, n')$ .

## 4 Exact controllability: the case of a simple matrix $A_{-1}$

### 4.1 The necessary condition of controllability

Let us study the resolvability of the systems of equalities (3.5) and (3.6). Consider the sequence of vectors

$$\left\{ \int_0^T e^{\lambda_m^k t} q_{m,k}^d u(t) dt, \quad |k| > N \right\} \quad (4.1)$$

for any fixed  $d$  and  $u(\cdot) \in L_2(0, T)$ . It follows from Theorem 3.6 that all non-zero functions of the collection  $\left\{ e^{\lambda_m^k t} q_{m,k}^d, |k| > N \right\}$  form a Riesz basis of their linear span in  $L_2(0, T')$  if  $T'$  is large enough. Therefore, by Proposition 3.2, the sequence (4.1) belongs to the class  $\ell_2$ . This gives the following proposition.

**Proposition 4.1** ([12]). If the state  $x_T$  is reachable from 0 by the system (1.2), then it satisfies the following equivalent conditions:

$$(C1) \sum_{|k| > N} \sum_{m=1}^n k^2 |\langle x_T, \psi_{m,k} \rangle|^2 < \infty, \quad (C2) \sum_{|k| > N} \sum_{m=1}^n k^2 \|P_m^k x_T\|^2 < \infty, \quad (C3) x_T \in \mathcal{D}(\mathcal{A}).$$

From Proposition 4.1 it follows (see also [5]), that the set  $R_T$  of the states reachable from 0 by virtue of the system (1.2) and controls from  $L_2(0, T)$  is always a subset of  $\mathcal{D}(\mathcal{A})$ . This also justifies Definition 2.1 given in the introduction. Next, we give the necessary conditions of null-controllability.

**Theorem 4.2** ([12]). If the system (1.2) is controllable by controls from  $L_2(0, T)$  for some  $T > 0$ , then the conditions (i) and (ii) from Theorem 2.2 hold.

We note that Theorem 4.2 is proved in [12] under the assumption  $\det A_{-1} \neq 0$  which is used in the proof of assertion (ii). Below, in Theorem 5.3, we prove that this assertion holds also if  $\det A_{-1} = 0$ .

## 4.2 The sufficient condition of controllability

The following proposition for abstract systems was proved in [12].

**Lemma 4.3** ([12]). *Assume that for an abstract system  $\dot{x} = \mathcal{A}x + \mathcal{B}u$  the following conditions hold:*

- (a)  $R_T \subset \mathcal{D}(\mathcal{A})$  for all  $T > 0$ ,
- (b) for some  $T_0 > 0$  the set  $R_{T_0}$  is a closed subspace of finite codimension in the space  $X_{\mathcal{A}} = \mathcal{D}(\mathcal{A})$ , with the standard graph norm  $\|x\|_{\mathcal{A}} = \sqrt{\|x\|^2 + \|\mathcal{A}x\|^2}$ .

Then for all  $T \geq T_0$  we have  $R_T = L$ , where  $L$  is a subspace of  $\mathcal{D}(\mathcal{A})$  invariant by the semigroup  $e^{At}$  and  $0 \leq \text{codim } L \leq \text{codim } R_{T_0} < \infty$ .

In the following, we denote by  $X_{\mathcal{A}}$  the space  $\mathcal{D}(\mathcal{A}) \subset M_2$  with the graph norm.

**Theorem 4.4** ([12]). *For the system (1.2), let us suppose that there is a natural  $N$  and  $T_0 > 0$  such that the moment problem (3.5) for  $T = T_0$  and  $|k| > N$  is solvable for all the vectors  $\{k \langle x_T, \psi_{m,k} \rangle\}_{|k| > N}$  satisfying the condition (C1). Then, from condition (i) of Theorem 2.2, it follows that  $R_T = \mathcal{D}(\mathcal{A})$  as  $T > T_0$ .*

Now we are ready to prove the sufficient condition for systems with single control.

**Theorem 4.5.** *Let the system (1.2) be of single control ( $r = 1$ ) and let conditions (i) and (ii) of Theorem 4.2 hold. Then*

- (1) the system (1.2) is null-controllable at the time  $T$  as  $T > n$ ;
- (2) the estimation of the time of controllability in (1) is exact, i.e., the system is not controllable at time  $T = n$ .

*Proof.* First of all, let us observe that for any matrix  $A_{-1}$  conditions (i) and (ii) of Theorem 4.2 imply, in the case of single control, that all the eigenspaces of  $\mathcal{A}^*$  and  $\widetilde{\mathcal{A}}^*$  are one-dimensional. Otherwise, there is an eigenvector  $g$  of  $\mathcal{A}^*$  (or  $\widetilde{\mathcal{A}}^*$ ) such that  $\langle \mathbf{b}, g \rangle_{M_2} = 0$ . We know that eigenvectors of the adjoint operator have the form  $g = (y, z(\theta))^T$ , where  $y$  is nonzero and satisfies  $\Delta_{\mathcal{A}}^*(\lambda_0)y = 0$  (or  $\Delta_{\widetilde{\mathcal{A}}}^*(\lambda_0)y = 0$ ) for some  $\lambda_0$ . Since  $\langle \mathbf{b}, g \rangle_{M_2} = 0$  gives  $\langle b, y \rangle_{\mathbb{C}^n} = 0$  we arrive at a contradiction with the conditions of Theorem 4.2.

Thus, equalities (3.5) and (3.6) take, in our case, the form

$$k \langle x_T, \psi_{m,k} \rangle = \int_0^T e^{\lambda_m^k t} q_{m,k} u(t) dt, \quad (4.2)$$

where  $|k| > N$ ,  $m = 1, \dots, n$ , and

$$\langle x_T, \widehat{\psi}_j^1 \rangle = \int_0^T e^{\widehat{\lambda}_m t} \widehat{q}_j^1(t) u(t) dt, \quad (4.3)$$

where  $j = 1, \dots, 2(N+1)n$ . From condition (i) it follows that  $q_{m,k} \neq 0$  for all  $|k| > N$ ,  $m = 1, \dots, n$ , and, moreover, all polynomials  $\{\widehat{q}_j^1(t)\}$  are nontrivial. Let us introduce the following notation

$$\begin{aligned} \Phi_1 &= \{e^{\lambda_m^k t} q_{m,k}, |k| > N, m = 1, \dots, n\}, \\ \widehat{\Phi} &= \{e^{\widehat{\lambda}_m t} \widehat{q}_j^1(t), j = 1, \dots, 2(N+1)n\}. \end{aligned}$$

Applying Theorem 3.6, we find that for a large enough  $N$ , the collection

$$\Phi = \Phi_1 \cup \widehat{\Phi}$$

forms a Riesz basis in  $\text{CILin}\Phi \subset L_2(0, T)$ . Then by Proposition 3.2 the moment problem (4.2) is solvable if and only if (C1) holds. Due to Theorem 4.4, this yields  $R_T = \mathcal{D}(\mathcal{A})$  for  $T > n$ .

To prove the assertion (2) we first recall that the total number of elements of the family  $\widehat{\Phi}$  equals  $\sum_{m=1}^{\ell} \widehat{p}_{m,1} = (2N+2)n$ . On the other hand, it follows from Theorem 3.6 that in  $L_2(0, n)$  we have

$$\text{codim CILin}\Phi_1 = (2N+1)n.$$

This means that the family  $\Phi = \Phi_1 \cup \widehat{\Phi}$  contains at least  $n$  functions, which are presented as linear combinations of the others. As a consequence, the set of reachability  $R_T$  for  $T = n$  cannot be equal to  $\mathcal{D}(\mathcal{A})$ . More precisely, the codimension of  $R_T$  in  $\mathcal{D}(\mathcal{A})$  satisfies the estimation  $n \leq \text{codim} R_T < \infty$ . The theorem is proved.  $\square$

*Remark 4.6.* It is clear that the system (1.2) is also uncontrollable at time  $T < n$ . Moreover, it follows from Proposition 3.3 that, in this case, the set  $\text{Cl}R_T$  is of infinite codimension in  $X_{\mathcal{A}}$ .

Let us now consider the multivariable case  $\dim B = r > 1$ . Let us recall that the Kalman matrix  $\begin{pmatrix} B & A_{-1}B & \cdots & A_{-1}^{n-1}B \end{pmatrix}$  plays an important role in the controllability theory and that its rank equals  $n$  iff  $\text{rank}(\lambda I - A_{-1} B) = n$  for all  $\lambda \in \mathbb{C}$ .

Let  $\{b_1, \dots, b_r\}$  be an arbitrary basis noted  $\beta$ . Let us introduce a set of integers. We denote  $B_i = (b_{i+1}, \dots, b_r)$ ,  $i = 0, 1, \dots, r-1$ , which gives in particular  $B_0 = B$  and  $B_{r-1} = (b_r)$  and we formally put  $B_r = 0$ . In the following, we need the integers

$$m_i^\beta = \text{rank}(B_{i-1} \quad A_{-1}B_{i-1} \cdots A_{-1}^{n-1}B_{i-1}) - \text{rank}(B_i \quad A_{-1}B_i \cdots A_{-1}^{n-1}B_i) \quad (4.4)$$

corresponding to the basis  $\beta$ . Let us denote

$$m_1 = \max_{\beta} m_1^\beta, \quad \bar{m} = \min_{\beta} \max_i m_i^\beta, \quad (4.5)$$

for all possible choices of basis  $\beta$ . It is easy to show that for all  $\beta$ , there is  $i$  such that  $m_i^\beta \geq m_1$  and then  $\bar{m} \geq m_1$ . Indeed, assume that  $m_1$  is realized on the basis  $\beta = \{b_1, \dots, b_r\}$ , and consider an arbitrary basis  $\beta_0 = \{b_1^0, \dots, b_r^0\}$ . Then there exists  $i$  such that  $\text{Lin}\{b_i^0, \dots, b_r^0\} \not\subset \text{Lin}\{b_2, \dots, b_r\}$  but  $\text{Lin}\{b_{i+1}^0, \dots, b_r^0\} \subset \text{Lin}\{b_2, \dots, b_r\}$ . For this integer  $i$  we have  $m_i^{\beta_0} \geq m_1$ .

Now we can formulate the main result of this section.

**Theorem 4.7.** *Let conditions (i) and (ii) of Theorem 4.2 hold. Then*

- (1) *the system (1.2) is null-controllable at the time  $T > \bar{m}$ ;*
- (2) *the system (1.2) is not null-controllable at the time  $T < m_1$ .*

*Proof.* The proof is based on the reorganization of the system of equations (3.5) into a new system where the first set of equations contains only the control  $u_1$ , the second the controls  $u_1$  and  $u_2$ , etc. The solutions are then obtained step by step. The interval of resolvability of corresponding moment problems then depends on the integers  $m_i^\beta$ , introduced above. Finally, this gives a general interval of resolvability  $[0, T]$  verifying the assessment i) and ii) of the theorem (see the detailed proof in [12]).

□

To complete our analysis, we next obtain the precise time of controllability. From Theorem 4.7 it is not clear what happens if the time  $T$  is such that  $m_1 \leq T \leq \bar{m}$  even if the conditions of controllability are satisfied. In order to give the exact time of controllability, we need the classical concept of the controllability indices. Recall that the first index  $n_1$  may be defined as the minimal integer  $\nu$  such that (see, for example, [23, Chapter 5])  $\text{rank}(B, A_{-1}B, \dots, A_{-1}^{\nu-1}B) = n$ .

**Lemma 4.8.** *Assume that the pair  $(A_{-1}, B)$  is controllable. Let  $n_1$  be the index of controllability of the couple  $(A_{-1}, B)$  and let  $\bar{m}, m_1$  be defined by (4.5). Then  $m_1 \leq n_1 \leq \bar{m}$ .*

It is well known that in contrast to indices  $m_1, \bar{m}$ , the controllability index  $n_1$  is invariant under feedback. This means that  $n_1$  is the same for all couples  $(A_{-1} + BP, B)$ , where  $P$  is an  $r \times n$  matrix. Then one can choose a feedback matrix  $P$  and a basis in  $\mathbb{C}^n$  such that  $A_{-1} + BP$  take the following form (see [23, Theorem 5.10 and Corollary 5.3]):  $F = \text{diag}\{F_1, \dots, F_r\}$ , where  $F_i$  are Frobenius blocks and  $B$  becomes  $G = \text{diag}\{g_1, \dots, g_r\}$ , where  $g_i = (0, 0, \dots, 1)^T$ , the dimension being  $n_i \times 1$ . It is easy to check that  $\bar{m}(F, G) = m_1(F, G) = n_1$ . Moreover, the spectrum of  $F$  may be chosen arbitrarily by means of an appropriate choice of  $P$ .

## 5 Controllability of systems with an arbitrary matrix $A_{-1}$

We have proved the main result for systems with a specific matrix  $A_{-1}$ . Now we consider the general case.

**Proposition 5.1.** *[23] If the pair  $(A_{-1}, B)$  is controllable, i.e.  $\text{rank}(B, A_{-1}B, \dots, A_{-1}^{n-1}B) = n$ , then for any set  $\{\mu_m\}_{m=1}^n$  there is a matrix  $P \in \mathbb{C}^{p \times n}$  such that*

$$\sigma(A_{-1} + BP) = \{\mu_m\}_{m=1}^n.$$

Let us now consider the following system

$$\dot{z}(t) = (A_{-1} + BP)\dot{z}(t-1) + \int_{-1}^0 A_2(\theta)\dot{z}(t+\theta)d\theta + \int_{-1}^0 A_3(\theta)z(t+\theta)d\theta + Bu. \quad (5.1)$$

It is easy to prove the following Lemma.

**Lemma 5.2.** *The system (1.1) is exactly null-controllable at the time  $T$  if and only if the perturbed system (5.1) is exactly null-controllable at the same time  $T$ .*

Let us observe that, in the conditions of controllability (i) and (ii) of Theorem 2.2, the matrix  $A_{-1}$  may be substituted by the matrix  $A_{-1} + BP$  for any  $P$ .

Indeed, let us denote by  $\mathcal{A}$  the operator corresponding to the system (5.1). Then, the relations  $\Delta_{\mathcal{A}}^*(\lambda)v = 0$  and  $B^*v = 0$  are equivalent to the relations  $\Delta_{\mathcal{A}}^*(\lambda)v = [\Delta_{\mathcal{A}}^*(\lambda) - \lambda e^{-\lambda}P^*B^*]v = 0$  and  $B^*v = 0$  with the same  $v$  and  $\lambda$ . This fact is the same as the equivalency of the conditions  $\text{rank}(B \ A_{-1}B \ \cdots \ A_{-1}^{n-1}B) = n$  and  $\text{rank}(B \ (A_{-1} + BP)B \ \cdots \ (A_{-1} + BP)^{n-1}B) = n$ , which is a well-known classic result (see e.g. [23]).

Now let us consider the situation when the pair  $(A_{-1}, B)$  is uncontrollable.

**Theorem 5.3.** *Assume that  $\det A_{-1} = 0$  and the pair  $(A_{-1}, B)$  is uncontrollable, i.e. there exists  $\mu \in \sigma(A_{-1})$  and  $v \in \mathbb{C}^n \setminus \{0\}$  such that  $A_{-1}^*v = \bar{\mu}v$  and  $B^*v = 0$ . Then, the system (1.1) is not exactly controllable.*

*Proof.* First we consider the situation when the uncontrollable eigenvalue of  $A_{-1}$  is  $\mu = 0$ , i.e. there exists  $v_0 \in \mathbb{C}^n \setminus \{0\}$  such that

$$A_{-1}^*v_0 = 0 \text{ and } B^*v_0 = 0. \quad (5.2)$$

Multiplying the system (1.1) by  $v_0^*$  we obtain

$$v_0^*\dot{z}(t) = v_0^*A_{-1}\dot{z}(t-1) + \int_{-1}^0 [v_0^*A_2(\theta)\dot{z}(t+\theta) + v_0^*A_3(\theta)z(t+\theta)] d\theta + v_0^*Bu.$$

Taking into account (5.2) we have

$$v_0^*\dot{z}(t) = \int_{-1}^0 [v_0^*A_2(\theta)\dot{z}(t+\theta) + v_0^*A_3(\theta)z(t+\theta)] d\theta. \quad (5.3)$$

If we assume that the system (1.1) is exactly null-controllable at time  $T > 0$ , then the set of solutions of (1.1) coincides with  $H^1([T-1, T]; \mathbb{C}^n)$ . The latter means that the set

$$\{v_0^*\dot{z}(t), t \in [T-1, T]\}$$

coincides with  $L_2([T-1, T]; \mathbb{C})$ .

On the other hand, the operator  $Q(z) = \int_{-1}^0 [v_0^*A_2(\theta)\dot{z}(t+\theta) + v_0^*A_3(\theta)z(t+\theta)] d\theta$  is a Fredholm operator from  $H^1([T-2, T]; \mathbb{C}^n)$  to  $L_2([T-1, T]; \mathbb{C})$ . Indeed, if we change the time  $\tau = t + \theta$  we obtain

$$Q(z) = \int_{t-1}^t [v_0^*A_2(\tau-t)\dot{z}(\tau) + v_0^*A_3(\tau-t)z(\tau)] d\tau.$$

Thus, the operator  $Q(z)$  is compact and then its image cannot coincide with  $L_2([T-1, T]; \mathbb{C})$ . We obtain the contradiction that proves the theorem in the case  $\mu = 0$ .

Now let us suppose that the uncontrollable eigenvalues of  $A_{-1}$  are nonzero and that 0 is a controllable eigenvalue. This means that  $\text{rank}(A_{-1} \ B) = n$  and then, by spectral assignment [23], there is a matrix  $P$  such that  $0 \notin \sigma(A_{-1} + BP)$ , that is this matrix is not singular and, obviously,  $(A_{-1} + BP, B)$  remains uncontrollable. Using now the result of Theorem 4.2 which was obtained in [12] under the assumption of a non-singular matrix  $A_{-1} + BP$ , we obtain that the system (5.1) is not exactly controllable and this implies that the original system (1.1) is not exactly controllable.  $\square$

*All the considerations of this section mean that Theorem 2.2 is verified for the case of an arbitrary matrix  $A_{-1}$ .*

## 6 Exact observability and duality with controllability

We give here a short description of the proof of the main result on observability. For more details see the paper [18]. It is based on the expression of the adjoint operator  $\mathcal{A}^*$ , on the neutral type system corresponding to this operator and on the relationship between this *adjoint* neutral type system and a dual neutral type system.

**Theorem 6.1.** *The adjoint operator  $\mathcal{A}^*$  is given by*

$$\mathcal{A}^* \begin{pmatrix} w \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} (A_2^*(0)w + \psi(0)) \\ -\frac{d[\psi(\theta) + A_2^*(\theta)w]}{d\theta} + A_3^*(\theta)w \end{pmatrix}, \quad (6.1)$$

with the domain

$$\mathcal{D}(\mathcal{A}^*) = \left\{ (w, \psi(\cdot)) : \psi(\theta) + A_2^*(\theta)w \in H^1, A_{-1}^* (A_2^*(0)w + \psi(0)) = \psi(-1) + A_2^*(-1)w \right\}.$$

As we can see, the expression of the adjoint operator  $\mathcal{A}^*$  is different (in its form) from that of the operator  $\mathcal{A}$ . It is natural to expect a difference for the “system” for which  $\mathcal{A}^*$  is the generator.

**Theorem 6.2.** *Let  $x$  be a solution of the abstract equation*

$$\dot{x} = \mathcal{A}^* x, \quad x(t) = \begin{pmatrix} w(t) \\ \psi_t(\theta) \end{pmatrix}. \quad (6.2)$$

Then the function  $w(t)$  is the solution of the neutral type equation

$$\dot{w}(t+1) = A_{-1}^* \dot{w}(t) + \int_{-1}^0 \left[ A_2^*(\tau) \dot{w}(t+1+\tau) + A_3^*(\tau) w(t+1+\tau) \right] d\tau.$$

Let us recall that, for the original system (1.2), the relation between the abstract state  $x(t)$  and the current state  $z(t)$  is  $x(t) = (v(t), z(t+\theta))$ . In other words, there is a shift in time, and a “castling” between components. This transformation can be represented by a linear operator  $\mathcal{F}$  acting on the initial conditions of the neutral type systems, from  $X_{\mathcal{A}}$  to  $M_2$ . It is important that this operator is bounded and bounded invertible [18] and this gives the relation needed for the conclusion: the observability operator  $\mathcal{K}$  is related to the reachability operator  $\mathcal{R}_T$  by the relation

$$\mathcal{K} F^{-1} = \begin{cases} \mathcal{R}_T^{\dagger*} & \text{if } \mathcal{C}x(t) = Cz(t-1), \\ e^{\mathcal{A}^{\dagger*}} \mathcal{R}_T^{\dagger*} & \text{if } \mathcal{C}x(t) = Cz(t), \end{cases} \quad (6.3)$$

where the dag  $\dagger$  represents the original system (1.1) with the matrices  $A_{-1}^*, A_2^*, A_3^*$  and  $C^*$  instead of  $B$ . This last relation (6.3) allows Theorem 2.4 to be proved.

## 7 Conclusion

We think that this description of the main result on exact controllability without technical details but with a sufficiently rigorous approach, explains our main contribution to the exact controllability of neutral type systems. The result on observability, which usually seems a simple application of the controllability result is, in this case less trivial. However, we have obtained a new illustration of the duality principle in this case. The technical proofs have also been omitted here to show the main result.

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## References

- [1] S. A. Avdonin and S. A. Ivanov. *Families of exponentials*. Cambridge University Press, Cambridge, 1995.
- [2] H. T. Banks, Marc Q. Jacobs, and C. E. Langenhop. Characterization of the controlled states in  $W_2^{(1)}$  of linear hereditary systems. *SIAM J. Control*, 13:611–649, 1975.
- [3] J. A. Burns, Terry L. Herdman, and H. W. Stech. Linear functional-differential equations as semigroups on product spaces. *SIAM J. Math. Anal.*, 14(1):98–116, 1983.
- [4] I. C. Gohberg and M. G. Krein. *Introduction to the theory of linear nonselfadjoint operators*. Translations of Mathematical Monographs, Vol. 18. American Mathematical Society, Providence, R.I., 1969.
- [5] K. Ito and T. J. Tarn. A linear quadratic optimal control for neutral systems. *Nonlinear Anal.*, 9(7):699–727, 1985.
- [6] M. Q. Jacobs and C. E. Langenhop. Criteria for function space controllability of linear neutral systems. *SIAM J. Control Optimization*, 14(6):1009–1048, 1976.
- [7] V. E. Khartovskii and A. T. Pavlovskaya. Complete controllability and controllability for linear autonomous systems of neutral type. *Automation and Remote Control*, (5):769–784, 2013.
- [8] A. V. Metel'skii and S. A. Minyuk. Criteria for the constructive identifiability and complete controllability of linear time-independent systems of neutral type. *Izv. Ross. Akad. Nauk Teor. Sist. Upr.*, (5):15–23, 2006.
- [9] D. A. O'Connor and T. J. Tarn. On stabilization by state feedback for neutral differential-difference equations. *IEEE Trans. Autom. Contr.*, 28(5):615–618, 1983.
- [10] D. A. O'Connor and T. J. Tarn. On the function space controllability of linear neutral systems. *SIAM J. Control Optim.*, 21(2):306–329, 1983.
- [11] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, vol. 44 of *Applied Mathematical Sciences*. Springer, New York, 1983.
- [12] R. Rabah and G. M. Sklyar. The analysis of exact controllability of neutral-type systems by the moment problem approach. *SIAM J. Contr. Optim.*, 46(6):2148–2181, 2007.
- [13] R. Rabah, G. M. Sklyar, and A. V. Rezounenko. Generalized Riesz basis property in the analysis of neutral type systems. *C. R. Math. Acad. Sci. Paris*, 337(1):19–24, 2003.



- [14] R. Rabah, G. M. Sklyar, and A. V. Rezounenko. Stability analysis of neutral type systems in Hilbert space. *J. Differential Equations*, 214(2):391–428, 2005.
- [15] R. Rabah, G. M. Sklyar, and A. V. Rezounenko. On strong regular stabilizability for linear neutral type systems. *J. Differential Equations*, 245(3):569–593, 2008.
- [16] R. Rabah and G. M. Sklyar. On exact controllability of linear time delay systems of neutral type. In *Applications of time delay systems*, vol. 352 of *Lect. Notes in Contr. and Inform. Sci.*, pp. 165–171. Springer, Berlin, 2007.
- [17] R. Rabah, G. M. Sklyar, and P. Yu. Barkhayev. Stability and stabilizability of mixed retarded-neutral type systems. *ESAIM Contr. Optim. Calc. Var.*, 18(3):656–692, 2012.
- [18] R. Rabah and G. M. Sklyar. Observability and controllability for linear neutral type systems. In *Proc. of the Symp. MTNS 2014*, pp. 223–229, Groningen, 2014.
- [19] H. Rivera Rodas and C. E. Langenhop. A sufficient condition for function space controllability of a linear neutral system. *SIAM J. Contr. Optim.*, 16(3):429–435, 1978.
- [20] D. Salamon. Neutral functional differential equations and semigroups of operators. In *Control theory for distributed parameter systems and applications (Vorau, 1982)*, volume 54 of *Lecture Notes in Control and Inform. Sci.*, pages 188–207. Springer, Berlin, 1983.
- [21] D. Salamon. *Control and observation of neutral systems*, volume 91 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1984.
- [22] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts. Basler Lehrbücher. Basel: Birkhäuser. xi, 483 p., 2009.
- [23] W. Murray Wonham. *Linear multivariable control: A geometric approach.*, volume 10 of *Applications of Mathematics*. Springer-Verlag, New York, third edition, 1985.
- [24] Y. Yamamoto. Reachability of a class of infinite-dimensional linear systems: an external approach with applications to general neutral systems. *SIAM J. Control Optim.*, 27(1):217–234, 1989.
- [25] Y. Yamamoto and Shinnichi Ueshima. A new model for neutral delay-differential systems. *Internat. J. Control*, 43(2):465–471, 1986.
- [26] R. M. Young. *An introduction to nonharmonic Fourier series*, vol. 93 of *Pure and Applied Mathematics*. Academic Press Inc., New York, 1980.