Physical Meaning of Physical-mesomechanical Formulation of Deformation and Fracture

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Abstract. Physical mesomechanics is a relatively recent theory of deformation and fracture. Based on the physical principle known as gauge invariance, it has derived field equations that describe the displacement behavior in plastically deforming materials. By the nature of the way the gauge invariance is applied, the theory is capable of describing all stages of deformation on the same theoretical basis. Previously, we analyzed this formalism from the mathematical point of view, and discussed the dynamics resulting from the field equations. In this paper, we focus on the physical meaning of the field equations and related dynamics. It is shown that plastic deformation is characterized as the dynamics where materials' longitudinal resistive force is proportional to the local velocity as opposed to the elastic regime where the longitudinal resistive force is proportional to the displacement. Consequently, in the plastic regime, part of the work done by external force is not stored in the material as potential energy but dissipated as heat. In this dynamics, a quantity called the deformation charge plays an important role. From the gauge theoretical viewpoint, the deformation charge is associated with the charge of symmetry. From the viewpoint of conventional mechanics, it is normal strain whose energy is unrecoverable to the mechanical field. Experimental results that support these arguments are present.

Keywords: Elastic deformation, plastic deformation, fracture, gauge invariance.

INTRODUCTION

Physical mesomechanics [1] is a relatively recent theory of deformation and fracture. The most distinctive feature of this theory over other conventional theories is its capability of describing all stages of deformation, from the elastic to the fracturing stage, on the same theoretical basis. This capability comes from the physical principle known as gauge invariance, which is the basis of physical mesomechanics [2], [3]. Gauge invariance [4] states that when a transformation matrix representing a certain physical process is coordinate dependent, the Lagrangian must be invariant under the transformation so that the physics can be formulated in the same fashion before and after the transformation. To ensure invariance, it is necessary that a vector field called the gauge field is included in the Lagrangian. Once an appropriate gauge field is introduced, the dynamics can be prescribed by applying the least action principle to the Lagrangian. If the transformation is valid only when it is coordinate independent, the theory is said to be globally symmetric; if the transformation is valid when it is coordinate dependent, the theory is said to be locally symmetric. The dynamics derived through gauge invariance represents the physics resulting from the extension of global symmetry to local symmetry. In the present context, the procedure to obtain the dynamics associated with local symmetry is as follows: express the distortion tensor with transformation matrices, find a gauge and its transformation that makes the Lagrangian invariant, and apply the least action principle to the Lagrangian. This leads to the physical-mesomechanical field equations that describe the law of displacement vectors in plastically deforming materials. Previously [5], we discussed the mathematical aspect of this procedure in some detail. From the physical viewpoint, the above formalism can be interpreted as follows: when a material is in the elastic regime, it follows the global transformation. When it enters the plastic regime, the transformation is still locally elastic but the parameters characterizing the transformation depend on the coordinates. The dynamics is prescribed through the interaction of local transformations with the field. In this dynamics, the longitudinal resistive force of materials in response to external force plays a key role; if the resistive force is proportional to the local stretch or compression, the dynamics is energy conservative and the deformation is recoverable. If it is proportional to the local velocity, the dynamics is energy dissipative and the deformation is unrecoverable. In this paper, we discuss the physical meaning of the dynamics focusing on the longitudinal force.

FORMULATION

Field Equation and Deformation Charge

The physical-mesomechanical field equation can be expressed in the following form [5], [6]

$$\nabla \cdot \vec{v} = \frac{\rho}{\varepsilon}, \tag{1}$$

$$\nabla \times \vec{\omega} = -\varepsilon\mu \frac{\partial \vec{v}}{\partial t} - \vec{j}, \tag{2}$$

$$\nabla \times \vec{v} = \frac{\partial \vec{\omega}}{\partial t}, \tag{3}$$

where $\vec{v}$ is the velocity of the deforming material at point (x,y,z), ε is the density of the medium ρ/ε and $\vec{j}$ are, respectively, the time and spatial component of the charge of symmetry, $\vec{\omega}$ is the angle of rotation of the local volume element from its rotational equilibrium, and $1/\mu$ is the shear modulus. The right-hand side of Eq. (1) can be viewed as a source term that causes divergence in the velocity field. By choosing coordinate x_s in the direction of the velocity vector at (x,y,z), we can rewrite the left-hand side of Eq. (1) as

$$\nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \frac{dv_s}{dx_s} \tag{4}$$

where v_x, v_y and v_z denote the components of the velocity vector. Apparently, the right-hand side of Eq. (4) represents the rate of normal strain in the direction of x_s. By multiplying the density ε on both hand sides of Eq. (1), we can express the quantity ρ as

$$\rho = \varepsilon \frac{dv_s}{dx_s}. \tag{5}$$

FIGURE 1. External force accelerates the whole volume (a) and stretches the volume (b)

On the other hand, the change in momentum of a unit volume flowing in the direction of x_s over unit time is

$$\frac{dp_s}{dt} = \varepsilon \frac{dv_s}{dt} = \varepsilon(\frac{\partial v_s}{\partial t} + \frac{\partial v_s}{\partial x_s}\frac{dx_s}{dt}) = \varepsilon(\frac{\partial v_s}{\partial t} + \frac{\partial v_s}{\partial x_s} v_s) = \varepsilon \frac{\partial v_s}{\partial t} + \rho v_s. \tag{6}$$

The first term on the rightmost-hand side represents the change in momentum of the unit volume due to the change in velocity over time as a whole; the whole volume gets accelerated. The second term represents the momentum change due to the fact that as the unit volume flows, the leading edge and trailing edge have different velocities due to the velocity gradient. Both are caused by external force acting on the unit volume. The work done by the external force associated with the first term contributes to a change in the kinetic energy of the unit volume. The work done by the external force associated with the second term is stored in the unit volume as energy of stretch or compression. If the material is elastic, this energy can be retrieved as work (free energy). Figure 1 illustrates the situation schematically. Figure 1(a) represents a case where the leading and trailing edge have the same velocity ($dv_s/dx_s = 0$). In this case, the unit volume is not stretched or compressed, and the work done by the external force simply changes the velocity of the unit volume. Figure 1(b) represents a case where the leading edge has greater velocity than the trailing edge ($dv_s/dx_s > 0$). In this case, part of the work done by the external force is stored as spring (elastic) energy.

Thus, we can interpret the charge density $\rho = \varepsilon(dv_s/dx_s)$ as a quantity representing stretch or compression of a unit volume. We call this the density of deformation charge.

Deformation Charge Current and Permanent Deformation

Application of divergence to Eq. (2) with the use of Eq. (1) leads to

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\frac{\vec{j}}{\mu}). \tag{7}$$

which can be viewed as an equation of continuity. Based on the above interpretation of ρ as representing stretch/compression, we can interpret Eq. (7) as follows. Imagine a volume element in a material and consider the total deformation charge (ρ times the volume). Eq. (7) states that the change in the total charge over time is solely determined by the divergence of $(-\vec{j}/\mu)$. As schematically illustrated in Figure 2, this can be interpreted as follows: the quantity $(-\vec{j}/\mu)$ represents normal force acting on the volume element. If $(-\vec{j}/\mu)$ is diverging, the total stretch inside the volume will increase ($\partial\rho/\partial t$ times volume >0). Thus, without the negative sign in front, $\vec{j}/\mu$ represents the reaction of the material to the external force, i.e., the material's resisting force. Eq. (7) can be interpreted as the case when the volume is a unit volume.

$$-\frac{j}{\mu} \Leftarrow \boxed{\frac{\partial\rho}{\partial t} > 0} \Rightarrow -\frac{j}{\mu}$$

FIGURE 2. Diverging normal force causing increase in charge density over time

When the divergence of $(-\vec{j}/\mu)$ is non-zero there are two ways for $\partial\rho/\partial t$ to be non-zero. The first is that the velocity gradient increases or decreases inside the unit volume over time; the continuous pattern of stretch/compression changes its density. The second is that an isolated pattern of stretch or compression comes into/goes out of the unit volume that is initially either stretched nor compressed. Figure 3 illustrates both cases schematically. The plastic deformation is associated with the second way. Naively speaking, this is equivalent to dislocation. In terms of more general mechanics, this is the case when a spring attached to a mass on one end and to a wall on the other end is detached from the wall. The work done by the external force is no longer stored as the spring's potential energy but dissipates via friction.

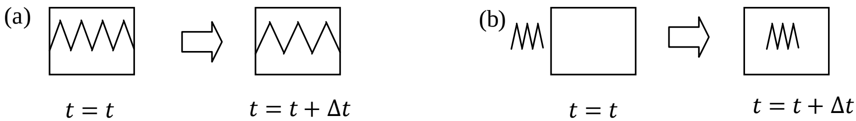

FIGURE 3. Two ways for charge density to increase over time. Energy conservative (a) and dissipative (b)

We can view the second case as the isolated charge flowing at drift velocity W_d, and express $\vec{j}/\mu$ as follows.

$$\frac{\vec{j}}{\mu} = \vec{W}_d \rho. \tag{8}$$

Comparison with Eq. (6) indicates that if W_d is in the same direction as v_s, the flow causes momentum loss. Putting $W_d = \sigma v_s$, we can rate the degree of momentum loss; when $\sigma = 1$, it is natural momentum change in the flow [Eq. (6) itself]. The greater the parameter σ, the greater the momentum loss. Figure 4 explains the momentum loss caused by a flow of positive charge in the direction of velocity v_s in the simplest case where the flow field is characterized by two levels of velocity; the high level at the leading edge and the low level at the trailing edge. The intermediate region is characterized by a positive charge $\varepsilon \partial v_s/\partial x_s > 0$. As the charge flows, the hatched part of the material loses its momentum.

$$\frac{\vec{j}}{\mu} = \sigma\rho\vec{v}_s. \tag{9}$$

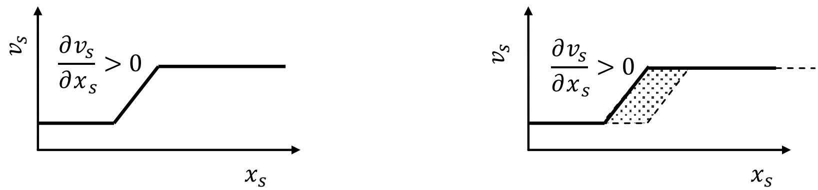

FIGURE 4. Positive charge flow causing momentum loss

Equation of Motion and Wave Characteristics

With the above interpretation of the source term and rearranging the terms, we may write Eq. (2) in the form of

$$\varepsilon\frac{\partial\vec{v}}{\partial t} = -\frac{1}{\mu}(\nabla\times\vec{\omega}) - \frac{\vec{j}}{\mu}. \tag{10}$$

We interpret the resultant equation as the equation of motion describing dynamics of a unit volume represented by mass ε. Here the left-hand side is the mass times the acceleration, the first and second term on the right-hand side represent the transverse and longitudinal external force acting on the unit volume respectively [6].

Eq. (10) essentially represents the wave nature of the displacement field [7]. Let us choose coordinates x_s and x_p where the former is perpendicular to the normal strain and the latter is parallel (Figure 5). The velocity components are v_s and v_p, and the displacement components are ξ_s and ξ_p. z is the third coordinate, perpendicular to both x_s and x_p. It is assumed that strain is purely normal in the direction of x_s. Thus, displacement components have only x_s dependence, i.e., $\partial/\partial x_p = \partial/\partial z = 0$. With this choice of coordinates and condition,

$$\omega_z = \frac{\partial\xi_s}{\partial x_p} - \frac{\partial\xi_p}{\partial x_s} = -\frac{\partial\xi_p}{\partial x_s}, \tag{11}$$

$$\omega_s = \frac{\partial\xi_p}{\partial z} - \frac{\partial\xi_z}{\partial x_p} = 0, \tag{12}$$

$$\omega_p = \frac{\partial\xi_z}{\partial x_s} - \frac{\partial\xi_s}{\partial z} = \frac{\partial\xi_z}{\partial x_s}, \tag{13}$$

$$(\nabla\times\vec{\omega})_p = \frac{\partial\omega_z}{\partial x_s} - \frac{\partial\omega_s}{\partial z} = -\frac{\partial^2\xi_p}{\partial^2 {x_s}^2}, \tag{14}$$

$$(\nabla\times\vec{\omega})_s = \frac{\partial\omega_p}{\partial z} - \frac{\partial\omega_z}{\partial p} = 0. \tag{15}$$

Utilizing Eq. (8), Eq. (10) becomes as follows for the s and p components

$$\varepsilon \frac{\partial^2 \xi_p}{\partial t^2} = \frac{1}{\mu} \frac{\partial^2 \xi_p}{\partial {x_s}^2} - W_{dp} \rho, \tag{16}$$

$$\varepsilon \frac{\partial^2 \xi_s}{\partial t^2} = -W_{ds} \rho, \tag{17}$$

where W_{dp} and W_{ds} are the p and s components of W_d, and $\vec{v} = \partial \vec{\xi} / \partial t$.

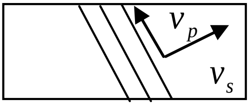

FIGURE 5. Components perpendicular (s) and parallel (p) to the normal strain representing a deformation charge

Elastic Limit

In this case, we know the solution to the wave equations (16) and (17). As a pure longitudinal wave, only Eq. (17) is meaningful, and its solution can be put in the form of Eq. (18). Also, the charge density can be expressed as Eq. (19).

$$\xi_s = \xi_{s0} e^{i(kx_s - \omega_{sw} t)}. \tag{18}$$

$$\rho = \varepsilon \frac{\partial v_s}{\partial x_s} = \varepsilon \frac{\partial}{\partial x_s} \frac{\partial \xi_s}{\partial t} = -\varepsilon \frac{\partial}{\partial x_s} \frac{\partial \xi_s}{\partial x_s} \frac{\omega_{sw}}{k} = -\varepsilon \frac{\partial^2 \xi_s}{\partial x_s^2} \frac{\omega_{sw}}{k}. \tag{19}$$

Here, ω_{sw} is the angular frequency of the ξ_s-wave. Substitution of Eqs. (18) and (19) into Eq. (17) leads to the following equation

$$\frac{\partial^2 \xi_s}{\partial t^2} = \frac{\partial^2 \xi_s}{\partial x_s^2} \frac{\omega_{sw}}{k} W_{ds}. \tag{20}$$

By interpreting the drift velocity W_{ds} as the phase velocity $v_{ph} = \omega_{sw}/k$, we obtain

$$\frac{\partial^2 \xi_s}{\partial t^2} = \frac{\partial^2 \xi_s}{\partial x_s^2} {v_{ph}}^2. \tag{21}$$

This is the well-known elastic wave equation.

In this case, in addition, the longitudinal resistance force takes the following form.

$$\begin{aligned} \frac{j_s}{\mu} =& W_d \rho = v_{ph} \varepsilon \frac{\partial v_s}{\partial x_s} = v_{ph} \varepsilon \frac{\partial}{\partial x_s} \frac{\partial \xi_s}{\partial t} \\ & = v_{ph} \varepsilon \frac{\partial}{\partial x_s} (-i\omega_{sw}) \xi_s = v_{ph} \varepsilon (-i\omega_{sw})(ik) \xi_s = \varepsilon {\omega_{sw}}^2 \xi_s. \end{aligned} \tag{22}$$

As expected, it is proportional to the displacement ξ_s.

Plastic Case

The plastic case can be argued as current j/μ being analogous to conduction current. In this case, ρ drifts without changing its shape (like a broken spring as mentioned above). Using Eq. (9) and noting that $v_p = \partial \xi_p / \partial t$, we can put Eqs. (16) and(17) in the following form

$$\frac{\partial^2 \xi_p}{\partial x_s^2} = \varepsilon\mu \frac{\partial^2 \xi_p}{\partial t^2} + \mu\sigma\rho \frac{\partial \xi_p}{\partial t}, \tag{23}$$

$$\varepsilon \frac{\partial \xi_s}{\partial t} = -\sigma\rho\xi_s. \tag{24}$$

The general solutions to these equations have the following form

$$\xi_p = \xi_{p0} e^{i(\tilde{k}x_s - \omega_{pw}t)}, \tag{25}$$

$$\xi_s = \xi_{s0} e^{-\frac{\sigma\rho}{\varepsilon}t}, \tag{26}$$

where the wave number $\tilde{k} = k + i\kappa$ is complex, and

$$k = \omega_{pw}\sqrt{\frac{\varepsilon\mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma\rho}{\varepsilon\omega_{pw}} \right)^2} + 1 \right]^{1/2}, \tag{27}$$

$$\kappa = \omega_{pw}\sqrt{\frac{\varepsilon\mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma\rho}{\varepsilon\omega_{pw}} \right)^2} - 1 \right]^{1/2}. \tag{28}$$

Eqs. (26) and (27) indicate that in the plastic case the longitudinal wave is not excited and the transverse wave decays [7]. Note that the greater the charge ρ, the faster the decay. This is consistent with the intuition that the higher the strain concentration, the closer to fracture.

SUPPORTING EXPERIMENTAL OBSERVATIONS

In this section, we present experimental results that support the preceding arguments.

Figure 6 shows optical interferometric patterns observed in a tensile experiment on a structural steel (SS400) specimen (a) along with the stress-strain characteristics (b) [8]. The interferometric fringes observed as dark stripes in this figure are contours of horizontal displacement of integral multiples of 0.45 μm. (In other words, the distance between neighboring stripes is 0.45 μm). In each image, there is a bright band whose intensity profile looks apparently different from the other part of the image (the band is circled on the leftmost image). These band patterns appear bright because a large number of fringes are concentrated within the width of the band. This has been confirmed in another similar experiment in which a second camera was used with a higher frame rate and greater magnification. Also shown to the right of the stress-strain characteristics

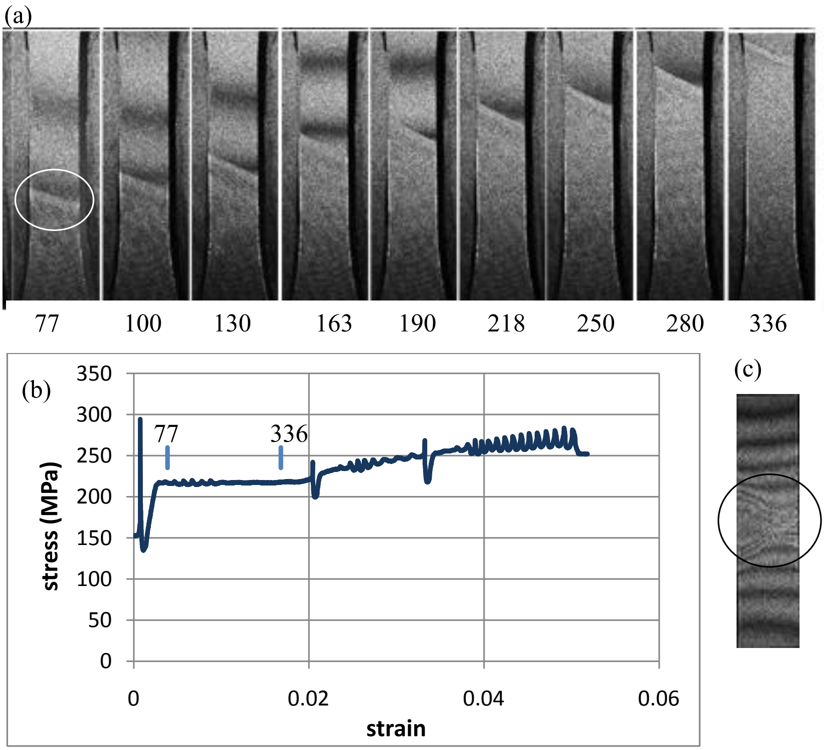

FIGURE 6. Optical interferometric pattern representing deformation charge drifting along the specimen (a). The corresponding stress-strain characteristics (b). Another pattern representing a charge observed in a different experiment (c)

in Figure 6(c) is a similar band pattern that shows the fine structure of concentrated fringes more explicitly. The fringes observed in this image also represent horizontal displacement. Notice that while the fringes above and below this band pattern are almost horizontal indicating that $\partial \xi_x / \partial x = 0$, the fine fringes observed within the band pattern indicate $\partial \xi_x / \partial x \neq 0$. Here x is the horizontal coordinate and ξ_x is the horizontal component of displacement. Apparently, the fine structure in the band represents a deformation charge defined by Eq. (5). The numbers 77 – 336 shown under Figure 6(a) are elapse times in s from a reference time prior to the yield point. The appearances of the band pattern in the first and last images are indicated in the stress-strain characteristics (labeled "77" and "336"). It is seen that the band pattern starts to appear near the lower end of the specimen and drifts at a constant velocity to the other end in the entire yield plateau of the stress-strain characteristics. This is perfectly consistent with the above-argued picture of the deformation charge representing the plastic regime.

In another experiment, it has been confirmed that the appearance of the interferometric bright band pattern coincides with acoustic emission (both in the timing and location on the specimen) [9]. This supports the above argument that drift of deformation charge is accompanied by some sort of "breaking a spring" like phenomenon.

Finally, Figure 7 presents another tensile experiment in which the temperature of the specimen (20 mm wide, 100 mm long, and 0.2 mm thick brass plate) is measured at the same time as the loading characteristics. This specimen has a notch at the center along the length of 100 mm. It is clearly seen that the temperature slightly decreases in the

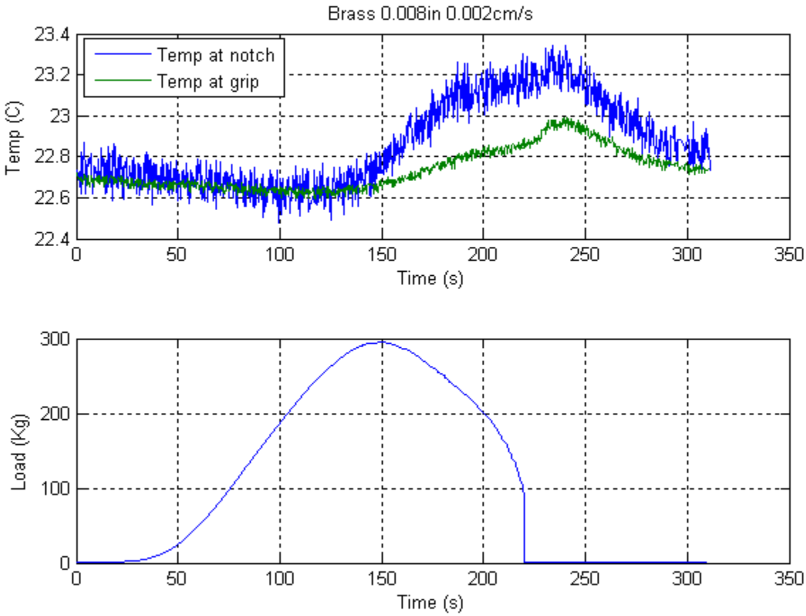

FIGURE 7. Temperature change (a) and loading characteristics (b) observed in a tensile experiment on a brass thin plate

region where the loading characteristics is linear, and remarkably rises when the load decreases having reached its highest value. The initial temperature decrease is due to the phenomenon known as the thermoelastic effect. The temperature rise accompanied by the load decrease is consistent with the above mentioned argument of energy dissipation. Notice that the temperatures measured near the notch and grip are more or less the same in the decrease, but the former is considerably higher than the latter in the rising phase, indicating that the heat source is where the deformation is highest.

SUMMARY

We have discussed the physical meaning of various formulae and expressions derived from the physical-mesomechanical field equations. It has been argued that the deformation charge has a crucial role in determining whether the deformation dynamics can be characterized as elastic or plastic. From the gauge theoretical viewpoint, this charge originates from the charge of symmetry. From a mechanical viewpoint, it can be understood as normal strain; when the strain is associated with the material's longitudinal resistive force proportional to the local displacement, the deformation is elastic; when it is associated with resistive force proportional to the local velocity, the deformation is plastic. The unrecoverable nature of plastic deformation has been explained as a flow of deformation charge, where the work done by the external force is stored in the elastic field but dissipated through the motion of the charge. Some experimental results that support these arguments have been presented.

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